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Stochastic Component Mode Synthesis

Mamadou T. Bah *, Prasanth B. Nair * *, Atul Bhaskar ♠ and Andy J. Keane ♠♠
University of Southampton, School of Engineering Sciences, Highfield, Southampton SO17 1BJ, U.K

Abstract

In this paper, a stochastic component mode synthesis method is developed for the dynamic analysis of large-scale structures with parameter uncertainties. The main idea is to represent each component displacement using a subspace spanned by a set of stochastic basis vectors in the same fashion as in stochastic reduced basis methods [1, 2]. These vectors represent however stochastic modes in contrast to the deterministic modes used in conventional substructuring methods [3]. The Craig-Bampton reduction procedure is used for illustration. A truncated set of stochastic fixed-free modes and a complete set of stochastic constraint modes are used to generate reduced matrices for each component. These are then coupled together through necessary compatibility constraints to form the global system matrices. The advantage of using stochastic component modes is that the Bubnov-Galerkin scheme can be applied for the computation of undetermined coefficients in the reduced approximation. Explicit expressions can be obtained for the responses in terms of the random parameters. Therefore the statistical moments of responses can be efficiently computed. The method is applied to a test case problem. Results obtained are compared with the traditional Craig-Bampton method, the first-order Taylor series and Monte Carlo Simulation benchmark results. We will refer to the proposed method as ROBUST or Reduced Order By Using Stochastic Techniques.

1. Introduction

Structural systems are often analyzed assuming that the modeled parameters are deterministic. However, parameter uncertainties exist in these systems and can make the prediction of the responses, based on nominal design invalid. Also, analyzing a system with uncertain parameters means that all possible physical realizations of these parameters must be numerically simulated. This allows for accurate statistical information and hence a better design. Stochastic methods of analysis are then required i.e., parameter uncertainties should be considered as random variables.

* Research Fellow, Member AIAA.

* * Senior Research Fellow, Member AIAA.

♠ Senior Lecturer

♠♠ Professor of Computational Engineering

In order to address this concern, new methods of analysis are to be developed. The review paper by Manohar and Ibrahim [4] of the state of the art in stochastic structural dynamics reveals that the perturbation methods appear to be the most popular technique for approximating the statistics of the eigenparameters and forced responses of uncertain systems.

The popularity of these methods can be primarily attributed to ease of implementation and computational efficiency. Analytical expressions for the mean and standard deviation of responses can be derived as functions of random inputs. However, the perturbation methods give reasonable quality results for the statistical moments only when the coefficients of variation of the random parameters are small. Further, the quality of the approximations often tends to deteriorate significantly when the frequency of excitation increases.

Reliability methods have also been developed for determining the statistical structural response characteristics. Most of these methods focus on the estimation of the probability of failure associated with a performance function and a most probable point search. A comprehensive review paper by Rackwitz [5] summarizes the theory and different methods of structural reliability, particularly the first and second order reliability methods. Numerical analyses in structural dynamics are usually conducted by using the finite element method. It is generally assumed that each term of the random coefficient matrices (mass and stiffness) can be expressed as a random polynomial in terms of a finite number of random variables. This assumption is not limiting. The rapid advances made in the area of stochastic finite element analysis make it possible to readily arrive at such representations of parameter uncertainty using techniques as the Karhunen-Loeve expansion scheme, see for example, Ghanem and Spanos [6]. The reader is also referred to a recent state-of-the-art report on computational stochastic mechanics by Schueller [7] for more details concerning parametric models of random uncertainties in finite element models.

However, in order to fully exploit such representations [5, 6, 7], efficient numerical schemes are required for statistical analysis. For large-scale analysis of structural systems, more adequate techniques must be derived. The research conducted here is motivated by the need of accurate and efficient reduced-order modelling that allows reliable statistical assessments of the effects of parameter uncertainties. Such methods are increasingly needed for example in the design of turbomachinery bladed disks [8, 9].

One class of reduced-order modelling techniques is the well-known Component Mode Synthesis (CMS), sometimes refered to as substructuring. CMS is widely used to study large-scale coupled component structural systems. These methods consist essentially of a separate determination of the modes of each component followed by synthesis of the entire system modes. Reduced-order models of substructures are then obtained using truncated sets of component modes and the original structure can be synthesized through necessary compatibility constraints. Different boundary conditions can be used to determine the component modes; the most commonly used being fixed and free interface modes. Since the finite element method is frequently used as the basis for modeling structural components, these methods are very convenient when the number of generalized coordinates and the computational cost are to be reduced for dynamic analysis of complex systems. But, a trade-off must be made between efficiency and accuracy.

The first paper to discuss CMS appears to be published by Hurty [10] in 1965. He presented the foundations of an approach based on fixed-interface modes and constraint modes. In 1968, Craig and Bampton [11] modified Hurty's method and the concept of CMS became readily usable in industry (here refered to as the CB method). They treated all interface degrees-of-freedom together in contrast to Hurty's method whereas the interface degrees-of-freedom are separated into rigid-body modes and static redundant constraint modes.

The paper by Benfield and Hruda [12] also presented a method to determine the vibration modes of a complex structural system by retaining only component vibration modes. In this method, component interface modes are not used. In [13] Rubin presented a free-interface method with residual flexibility and inertia where the physical interface degree-of-freedom are retained to solve for an arbitrary force.

In contrast to Rubin's method, Craig and Chang proposed in [14] a free-interface method with residual flexibility. Here, all the interface degree-of-freedom are free and a complete set of residual attachment modes are used to supplement the normal modes. A recent paper by Craig [1] reviewed the different procedures used for the formulation of component modes for substructures and the coupling of substructure models to form reduced-order models of the original system. These models have been, in general, successful in the prediction of response of complex coupled structural systems: bladed disks represent an excellent illustrative application [15, 16].

In this paper, we will focus on the Craig-Bampton (CB) method due to its stability [17] although it had been found that it presents a slow modal convergence compared to the free-interface CMS methods. The CB method is a straight-forward technique and suitable for the reduction of the size of matrices of large scale finite-element structural components. It essentially requires in addition to the truncated set of fixed-free modes ($\mathbf{\Phi}$ formed by s selected modes for each component) a complete set of static elastic constraint modes ($\mathbf{\Psi}$). The additional and unavoidable set of constraint modes allows the reduced model to span the space of possible motions of the entire coupled system. Note that the fundamental step for reducing the original degrees-of-freedom consists only of the truncation of each component modes at some level assuming that the higher modes will not have an effect on the combined system modes.

This paper combines the CB reduction procedure with probabilistic methods for analysis of large-scale coupled structural systems with parameter uncertainties. A reduced order model of each component is first obtained by using a truncated set of stochastic component modes in the spirit of Stochastic Reduced Basis Methods [1, 2]. The component modes are obtained by solving algebraic random eigenvalue problems with randomly parametrized coefficient matrices. They are assumed to be a linear combination of a finite number of random variables.

Subsequently, the Bubnov-Galerkin scheme is employed to arrive at a reduced-order deterministic eigenproblem to compute the undetermined coefficients in the approximation. This enables the statistics of the original random eigenvalue problem and the frequency response to be efficiently computed. Some numerical studies are presented on a spring-mass test case to assess the accuracy of the proposed method. They are compared with benchmark results computed using the Monte Carlo Simulation, the traditional CB method and a first order Taylor series (TS1) applied on the original system.

2. Craig-Bampton Method

To apply this method, the original system is first divided into p components. For the j-th component, the mass and stiffness matrices and the physical displacement vector are written in a partitioned form as follows

$$\mathbf{M}^j = \begin{bmatrix} \mathbf{M}_{ii} & \mathbf{M}_{ib} \\ \mathbf{M}_{bi} & \mathbf{M}_{bb} \end{bmatrix}, \ \mathbf{K}^j = \begin{bmatrix} \mathbf{K}_{ii} & \mathbf{K}_{ib} \\ \mathbf{K}_{bi} & \mathbf{K}_{bb} \end{bmatrix}, \ \mathbf{x}^j = \begin{Bmatrix} \mathbf{x}_i^j \\ \mathbf{x}_b^j \end{Bmatrix}, \quad (1)$$

where i and b denote the internal and boundary degree-of-freedom respectively. The physical displacement vector of the

j-th component is transformed using the so-called "CB transformation matrix", which is expressed in terms of two sets of component modes

$$\mathbf{U}_{\mathrm{CB}}^{j} = \begin{bmatrix} \mathbf{\Phi} & \mathbf{\Psi} \\ \mathbf{0} & \mathbf{I}_{bb} \end{bmatrix}, \tag{2}$$

where $\mathbf{\Phi}$ is the interior partition of the fixed-interface modal matrix. In other words the mode shapes that form the matrix $\mathbf{\Phi}$ are obtained by solving the eigenvalue problem defined by the matrices $\mathbf{M}_{ii}$ and $\mathbf{K}_{ii}$ and selecting the s vectors of interest, i.e.,

$$\mathbf{K}_{ii}\boldsymbol{\varphi}_{l} = \lambda_{l}\mathbf{M}_{ii}\boldsymbol{\varphi}_{l} \tag{3}$$

and

$$\mathbf{\Phi} = \left[\boldsymbol{\varphi}_{1}, \boldsymbol{\varphi}_{2}, \ldots, \boldsymbol{\varphi}_{s}\right]. \tag{4}$$

The corresponding eigenvalues λ_l, $l = 1,\ldots,s$ are also collected into a diagonal matrix $\mathbf{\Lambda}^{j}$. $\mathbf{\Psi}$ is the constraint-mode matrix, obtained by solving the first block of equations from the following static problem

$$\begin{bmatrix} \mathbf{K}_{ii} & \mathbf{K}_{ib} \\ \mathbf{K}_{bi} & \mathbf{K}_{bb} \end{bmatrix} \begin{bmatrix} \mathbf{\Psi} \\ \mathbf{I}_{bb} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{R} \end{bmatrix}, \tag{5}$$

where $\mathbf{R}$ is the reaction force matrix due to the imposed unit displacements and

$$\mathbf{\Psi} = -\mathbf{K}_{ii}^{-1}\mathbf{K}_{ib}. \tag{6}$$

Note that the constraint modes in $\mathbf{\Psi}$ are obtained by successive unit deflections of each interface degree-of-freedom while all others are held fixed. The matrices $\mathbf{0}$ and $\mathbf{I}_{bb}$ in equation (2) or (5) are null and identity matrices respectively. Using equation (2), the physical displacement vector of the j-th component can be now transformed to a reduced vector of generalized and physical coordinates, i.e.,

$$\mathbf{x}^{j} = \begin{Bmatrix} \mathbf{x}_{i}^{j} \\ \mathbf{x}_{b}^{j} \end{Bmatrix} = \mathbf{U}_{\mathrm{CB}}^{j} \begin{Bmatrix} \mathbf{q}^{j} \\ \mathbf{x}_{b}^{j} \end{Bmatrix} = \begin{Bmatrix} \mathbf{\Phi}\,\mathbf{q}^{j} + \mathbf{\Psi}\,\mathbf{x}_{b}^{j} \\ \mathbf{x}_{b}^{j} \end{Bmatrix}, \tag{7}$$

where $\mathbf{q}^{j}$ is the reduced set of generalized coordinates for the j-th component. Combining equations (1) and (2), the CB reduced mass and stiffness matrices of the j-th component are given by

$$\mathbf{M}_{\mathrm{CB}}^{j} = \mathbf{U}_{\mathrm{CB}}^{j\;\mathrm{T}}\mathbf{M}^{j}\mathbf{U}_{\mathrm{CB}}^{j} = \begin{bmatrix} \overbrace{\mathbf{\Phi}^{\mathrm{T}}\mathbf{M}_{ii}\mathbf{\Phi}}^{\mathbf{I}} & \mathbf{\Phi}^{\mathrm{T}}\left(\mathbf{M}_{ii}\mathbf{\Psi}+\mathbf{M}_{ib}\right) \\ \mathrm{sym} & \mathbf{\Psi}^{\mathrm{T}}\left(\mathbf{M}_{ii}\mathbf{\Psi}+\mathbf{M}_{ib}\right)+\mathbf{M}_{ib}^{\mathrm{T}}\mathbf{\Psi}+\mathbf{M}_{bb} \end{bmatrix} \tag{8}$$

and

$$\mathbf{K}_{\mathrm{CB}}^{j} = \mathbf{U}_{\mathrm{CB}}^{j\;\mathrm{T}}\mathbf{K}^{j}\mathbf{U}_{\mathrm{CB}}^{j} = \begin{bmatrix} \overbrace{\mathbf{\Phi}^{\mathrm{T}}\mathbf{K}_{ii}\mathbf{\Phi}}^{\mathbf{\Lambda}^{j}} & \mathbf{0} \\ \mathbf{0} & \mathbf{K}_{bb}-\mathbf{K}_{ib}^{\mathrm{T}}\mathbf{K}_{ii}^{-1}\mathbf{K}_{ib} \end{bmatrix} \tag{9}$$

Next a transformation matrix $\mathbf{TM}_j$ is defined between the reduced vector of generalized coordinates of each j-th component and the reduced vector of generalized coordinates of the entire system. If we consider for example a two-component system, the following expressions can be written

$$\begin{Bmatrix} \mathrm{q}^{1} \\ \mathrm{x}_{b}^{1} \end{Bmatrix} = \mathbf{TM}_{1} \begin{Bmatrix} \mathrm{q}^{1} \\ \mathrm{q}^{2} \\ \mathrm{x}_{b}^{1} \\ \mathrm{x}_{b}^{2} \end{Bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \mathrm{q}^{1} \\ \mathrm{q}^{2} \\ \mathrm{x}_{b}^{1} \\ \mathrm{x}_{b}^{2} \end{Bmatrix} \quad \text{and}$$

$$\begin{Bmatrix} \mathrm{q}^{2} \\ \mathrm{x}_{b}^{2} \end{Bmatrix} = \mathbf{TM}_{2} \begin{Bmatrix} \mathrm{q}^{1} \\ \mathrm{q}^{2} \\ \mathrm{x}_{b}^{1} \\ \mathrm{x}_{b}^{2} \end{Bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{Bmatrix} \mathrm{q}^{1} \\ \mathrm{q}^{2} \\ \mathrm{x}_{b}^{1} \\ \mathrm{x}_{b}^{2} \end{Bmatrix}.$$

Finally the interface compatibility condition is used to couple the component matrices together. For a two-component system and if the interface compatibility is expressed as $\mathrm{x}_{b}^{1} = \mathrm{x}_{b}^{2} = \mathrm{x}_{b}$, then the reduced vector of the system can be transformed as follows

$$\begin{Bmatrix} q^1 \\ q^2 \\ x_b^1 \\ x_b^2 \end{Bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{Bmatrix} q^1 \\ q^2 \\ x_b \end{Bmatrix} = \mathbf{T}\mathbf{M}_{\mathrm{CB}}\mathbf{q}_{\mathrm{CB}}, \qquad (10)$$

Then the final synthesized CB mass and stiffness matrices of the coupled two-component system are given as

$$\mathbf{M}_{\mathrm{CB}} = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{M}^1_{\mathrm{CB}_{ib}} \\ \mathbf{0} & \mathbf{I} & \mathbf{M}^2_{\mathrm{CB}_{ib}} \\ \mathbf{M}^1_{\mathrm{CB}_{bi}} & \mathbf{M}^1_{\mathrm{CB}_{bi}} & \mathbf{M}^1_{\mathrm{CB}bb} + \mathbf{M}^1_{\mathrm{CB}_{bb}} \end{bmatrix} \qquad (11)$$

and

$$\mathbf{K}_{\mathrm{CB}} = \begin{bmatrix} \mathbf{\Lambda}^1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{\Lambda}^2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{K}^1_{\mathrm{CB}_{bb}} + \mathbf{K}^2_{\mathrm{CB}_{bb}} \end{bmatrix}. \qquad (12)$$

Using equations (11) and (12), the complete reduced model of the two-component system can be set up as follows

$$\mathbf{M}_{\mathrm{CB}}\ddot{\mathbf{q}}_{\mathrm{CB}} + \mathbf{C}_{\mathrm{CB}}\dot{\mathbf{q}}_{\mathrm{CB}} + \mathbf{K}_{\mathrm{CB}}\mathbf{q}_{\mathrm{CB}} = \mathbf{f}_{\mathrm{CB}}, \qquad (13)$$

where $\mathbf{C}_{\mathrm{CB}}$ is the reduced modal damping matrix, whose elements are given by $2\xi\omega_{\mathrm{CB}_i}$; ξ is the damping coefficient and ω_{CB_i} is the i-th natural frequency obtained by solving the reduced eigenvalue problem defined by $\mathbf{M}_{\mathrm{CB}}$ and $\mathbf{K}_{\mathrm{CB}}$. $\mathbf{f}_{\mathrm{CB}}$ is the reduced modal forced vector given by $\left\{\mathbf{\Phi}^{1^{\mathrm{T}}}\mathbf{f}^1_{ii},\ \mathbf{\Phi}^{2^{\mathrm{T}}}\mathbf{f}^2_{ii},\ \mathbf{\Psi}^{1^{\mathrm{T}}}\mathbf{f}^1_{ii} + \mathbf{\Psi}^{2^{\mathrm{T}}}\mathbf{f}^2_{ii} + \mathbf{f}_b\right\}^{\mathrm{T}}$, where $\mathbf{f}^j_{ii}$ is the force vector on the internal boundary degrees-of-freedom of the j-th component and $\mathbf{f}_b$ is the force vector on the boundary degrees-of-freedom for the two-component system. $\mathbf{\Phi}^j$ and $\mathbf{\Psi}^j$ are the fixed-interface modal matrix (see Eqns. (3) and (4)) and constraint-mode matrix (see Eqn. (6)) of the j-th component respectively.

By examining equations (11) and (12), it can be seen that partitioned matrices can be obtained in a quite straight-forward manner and they appear in special and compact forms. However, there is one disadvantage when using the CB method. In the formulation, all the interface modes are retained in the component's reduced vector. These 'additional modes' can effectively lead to an unsatisfactorily large number of degree-of-freedom in the reduced model. This is a major limitation of the CB method i.e., the use of all interface modes in the reduced model. Some researchers attempted to circumvent this difficulty.

When analyzing structures such as mistuned bladed disk systems, the blades and disk are usually treated as distinct substructures. Castanier *et al.* presented a component-mode approach [18], in which the motion of the blade is described as the finite element mode shapes of a blade fixed at the disk-blade interface and the summation of disk mode motions at the disk-blade interface. In other words, the blade motion is a sum of disk induced motion and the motion of a cantilevered blade. Therefore, no constraint modes are needed.

Bladh *et al.* [16] proposed a secondary modal analysis on a CB-based reduced model of bladed disks (where disk and blades are substructures). In other words, only modes of the intermediate model that fall within the frequency band of interest are retained. Another difficulty when using the CB method is the truncation of fixed-interface modes. Admire and Tinker [19], proposed to use the Residual Flexibility (RF) method for each component. RF is used to partially account for the higher modes by determining their flexibility.

3. Uncertainty and CMS

One very important feature of CMS methods is that the modes used for generating the reduced-order models are deterministic in nature. Parameter uncertainty or randomness(or mistuning in the case of bladed disks) is therefore introduced in a convenient manner once the coupled system reduced matrices are obtained. This involves, in practice, only examining the reduced model partitioned matrices. In other words, uncertainty is introduced in the modal domain rather than the physical domain, see equations (11) and (12).

A driving factor behind this is perhaps that most of the elements of the component partitioned matrices (only stiffness matrix in practice) are readily available from modal test (experimental modes). Also it is frequently assumed that uncertainty affects only the component internal degree-of-freedom but not the interface modes. In other words only $\mathbf{\Lambda}_j$, $j = 1,\ldots,p$ are perturbed i.e. a certain distribution pattern is assumed or selected from a large population of components.

The RF used in [19] for example leads to an approximately unity mass and all the elements of the system matrix are available from modal test. Once a reduced model is obtained, a complete incorporation of the probabilistic data is possible. Brown and Ferri [20] for instance, combined the RF method with reliability methods to efficiently obtain the statistical characteristics of the desired response variable. Note that in this approach, the component eigenvalues, eigenvectors and residual flexibility are used as input random variables.

A recent paper by Mace and Shorter [21] described a local modal/perturbational method for estimating the statistics of the forced response of a system with uncertain properties. In their approach, first they found the global modes of vibration in terms of the local modes using a fixed-interface component mode synthesis. Assuming that the modal properties of the subsystems are assumed to be random, they applied a perturbation technique that relates small changes in the subsystem modal properties to changes in the global properties. Therefore, solving the global eigenvalue problem is avoided. Finally MCS is performed on the perturbation results.

Yang and Griffin [22] proposed to develop reduced-order models for bladed disks, in which the modes of the mistuned system are represented in terms of a subset of nominal system modes. A statistical analysis will therefore involve the simulation of large number of mistuned systems. However, it was concluded in [22] that the nominal modes used for reduction do not have to be tuned ones. In fact, the modes of a completely mistuned system could be used as nominal and their approach could be applied to determine the effects of additional random mistuning. Also, since reduced-order models allow for an efficient modeling of the effects of real physical variations, it is an important issue of translating the physical variations into these models.

From the above discussion, it can be concluded that solving the dynamics of large-scale structural systems with uncertain parameters will involve a combination of substructuring and efficient probabilistic numerical schemes. Clearly, computing exact solutions is a non-trivial task. To overcome this, stochastic component modes are used here to arrive at accurate reduced order models. Given the statistics of the randomness, the statistics of the system response are computed in a more efficient manner than in the traditional CB method while preserving the same size of the reduced model. This approach is presented in the next section.

4. Stochastic CB Method

On the lines of the classical CB method, we begin by finding the first set of modes by reconsidering equation (3) i.e., $\mathbf{K}_{ii}\boldsymbol{\varphi}_l = \lambda_l \mathbf{M}_{ii}\boldsymbol{\varphi}_l$, where $\mathbf{K}_{ii}$ is now a random matrix. $\mathbf{K}_{ii} = \mathbf{K}_{ii,\mathrm{o}} + \sum_{k=1}^{n} \frac{\partial \mathbf{K}_{ii}}{\partial \theta_k}\delta\theta_k$. $\mathbf{K}_{ii,\mathrm{o}}$ denotes the stiffness of the unperturbed component system and $\theta_k, k = 1,\ldots,n$ denotes the k-th random variable in the j-th component. For simplicity, we assume that parameter uncertainties affect only the stiffness matrices. For each eigenvector, we can write the following

$$\boldsymbol{\varphi}_l = \boldsymbol{\varphi}_{l,\mathrm{o}} + \delta\boldsymbol{\varphi}_l = \boldsymbol{\varphi}_{l,\mathrm{o}} + \sum_{k=1}^{n} \frac{\partial \boldsymbol{\varphi}_l}{\partial \theta_k}\delta\theta_k, \tag{14}$$

where n is the total number of random variables in the j-th component and $\frac{\partial \boldsymbol{\varphi}_l}{\partial \theta_k}$ is the sensitivity of the j-th eigenvector.

Therefore the matrix of normal fixed-free modes $\boldsymbol{\Phi}$ is now written as

$$\boldsymbol{\Phi} = \boldsymbol{\Phi}_\mathrm{o} + \delta\boldsymbol{\Phi} = \boldsymbol{\Phi}_\mathrm{o} + \sum_{k=1}^{n} \frac{\partial \boldsymbol{\Phi}}{\partial \theta_k}\delta\theta_k. \tag{15}$$

Also the matrix of constraint modes that was given by solving equation (6) i.e. $\boldsymbol{\Psi} = -\mathbf{K}_{ii}^{-1}\mathbf{K}_{ib}$ is now given by

$$\boldsymbol{\Psi} = \boldsymbol{\Psi}_\mathrm{o} + \delta\boldsymbol{\Psi} = \boldsymbol{\Psi}_\mathrm{o} + \sum_{k=1}^{n} \frac{\partial \boldsymbol{\Psi}}{\partial \theta_k}\delta\theta_k. \tag{16}$$

Note that $\mathbf{K}_{ib} = \mathbf{K}_{ib,\mathrm{o}} + \sum_{k=1}^{n} \frac{\partial \mathbf{K}_{ib}}{\partial \theta_k}\delta\theta_k$ and $\frac{\partial \boldsymbol{\Psi}}{\partial \theta_k}$ is the sensitivity of the random matrix of constraint modes. $\frac{\partial \boldsymbol{\Psi}}{\partial \theta_k}$ is obtained by taking the first derivative of both sides of equation (6), then we obtain

$$\frac{\partial \boldsymbol{\Psi}}{\partial \theta_k} = -\mathbf{K}_{ii,\mathrm{o}}^{-1}\left(\frac{\partial \mathbf{K}_{ib}}{\partial \theta_k} + \frac{\partial \mathbf{K}_{ii}}{\partial \theta_k}\boldsymbol{\Psi}_\mathrm{o}\right). \tag{17}$$

The transformation matrix between the physical displacements of the j-th component and the modal coordinates is now

$$\mathbf{U}_\mathrm{S}^j = \begin{bmatrix} \boldsymbol{\Phi}_\mathrm{o} + \delta\boldsymbol{\Phi} & \boldsymbol{\Psi}_\mathrm{o} + \delta\boldsymbol{\Psi} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}. \tag{18}$$

4.1 Equations of motion

Using equation (18), the physical displacement vector of the j-th component can be now transformed to a reduced vector of generalized and physical coordinates as in equation (7). Similar to the classical CB method, the complete reduced model can be set up as in equation (13). The generalized coordinates of the j-th component are first transformed into the reduced model's vector of generalized coordinates using the transformation matrices $\mathbf{TM}_j$. Then, the compatibility condition is applied to synthesize the entire coupled model using $\mathbf{TM}_{CB}$. The following equations are obtained

$$\mathbf{M}_S\ddot{\mathbf{q}}_S + \mathbf{C}_S\dot{\mathbf{q}}_S + \mathbf{K}_S\mathbf{q}_S = \mathbf{f}_S, \qquad (19)$$

where $\mathbf{M}_S$, $\mathbf{C}_S$, $\mathbf{K}_S$ and $\mathbf{F}_S$ and $\mathbf{q}_S$ are the random reduced mass, damping, stiffness matrices, force vector and displacement vector respectively. Note that in equation (19), all the possible physical variations are translated in the reduced model, leading to a better capture of their effects. When the reduced eigenvalue problem defined by $\mathbf{K}_S$ and $\mathbf{M}_S$ is considered, the statistics of eigenvalues can be computed by using simulation schemes i.e. by sampling the reduced representation (equation 19) and repeatedly solving for each realization of the random parameters.

For the frequency response problem, a two-component system is considered for the sake of clarity. All the equations derived here can be readily extended to multi-component systems. Note that a compact expression can be written for the global transformation matrix (between the physical coordinates of the original coupled and the modal coordinates of the reduced system). For convenience, the original system mass and stiffness matrices are partitioned as follows

$$\begin{bmatrix} \mathbf{M}^1_{ii} & \mathbf{M}^1_{ib} & \mathbf{0} \\ \mathbf{M}^1_{bi} & \mathbf{M}^1_{bb}+\mathbf{M}^2_{bb} & \mathbf{M}^2_{bi} \\ \mathbf{0} & \mathbf{M}^2_{ib} & \mathbf{M}^2_{ii} \end{bmatrix},$$

$$\begin{bmatrix} \mathbf{K}^1_{ii} & \mathbf{K}^1_{ib} & \mathbf{0} \\ \mathbf{K}^1_{bi} & \mathbf{K}^1_{bb}+\mathbf{K}^2_{bb} & \mathbf{K}^2_{bi} \\ \mathbf{0} & \mathbf{K}^2_{ib} & \mathbf{K}^2_{ii} \end{bmatrix}, \qquad (20)$$

and the global transformation matrix has the form

$$\mathbf{TM}_S = \begin{bmatrix} \mathbf{\Phi}^1 & \mathbf{0} & \mathbf{\Psi}^1 \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \\ \mathbf{0} & \mathbf{\Phi}^2 & \mathbf{\Psi}^2 \end{bmatrix} \qquad (21)$$

where $\mathbf{\Phi}^j$ and $\mathbf{\Psi}^j$ are the matrices of m mode shapes and constraint modes for the j-th component, given by equations (15) and (16), respectively. $\mathbf{TM}_S$ is a matrix of stochastic basis vectors in contrast to the CB method, where deterministic modes are used. $\mathbf{TM}_S$ is used to approximate the random displacement vector of the original system. In other words, the vectors in $\mathbf{TM}_S$ span a subspace in which an approximation is being made. This approximation is expressed as

$$\hat{\mathbf{q}}(\boldsymbol{\theta}) = \mathbf{TM}_S\,\boldsymbol{\xi}, \qquad (22)$$

where $\boldsymbol{\xi}$ is the set of undetermined coefficients in the reduced basis representation and $\boldsymbol{\theta} = \{\theta_1,\ldots,\theta_N\}^T$ is the set of N random variables. For the computation of the undetermined coefficients, two variants of the Bubnov-Galerkin (BG) scheme [1, 2]. For the sake of simplicity, we will present here only the zero-order BG scheme although the exact BG scheme and MCS schemes can be applied (equation (22)) for an efficient computation of the response statistics. The stochastic reduced basis representation is sampled with random function models for the undetermined coefficients.

4.2 Zero-order BG scheme

Assuming that the dynamic stiffness matrix of the original system is $\mathbf{A}(\boldsymbol{\theta})$ and $\mathbf{f}$ is the external excitation force vector, a stochastic residual error vector is defined as

$$\mathbf{r}(\boldsymbol{\theta}) = \mathbf{A}(\boldsymbol{\theta})\,\mathbf{TM}_S\,\boldsymbol{\xi} - \mathbf{f}. \qquad (23)$$

$\boldsymbol{\xi}$ is determined by enforcing that $\mathbf{r}(\boldsymbol{\theta})$ is orthogonal to $\mathbf{TM}_S$ in an average sense. By considering the inner product of two random vectors in the Hilbert space of random variables, the following condition results

$$\left\langle \mathbf{TM}_S^*\ \mathbf{r}(\boldsymbol{\theta}) \right\rangle, \qquad (24)$$

where * denotes the complex conjugate transpose and $\langle\cdot\rangle$ denotes the complex conjugate transpose. By combining equations (23) and (24), it is obtained

$$\left\langle \mathbf{TM}_S^* \mathbf{A}(\theta) \mathbf{TM}_S \xi - \mathbf{TM}_S^* \mathbf{f} \right\rangle = 0 . \quad (25)$$

Equation (25) can be written in a compact form as

$$\mathbf{A}_{\text{S-BG}_0} \xi = \mathbf{f}_{\text{S-BG}_0} , \quad (26)$$

$\mathbf{A}_{\text{S-BG}_0} = \left\langle \mathbf{TM}_S^* \mathbf{A}(\theta) \mathbf{TM}_S \right\rangle$ and $\mathbf{f}_{\text{S-BG}_0} = \left\langle \mathbf{TM}_S^* \mathbf{f} \right\rangle$ denote the 3×3 reduced dynamic stiffness matrix and the 3×1 force vector, respectively. For the reduced two-component model, the partitioned mass and stiffness matrices are shown below

$$\mathbf{M}_{\text{S-BG}_0} = \begin{bmatrix} \mathbf{M}^1_{\text{S-BG}_0}(1,1) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}^2_{\text{S-BG}_0}(1,1) & \mathbf{M}^2_{\text{S-BG}_0}(1,2) \\ \mathbf{0} & \mathbf{M}^1_{\text{S-BG}_0}(1,2) & \mathbf{M}^1_{\text{S-BG}_0}(2,2) + \mathbf{M}^2_{\text{S-BG}_0}(2,2) \end{bmatrix} \quad (27)$$

and

$$\mathbf{K}_{\text{S-BG}_0} = \begin{bmatrix} \mathbf{K}^1_{\text{S-BG}_0}(1,1) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{K}^2_{\text{S-BG}_0}(1,1) & \mathbf{K}^2_{\text{S-BG}_0}(1,2) \\ \mathbf{0} & \mathbf{K}^1_{\text{S-BG}_0}(1,2) & \mathbf{K}^1_{\text{S-BG}_0}(2,2) + \mathbf{K}^2_{\text{S-BG}_0}(2,2) \end{bmatrix}, \quad (28)$$

where $\mathbf{M}^j_{\text{S-BG}_0} = \left\langle \mathbf{U}_S^{j\text{T}} \mathbf{M}^j \mathbf{U}_S^j \right\rangle$, $\mathbf{K}^j_{\text{S-BG}_0} = \left\langle \mathbf{U}_S^{j\text{T}} \mathbf{K}^j \mathbf{U}_S^j \right\rangle$.

Note that $\mathbf{M}^j$ and $\mathbf{K}^j$ are the mass and stiffness matrices of the *j*-th component. The explicit expressions of the elements of matrices $\mathbf{M}_{\text{S-BG}_0}$ and $\mathbf{K}_{\text{S-BG}_0}$ are given in the Appendix. Once the set of coefficients are computed by solving the reduced-order problem (26), the mean and covariance of the system response can be evaluated at each frequency point.

5. Example and Results

For simplicity, the stochastic component mode synthesis approach is applied to a spring-mass problem. It was used in [20] and consists of two substructures or components. Each of them has four degree-of-freedom, see Figure 1. Note that this example is simple but allows gaining an understanding on the performance of the method. For the full model represented in Figure 1, the mass matrix is a diagonal matrix and the stiffness matrix is given by

$$\begin{bmatrix} k_1+k_2 & -k_2 & & & & & \\ -k_2 & k_2+k_3 & -k_3 & & & \mathbf{0} & \\ & -k_3 & k_3+k_4 & -k_4 & & & \\ & & -k_4 & k_4+k_5 & -k_5 & & \\ & & & -k_5 & k_5+k_6 & -k_6 & \\ & \mathbf{0} & & & -k_6 & k_6+k_7 & -k_7 \\ & & & & & -k_7 & k_7 \end{bmatrix}.$$

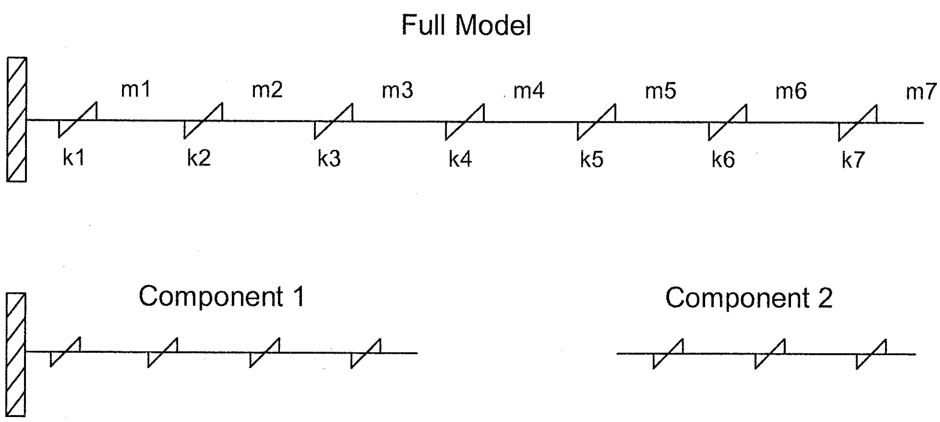

Figure 1: spring-mass system

For the components 1 and 2, the mass and stiffness matrices are given by

$$\begin{bmatrix} m_1 & & & \\ & m_2 & & \\ & & m_3 & \\ & & & m_4/2 \end{bmatrix}, \begin{bmatrix} k_1+k_2 & -k_2 & & \mathbf{0} \\ -k_2 & k_2+k_3 & -k_3 & \\ & -k_3 & k_3+k_4 & -k_4 \\ \mathbf{0} & & -k_4 & k_4 \end{bmatrix}$$

and

$$\begin{bmatrix} m_4/2 & & & \\ & m_5 & & \\ & & m_6 & \\ & & & m_7 \end{bmatrix}, \begin{bmatrix} k_5 & -k_5 & & \mathbf{0} \\ -k_5 & k_5+k_6 & -k_6 & \\ & -k_6 & k_6+k_7 & -k_7 \\ \mathbf{0} & & -k_7 & k_7 \end{bmatrix}$$

respectively.

For the sake of probabilistic generality, all the springs have random stiffnesses. Each spring in the system was assigned a normal distribution with a mean of 200 and standard deviation of $\sigma = 5\%$ i.e. $k_i = \bar{k}(1+\theta_i)$, where $\bar{k}$ is the mean value and θ_i is a random number. Note that results obtained here i.e. the proposed approach referred to as ROBUST for the statistics of eigenvalues and frequency responses are obtained by using simulation schemes. Results are compared with the traditional Craig-Bampton (CB) method, the first-order Taylor series (TS1) and Monte Carlo Simulation (benchmark results), the last two both being applied on the full model.

Also, note that in this work, for the eigensensitivity analysis, we use the Fox's method [23, 24] although there exist many other methods [25-28]. Fox's method is chosen because of its efficiency for large-scale problems. The desired eigenvector derivative is assumed to be a superposition of the mode shapes computed from the eigenvalue problem of the tuned system. When using CB or ROBUST, two of the four mode shapes and one constraint mode for each component are kept for analysis, leading to reduced models of five degree-of-freedom. One thousand samples are generated for each random stiffness. Figure 2 displays the mean of natural frequencies. The solid line represents the exact MCS results, the plusses represent TS1, the circles represent CB and the stars represent ROBUST. Both CB and ROBUST accurately predict the means of the first four natural frequencies compared to TS1.

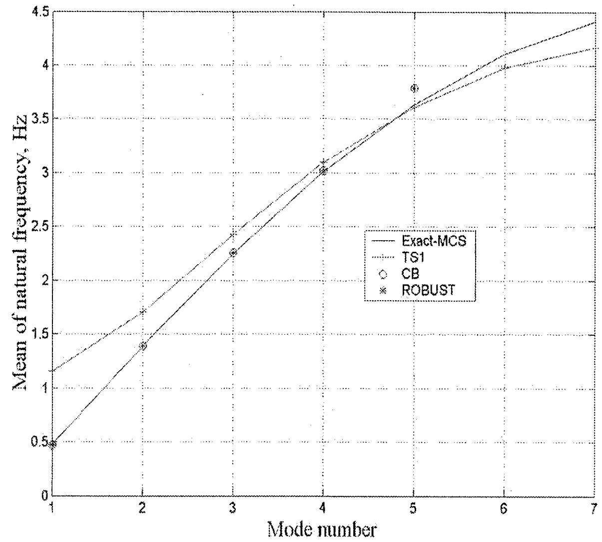

Figure 2: Mean of natural frequency.

The same story is seen in Figure 3 for the standard deviation of natural frequencies. Once again ROBUST performs very well and accurately predicts the first four natural frequencies.

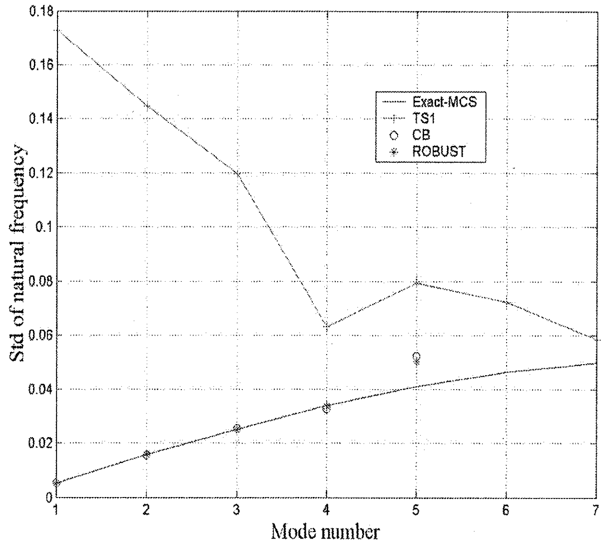

Figure 3: Standard deviation of natural frequency.

For the frequency response problem, a single force was applied at the tip diplacement of the full model on Figure 1. The range of excitation frequencies span 2 to 12 rad/s i.e., a region covering the two first modes. At each frequency point the statistics of the amplitude of tip displacement were generated using MCS, TS1, CB and ROBUST.

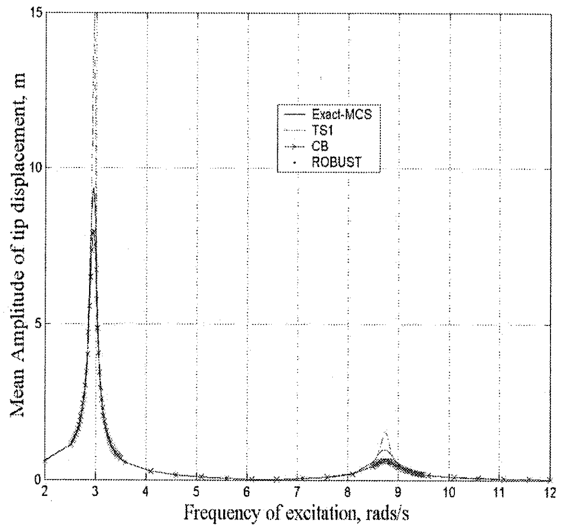

Figure 4: Mean amplitude of tip displacement.

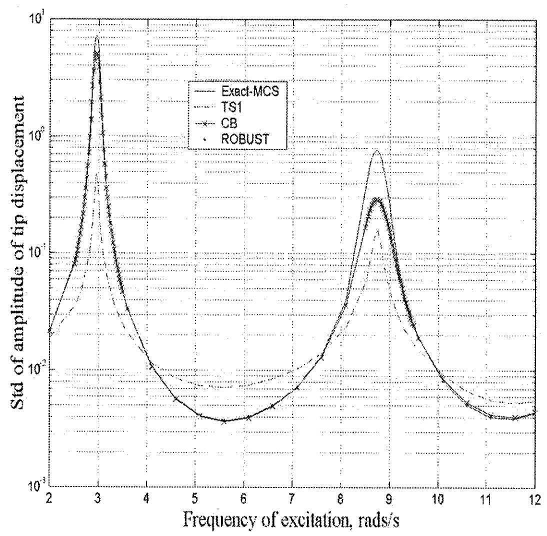

Figure 5: Standard deviation of tip displacement.

Figure 4 displays the mean amplitude of the tip displacement. The straight line is MCS, the dashed line is TS1, the crosses represent CB and the dots represent ROBUST. The pattern is well captured by the proposed method and the two peak responses at the resonance frequencies are also well represented. Note that at these frequencies, CB and ROBUST do not exactly reach the maximum responses, TS1 always over predicts them.

The standard deviation of amplitude of tip displacement is shown on Figure 5. The same legend as in Figure 4 is used. ROBUST and CB predict reasonably well the trend shown by the exact MCS results. Two peaks are displayed on this Figure. They represent the responses, when the system is excited around the two first modes. The first peak is very well captured by both CB and ROBUST. The second peak is however not exactly captured but results obtained are still better than TS1.

6. Concluding Remarks

In this paper, a stochastic component mode approach was presented for efficient dynamic analysis of structural systems with parameter uncertainties. The system is divided into components for which stochastic modes are computed separately. Reduced random matrices are then obtained for each component, which can be coupled together in the same fashion as in traditional substructuring techniques. The main idea of using stochastic modes can be in fact applied to any component-mode-base reduced order technique when the studied system has uncertain parameters. An advantage is that randomness can be introduced in all component partitioned matrices and vectors, leading to a complete stochastic analysis of the effects of uncertain parameters. Using the zero-order BG scheme, compact expressions can be obtained for the statistical moments of response. The formulation was applied to a simple model of a spring-mass structure to estimate the mean and standard deviation of eigenvalues and frequency responses. Results obtained are accurate for both problems. It remains to apply the proposed method to more challenging problems (large finite element models) to explore both accuracy and efficiency. Periodic systems could be a good example, where small perturbations in the structural properties can lead to significant changes in the dynamic responses.

Acknowledgments

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Appendix

Here we present the Explicit expressions of the elements of matrices the reduced matrices $\mathbf{M}_{\text{S-BG}_0}$ and $\mathbf{K}_{\text{S-BG}_0}$

$$\mathbf{M}^j_{\text{S-BG}_0}=\left\langle \mathbf{U}_S^{j\,\mathrm{T}}\mathbf{M}^j\mathbf{U}_S^j\right\rangle=\begin{bmatrix}\left\langle\mathbf{\Phi}^{\mathrm{T}}\mathbf{M}_{ii}\mathbf{\Phi}\right\rangle & \left\langle\mathbf{\Phi}^{\mathrm{T}}\left(\mathbf{M}_{ii}\mathbf{\Psi}+\mathbf{M}_{ib}\right)\right\rangle\\ \text{sym} & \left\langle\mathbf{\Psi}^{\mathrm{T}}\left(\mathbf{M}_{ii}\mathbf{\Psi}+\mathbf{M}_{ib}\right)+\mathbf{M}_{ib}^{\mathrm{T}}\mathbf{\Psi}+\mathbf{M}_{bb}\right\rangle\end{bmatrix}$$

$$\mathbf{M}^j_{\text{S-BG}_0}(1,1)=\mathbf{\Phi}_o^{\mathrm{T}}\mathbf{M}_{ii}\mathbf{\Phi}_o+\sigma^2\left(\sum_{k=1}^{n}\frac{\partial\mathbf{\Phi}^{\mathrm{T}}}{\partial\theta_k}\mathbf{M}_{ii}\frac{\partial\mathbf{\Phi}}{\partial\theta_k}\right)$$

$$\mathbf{M}^j_{\text{S-BG}_0}(1,2)=\mathbf{\Phi}_o^{\mathrm{T}}\mathbf{M}_{ii}\mathbf{\Psi}_o+\mathbf{\Phi}_o^{\mathrm{T}}\mathbf{M}_{ib}+\sigma^2\left(\sum_{k=1}^{n}\frac{\partial\mathbf{\Phi}^{\mathrm{T}}}{\partial\theta_k}\mathbf{M}_{ii}\frac{\partial\mathbf{\Psi}}{\partial\theta_k}\right)$$

$$\mathbf{M}^j_{\text{S-BG}_0}(2,1)=\mathbf{M}^j_{\text{S-BG}_0}(1,2)^{\mathrm{T}}$$

$$\mathbf{M}^j_{\text{S-BG}_0}(2,2)=\mathbf{\Psi}_o^{\mathrm{T}}\mathbf{M}_{ii}\mathbf{\Psi}_o+\mathbf{\Psi}_o^{\mathrm{T}}\mathbf{M}_{ib}+\mathbf{M}_{ib}^{\mathrm{T}}\mathbf{\Psi}_o+\mathbf{M}_{bb}+\sigma^2\left(\sum_{k=1}^{n}\frac{\partial\mathbf{\Psi}^{\mathrm{T}}}{\partial\theta_k}\mathbf{M}_{ii}\frac{\partial\mathbf{\Psi}}{\partial\theta_k}\right)$$

$$\mathbf{K}^j_{\text{S-BG}_0}=\left\langle \mathbf{U}_S^{j\,\mathrm{T}}\mathbf{K}^j\mathbf{U}_S^j\right\rangle=\begin{bmatrix}\left\langle\mathbf{\Phi}^{\mathrm{T}}\mathbf{K}_{ii}\mathbf{\Phi}\right\rangle & \left\langle\mathbf{\Phi}^{\mathrm{T}}\left(\mathbf{K}_{ii}\mathbf{\Psi}+\mathbf{K}_{ib}\right)\right\rangle\\ \text{sym} & \left\langle\mathbf{\Psi}^{\mathrm{T}}\left(\mathbf{K}_{ii}\mathbf{\Psi}+\mathbf{K}_{ib}\right)+\mathbf{K}_{ib}^{\mathrm{T}}\mathbf{\Psi}+\mathbf{K}_{bb}\right\rangle\end{bmatrix}$$

$$\mathbf{K}^j_{\text{S-BG}_0}(1,1)=\mathbf{\Phi}_o^{\mathrm{T}}\mathbf{K}_{ii,o}\mathbf{\Phi}_o+\sigma^2\left(\sum_{k=1}^{n}\frac{\partial\mathbf{\Phi}^{\mathrm{T}}}{\partial\theta_k}\frac{\partial\mathbf{K}_{ii}}{\partial\theta_k}\right)\mathbf{\Phi}_o+\sigma^2\mathbf{\Phi}_o^{\mathrm{T}}\left(\sum_{k=1}^{n}\frac{\partial\mathbf{K}_{ii}}{\partial\theta_k}\frac{\partial\mathbf{\Phi}}{\partial\theta_k}\right)+$$
$$\sigma^2\left(\sum_{k=1}^{n}\frac{\partial\mathbf{\Phi}^{\mathrm{T}}}{\partial\theta_k}\mathbf{K}_{ii,o}\frac{\partial\mathbf{\Phi}}{\partial\theta_k}\right)$$

$$\mathbf{K}^j_{\text{S-BG}_0}(1,2)=\sigma^2\left(\sum_{k=1}^{n}\frac{\partial\mathbf{\Phi}^{\mathrm{T}}}{\partial\theta_k}\frac{\partial\mathbf{K}_{ii}}{\partial\theta_k}\right)\mathbf{\Psi}_o+\sigma^2\mathbf{\Phi}_o^{\mathrm{T}}\left(\sum_{k=1}^{n}\frac{\partial\mathbf{K}_{ii}}{\partial\theta_k}\frac{\partial\mathbf{\Psi}}{\partial\theta_k}\right)+$$
$$\sigma^2\left(\sum_{k=1}^{n}\frac{\partial\mathbf{\Phi}^{\mathrm{T}}}{\partial\theta_k}\mathbf{K}_{ii,o}\frac{\partial\mathbf{\Psi}}{\partial\theta_k}\right)+\sigma^2\left(\sum_{k=1}^{n'}\frac{\partial\mathbf{\Phi}^{\mathrm{T}}}{\partial\theta_k}\frac{\partial\mathbf{K}_{ib}}{\partial\theta_k}\right)$$

$$\mathbf{K}^j_{\text{S-BG}_0}(2,2)=\mathbf{K}_{ib,o}^{\mathrm{T}}\mathbf{\Psi}_o+\mathbf{K}_{bb}+\sigma^2\left(\sum_{k=1}^{n}\frac{\partial\mathbf{\Psi}^{\mathrm{T}}}{\partial\theta_k}\frac{\partial\mathbf{K}_{ii}}{\partial\theta_k}\right)\mathbf{\Psi}_o+\sigma^2\mathbf{\Psi}_o^{\mathrm{T}}\left(\sum_{k=1}^{n}\frac{\partial\mathbf{K}_{ii}}{\partial\theta_k}\frac{\partial\mathbf{\Psi}}{\partial\theta_k}\right)+$$
$$\sigma^2\left(\sum_{k=1}^{n}\frac{\partial\mathbf{\Psi}^{\mathrm{T}}}{\partial\theta_k}\mathbf{K}_{ii,o}\frac{\partial\mathbf{\Psi}}{\partial\theta_k}\right)+\sigma^2\left(\sum_{k=1}^{n}\frac{\partial\mathbf{\Psi}^{\mathrm{T}}}{\partial\theta_k}\frac{\partial\mathbf{K}_{ib}}{\partial\theta_k}\right)+\sigma^2\left(\sum_{k=1}^{n}\frac{\partial\mathbf{K}_{ib}}{\partial\theta_k}^{\mathrm{T}}\frac{\partial\mathbf{\Psi}}{\partial\theta_k}\right)$$

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