Efficient Prediction of the Forced Response Statistics of Mistuned Bladed Discs

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ABSTRACT

This paper presents two efficient reduced-order modelling techniques for predicting the forced response statistics of bladed disc assemblies. First, the formulation presented in [1] is extended to the forced response problem. Component modes for a blade-disc sector are used as basis vectors, leading to a reduced model of the same size as the number of sectors and allowing for pass-band based calculations. For each realization of the random system parameters, a reduced system of equations is solved to compute the displacement vector for each frequency band of interest. Statistics of responses at each frequency point can be therefore estimated by performing Monte Carlo Simulations of cost comparable to single degree-of-freedom mass-spring systems. Second, a stochastic reduced basis approach is applied to the mistuning analysis problem. Here, the system response in the frequency domain is represented using a linear combination of complex stochastic basis vectors which span the preconditioned stochastic Krylov Subspace [2, 3]. Orthogonal stochastic projection schemes are employed for computing the undetermined coefficients in the stochastic reduced basis representation. These schemes lead to explicit expressions for the response to be obtained, thereby allowing the efficient computation of the response statistics.

INTRODUCTION

The dynamic analysis of a perfectly periodic structure such as a bladed disc requires modeling of only a single substructure or sector i.e. one blade with the corresponding portion of the disc. Using the theory of cyclic symmetry, the dynamics of the entire structure can be derived. The computational time and effort are consequently considerably reduced. The assumption that blades are identical is often considered as the basis for dynamic analysis.

In practice, the existence of small blade-to-blade differences in the structural properties is inevitable due to material or manufacturing tolerances or in-service degradation. This phenomenon, random in nature and commonly referred to as mistuning is a practical industrial problem since blades can exhibit excessive stress levels leading to unexpected failure. This situation can correspond to the severe concentration of vibration around one or a few blades while the others practically do not participate in the vibration of the entire system. This can also lead to significant increases in the forced responses amplitudes compared to when the system is perfectly tuned (i.e. when the blades' properties are identical). Mistuning effects must be included in the analysis if accurate predictions of blades' amplitudes, stresses or fatigue life are to be estimated [4, 5, 6].

Unfortunately, when the Finite Element Method is used to model these structures and thousands of nominally identical bladed disc assemblies are to be simulated using the Monte Carlo Simulation (MCS) method, the computational task becomes very difficult. Unless the number of degrees-of-freedom is kept relatively low within a blade/disc sector, the computational task is unusually demanding. One way to solve this problem is to build efficient and accurate computational models for the prediction of blade-response statistics, therefore allowing for reliable statistical assessments during turbomachinery design process [7, 8].

Two ways of dealing with mistuning problem are presented in this paper. In the first one, it is assumed that the deformed shape of the entire structure can be described using different scaling for each tuned sector mode shape [1]. This leads to a reduced model of the same size as the number of coupled blades and allows to generate results for a specific pass-band of interest at a substantially low cost. In the second one, stochastic reduced basis methods [2] are investigated to develop an efficient numerical scheme for statistical analysis [9, 10] of periodic structures such as bladed discs. The terms of the preconditioned Krylov subspace are used as basis vectors and the undetermined coefficients in the reduced basis approximation are computed using two variants of the stochastic Bubnov-Galerkin scheme [11]. This approach leads to an explicit expression for the response as a function of the random system parameters and enables to characterize statistically the system response in a computationally efficient fashion.

The system considered is a simplified model of a bladed disc assembly where each blade is modelled as an elastic beam. Stiffness coupling between blades is provided in the plane of the disc assumed to be rigid. Mistuning originates from the stiffness at the roots of the blades via rotational springs at the hub ends of the beams i.e. at the blade root. The boundary conditions at the root are zero transverse displacement of the beam and a fairly stiff rotational spring to simulate the 'fir-tree' structure and the flexibility of the mounting of a turbo-machinery-blade fixity. The finite element model of the cyclic beam studied accounts for bending modes only. Numerical simulations are carried out for weak, moderate and strong interblade coupling cases and the standard deviation of mistuning is fixed at 5%. Calculations are targeted to the family of modes corresponding to the first passband i.e., excitation frequencies are chosen around this cluster of modes. Numerical results obtained using both component mode and stochastic reduced basis methods are compared with MCS performed on the original system. It is shown that the present methods give highly accurate results at a computational cost lower than MCS.

EQUATIONS OF MOTION

The system equations of motion in the frequency domain are given by

$$\mathbf{A}(\boldsymbol{\theta})\mathbf{q}(\boldsymbol{\theta})=\mathbf{f}, \quad (1)$$

where $\mathbf{A}(\boldsymbol{\theta})=-\omega^2\mathbf{M}+j\omega\mathbf{C}+\mathbf{K}(\boldsymbol{\theta})$ is the random dynamic stiffness matrix of size N i.e. the system total number of degrees of freedom (dof). If q is the number of dof per sector and p is the number of sectors in the assembly, then $N = p\times q$. Generally, $p << q$. $\boldsymbol{\theta}=\{\theta_i\}$, $i=1, \ldots, p$ is the vector of p random system parameters, and $\mathbf{q}(\boldsymbol{\theta})$ is the random displacement response. ω is the external excitation frequency and $j=\sqrt{-1}$. $\mathbf{M}$, $\mathbf{K}$ and $\mathbf{C}$ denote the system mass, stiffness and damping matrices, respectively. **M** and **K** are written in a compact form using the Kronecker product notation as

$$\mathbf{M} = \mathbf{I}_{\mathrm{p}} \otimes \mathbf{M}_{\mathrm{B}}, \tag{2}$$

where $\mathbf{I}_{\mathrm{p}}$ is identity matrix of size p and $\mathbf{M}_{\mathrm{B}}$ is mass matrix of a sector i.e. a single beam with a rotational spring at its first end, of size q. $\mathbf{K}(\boldsymbol{\theta})$ is written as a sum of three components

$$\mathbf{K}(\boldsymbol{\theta}) = \mathbf{K}_{\mathrm{unc.}} + \mathbf{K}_{\mathrm{unc.}} + \mathbf{K}_{\mathrm{mist.}}, \tag{3}$$

where $\mathbf{K}_{\mathrm{unc.}} = \mathbf{I}_{\mathrm{p}} \otimes \mathbf{K}_{\mathrm{B}}$ is a block diagonal matrix, representing the stiffness matrix of the perfectly tuned but uncoupled system; $\mathbf{K}_{\mathrm{B}}$ is stiffness matrix of a sector, of size q; $\mathbf{K}_{\mathrm{coup.}} = \mathbf{C}_{\mathrm{K}} \otimes \mathbf{K}_{\mathrm{spr.}}$ is a block-circulant matrix representing the contribution of the coupling linear springs to the system's potential energy; $\mathbf{C}_{\mathrm{K}} = \begin{bmatrix} 2 & -1 & 0 & \cdots & -1 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & -1 \\ -1 & 0 & \cdots & -1 & 2 \end{bmatrix}$ is the connectivity matrix of the cyclic system and $\mathbf{K}_{\mathrm{spr.}}$ is a null stiffness matrix except one element whose location depends on the location of the linear spring connecting two adjacent beams. It is chosen here the q-1-th element; $\mathbf{K}_{\mathrm{mist.}} = \mathbf{diag}(k_{ri}) \otimes \mathbf{I}_1$ is the mistuning matrix, where $k_{ri} = k_{rot}\theta_i$; k_{rot} is the nominal root stiffness and θ_i is the non-dimensional mistuning parameter for the i-th blade defined by a number given by a random generator. $\mathbf{I}_1$ is a null stiffness matrix except the first element. **f** is the external excitation force vector chosen to be the engine order excitation force. **f** defines the forcing on all blade degrees-of-freedom of the system. It is expressed as

$$\mathbf{f} = \left\{ e^{j\phi_i} \right\}^{\mathrm{T}} \otimes \mathbf{f}_{\mathrm{B}}, \tag{4}$$

where $\phi_i = 2\pi n(i-1)/p, \quad i = 1,\ldots,p$ is the phase angle of force for the i-th blade component, n is the engine order and $\mathbf{f}_{\mathrm{B}}$ is the force vector on a single blade, of the same size of the number of degrees-of-freedom of one blade i.e. q.

Pass-band based Reduced Order Model (PBROM)

Using PBROM, it is assumed that the deformed shape of the entire structure can be described using different scaling for each tuned sector mode shape [1], i.e.,

$$\hat{\mathbf{q}} = \left\{ a_1 \boldsymbol{\psi}^{(i)\mathrm{T}}, a_2 \boldsymbol{\psi}^{(i)\mathrm{T}}, \ldots, a_p \boldsymbol{\psi}^{(i)\mathrm{T}} \right\}^{\mathrm{T}}, \tag{5}$$

where $\boldsymbol{\psi}^{(i)}$ is the i-th mode shape of a single tuned sector and $\mathbf{a} = \{a_1, a_2, \ldots, a_p\}^{\mathrm{T}}$ is the set of new generalized coordinates. Eqn. (5) can be rewritten using a transformation matrix **T** as follows

$$\hat{\mathbf{q}} = \mathbf{T}\,\mathbf{a} = \left(\mathbf{I}_{\mathrm{p}} \otimes \boldsymbol{\psi}^{(i)}\right)\mathbf{a}, \tag{6}$$

Inserting (6) into (1), replacing the mass and stiffness matrices by their compact expressions, multiplying both sides of (1) by the transpose of $\mathbf{T}$ and using the properties of Kronecker product, we arrive at the following reduced matrices

$$\bar{\mathbf{M}} = \left(\mathbf{I}_{\mathrm{p}} \otimes \psi^{(i)\mathrm{T}}\right)\left(\mathbf{I}_{\mathrm{p}} \otimes \mathbf{M}_{\mathrm{B}}\right)\left(\mathbf{I}_{\mathrm{p}} \otimes \psi^{(i)}\right) = \mathbf{I}_{\mathrm{p}}\left(\psi^{(i)\mathrm{T}}\mathbf{M}_{\mathrm{B}}\psi^{(i)}\right), \tag{7}$$

$$\bar{\mathbf{K}}_{\mathrm{unc.}} = \left(\mathbf{I}_{\mathrm{p}} \otimes \psi^{(i)\mathrm{T}}\right)\left(\mathbf{I}_{\mathrm{p}} \otimes \mathbf{K}_{\mathrm{B}}\right)\left(\mathbf{I}_{\mathrm{p}} \otimes \psi^{(i)}\right) = \mathbf{I}_{\mathrm{p}}\left(\psi^{(i)\mathrm{T}}\mathbf{K}_{\mathrm{B}}\psi^{(i)}\right), \tag{8}$$

$$\bar{\mathbf{K}}_{\mathrm{coup.}} = \left(\mathbf{I}_{\mathrm{p}} \otimes \psi^{(i)\mathrm{T}}\right)\left(\mathbf{C}_{\mathrm{K}} \otimes \mathbf{K}_{\mathrm{spr.}}\right)\left(\mathbf{I}_{\mathrm{p}} \otimes \psi^{(i)}\right) = \mathbf{C}_{\mathrm{K}}\left(\psi^{(i)\mathrm{T}}\mathbf{K}_{\mathrm{spr.}}\psi^{(i)}\right), \tag{9}$$

$$\bar{\mathbf{K}}_{\mathrm{mist.}} = \left(\mathbf{I}_{\mathrm{p}} \otimes \psi^{(i)\mathrm{T}}\right)\left(\mathbf{diag}(k_{ri}) \otimes \mathbf{I}_{1}\right)\left(\mathbf{I}_{\mathrm{p}} \otimes \psi^{(i)}\right) = \mathbf{diag}(k_{ri})\left(\psi^{(i)\mathrm{T}}\mathbf{I}_{1}\psi^{(i)}\right). \tag{10}$$

Note that in the Equations (7)-(10), the expressions $\left(\psi^{(i)\mathrm{T}}\mathbf{M}_{\mathrm{B}}\psi^{(i)}\right)$, $\left(\psi^{(i)\mathrm{T}}\mathbf{K}_{\mathrm{B}}\psi^{(i)}\right)$, $\left(\psi^{(i)\mathrm{T}}\mathbf{K}_{\mathrm{spr.}}\psi^{(i)}\right)$ and $\left(\psi^{(i)\mathrm{T}}\mathbf{I}_{1}\psi^{(i)}\right)$ are known coefficients.

The modal reduced viscous damping matrix $\bar{\mathbf{C}}$ is assumed to be diagonal and its elements are given by

$$\bar{c}_{ii} = 2\zeta\omega_i, \tag{11}$$

where ω_i is the i-th natural frequency obtained by solving the reduced eigenvalue problem defined by $\bar{\mathbf{M}}$ and $\bar{\mathbf{K}}_{\mathrm{unc.}} + \bar{\mathbf{K}}_{\mathrm{coup.}}$. ζ represents the viscous damping coeffcient. The reduced force vector is

$$\bar{\mathbf{f}} = \left(\mathbf{I}_{\mathrm{p}} \otimes \psi^{(i)\mathrm{T}}\right)\left(\left\{e^{j\phi_i}\right\}^{\mathrm{T}} \otimes \mathbf{f}_{\mathrm{B}}\right) = \left\{e^{j\phi_i}\right\}^{\mathrm{T}}\left(\psi^{(i)\mathrm{T}}\mathbf{f}_{\mathrm{B}}\right) \tag{12}$$

Note that the reduced modal matrices and vector are of the same size of the number of blades, i.e., p. This means that for one realization of the system random parameters, the following reduced set of p differential equations will be computed to find the new displacement vector and study the effects of random mistuning, i.e.,

$$\left(-\omega^2\bar{\mathbf{M}} + j\omega\bar{\mathbf{C}} + \bar{\mathbf{K}}\right)\mathbf{a} = \bar{\mathbf{f}}. \tag{13}$$

Statistics of the responses at each frequency point can be therefore estimated by averaging over a large ensemble of realizations of the mistuned system. The advantage is that these statistics are computed at a cost of mass-spring models of the same size as the number of blades. It can be also noted that the reduced matrices and vector in (13) have the same structure as those of a mono-coupled single-degree-of-freedom component model of a bladed disc [6].

STOCHASTIC REDUCED BASIS METHODS (SRBMs)

The fundamental idea of SRBMs is to approximate the solution of Equation (1) using a subspace spanned by a set of stochastic basis vectors. A theoretical justification was presented in [2, 3] for employing the terms of the preconditioned stochastic Krylov subspace as basis vectors. It was shown that the solution of a linear random algebraic system of equations can be approximated to an arbitrary degree of accuracy using this set of basis vectors [9, 10]. If $\mathbf{K}_0$ is the stiffness matrix of the perfectly tuned system, i.e., $\mathbf{K}_0 = \mathbf{K}_{\mathrm{coup.}} + \mathbf{K}_{\mathrm{unc.}}$, then $\mathbf{K}(\mathbf{0}) = \mathbf{K}_{\mathrm{mist.}}$ can be expanded as

$$\mathbf{K}(\boldsymbol{\theta}) = \mathbf{K}_0 + \Delta\mathbf{K} = \mathbf{K}_0 + \sum_{i=1}^{p} \mathbf{K}_i \theta_i , \tag{14}$$

where $\Delta\mathbf{K}$ is the deviation of the stiffness matrix due to mistuning and $\mathbf{K}_i$ is a deterministic matrix related to the baseline tuned structure. Note that this representation is chosen here for the sake of convenience. This results in the following expression of the dynamic stiffness

$$\mathbf{A}(\boldsymbol{\theta}) = \mathbf{A}_0 + \Delta\mathbf{A} , \tag{15}$$

where $\mathbf{A}_0 = -\omega^2\mathbf{M} + j\omega\mathbf{C} + \mathbf{K}_0$ and $\Delta\mathbf{A} = \Delta\mathbf{K}$. For the representation of the random stiffness matrix in Eqn. (14), and further by employing the matrix $\mathbf{A}_0$ as a preconditioner, it can be shown that the terms of the preconditioned stochastic Krylov subspace coincides with the perturbation series. This implies that the same results can be obtained by using the terms of the perturbation series as stochastic basis vectors. Here, three basis vectors are used to represent the solution of Eqn. (1) as

$$\hat{\mathbf{q}}(\boldsymbol{\theta}) = \sum_{i=0}^{2} \xi_i \, \boldsymbol{\psi}_i(\boldsymbol{\theta}) = \boldsymbol{\Psi}_{SRBM}(\boldsymbol{\theta})\boldsymbol{\xi} , \tag{16}$$

where $\boldsymbol{\Psi}_{SRBM}(\boldsymbol{\theta}) = [\boldsymbol{\psi}_0(\boldsymbol{\theta})\, \boldsymbol{\psi}_1(\boldsymbol{\theta})\, \boldsymbol{\psi}_2(\boldsymbol{\theta})] \in \mathbb{C}^{Nx3}$ and $\boldsymbol{\xi} = \{\xi_0, \xi_1, \xi_2\}^{\mathrm{T}} \in \mathbb{C}^{3x1}$ denote the matrix of complex stochastic basic vectors and the vector of undetermined coefficients, respectively. The first basis vector ψ_0 is obtained by solving for the frequency response of the tuned system, i.e.,

$$\boldsymbol{\psi}_0 = \mathbf{A}_0^{-1}\mathbf{f} \, . \tag{17}$$

The other two basis vectors are given by

$$\boldsymbol{\psi}_1(\boldsymbol{\theta}) = \sum_{i=1}^{p} \frac{\partial \mathbf{q}}{\partial \theta_i} \theta_i \tag{18}$$

$$\boldsymbol{\psi}_2(\boldsymbol{\theta}) = \sum_{i=1}^{p} \sum_{j=1}^{p} \frac{\partial^2 \mathbf{q}}{\partial \theta_i \partial \theta_j} \theta_i \theta_j \, . \tag{19}$$

The response sensitivities appearing in Eqns. (18, 19) are computed as

$$\frac{\partial \mathbf{q}}{\partial \theta_i} = -\mathbf{A}_0^{-1} \frac{\partial \mathbf{K}}{\partial \theta_i} \boldsymbol{\psi}_0 \tag{20}$$

$$\frac{\partial^2 \mathbf{q}}{\partial \theta_i \partial \theta_j} = \mathbf{A}_0^{-1} \left(\frac{\partial \mathbf{K}}{\partial \theta_i} \mathbf{A}_0^{-1} \frac{\partial \mathbf{K}}{\partial \theta_j} \mathbf{q}_0 + \frac{\partial \mathbf{K}}{\partial \theta_j} \mathbf{A}_0^{-1} \frac{\partial \mathbf{K}}{\partial \theta_i} \mathbf{q}_0 \right) . \tag{21}$$

Eqns. (18)-(21) imply that the computational cost can be very high (if the sensitivity analysis of large-scale systems across a broad range of excitation frequencies must be conducted), specially for large-scale systems. At each frequency point, an independent set of stochastic basis vectors needs to be computed. Also, this means that $\mathbf{A}_0$ needs to be repeatedly inverted at each frequency of interest. Fortunately for cyclic structures, $\mathbf{A}_0$ is a block-circulant matrix that can be block-diagonalized using the transformation $(\mathbf{E}^* \otimes \mathbf{I})\mathbf{A}_0(\mathbf{E} \otimes \mathbf{I})$, $\mathbf{E}$ is the so-called Fourier matrix of size p; $*$ and $\mathbf{I}$ denote the complex conjugate transpose of a matrix and an identity matrix of size equal to that of a block in $\mathbf{A}_0$ (i.e., the number of degrees-of-freedom of a blade-disc sector q), respectively.

Note that the computation of the sensitivities of the displacement $\mathbf{q}$ in the modal domain can be even more efficient if a smaller set of nominal modes (eigenvectors of the tuned system) to approximately compute the sensitivities of $\mathbf{q}$ in the modal basis. Furthermore, the basis vectors used in PBROM can be employed instead. This means that the modes for one specific passband can be used to arrive at a reduced dynamic stiffness matrix of the tuned system. Therefore, in the sensitivity computation, for a large number of frequency points, this reduced matrix of size of the number of blades is inverted at a low cost.

To compute the undetermined coefficient vector $\boldsymbol{\xi}$ in the stochastic reduced basis representation, two stochastic variants of the Bubnov-Galerkin (BG) scheme are used [11]. First a stochastic residual error vector is defined as

$$\mathbf{r}(\boldsymbol{\theta}) = \mathbf{A}(\boldsymbol{\theta})\boldsymbol{\Psi}_{SRBM}(\boldsymbol{\theta})\boldsymbol{\xi} - \mathbf{f}. \tag{22}$$

In the first variant $\boldsymbol{\xi}$ is determined by enforcing that the stochastic residual $\mathbf{r}(\boldsymbol{\theta})$ is orthogonal to $\boldsymbol{\Psi}_{SRBM}(\boldsymbol{\theta})$ in an average sense. By considering the inner product of two random vectors in the Hilbert space of random variables, it is obtained

$$\left\langle \boldsymbol{\Psi}^*_{SRBM}(\boldsymbol{\theta})\,\mathbf{r}(\boldsymbol{\theta})\right\rangle = 0, \tag{23}$$

where $\langle\cdot\rangle$ denotes the ensemble average. Equation (23) leads to the following 3×3 reduced deterministic system of equations for the coefficients ξ_0, ξ_1 and ξ_2

$$\left\langle \boldsymbol{\Psi}^*_{SRBM}(\boldsymbol{\theta})\mathbf{A}(\boldsymbol{\theta})\boldsymbol{\Psi}_{SRBM}(\boldsymbol{\theta})\boldsymbol{\xi} - \boldsymbol{\Psi}^*_{SRBM}(\boldsymbol{\theta})\mathbf{f}\right\rangle = \left\langle \boldsymbol{\Psi}^*_{SRBM}(\boldsymbol{\theta})\mathbf{A}(\boldsymbol{\theta})\boldsymbol{\Psi}_{SRBM}(\boldsymbol{\theta})\right\rangle \boldsymbol{\xi} - \left\langle \boldsymbol{\Psi}^*_{SRBM}(\boldsymbol{\theta})\mathbf{f}\right\rangle = 0. \tag{24}$$

Eqn. (23) is interpreted as a zero-order condition and this formulation is henceforth referred to as SRBM-BG$_0$. The deterministic system of equations to be solved for the vector of undetermined coefficients $\boldsymbol{\xi}$ can be written in a compact form as

$$\mathbf{A}_{SRBM-BG_0}\boldsymbol{\xi} = \mathbf{f}_{SRBM-BG_0}, \tag{25}$$

where $\mathbf{A}_{SRBM-BG_0} = \left\langle \boldsymbol{\Psi}^*_{SRBM}(\boldsymbol{\theta})\mathbf{A}(\boldsymbol{\theta})\boldsymbol{\Psi}_{SRBM}(\boldsymbol{\theta})\right\rangle$ and $\mathbf{f}_{SRBM-BG_0} = \left\langle \boldsymbol{\Psi}^*_{SRBM}(\boldsymbol{\theta})\mathbf{f}\right\rangle$ denote the reduced dynamic stiffness matrix and the reduced force vector, respectively. Explicit expressions for their elements are given in [9, 10]. Once the coefficients ξ_0, ξ_1 and ξ_2 are computed by solving the reduced-order problem in Equation (25), the mean $(\mu_{\hat{\mathbf{q}}})$ and covariance matrix $(\Sigma_{\hat{\mathbf{q}}})$ of the system response at each excitation frequency can be computed as

$$(\mu_{\hat{\mathbf{q}}}) = \langle\hat{\mathbf{q}}\rangle = \langle \xi_0\psi_0 + \xi_1\psi_1 + \xi_2\psi_2\rangle \tag{26}$$

$$(\Sigma_{\hat{\mathbf{q}}}) = \left\langle \hat{\mathbf{q}}(\boldsymbol{\theta})\hat{\mathbf{q}}^*(\boldsymbol{\theta})\right\rangle = \left\langle \boldsymbol{\Psi}(\boldsymbol{\theta})\boldsymbol{\xi}\boldsymbol{\xi}^*\boldsymbol{\Psi}(\boldsymbol{\theta})^*\right\rangle = \sum_{i=0}^{p}\sum_{j=0}^{p}\xi_i\xi_j^*\left\langle \psi_i(\boldsymbol{\theta})\psi_j^*(\boldsymbol{\theta})\right\rangle. \tag{27}$$

Compact expressions for the mean and covariance matrix for the case when the elements of $\boldsymbol{\theta}$ are uncorrelated zero-mean Gaussian random variables are given in [9, 10].

In the second stochastic variant of the BG scheme, it is demanded that the stochastic residual error is orthogonal to the approximating space of basis vectors with probability one. In contrast to the SRBM-BG$_0$, this alternative formulation referred to as SRBM-BG is exact because $\mathbf{r}(\boldsymbol{\theta})$ is orthogonal to $\boldsymbol{\Psi}^*_{SRBM}$ in an exact sense. This leads to the following reduced–order system of random equations to be solved for the undetermined coefficients

$$\mathbf{A}_{SRBM-BG}\boldsymbol{\xi} = \mathbf{f}_{SRBM-BG}, \tag{28}$$

where $\mathbf{A}_{SRBM-BG} = \boldsymbol{\Psi}^*_{SRBM}(\boldsymbol{\theta})\mathbf{A}(\boldsymbol{\theta})\boldsymbol{\Psi}_{SRBM}(\boldsymbol{\theta})$ and $\mathbf{f}_{SRBM-BG} = \boldsymbol{\Psi}^*_{SRBM}(\boldsymbol{\theta})\mathbf{f}$ are the reduced-order random matrix and the force vector.

$$\mathbf{A}_{SRBM-BG}\xi = \mathbf{F}_{SRBM-BG}, \quad (29)$$

where $\mathbf{A}_{SRBM-BG} = \mathbf{\Psi}^*(\mathbf{\theta})\mathbf{A}(\mathbf{\theta})\mathbf{\Psi}(\mathbf{\theta})$ and $\mathbf{F}_{SRBM-BG} = \mathbf{\Psi}^*(\mathbf{\theta})\mathbf{F}$ are the reduced-order random matrix and the force vector, respectively. Explicit computation of the undetermined coefficients will involve the symbolic inversion of $\mathbf{A}_{SRBM-BG}$. Since this can lead to a complicated expression for $\hat{\mathbf{q}}(\mathbf{\theta})$ and the resulting approximation is a highly nonlinear function of the system parameters, analytical characterization of the response statistics is no longer readily possible. Therefore, MCS schemes can be applied to efficiently compute the response statistics by sampling the stochastic reduced basis representation with random function models for the undetermined coefficients.

SIMULATION RESULTS

A 10-beam discretized model is used to compare the performance of the proposed methods. Note that in this paper only results obtained by SRBM-BG and PBROM are presented. In particular, at each excitation frequency, the first two moments of the frequency response maximum amplitude among the blades are computed in the post-processing stage. These moments are compared to benchmark MCS results on the full equations. The parameters of the beam used here for simulations are given below:

Material: Aluminum — Young's modulus, E: 7×10^{10} Pa
Density, ρ: 2700 kg m^{-3} — Length, L: 0.21 m
Height, h: 0.020 m — Breadth, b: 0.012 m
Second moment of area, $I = bh^3/12$: 8×10^{-9} m^4, Mass per unit length, $m = \rho bh$: 0.648 kg m^{-1}

At each excitation frequency, results for the system response are generated using the two proposed methods. As mistuning originates only from the stiffness of the blade root (of the rotational spring at the first end of beam), a large population of random numbers for each blade root is generated and the standard deviation of the random parameters $\mathbf{\theta} = \{\theta_i\}$, $i = 1, \ldots, p$ is fixed at 5%. The viscous damping coefficients is 1%. Three cases of interblade coupling are considered: weak, moderate and strong. To illustrate the strength of interblade coupling, one deterministic analysis of a single mistuned system is made. This means that only one mistuning pattern is randomly selected and simulated. The natural frequencies and mode shapes of both tuned and mistuned systems are then plotted for each case of coupling strength. The natural frequencies obtained by PBROM are also plotted to make sure that modes are accurately approximated using the reduced order model. This also allows the possibility of mode localization [1] occurrence to be predicted. Note that simulations are targeted to a cluster of frequencies corresponding to the first passband i.e. the first family of modes.

Figures 1 and 2 correspond to very weak coupling, when the modal density is very high and all mistuned modes are strongly localized. The mean of maximum frequency responses is plotted as function of the frequency of the first engine order excitation, see Figure 3. Note that the excitation frequencies span the range 2343 to 2344 rad/s. Although this is a small range, it covers all the possible resonance frequencies of the simulated mistuned systems. Both SRBM-BG and PBROM capture well (compared to respect to MCS results) the behaviour of the mean of maximum amplitude except the region between 2343.6 and 2343.75 Hz, where PBROM is performing better than SRBM-BG. Note that PBROM uses the modes of the uncoupled tuned system as basis vectors for the approximation of the system response. Since the mistuned mode shapes of the weakly coupled system (strong localization in Figure 2) look like the modes of the uncoupled (tuned) system, it is expected that PBROM will perform well in high modal density regions. Figure 4 displays the variance of maximum amplitude of the weakly coupled blade system. In the mid region, this quantity is accurately predicted by both approximations. But, in the low and high regions, the prediction made by using SRBM-BG is more accurate than PBROM.

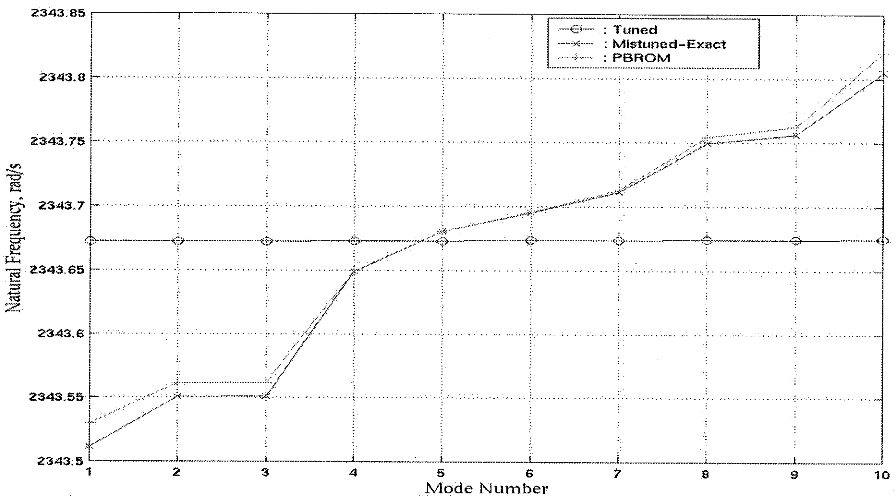

Figure 1: Natural frequencies of the tuned and mistuned systems. Weak coupling.

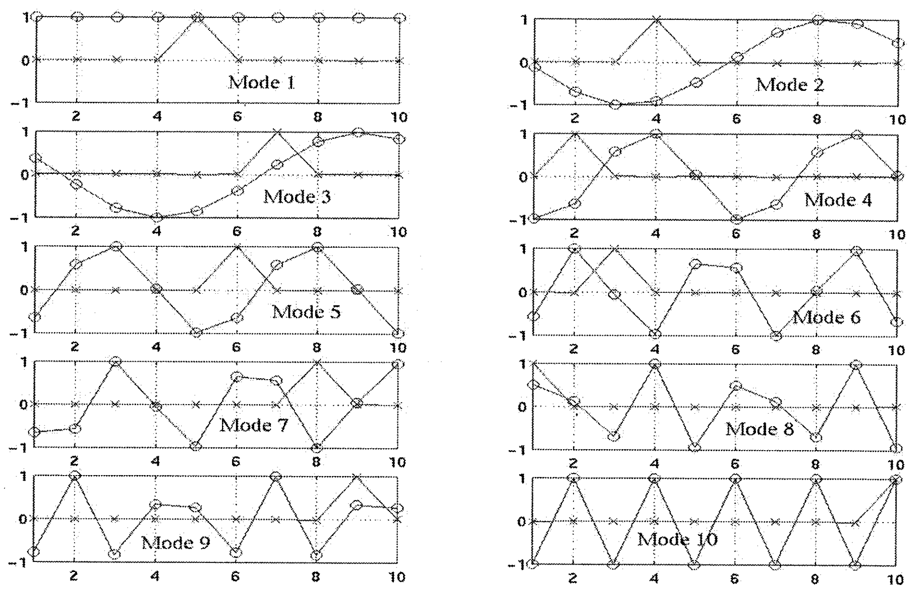

Figure 2: Mode shapes of the tuned (o) and mistuned (x) systems. x-axis: blade number; y- axis:normalized amplitude. Weak coupling.

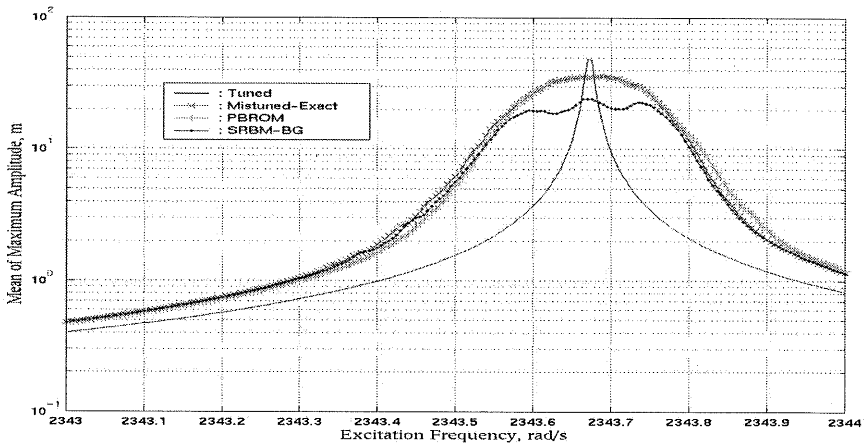

Figure 3: Mean of maximum amplitude. Weak Coupling. $\xi = 1\%$.

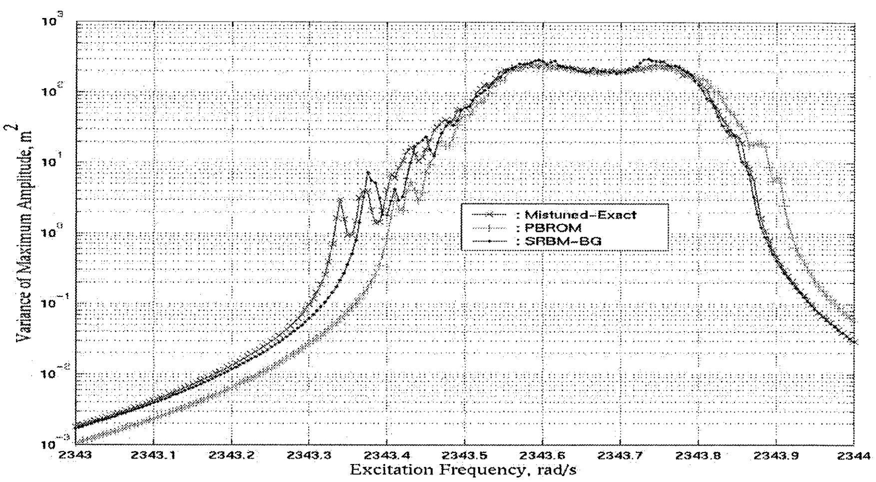

Figure 4: Variance of maximum amplitude. Weak Coupling. $\xi = 1\%$.

When the interblade coupling is moderately weak (i.e. now increased), the width of the natural frequencies of the tuned sytem increases, see Figure 5. The mistuned system shows that the natural frequencies are now split, but the corresponding mode shapes are still strongly localized, see Figure 6. Note that the mistuning pattern used for simulation is the same as earlier. The mean of maximum amplitude across the range 2343 to 2344 Hz is presented on Figure 7. Once again it is seen that when strong localisation occurs, in the mid region (encompassing highly clustered resonance frequencies), PBROM performs better than SRBM-BG. However, the results displayed on Figure 8 for the variance of the maximum amplidude show that in the low and high regions, while SRBM-BG results match the MCS results well, PBROM can slightly under or over predict the effects of mistuning.

Finally, the interblade coupling is increased such that mistuning does not have a strong effect on the modes of free vibration. This is referred to as strong coupling case, where no localization occurs, see Figures 9 and 10. The modes of the mistuned system are then extended throughout the structure i.e. each blade participates in the vibration.

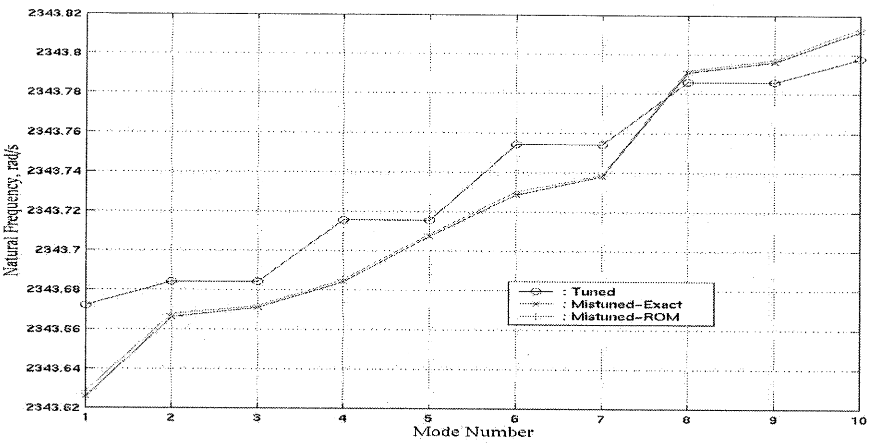

Figure 5: Natural frequencies of the tuned and mistuned systems. Moderate Coupling.

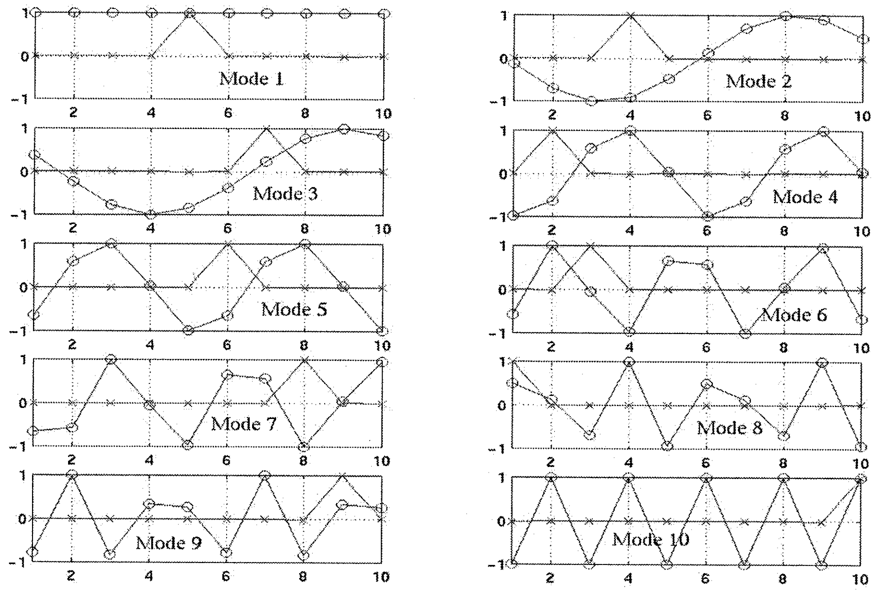

Figure 6: Mode shapes of the tuned (o) and mistuned (x) systems. x-axis: blade number; y-axis: normalized amplitude. Moderate coupling.

For the forced vibration, the chosen excitation frequencies span the range 2343 to 2344.2 rad/s. Both SRBM-BG and PBROM predict well the behaviour of the mean of maximum amplitude, which is displayed on Figure 11. In the middle region however, at certain frequencies, SRBM-BG does not reach the exact benchmark results responses. The variance of the maximum amplitude is plotted on Figure 12. Here again, the trend is accurately predicted by the proposed approximations alhough in the low and high region of excitation frequencies, PBROM can slightly deviate from the actual responses.

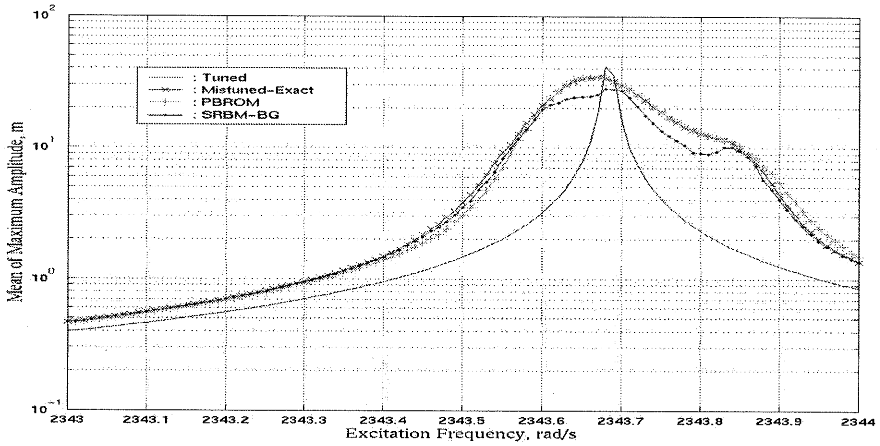

Figure 7: Mean of maximum amplitude. Moderate Coupling. $\xi = 1\%$.

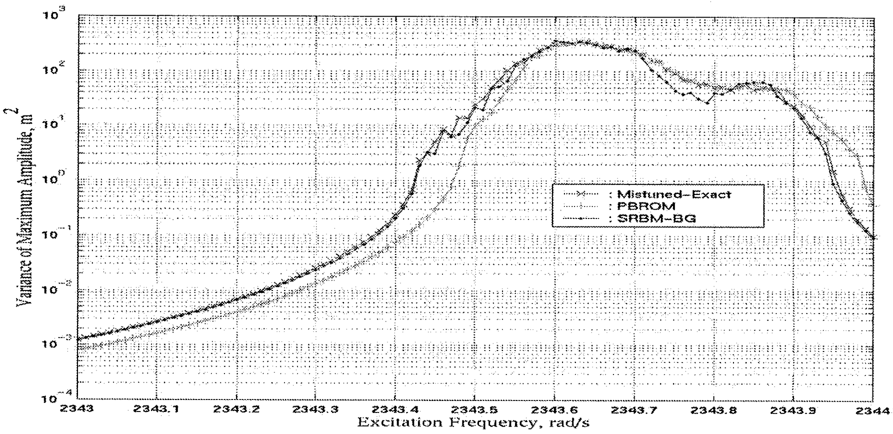

Figure 8: Variance of maximum amplitude. Moderate Coupling. $\xi = 1\%$.

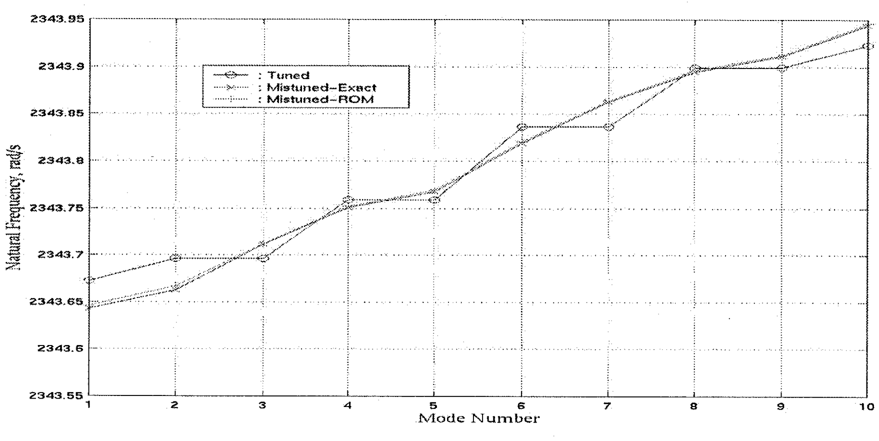

Figure 9: Natural frequencies of the tuned and mistuned systems. Strong coupling.

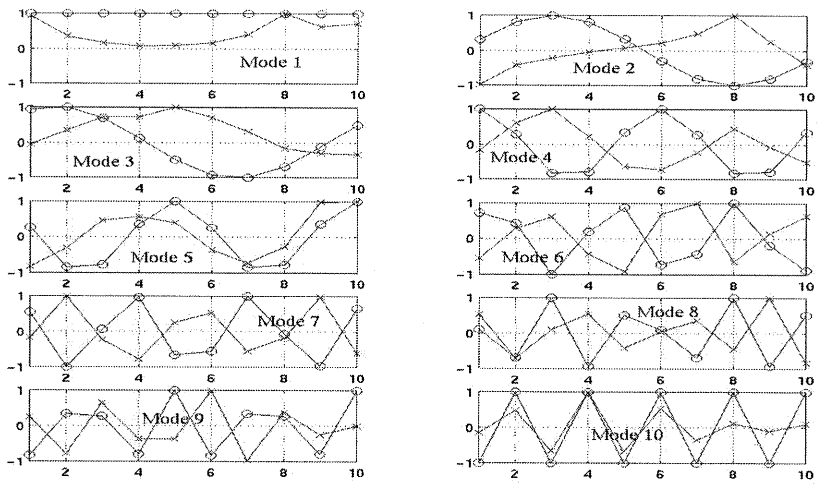

Figure 10: Mode shapes of the tuned (o) and mistuned (x) systems. x-axis: blade number; y-axis: normalized amplitude. Strong coupling.

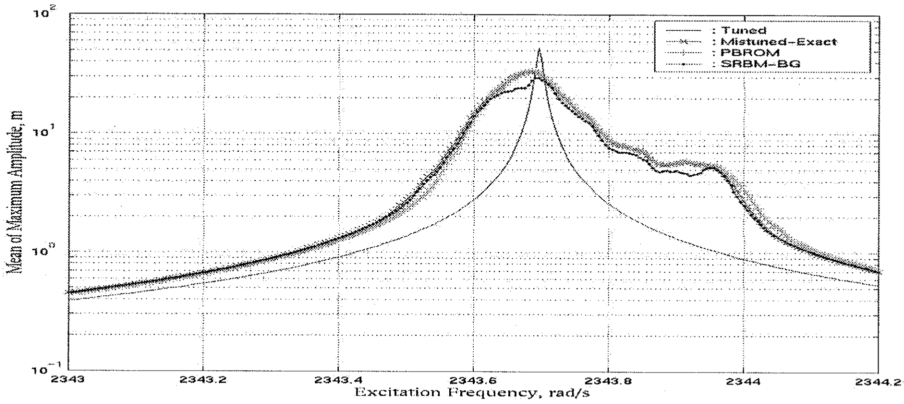

Figure 11: Mean of maximum amplitude. Strong Coupling. $\xi = 1\%$.

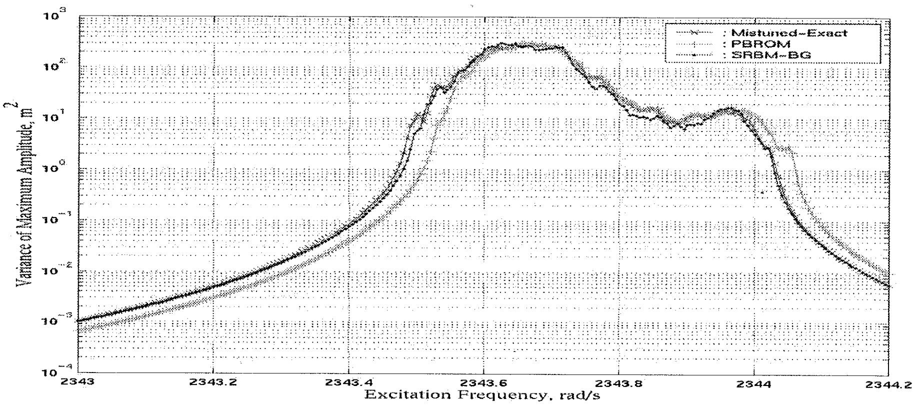

Figure 12: Variance of maximum amplitude. Strong Coupling. $\xi = 1\%$.

CONCLUSIONS

In this paper, the PBROM method was extended to the forced vibration problem for computing the frequency response of mistuned bladed disc assemblies. The fundamental idea is to approximate the displacement vector by using component modes of a tuned sector as basis vectors. The statistics of response amplitudes are then estimated by applying Monte Carlo Simulations. In particular the first two moments of the maximum amplitude among the blades at each frequency point are computed. Numerical studies on a multi-degree-of-freedom coupled beam model problem are presented to test the level of accuracy for different coupling cases. Results were compared with SRBM-BG and benchmark results generated using Monte Carlo Simulation applied on the original system. Both PBROM and SRBM-BG give reasonably accurate results for a standard deviation of mistuning of 5%.

Two main conclusions can be drawn from this study. The first one is that in high modal density regions (where frequencies are highly clustered and mode localization does occur) PBROM is more likely to work better than SRBM-BG. The reason is the transformation matrix used for approximating the modes of the uncoupled tuned system (see the block-diagonal form of this transformation matrix in Eqn. (5)). Hence, any localized mode shape of the entire system will look like one of the basis vectors. However when the modes are strongly coupled (or are outside the high modal density regions), the mistuned system yields tuned-like system. In this case, PBROM can be improved by using the modes of the coupled system as basis vectors. The second conclusion is that the stochastic approximation SRBM-BG in contrast to PBROM gives accurate results when the modes are strongly coupled. The reason behind is perhaps the choice of the preconditionner $\mathbf{A}_0$, i.e., the dynamic stiffness of the coupled tuned system, see Equations (17), (20) and (21). But in high modal density regions, it could be improved by defining the preconditionner $\mathbf{A}_0$ as the dynamic stiffness of the uncoupled tuned.

For real bladed discs, the strength of interblade coupling is measured by the amount of interaction i.e., width of veering between families of modes. If the modal density of a specific cluster of frequencies of interest is high, the basic mode shape can be used to construct a reduced model in the same fashion as PBROM. If not, PBROM must be based on the tuned mode shapes of the coupled system. Similarly if the stochastic reduced basis method is to be applied, the accuracy of results for the responses will depend on whether $\mathbf{A}_0$ represents the uncoupled or coupled tuned system. Research is underway to overcome this by combining the two approaches together to form a stochastic component mode synthesis method [12]. This involves using stochastic modes as basis vectors in the same fashion as in SRBMs and applying the BG scheme for the computation of the undetermined coefficients in the reduced approximation.

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