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New Developments in Computational Stochastic Mechanics, Part II: Applications

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Abstract

In a companion paper (Nair, P. B., and Keane, A. J., "New Developments in Computational Stochastic Mechanics, Part I: Theory", *AIAA-2000-1827*), stochastic reduced basis approximation (SRBA) methods were presented for analysis of systems governed by stochastic partial differential equations (PDEs). The fundamental idea proposed was to use the terms of the Neumann expansion series as stochastic basis vectors along with undetermined coefficients for representing the response process. Solution procedures based on variants of the stochastic Bubnov-Galerkin scheme were developed for determining the coefficients of the reduced basis representation. This paper presents detailed numerical studies for two example problems from the domain of stochastic structural mechanics. The main objective here is to study the numerical characteristics of SRBA methods, and to compare the results with the Neumann expansion scheme. It is demonstrated that the SRBA methods give significantly better results as compared to the Neumann expansion scheme, particularly for large stochastic variations in the random system parameters.

Introduction

In a companion paper[1], stochastic reduced basis approximation (SRBA) methods were proposed for analysis of systems governed by stochastic PDEs. In particular, efficient numerical schemes were developed for solution of large-scale linear algebraic systems of equations with random coefficients, such as those obtained from discretizing linear stochastic PDEs in space and the random dimension of the problem. Nonlinear stochastic PDEs can also be ultimately reduced to this form by employing stochastic linearization techniques.

The fundamental proposition made in the SRBA formulation was that the response process can be represented in the subspace spanned by the terms of the Neumann expansion series. This leads to a reduced basis representation of the system response in terms of stochastic basis vectors and undetermined coefficients. An attractive feature of this formulation is that a complete probabilistic description of the response can be computed to an arbitrary degree of accuracy by increasing the number of stochastic basis vectors. Further, since only a reduced-order system of equations are solved in the SRBA formulation, the response statistics can be computed in a highly efficient fashion.

Two variants of the Bubnov-Galerkin scheme were developed for computing the undetermined coefficients in the reduced basis representation. In the first formulation, the coefficients are considered as deterministic scalars which are computed by solving an ensemble averaged reduced-order deterministic system of equations. This leads to an explicit expression for the stochastic response as a polynomial in the random system parameters. Hence, all the statistical moments of the response can be analytically computed; see, McGullagh[2] for a detailed exposition on statistical analysis of random polynomials. This formulation was referred to as the approximate SRBA (ASRBA) method. A simplified ASRBA method was also presented which involved rearranging the first-order Neumann expansion term to expand the matrix of basis vectors.

The second formulation referred to as the SRBA method considers the coefficients of the reduced basis as random functions. This ensures that the stochastic residual error is orthogonal with respect to the approximating space of basis vectors with probability one. However, in order to derive explicit expressions for the random functions, a reduced-order matrix must be symbolic inverted. As shown in the companion paper[1], this can be readily done when only two or three terms of the Neumann expansion series are used as basis vectors. In contrast

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to the ASRBA method, the SRBA formulation leads to a very complicated (albeit explicit) expression for the response process. Hence, a complete statistical description of the response would involve the use of a simulation procedure.

This paper presents some studies on the application of the SRBA methods proposed in the companion paper[1] to problems in structural mechanics. The objectives of this paper are to : (1) conduct detailed studies on the convergence characteristics and accuracy of SRBA and ASRBA methods, and (2) compare the results with the Neumann expansion scheme.

Two classes of problems are considered: Class I: linear stochastic structures subjected to static loads, and Class II: linear stochastic structural dynamical systems subjected to harmonic forcing. The first example involves static response analysis of a 40 member frame structure with stochastic Young's modulus subject to a deterministic force. The second example involves frequency response analysis of 20 member frame structure with stochastic Young's modulus and mass density.

Monte Carlo simulation studies are conducted to generate benchmark results against which the various approximate methods are compared. Results are presented for the first, second, and third-order SRBA methods which are henceforth referred to as SRBA1, SRBA2, and SRBA3, respectively. The approximate SRBA (ASRBA) methods of order one, two, and three referred to as ASRBA1, ASRBA2, and ASRBA3 have also been applied to the example problems. Recollect that the number of basis vectors used in the formulations is one greater than the order of the method. The results obtained using the SRBA and ASRBA methods are compared with those obtained using the first, second, and third-order Neumann expansion scheme, referred to as NEU1, NEU2, and NEU3.

The effect of increase in the standard deviation of the random system parameters on the accuracy and convergence of the various methods are studied in detail. It is demonstrated that the SRBA and ASRBA methods give significantly better results as compared to the Neumann expansion scheme, particularly for large stochastic variations in the random system parameters.

Example 1: Static Analysis of a Linear Stochastic Structural System

The first example considered here is a 40 member cantilevered frame structure with random Young's modulus shown in Figure 1. Each structural member is modeled using two Euler-Bernoulli beam elements, which leads to a total of 180 degrees of freedom (dof). The Young's modulus of each structural member is modeled as $\text{Eo}(1+\theta_i)$, $i = 1, 2, \ldots, 42$; where θ_i are uncorrelated zero-mean Gaussian random variables with standard deviation of σ_θ. The nominal values of the axial and flexural rigidity are taken as $EoA = 6.987\times10^6$ N, and $EoI = 1.286\times10^3$ Nm^2, respectively. The structure is subject to a deterministic static force at the tip of the structure.

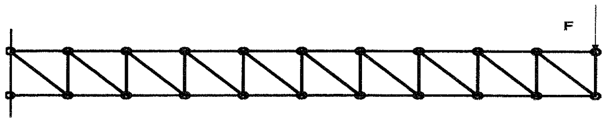

Figure 1 : 40 Member Frame Structure

Using a Taylor series approximation for the structural stiffness matrix, the stochastic linear algebraic system of equations for static equilibrium can be written as

$$\left(\mathbf{L} + \sum_{i=1}^{40} \theta_i \mathbf{\Pi}_i\right) \mathbf{u}(\mathbf{\Theta}) = \mathbf{f} \tag{1}$$

where $\mathbf{L}$ denotes the stiffness matrix computed using the mean value of the Young's modulus, and $\mathbf{\Pi}_i$ can be interpreted as the sensitivity of the stiffness matrix with respect to the Young's modulus of member i. A summary of the approximate methods compared in this paper is presented below for the sake of completeness.

The Neumann expansion scheme for the displacement can be written as

$$\mathbf{u}(\mathbf{\Theta}) = \sum_{i=1}^{40} (-1)^{i+1}\, \mathbf{\Psi}_i(\mathbf{\Theta}) \tag{2}$$

where

$$\mathbf{\Psi}_i(\mathbf{\Theta}) = \left(\mathbf{L}^{-1} \sum_{j=1}^{40} \theta_j \mathbf{\Pi}_j\right)^{i-1}. \tag{3}$$

It can be seen that equation (3) is a polynomial in θ_i. As mentioned earlier, the fundamental idea used in the SRBA formulation is to postulate an approximation for $\mathbf{u}(\mathbf{\Theta})$ of the form

$$\hat{\mathbf{u}}(\mathbf{\Theta}) = \sum_{i=1}^{40} \zeta_i(\mathbf{\Theta}) \mathbf{\Psi}_i(\mathbf{\Theta}) \tag{4}$$

where $\zeta_i(\Theta)$ denotes the undetermined coefficients in the reduced basis approximation.

The coefficients in equation (4) are computed using equation (1) and (4) via a stochastic Bubnov-Galerkin projection scheme. Recollect that in the ASRBA formulation, the coefficients are considered as deterministic scalars, whereas in the SRBA formulation the coefficients are considered as random functions. The implication of this being that the ASRBA method allows all the statistical moments of the response to be computed analytically since equation (4) will give a polynomial expression in terms of θ_i for the response process. In contrast, since the resulting expression using the SRBA method is a highly nonlinear function of θ_i, simulation schemes have to be employed to compute the response statistics.

Three cases are taken up to study the accuracy of the various methods for increasing coefficient of variation of the random Young's modulus. The value of σ_θ is set at 0.05, 0.10, and 0.20 for case 1, case 2, and case 3, respectively.

For each case, Monte Carlo simulation (MCS) using exact static analysis with a sample size of 10000 is used to generate benchmark results. The results obtained using the approximate methods are compared with these benchmark results, referred to as Exact MCS. The same sample size was also used to compute the pdf of the response using the explicit expressions obtained for all the approximate methods. Note that the pdfs are normalized with respect to the mean value predicted using exact MCS.

Table 1 : Comparison of Methods for Case 1; $\sigma_\theta = 0.05$

Method	Error in Mean (%)		Error in STD (%)	
	Max	Avg.	Max	Avg.
SRBA1	0.1932	0.0074	0.7301	0.2798
SRBA2	0.0017	0.0001	0.0692	0.0103
SRBA3	0.0001	0.0000	0.0010	0.0002
ASRBA1	0.2558	0.0081	1.2473	0.8963
ASRBA2	0.0075	0.0004	0.0559	0.0113
ASRBA3	0.0004	0.0000	0.0044	0.0007
NEU1	0.5037	0.2555	1.5818	1.2340
NEU2	0.0123	0.0026	0.8051	0.7582
NEU3	0.0052	0.0019	0.0297	0.0207

A comparison of the percentage errors in the mean and standard deviation of the static response computed using all the methods for case 1 ($\sigma_\theta = 0.05$) is summarized in Table 1. As mentioned earlier, these errors are computed with respect to the results obtained using Exact MCS. The maximum and average errors in the mean and standard deviation (STD) across all the 180 dofs are shown in the Table. It can be clearly seen that when $\sigma_\theta = 0.05$, all the approximate methods give excellent results for both statistical moments of the response. SRBA methods are seen to show much faster convergence to the exact statistics when the number of basis vectors are increased as compared to the ASRBA methods, and the Neumann expansion scheme.

It is important to note that, in spite of considering the coefficients in the reduced basis as deterministic constants, the ASRBA methods give significantly improved results as compared to the Neumann expansion scheme. Further, the errors in the statistics becomes nearly zero when four basis vectors are used in the ASRBA formulation.

The probability density function (pdf) of the transverse component of the tip displacement computed using all the first-order methods are compared with the pdf obtained using exact MCS in Figure 2. Note that all the approximate responses are normalized with respect to the mean value predicted by exact MCS in this figure. It can be clearly seen that both ASRBA1 and SRBA1 give better approximations for the tails of the pdf as compared to NEU1. For all the second and third-order methods, the pdf was observed to show nearly exact agreement with the results obtained using Exact MCS.

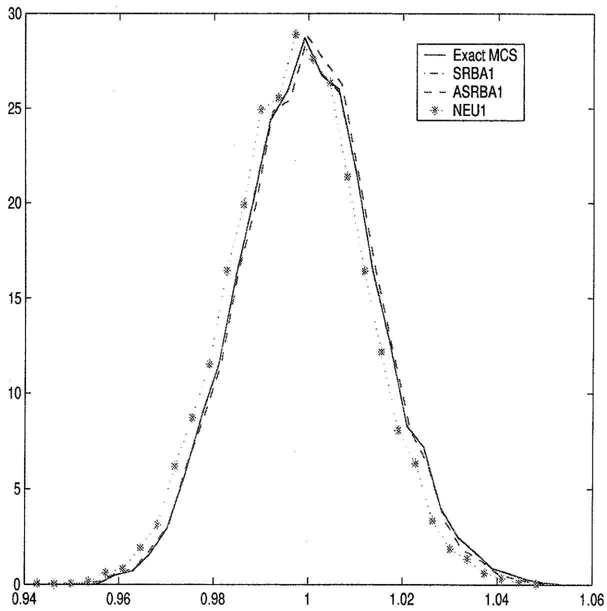

Figure 2 : Comparison of pdf of Tip Displacement Obtained Using First-order Methods for Case 1; $\sigma_\theta = 0.05$

For case 2, σ_θ was chosen to be 0.10. The trends of the percentage error in the mean and standard deviation computed using various methods for this case are summarized in Table 2. It can seen that the general trends for the mean displacement appear to be rather similar to those observed earlier for case 1. However, the errors are seen to be higher for the standard deviation of the response when the first-order methods are used, with SRBA1 giving the best results.

Table 2 : Comparison of Methods for Case 2 $\sigma_\theta = 0.10$

Method	Error in Mean (%)		Error in STD (%)	
	Max	Avg.	Max	Avg.
SRBA1	0.8199	0.0297	2.5895	1.1029
SRBA2	0.0150	0.0009	0.2601	0.0311
SRBA3	0.0056	0.0003	0.0059	0.0018
ASRBA1	1.0867	0.0496	4.3583	3.5258
ASRBA2	0.0715	0.0048	0.4238	0.1938
ASRBA3	0.0107	0.0006	0.0474	0.0147
NEU1	2.0697	1.0406	5.3869	4.5627
NEU2	0.1379	0.0372	3.3933	3.1686
NEU3	0.0833	0.0322	0.3952	0.3117

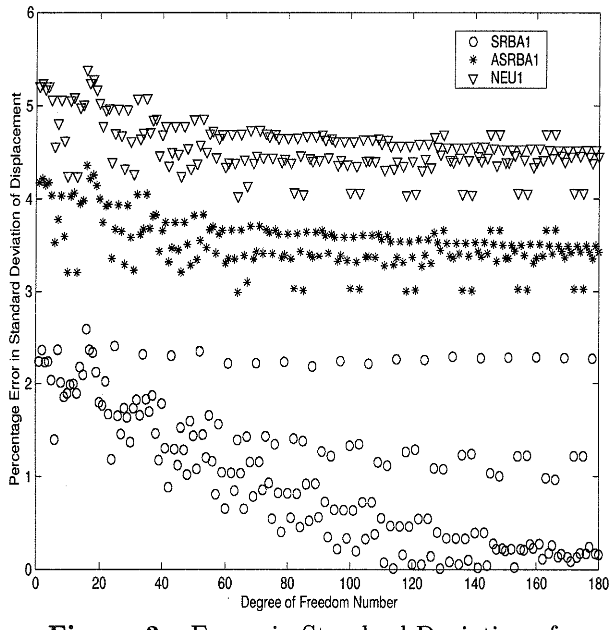

Figure 3 : Errors in Standard Deviation of Displacement Using the First-order Methods for Case 2; $\sigma_\theta = 0.10$

As observed earlier for case 1, the SRBA methods give the best results for this problem, with the results becoming nearly exact for SRBA3. Both the SRBA and ASRBA methods converge much more rapidly to the exact MCS results as compared to the Neumann expansion scheme.

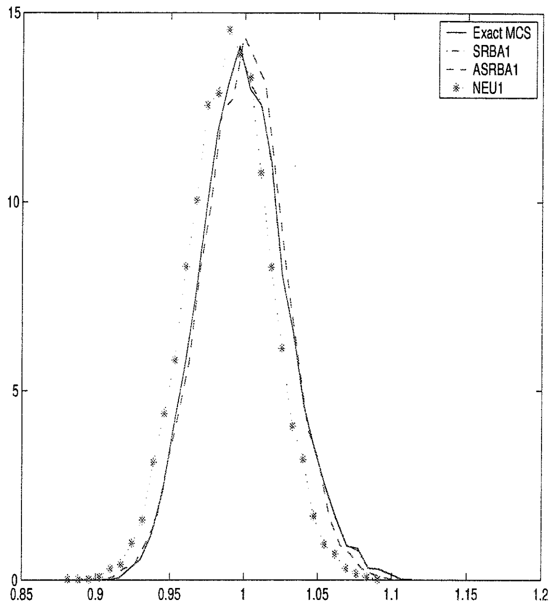

Figure 4 : Comparison of pdf of Tip Displacement Obtained Using First-order Methods for Case 2; $\sigma_\theta = 0.10$

The errors in the standard deviation of the displacement at all dofs for the first-order methods are shown in Figure 3. As expected, it can be seen that the pattern of the error distribution across the dof are identical for the ASRBA methods and the Neumann expansion scheme of same order, although the magnitude of errors are rather different. The pdf of the transverse displacement component at the tip of the structure is shown for all the first-order methods in Figure 4, along with the pdf obtained using exact MCS. It can be seen that SRBA1 and ASRBA1 show better agreement with exact MCS, particularly for the tails of the pdf.

The percentage errors in the standard deviation across all the dofs for the second-order methods are shown in Figure 5. This figure clearly illustrates the improvements which can be achieved over the Neumann expansion using a stochastic reduced basis approximation.

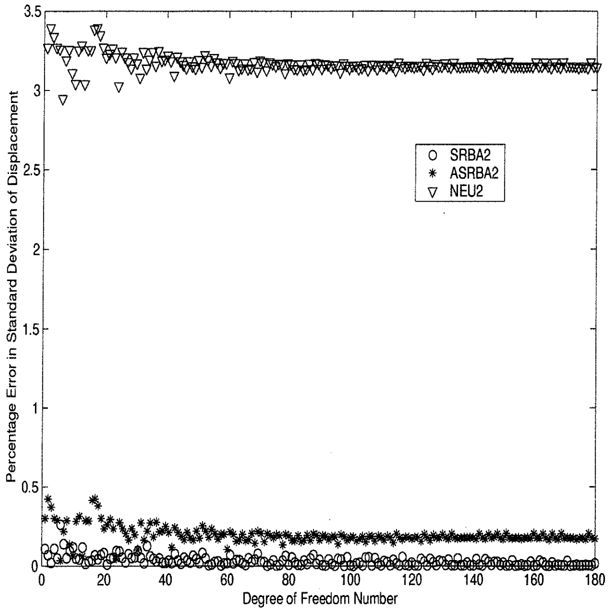

Figure 5 : Errors in Standard Deviation of Displacement Using the Second-order Methods for Case 2; $\sigma_\theta = 0.10$

In case 3, σ_θ is chosen to be 0.20. This corresponds to a pathological scenario involving large randomness in the Young's modulus, and is expected to give insights in to when the various approximate methods may break down or give unacceptable results. The trends of the percentage error in the mean and standard deviation of the displacement across all the dof are summarized in Table 3.

Table 3 : Comparison of Methods for Case 3 $\sigma_\theta = 0.20$

Method	Error in Mean (%)		Error in STD (%)	
	Max	Avg.	Max	Avg.
SRBA1	4.1740	0.3077	12.256	6.8405
SRBA2	0.2468	0.0251	1.3136	0.5975
SRBA3	0.0864	0.0055	0.2333	0.0499
ASRBA1	5.4806	0.5610	18.378	16.043
ASRBA2	1.0590	0.1131	5.6885	4.2522
ASRBA3	0.0765	0.0226	1.8173	1.3536
NEU1	9.2472	4.5225	21.845	19.610
NEU2	2.1011	0.6525	16.981	15.713
NEU3	1.6954	0.6144	6.9611	6.0466

From these trends a number of observations can be made. First, it can be clearly seen that the response statistics computed using all the first-order methods are unacceptable for the purposes of reliability analysis. The percentage error in the mean and standard deviation across all the dofs for the first-order methods are shown in Figures 6 and 7.

The pdf of the tip displacement computed using the first-order methods are compared with that obtained using exact MCS in Figure 8.

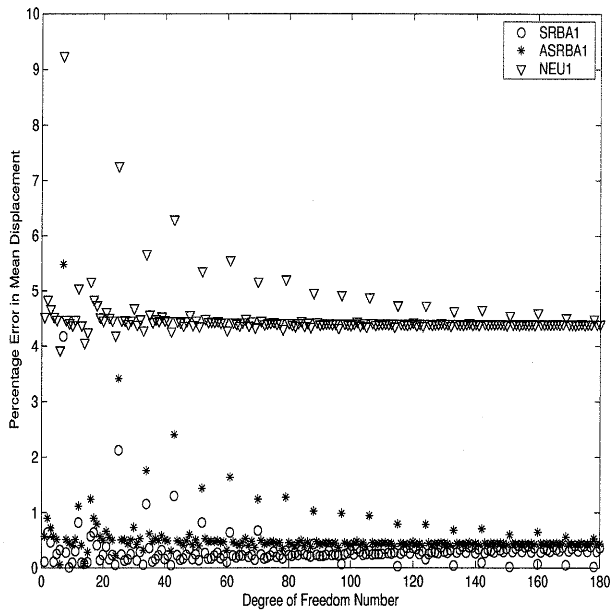

Figure 6 : Errors in Mean Displacement Using the First-order Methods for Case 3; $\sigma_\theta = 0.20$

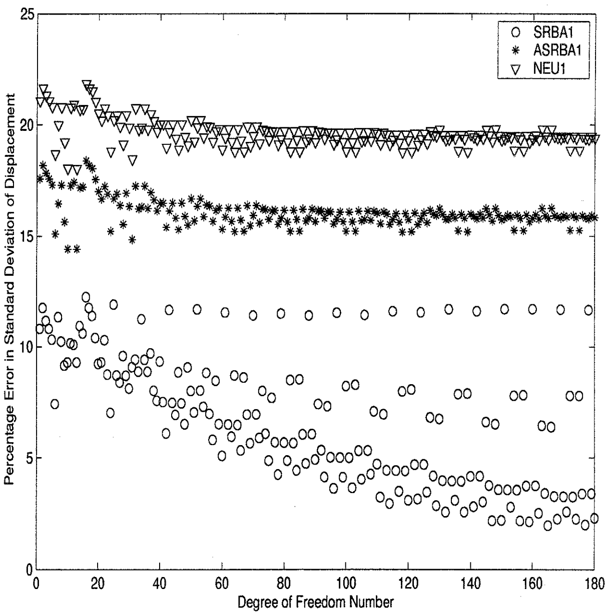

Figure 7 : Errors in Standard Deviation of Displacement Using the First-order Methods for Case 3; $\sigma_\theta = 0.20$

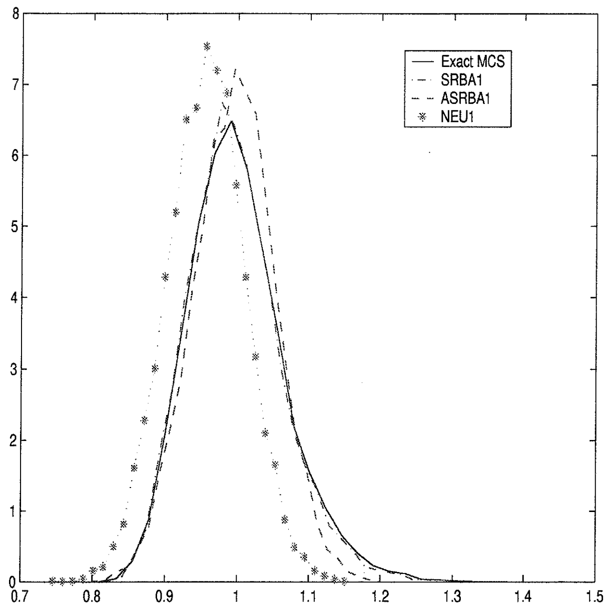

Figure 8 : Comparison of pdf of Tip Displacement Obtained Using First-order Methods for Case 3; $\sigma_\theta = 0.20$

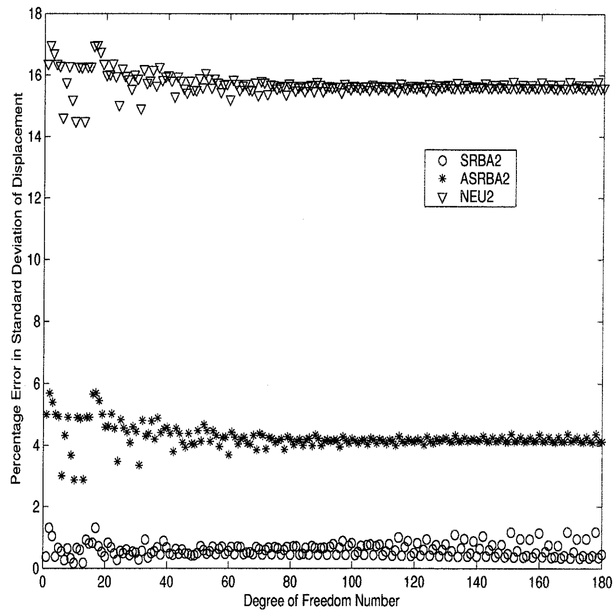

Figure 9 : Errors in Standard Deviation of Displacement Using the Second-order Methods for Case 3; $\sigma_\theta = 0.20$

It can also be observed from Table 3 that all the second-order methods show good agreement with exact MCS for the mean displacement. However, the maximum error in the standard deviation is of order of 17 % for NEU2. In comparison, the errors using SRBA2 and ASRBA2 is just of the order of 1.3 % and 5.7 %, respectively. The percentage error in the standard deviation of the displacement across all the dof using the second-order methods are compared in Figure 9.

The pdf of the tip displacement computed using the second-order methods are compared with exact MCS in Figure 10. It can be clearly seen that SRBA2 shows excellent correlation with Exact MCS. This clearly demonstrates that SRBA2 gives excellent results for large coefficient of variation of θ_i.

When the order of the methods are increased, the SRBA methods show faster rate of convergence as compared to both the other methods. It can be seen from Table 3 that SRBA3 gives nearly exact results for both statistical moments of the response. However, the expressions for SRBA3 are rather cumbersome as compared to the family of ASRBA methods. In contrast, the order of the ASRBA method can be increased to arbitrary degree with negligible increment in the computational effort. In general, it can be observed that results from ASRBA3 are comparable in accuracy to those from SRBA2.

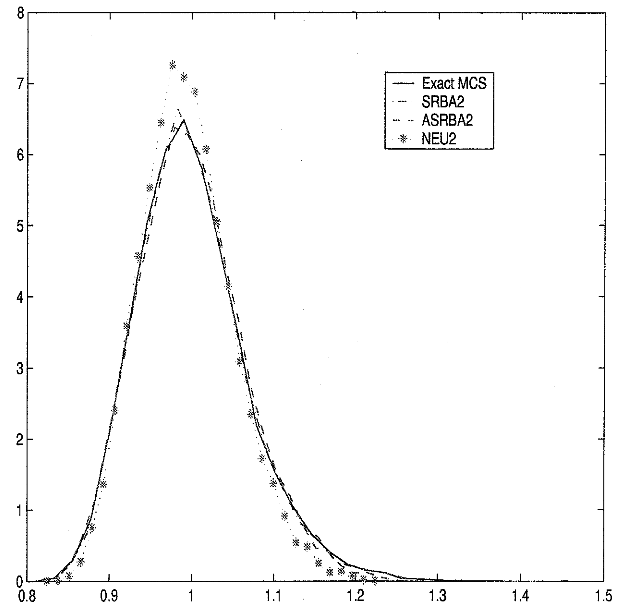

Figure 10 : Comparison of pdf of Tip Displacement Obtained Using Second-order Methods for Case 3; $\sigma_\theta = 0.20$

The percentage errors in the mean and standard deviation for all the third-order methods are compared in Figures 11 and 12. It can be clearly seen that both SRBA3 and ASRBA3 gives significantly better results as compared to NEU3. The pdf of the tip displacement for all the third-order methods are compared in Figure 13.

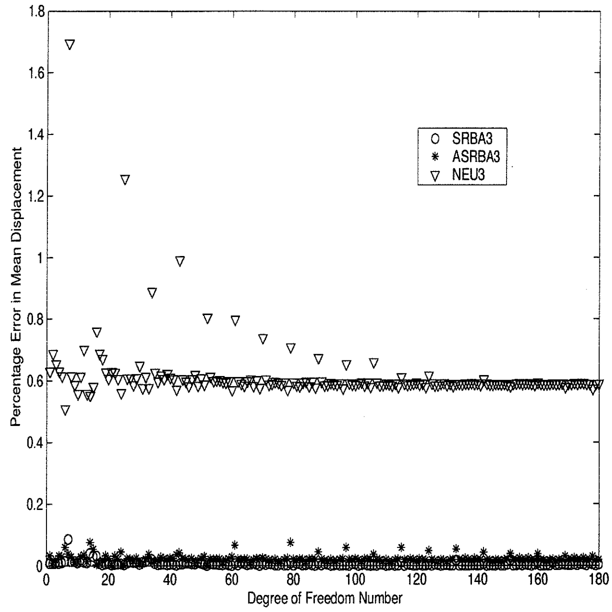

Figure 11 : Errors in Mean Displacement Using the Third-order Methods for Case 3; $\sigma_\theta = 0.20$

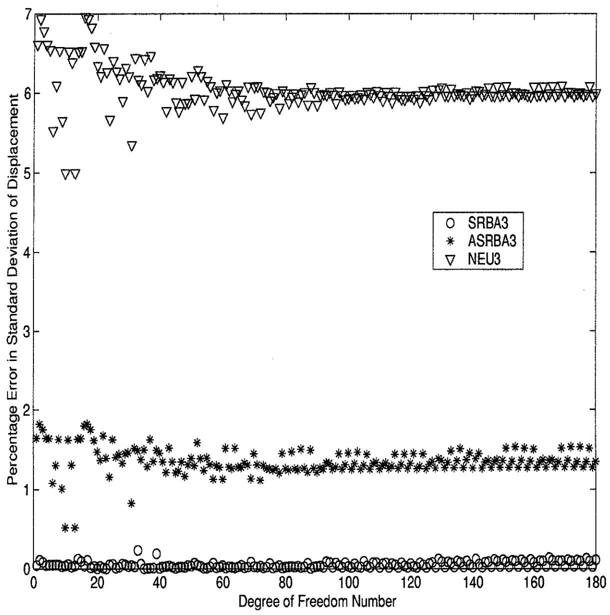

Figure 12 : Errors in Standard Deviation of Displacement Using the Third-order Methods for Case 3; $\sigma_\theta = 0.20$

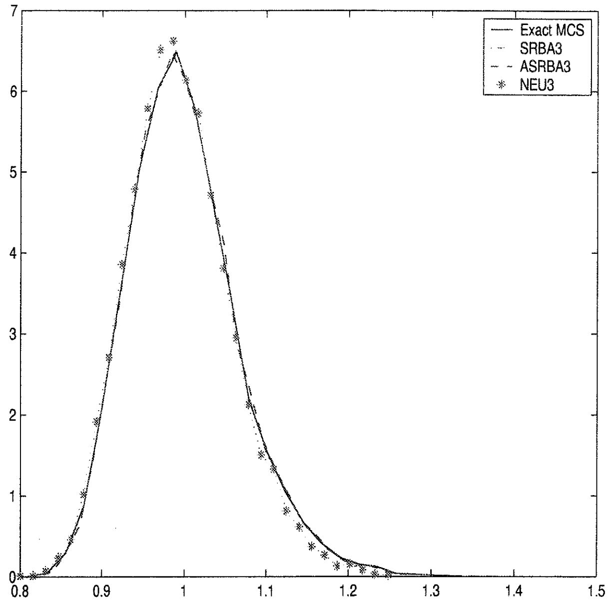

Figure 13 : Comparison of pdf of Tip Displacement Obtained Using Third-order Methods for Case 3; $\sigma_\theta = 0.20$

Frequency Response Analysis of a Stochastic System

This section presents results for frequency response analysis of a 20 member frame structure with random Young's modulus and mass density shown in Figure 14. The structure is modeled using 4 elements for each beam member, which leads to a finite element model with a total of 210 dof. The axial and flexural rigidity of each structural member are modeled as $EoA(1+\eta_i)$ and $EoI(1+\eta_i)$, $i = 1, 2, \ldots, 20$, and the mass density of each member is modeled as $\rho = \rho_o(1+\eta_i)$, $i = 21, 22, \ldots, 40$. η_i are considered as uncorrelated zero-mean Gaussian random variables, while $EoA = 6.987 \times 10^6$ N, $EoI = 1.286 \times 10^3$ Nm^2, and $\rho_o = 2.74$ kg/m. This leads to a total of 40 random system parameters for this problem.

The structure subjected to transverse harmonic excitation at node 1. The transverse component of the displacement response at node 9 is studied in the region of 0-500 Hz.

The governing equations for the frequency response of this linear system can be written as

$$\left(\mathbf{L}(\omega) + \sum_{i=1}^{40} \eta_i \mathbf{\Pi}_i(\omega)\right) \mathbf{u}(\Theta, \omega) = \mathbf{f} \qquad (5)$$

where $\mathbf{L}(\omega)$ denotes the dynamic stiffness matrix computed using the nominal values of the Young's modulus and mass density. $\mathbf{\Pi}_i(\omega)$ can be interpreted as the sensitivity of the dynamic stiffness matrix with respect to η_i. Equation (5) is required to be solved for each excitation frequency of interest.

In contrast to equation (1), equation (5) denotes a system of complex linear algebraic equations with random coefficients. Further, the stochastic basis vectors are complex, which in turn leads to the requirement of considering the coefficients in the SRBA formulation as complex quantities. It is important to note that the Galerkin projection scheme has to be appropriately modified to tackle complex vectors. In particular, the stochastic complex residual error has to be orthogonalized with respect to the approximating space by using the definition for the inner product in the field of complex numbers, i.e., $\mathbf{x}^H\mathbf{x} = 0$; where the superscript H denotes the complex conjugate transpose.

Two cases corresponding to standard deviation of η_i of 0.05 and 0.15 are considered. Results are presented for the SRBA1, SRBA2, NEU1, and NEU2 methods. The response for all the methods was computed at 150 equally spaced points in the region of 0-500 Hz. A sample size of 5000 was used for all the simulations. Again the results are compared to an "exact" MCS.

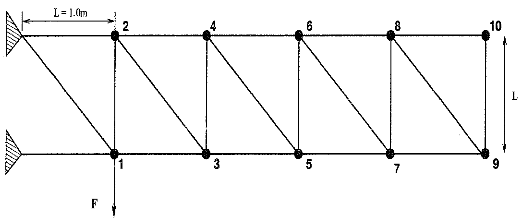

Figure 14 : 20 Member Frame Structure

The mean and standard deviation of the FRF predicted by the various methods for Case 1 are shown in Figures 15 and 16. Similar to the earlier example, it can be clearly seen that the SRBA methods allow significantly better approximations to be obtained as compared to the Neumann expansion scheme. In particular, SRBA2 gives high-quality approximations for both statistical moments of the displacement response.

It can be observed that in general that NEU2 gives more erroneous results as compared NEU1 for both statistical moments of the frequency response, particularly near resonance frequencies. This indicates that the Neumann expansion scheme fails to converge for this problem, particularly for high excitation frequencies. However, the terms of the Neumann series gives a good subspace for approximating the response process.

The mean and standard deviation of the displacement response for $\sigma_\eta = 0.15$ is shown in Figures 17 and 18. It can be seen that the errors in the standard deviation are much higher as compared to Case 1. The high error in the Neumann expansion scheme is seen to have a knock-on effect on the quality of the basis vectors used in the SRBA formulation. This leads to a slow rate of convergence for the SRBA methods, particularly for the standard deviation of the response at high excitation frequencies.

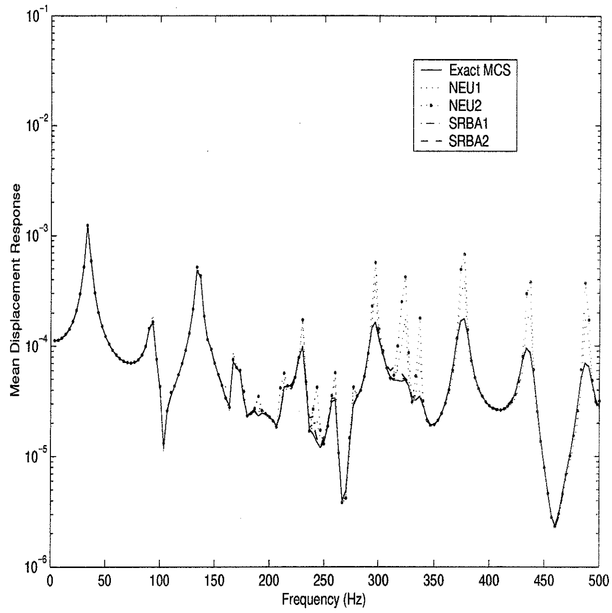

Figure 15 : Comparison of Mean FRF for Case 1 $\sigma_\eta = 0.05$

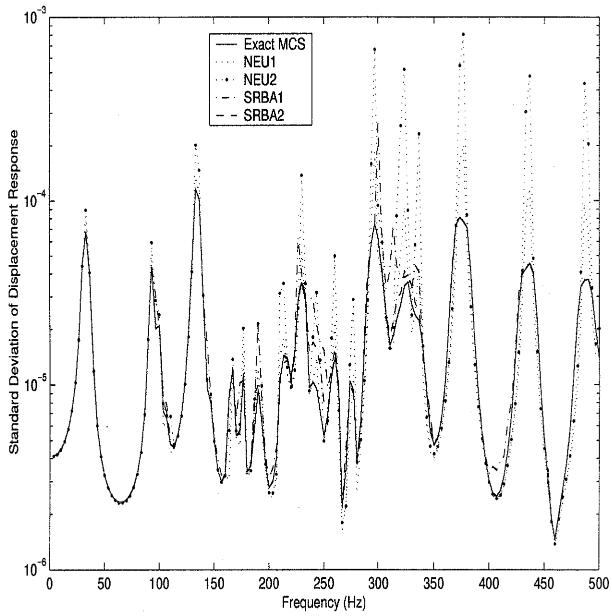

Figure 16 : Comparison of Standard Deviation of FRF for Case 1 $\sigma_\eta = 0.05$

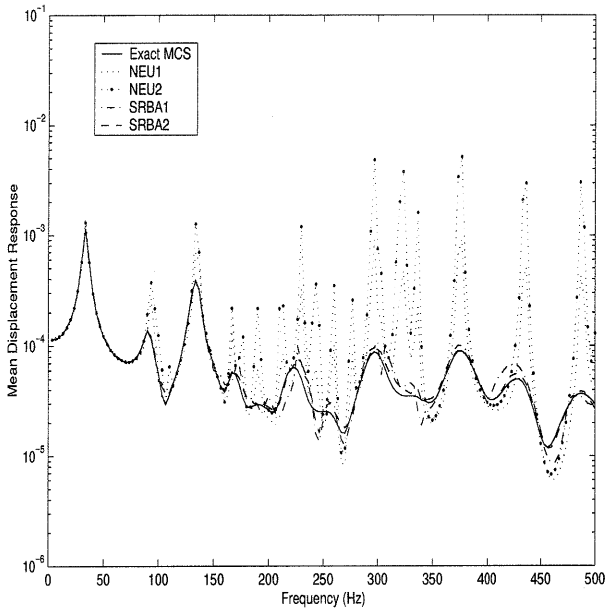

Figure 17 : Comparison of Mean FRF for Case 2 $\sigma_\eta = 0.15$

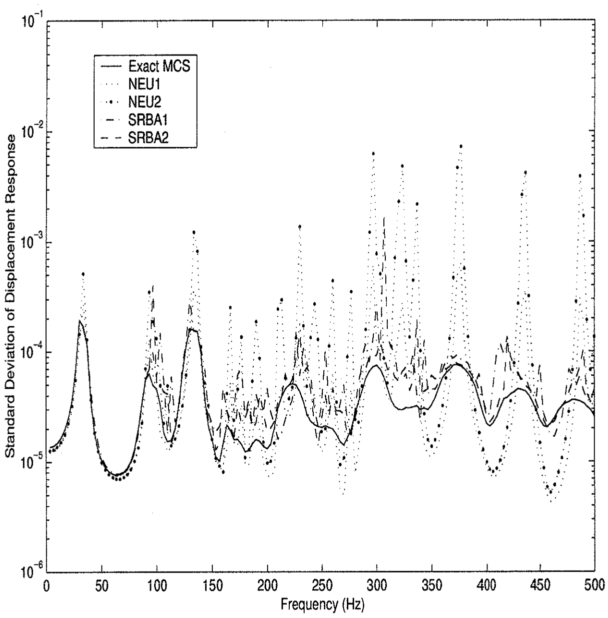

Figure 18 : Comparison of Standard Deviation of FRF for Case 2 $\sigma_\eta = 0.15$

Concluding Remarks

Detailed numerical studies have been presented for two example problems to test and validate the formulations presented in the companion paper[1]. These studies clearly demonstrate that the convergence rates of the SRBA and ASRBA methods are significantly better than for the Neumann expansion scheme. It is also shown that both variants of the SRBA methods give significantly better results for the pdf of the displacement response. In particular, for the statics problem, it was shown that high-quality approximations can be obtained using the second-order SRBA method and the third-order ASRBA method when the coefficient of variation of the random system parameters are of the order of 20%.

The results obtained for the structural dynamics problem suggests that high-quality approximations can be obtained for both the mean and standard deviation of the frequency response for small coefficient of variation of the system parameters. In contrast, the Neumann expansion does not converge at higher frequencies of excitation even for cases involving small randomness. The accuracy and convergence rate of the SRBA methods is seen to worsen at higher frequencies for moderate to large coefficient of variation of the random system parameters. This can be attributed to the divergence of the Neumann expansion scheme, which in turn calls for the requirement of increasing the number of basis vectors to achieve good quality approximations.

However, the process of testing the formulations proposed in this research is far from complete. The issues which remain to be investigated are summarized below :

- Implementation and testing of the simplified ASRBA method proposed in the companion paper[1].
- Comparison of SRBA methods with the Polynomial chaos expansion scheme presented in Ghanem and Spanos[3] in terms of accuracy, convergence characteristics, and computational cost.
- Application of SRBA methods to random vibration analysis of nonlinear stochastic systems in conjunction with stochastic linearization techniques.
- Studies on the effect of simplifying the terms in the SRBA formulation on the accuracy of the results.

As computational experience accumulates on a variety of problem domains, better insight will be obtained into the characteristics of the SRBA methods.

It is hoped that the formulations developed in this research will accelerate the development of efficient solution schemes for tackling a wide variety of problems in computational stochastic mechanics.

Acknowledgments

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