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Transfer matrix analysis of the elastostatics of one-dimensional repetitive structures

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Transfer matrices are used widely for the dynamic analysis of engineering structures, increasingly so for static analysis, and are particularly useful in the treatment of repetitive structures for which, in general, the behaviour of a complete structure can be determined through the analysis of a single repeating cell, together with boundary conditions if the structure is not of infinite extent. For elastostatic analyses, non-unity eigenvalues of the transfer matrix of a repeating cell are the rates of decay of self-equilibrated loading, as anticipated by Saint-Venant's principle. Multiple unity eigenvalues pertain to the transmission of load, e.g. tension, or bending moment, and equivalent (homogenized) continuum properties, such as cross-sectional area, second moment of area and Poisson's ratio, can be determined from the associated eigen- and principal vectors. Various disparate results, the majority new, others drawn from diverse sources, are presented. These include calculation of principal vectors using the Moore–Penrose inverse, bi- and symplectic orthogonality and relationship with the reciprocal theorem, restrictions on complex unity eigenvalues, effect of cell left-to-right symmetry on both the stiffness and transfer matrices, eigenvalue veering in the absence of translational symmetry and limitations on possible Jordan canonical forms. It is shown that only a repeating unity eigenvalue can lead to a non-trivial Jordan block form, so degenerate decay modes cannot exist. The present elastostatic analysis complements Langley's (Langley 1996 *Proc. R. Soc. A* **452**, 1631–1648) transfer matrix analysis of wave motion energetics.

Keywords: transfer; symplectic; matrix; elastostatic; pseudo-inverse; Jordan canonical form

1. Introduction

Repetitive (or periodic) structures are analysed most efficiently when such periodicity is taken into account. A typical approach (Langley 1996) relates a state vector of displacement and force components on either side of a generic repeating cell by a transfer matrix, $\boldsymbol{G}$, which may be determined from the stiffness matrix $\boldsymbol{K}$, although discrete displacement field (Karpov *et al.* 2002*a*) and discrete Fourier transform (Karpov *et al.* 2002*b*) formulations are also possible. An eigenvector of the transfer matrix describes a pattern of displacement and force components which is unique to within a scalar multiplier; translational

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symmetry demands that this pattern is preserved as one moves from the left-hand to the right-hand side of the cell, allowing one to write $\boldsymbol{s}_{\mathrm{R}} = \lambda \boldsymbol{s}_{\mathrm{L}}$; this leads to the standard eigenvalue problem $\lambda \boldsymbol{s}_{\mathrm{L}} = \boldsymbol{G}\boldsymbol{s}_{\mathrm{L}}$, or $(\boldsymbol{G} - \lambda \boldsymbol{I})\boldsymbol{s}_{\mathrm{L}} = \boldsymbol{0}$. For dynamic problems, this is an application of Bloch's theorem (see Brillouin 1953), and leads to an eigenvalue problem for the propagation constants or, equivalently, the natural frequencies; the approach is highly developed and has been applied to both one-dimensional (beam-like) and two-dimensional (plate-like) problems (Mead 1970, 1996; Meirowitz & Engels 1977; Yong & Lin 1989*a*,*b*; Zhong & Williams 1992, 1995; Langley 1996). The theory is less well developed for static analysis, but has been applied to one-dimensional prismatic planar (Stephen & Wang 1996*a*), asymmetric and pre-twisted (Stephen & Zhang 2004, 2006) and curved repetitive structures (Stephen & Ghosh 2005).

This paper is concerned largely with the static problem, and presents numerous results concerning the eigenanalysis of a (real) transfer matrix $\boldsymbol{G}$; some of these results are not new, but are drawn from diverse references in order to provide a convenient resource. As a consequence of the symmetry of the stiffness matrix $\boldsymbol{K}$, the transfer matrix has the property of being symplectic (Pease 1965), so its eigenvalues occur as reciprocals, and fall into five possible classes (Meyer & Hall 1991):

(i) The real unity eigenvalue $\lambda = 1$, which must occur an even number of times; the inverse is a repeat.
(ii) The negative real unity eigenvalue $\lambda = -1$, which must occur an even number of times; again the inverse is a repeat.
(iii) The real non-unity eigenvalues occur as a pair λ and λ^{-1}.
(iv) The complex unity eigenvalues occur as a unitary pair $\lambda = \mathrm{e}^{\mathrm{i}\alpha}$ and $\lambda = \mathrm{e}^{-\mathrm{i}\alpha}$; the inverse is simultaneously the complex conjugate.
(v) The general complex eigenvalues occur as a quartet of reciprocals and complex conjugates, that is $\lambda = a + \mathrm{i}b$, $\lambda = a - \mathrm{i}b$, $\lambda = (a + \mathrm{i}b)^{-1}$ and $\lambda = (a - \mathrm{i}b)^{-1}$ are all, as an ensemble, possible eigenvalues.

A consequence of the above is that the determinant of a symplectic matrix is equal to +1. Suppose that λ is an eigenvalue having multiplicity k, then λ^{-1} is an eigenvalue also having multiplicity k; thus, the Jordan blocks corresponding to λ and λ^{-1} have the same structure. The group of real symplectic matrices of size $2n \times 2n$, denoted $\mathrm{Sp}(n, \mathbb{R})$, is the fundamental group underlying classical mechanics yet, according to Abraham & Marsden (1978), 'very little application of its structure seems to have been made beyond these elementary eigenvalue properties'. Symplectic matrices have the property of preserving Hamiltonian structure, and are often employed as similarity matrices within the field of Floquet (periodic) dynamic systems, such as one finds in the field of celestial mechanics (Meyer & Hall 1991); they also find application within optimal control engineering (Stengel 1986), and time-series analysis (Aoki 1987). Their introduction into the field of solid mechanics is largely due to Zhong *et al.* (1992), Zhong & Williams (1992, 1995) and Zhong (1995).

For a one-dimensional (beam-like) repetitive structure, non-unity eigenvalues pertain to the decay of self-equilibrated loading, as anticipated by Saint-Venant's principle. Multiple unity eigenvalues pertain to transmission of end loading, as in Saint-Venant's problem, and the associated eigen- and principal (or generalized) vectors transform the transfer matrix to a Jordan canonical form (JCF), revealing

Table 1. Notation.

a, A, $\boldsymbol{A}$	constant, cross-sectional area, matrix
b, $\boldsymbol{b}$, $\boldsymbol{B}$	constant, column vector, similarity matrix
d, $\boldsymbol{d}$	component of, nodal displacement vector
$\boldsymbol{e}$, E	eigenvector, Young's modulus
F, $\boldsymbol{F}$	component of, nodal force vector
$\boldsymbol{G}$, G	transfer matrix, shear modulus
$\boldsymbol{H}$	Hamiltonian system matrix
$\boldsymbol{I}$, I	identity matrix, second moment of area
$\boldsymbol{J}$, $\boldsymbol{J}_{\mathrm{m}}$	Jordan canonical form, metric
L	left, length
$\boldsymbol{K}$, k	stiffness matrix, dimension of Jordan block, eigenvalue
M	nodal moment component
$\boldsymbol{N}$, n, N	nilpotent matrix, index, number of cells, nodal moment component
p, $\boldsymbol{p}$	component of, nodal force vector, principal vector
R, $\boldsymbol{R}$, $\mathbb{R}$	right, reflection matrix, real numbers
$\boldsymbol{s}$, $\boldsymbol{S}$	state vector, real symmetric matrix
$\boldsymbol{V}$	similarity matrix of eigen- and principal vectors
x, y, z	Cartesian coordinates
$\boldsymbol{x}$	vector
$\boldsymbol{X}$, $\boldsymbol{Y}$	(right) eigenvectors of $\boldsymbol{G}$ and $\boldsymbol{G}^{\mathrm{T}}$
α	arbitrary angle
β	arbitrary multiple
λ	eigenvalue
κ	shear coefficient (in Timoshenko beam theory)
ν	Poisson's ratio
θ	arbitrary angle
gld	greatest linear dimension
JCF	Jordan canonical form
WD	work done

couplings between the various modes; for example, a shearing force is inevitably coupled to a bending moment. Moreover, equivalent continuum (homogenized) properties of the reticulated structure may be calculated.

Studies by the present author and co-workers have previously concentrated on pin-jointed repetitive structures; this choice was made because the finite-element analysis (FEA) of such structures may be regarded as exact; in turn, predictions from the eigenanalysis can be verified by comparison with what may be regarded as exact FEA results. This does not imply that pin-jointed structures are not of interest in their own right; indeed, the removal of members from such a structure can reduce it to a mechanism, which in turn allows its transportation in a very compact form, an attribute likely to find favour in aerospace application. Here we record the effect of rigid- rather than pin-jointing; the continuum properties are virtually unchanged, and one has new decay modes associated with self-equilibrated moments applied at the nodes. Repetitive structures are not limited to frameworks, or lattice structures; a continuum structure such as a metre rule would be perfectly repetitive were it not for the progressive numbering along its length—each centimetre of rule is identical to that preceding and following. Thus, Stephen & Wang (1996*b*) have developed a hybrid finite element/transfer matrix

method that allows one to calculate the Saint-Venant decay rates of self-equilibrated loading for rods or beams of general cross-section. On the other hand, if there is no discretization of the continuum, then the symplectic transfer matrix $\boldsymbol{G}$ is replaced by a Hamiltonian system matrix $\boldsymbol{H}$, within a relationship of the form

$$\frac{\mathrm{d}\boldsymbol{s}}{\mathrm{d}x} = \boldsymbol{H}\boldsymbol{s}, \tag{1.1}$$

where the state vector $\boldsymbol{s}$ consists of the displacement components and the cross-sectional stress components. This state-space approach to the *Theory of Elasticity* was developed by Zhong (1995), and an exposition was presented recently by Stephen (2004) for the elastostatics of a prismatic rod or beam, where it was also shown that only a repeating zero eigenvalue can lead to a non-trivial Jordan block form.

The present paper may be seen as complementary to a recent analysis of wave motion energetics using transfer matrices (Langley 1996). However, besides some introductory definitions and results, the focus is quite different; the latter specifically excludes issues relating to principal eigenvectors attendant upon the multiple (unity) eigenvalues, which is a particular feature of the static analysis. Moreover, dynamic analyses typically assume the structure to be of infinite extent, which avoids issues relating to boundary conditions. For the static case, while general results may be gleaned from a single cell, boundary conditions at both ends of the structure must be taken into account in order to provide a complete solution to any particular problem. The introductory material includes previously known results, including the reciprocal eigenvalue properties as a consequence of the symplectic nature of the transfer matrix, bi- and symplectic orthogonality and the relationship with the Betti–Maxwell reciprocal theorem, and the impossibility of complex unity eigenvalues for prismatic repetitive structures. The Moore–Penrose pseudo-inverse is introduced as a rational approach to the computation of principal vectors associated with the multiple unity eigenvalues. Employing a strain energy argument, it is shown that only the eigenvalues $\lambda = \pm 1$ can give rise to a non-trivial JCF, at least for the prismatic structure. An example of a structure for which the transfer matrix has repeating negative unity eigenvalue is one possessing a scissor-like mechanism, and this possibility can be gleaned through simple arguments regarding the dimension of the transfer matrix. A planar structure, previously treated as pin-jointed, is reconsidered as rigid-jointed; the additional rotational nodal degrees of freedom give rise to new Saint-Venant decay modes—the number of transmission modes associated with unity eigenvalues is fixed—while the equivalent continuum properties are practically unaffected. This confirms the practice of treating real, rigid-jointed structures as pin-jointed, at least for small deflection elastic analysis. A variety of relationships between partitions of both the stiffness and transfer matrices results are presented for a cell possessing left-to-right symmetry. On the other hand, lack of symmetry implies less restriction, and one has splitting of unity eigenvalues for a tapered cell that lacks translational symmetry.

2. Transfer matrix formulation

The rigid- or pin-jointed planar framework under consideration is shown in figure 1. The geometric and material properties are identical to those employed by

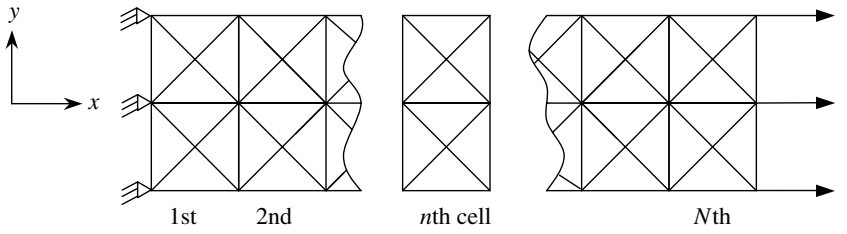

Figure 1. Rigid- or pin-jointed planar framework, fixed at left-hand end and subject to tensile force at the right; the length of the truss is equal to the number of the cells, N.

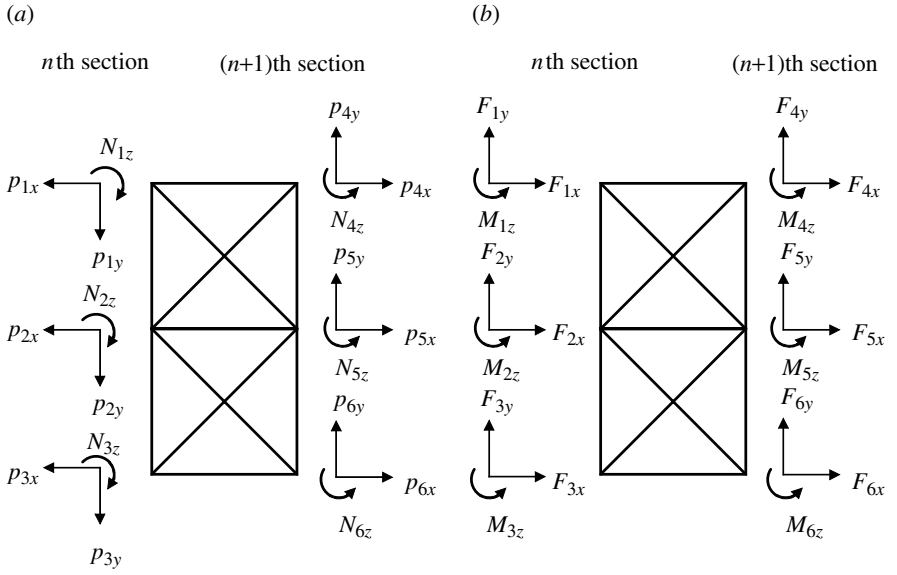

Figure 2. Single, $(n+1)$th, cell of rigid-jointed framework in figure 1; (a) and (b) show positive joint force and moment according to transfer matrix and FEA sign conventions, respectively.

Stephen & Wang (1996*a*): the Young modulus of each member is $E = 200 \times 10^9$ N/m^2, horizontal and vertical members are of length $L = 1$ m and have (circular) cross-sectional area $A = 1$ cm^2; the diagonal members have length $\sqrt{2}$ m and cross-sectional area 0.5 cm^2. However, since vertical members are regarded as being shared between adjacent cells, the repeating single cell must have vertical members with one-half stiffness; for the pin-jointed structure, this just requires that the cross-sectional areas should be $A/2$. For the rigid-jointed structure, the members can also carry bending moment, so it is also required that the bending stiffness and, hence, the second moment of area should be halved.

A typical cell located between the nth and $(n+1)$th sections of the structure in figure 1 is shown in figure 2. Let $\boldsymbol{p}_n$ and $\boldsymbol{d}_n$ denote the generalized nodal force and displacement vectors, respectively, associated with the nth section; the state vectors at the section nth and $(n+1)$th sections are then $\boldsymbol{s}_n = [\boldsymbol{d}_{\mathrm{n}}^{\mathrm{T}} \quad \boldsymbol{p}_n^{\mathrm{T}}]^{\mathrm{T}}$ and $\boldsymbol{s}_{n+1} = [\boldsymbol{d}_{n+1}^{\mathrm{T}} \quad \boldsymbol{p}_{n+1}^{\mathrm{T}}]^{\mathrm{T}}$, and they are related by the transfer matrix $\boldsymbol{G}$ through the equation

$$\boldsymbol{s}_{n+1} = \boldsymbol{G}\boldsymbol{s}_n, \tag{2.1}$$

or in partitioned form

$$\begin{bmatrix} \boldsymbol{d}_{n+1} \\ \boldsymbol{p}_{n+1} \end{bmatrix} = \begin{bmatrix} \boldsymbol{G}_{dd} & \boldsymbol{G}_{dp} \\ \boldsymbol{G}_{pd} & \boldsymbol{G}_{pp} \end{bmatrix} \begin{bmatrix} \boldsymbol{d}_{n} \\ \boldsymbol{p}_{n} \end{bmatrix}. \tag{2.2}$$

Two consecutive state vectors are also related by a scalar λ as

$$\boldsymbol{s}_{n+1} = \lambda \boldsymbol{s}_n; \tag{2.3}$$

this is the static equivalent of an application of Bloch's theorem for systems possessing translational symmetry. Substitution of the above into equation (2.1) leads directly to the standard eigenproblem

$$[\boldsymbol{G} - \lambda \boldsymbol{I}]\boldsymbol{s}_n = 0, \tag{2.4}$$

where $\boldsymbol{I}$ is the identity matrix of the appropriate size. The eigenvalues of the transfer matrix describe how associated eigenvectors scale as one moves from one nodal section to the next. A unity eigenvalue implies that it is transmitted unchanged, while a non-unity eigenvalue $|\lambda| < 1$ implies that the nodal displacements and forces decay as one moves from cell to cell, left-to-right; the reciprocal, $|\lambda^{-1}| > 1$ represents an increase from left-to-right, hence a decay from right-to-left.

The transfer matrix $\boldsymbol{G}$ is obtained from the stiffness matrix $\boldsymbol{K}$ of the single repeating cell; referring to figure 2*b*, the generalized force and displacement vectors $\boldsymbol{F}$ and $\boldsymbol{d}$ are related by the stiffness matrix equation $\boldsymbol{F} = \boldsymbol{K}\boldsymbol{d}$, or in partitioned form

$$\begin{bmatrix} \boldsymbol{F}_{n} \\ \boldsymbol{F}_{n+1} \end{bmatrix} = \begin{bmatrix} \boldsymbol{K}_{\mathrm{LL}} & \boldsymbol{K}_{\mathrm{LR}} \\ \boldsymbol{K}_{\mathrm{RL}} & \boldsymbol{K}_{\mathrm{RR}} \end{bmatrix} \begin{bmatrix} \boldsymbol{d}_{n} \\ \boldsymbol{d}_{n+1} \end{bmatrix}. \tag{2.5}$$

Transfer matrix analysis employs the sign conventions of the *Theory of Elasticity*, so set $\boldsymbol{F}_n = -\boldsymbol{p}_n$, $\boldsymbol{F}_{n+1} = \boldsymbol{p}_{n+1}$, and substitute into equation (2.5), expand and rearrange to give

$$\begin{bmatrix} \boldsymbol{d}_{n+1} \\ \boldsymbol{p}_{n+1} \end{bmatrix} = \boldsymbol{G} \begin{bmatrix} \boldsymbol{d}_{n} \\ \boldsymbol{p}_{n} \end{bmatrix}, \tag{2.6}$$

when the transfer matrix $\boldsymbol{G}$ becomes

$$\boldsymbol{G} = \begin{bmatrix} -\boldsymbol{K}_{\mathrm{LR}}^{-1}\boldsymbol{K}_{\mathrm{LL}} & -\boldsymbol{K}_{\mathrm{LR}}^{-1} \\ \boldsymbol{K}_{\mathrm{RL}} - \boldsymbol{K}_{\mathrm{RR}}\boldsymbol{K}_{\mathrm{LR}}^{-1}\boldsymbol{K}_{\mathrm{LL}} & -\boldsymbol{K}_{\mathrm{RR}}\boldsymbol{K}_{\mathrm{LR}}^{-1} \end{bmatrix}. \tag{2.7}$$

Having performed the eigenanalysis, a similarity matrix $\boldsymbol{V}$ consisting of all eigen- and principal vectors including both decay and transmission modes can be constructed and this transforms the transfer matrix $\boldsymbol{G}$ to JCF according to

$$\boldsymbol{V}^{-1}\boldsymbol{G}\boldsymbol{V} = \boldsymbol{J}, \tag{2.8}$$

where $\boldsymbol{J}$ is the JCF. The multiple unity eigenvalue typically appears in two or more distinct Jordan blocks, so the transfer matrix $\boldsymbol{G}$ is both *defective* and *derogatory*. Not only does the JCF reveals the coupling between the various modes, e.g. the shearing force principal vector is coupled to the bending moment vector, it also allows one to calculate powers of the transfer matrix $\boldsymbol{G}$ in the most efficient and accurate manner; suppose one knows the applied state vector $\boldsymbol{s}(0)$ on the zero$^{\mathrm{th}}$ left-hand end of the structure in figure 1, the state vector on the right-hand side of this first cell is given by

$$\boldsymbol{s}(1) = \boldsymbol{G}\boldsymbol{s}(0), \tag{2.9}$$

and the state vector on the right-hand side of the nth cell is then

$$\boldsymbol{s}(n) = \boldsymbol{G}^n \boldsymbol{s}(0). \tag{2.10}$$

Powers of the transfer matrix (the cumulative transfer matrix) are evaluated according to

$$\boldsymbol{G}^n = (\boldsymbol{V}\boldsymbol{J}\boldsymbol{V}^{-1})^n = (\boldsymbol{V}\boldsymbol{J}\boldsymbol{V}^{-1})(\boldsymbol{V}\boldsymbol{J}\boldsymbol{V}^{-1})\ldots(\boldsymbol{V}\boldsymbol{J}\boldsymbol{V}^{-1}) = \boldsymbol{V}\boldsymbol{J}^n\boldsymbol{V}^{-1}. \tag{2.11}$$

Moreover, the nth power of the JCF simply requires evaluation of the nth power of the diagonal elements, although for the non-trivial Jordan blocks, a more involved treatment is required. Let $\boldsymbol{J}_i$ be a Jordan block pertaining to eigenvalue λ_i, having dimension $k\times k$, written as

$$\boldsymbol{J}_i = \begin{bmatrix} \lambda_i & 1 & & \\ & \ddots & \ddots & \\ & & \lambda_i & 1 \\ & & & \lambda_i \end{bmatrix} = \lambda_i \boldsymbol{I} + \boldsymbol{N}_i, \tag{2.12}$$

where $\boldsymbol{N}_i$ is the nilpotent matrix

$$\boldsymbol{N}_i = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ & & & 0 \end{bmatrix}. \tag{2.13}$$

The kth power of the nilpotent matrix is zero, so the binomial expansion of $\boldsymbol{J}_i^n = (\lambda_i \boldsymbol{I} + \boldsymbol{N}_i)^n$ has a finite number of terms as

$$(\lambda_i \boldsymbol{I} + \boldsymbol{N}_i)^n = \lambda_i^n \boldsymbol{I} + \binom{n}{1}\lambda_i^{n-1}\boldsymbol{N}_i + \binom{n}{2}\lambda_i^{n-2}\boldsymbol{N}_i^2 + \cdots + \binom{n}{k-1}\lambda_i^{n-k+1}\boldsymbol{N}_i^{k-1}, \tag{2.14}$$

since higher powers of $\boldsymbol{N}_i$ are zero; in the above, the binomial coefficients are given by

$$\binom{n}{b} = \frac{n(n-1)(n-2)\cdots(n-b+1)}{b!}. \tag{2.15}$$

The nth power of the Jordan block becomes

$$\boldsymbol{J}_i^n = \begin{bmatrix} \lambda_i^n & n\lambda_i^{n-1} & \dfrac{n(n-1)}{2}\lambda_i^{n-2} & \cdots \\ 0 & \lambda_i^n & n\lambda_i^{n-1} & \cdots \\ 0 & 0 & \lambda_i^n & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}. \tag{2.16}$$

Expression (2.10) becomes

$$\boldsymbol{s}(n) = \boldsymbol{V}\boldsymbol{J}^n\boldsymbol{V}^{-1}\boldsymbol{s}(0), \tag{2.17}$$

but before it can be applied one requires complete knowledge of the state vector $\boldsymbol{s}(0)$; typically one has partial information at both ends of the structure. Referring

to figure 1, one knows the force vector $\boldsymbol{p}_N$ at the Nth nodal cross-section, but not the displacement components. On the other hand, at the fully fixed left-hand end, one knows the displacement vector $\boldsymbol{d}_0$ (equal to a zero column) but not the force vector $\boldsymbol{p}_0$ which, besides the reaction to the tensile force, must contain self-equilibrating load sufficient to suppress Poisson's ratio contraction effects. The state vectors at either end are related by a cumulative transfer matrix as

$$\boldsymbol{s}(N) = \boldsymbol{V}\boldsymbol{J}^N\boldsymbol{V}^{-1}\boldsymbol{s}(0), \tag{2.18}$$

or in more detail

$$\begin{bmatrix}\boldsymbol{d}_N\\ \boldsymbol{p}_N\end{bmatrix} = \begin{bmatrix}\boldsymbol{G}_{ddN} & \boldsymbol{G}_{dpN}\\ \boldsymbol{G}_{pdN} & \boldsymbol{G}_{ppN}\end{bmatrix}\begin{bmatrix}\boldsymbol{d}_0\\ \boldsymbol{p}_0\end{bmatrix}, \tag{2.19}$$

where $\boldsymbol{G}_{ddN}$, $\boldsymbol{G}_{dpN}$, $\boldsymbol{G}_{pdN}$ and $\boldsymbol{G}_{ppN}$ are square partitions of $\boldsymbol{V}\boldsymbol{J}^N\boldsymbol{V}^{-1}$. With the exception of $\boldsymbol{G}_{pdN}$, these partitions are invertible; partition $\boldsymbol{G}_{pdN}$ must be singular, as the displacement vector $\boldsymbol{d}_0$ cannot be calculated from knowledge of the force vectors at each end—one can always add rigid body displacements or rotations. From the second row, one has $\boldsymbol{p}_0 = \boldsymbol{G}_{ppN}^{-1}\boldsymbol{p}_N - \boldsymbol{G}_{ppN}^{-1}\boldsymbol{G}_{pdN}\boldsymbol{d}_0$, and substituting this into the second row yields $\boldsymbol{d}_N = (\boldsymbol{G}_{ddN} - \boldsymbol{G}_{dpN}\boldsymbol{G}_{ppN}^{-1}\boldsymbol{G}_{pdN})\boldsymbol{d}_0 + \boldsymbol{G}_{dpN}\boldsymbol{G}_{ppN}^{-1}\boldsymbol{p}_N$. One now has complete knowledge of the state vectors at each end. This process may be extended to other end conditions: for example, suppose that the uppermost support at the left-hand of the structure, figure 1, is removed; now only four of the elements of the vector $\boldsymbol{d}_0$ are equal to zero, but two of the elements of $\boldsymbol{p}_0$ would now also be equal to zero. Thus, the state vector $\boldsymbol{s}(0)$ can be re-ordered as six known (subscript k) and six unknown (subscript u) components as

$$\begin{bmatrix}\boldsymbol{d}_N\\ \boldsymbol{p}_N\end{bmatrix} = \begin{bmatrix}\boldsymbol{G}_{dkN} & \boldsymbol{G}_{duN}\\ \boldsymbol{G}_{pkN} & \boldsymbol{G}_{puN}\end{bmatrix}\begin{bmatrix}\boldsymbol{s}_k(0)\\ \boldsymbol{s}_u(0)\end{bmatrix}, \tag{2.20}$$

with rows and columns of the complete structure transfer matrix being rearranged accordingly; the above manipulations are now applied to equation (2.20) rather than (2.19).

3. Calculation of principal vectors: the pseudo-inverse

Numerical determination of a non-trivial JCF of a matrix is generally unstable, and is possible only if the matrix elements are known exactly, for example, as integers or integer fractions, or if the repeating eigenvalues are known exactly, for example, on physical grounds (Kailath 1980); fortunately, the latter is true for the multiple real and complex unity eigenvalues that arise in the present approach. The majority of textbooks (e.g. Ogata 1990) suggest that one should start by calculating the principal vector of highest grade, that is, for a chain of k eigen- and principal vectors, one should first determine the principal vector $\boldsymbol{p}_k$ from the equation

$$[\boldsymbol{G} - \lambda\boldsymbol{I}]^k\boldsymbol{p}_k = 0; \tag{3.1}$$

this entails an arbitrary choice of some elements of $\boldsymbol{p}_k$, since $[\boldsymbol{G} - \lambda\boldsymbol{I}]^k$ is rank deficient, but once this choice has been made, all of the principal vectors of lower grade (including the eigenvector) follow unambiguously. Instead of this procedure, the present author has previously calculated the chain in reverse order, primarily because the eigenvectors are obviously the rigid body displacements and possibly

rotation; essentially, this is a trade between choosing arbitrary elements of the principal vector of highest grade just once (but with no guarantee that this will lead to the simplest eigenvector) and making a simple choice several times over as one works along the chain toward the highest grade principal vector. The Moore–Penrose or pseudo-inverse of a rank-deficient matrix removes this element of choice and assigns values to the arbitrary elements in a rational manner.

Consider the matrix equation $\boldsymbol{A}\boldsymbol{x}=\boldsymbol{b}$; the most common application of the pseudo-inverse is when one has more equations than unknowns, which is typical of linear regression of experimental data and there is no solution in the classical sense. Matrix $\boldsymbol{A}$ has more rows than columns, so is obviously not invertible. Pre-multiply by $\boldsymbol{A}^{\mathrm{T}}$ to give

$$\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{x}=\boldsymbol{A}^{\mathrm{T}}\boldsymbol{b}. \tag{3.2}$$

Matrix $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}$ is now square and may be inverted to give $\boldsymbol{x}=(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A})^{-1}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{b}$, or $\boldsymbol{x}=\boldsymbol{A}^{\mathrm{LM}}\boldsymbol{b}$, where the *left* pseudo-inverse is $\boldsymbol{A}^{\mathrm{LM}}=(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A})^{-1}\boldsymbol{A}^{\mathrm{T}}$. This *solution* is often denoted as $\boldsymbol{x}^{\circ}$ and has the property of minimizing the norm $\|\boldsymbol{A}\boldsymbol{x}^{\circ}-\boldsymbol{b}\|$.

On the other hand, when $\boldsymbol{A}$ has more columns than rows (rank deficient), which is typical of the situation when calculating eigen- and principal vectors, there are an infinite number of solutions (recall that an eigenvector is a unique pattern which may be multiplied by a scalar to give an equally valid eigenvector; similarly, one may add an arbitrary multiple of a generating eigenvector to a principal vector and it is still a principal vector). Now, one requires the *right* pseudo-inverse defined as $\boldsymbol{A}^{\mathrm{RM}}=\boldsymbol{A}^{\mathrm{T}}(\boldsymbol{A}\boldsymbol{A}^{\mathrm{T}})^{-1}$, for which

$$\boldsymbol{A}\boldsymbol{A}^{\mathrm{RM}}=\boldsymbol{A}\boldsymbol{A}^{\mathrm{T}}(\boldsymbol{A}\boldsymbol{A}^{\mathrm{T}})^{-1}=\boldsymbol{I}; \tag{3.3}$$

the above expression (which is equal to a conforming identity matrix) is now *shoe-horned* into equation (3.2) to give

$$\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{x}^{\circ}=\boldsymbol{A}^{\mathrm{T}}\left(\underbrace{\boldsymbol{A}\boldsymbol{A}^{\mathrm{T}}(\boldsymbol{A}\boldsymbol{A}^{\mathrm{T}})^{-1}}_{\boldsymbol{I}}\right)\boldsymbol{b}. \tag{3.4}$$

As with the left pseudo-inverse, now pre-multiply by $(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A})^{-1}$ to give

$$\boldsymbol{x}^{\circ}=\underbrace{\boldsymbol{A}^{\mathrm{T}}(\boldsymbol{A}\boldsymbol{A}^{\mathrm{T}})^{-1}\boldsymbol{b}}_{\boldsymbol{A}^{\mathrm{RM}}}. \tag{3.5}$$

Of the infinite number of solutions, the above is that which has minimum norm $\|\boldsymbol{x}^{\circ}\|$, i.e. $\boldsymbol{x}^{\circ}$ is closest to the origin. Physically, one is adding multiples of the generating eigenvector (the rigid body displacement), such that the principal vector has displacements that are, on average, closest to the origin. For the symmetric cell shown in figure 1, when calculating the principal vector describing tension, the minimum norm solution provided by the pseudo-inverse is such that the axial (x-direction) displacement is zero, and the average of nodal displacements in the y-direction is also zero. Node 2 has zero vertical displacement, while the Poisson ratio effects on nodes 1 and 3 are equal and opposite; this is exactly what one would have chosen.

4. Rigid- versus pin-jointing

Reconsider the planar structure treated by Stephen & Wang (1996*a*), but with the addition of rigid joints. The transfer matrix is now of size 18×18, and since the

Table 2. Saint-Venant decay factors.

pin-jointed	rigid-jointed
0.28286	0.28292
−0.070207	−0.069779
0.059596	0.059597
	−0.15548
	−0.13734
	−0.063740

number of transmission modes having unity eigenvalue is unchanged at six, the immediate effect of rigid-jointing is to double the number of (left-to-right) Saint-Venant decay modes from three to six, as shown in table 2, together with their reciprocals which are not shown.

Rigid jointing has negligible effect on the three left-to-right decay rates of the pin-jointed cell; the main effect is the introduction of three new decay modes, two of which ($\lambda = -0.15548$ and -0.13734) decay at approximately twice the rate of the dominant (slowest rate of decay) mode ($\lambda = 0.28292$); the minus sign indicates that the decay is oscillatory from cell to cell. For the pin-jointed structure, the decay eigenvectors consist of nodal forces which self-equilibrate in the x- and y-directions, it being impossible to apply a moment at a pin-joint. With rigid-joints, these modes now have a very small additional self-equilibrated moment—indeed, just sufficient that the displacement components of the eigenvector should decay with the specified eigenvalue. The new decay modes still have self-equilibrated force loading in the x- and y-directions, but with the addition of comparatively large self-equilibrating nodal moments. The force and moment components of the left-to-right decay eigenvectors are shown in figure 3.

A comparison of the equivalent continuum properties is shown in table 3. As might be expected, the effect of rigid-jointing is to increase all of the equivalent stiffness'—note that a decrease in Poisson's ratio is equivalent to an increase in the shear modulus G since the Young modulus E is regarded as fixed; however, these increases are quite negligible.

5. Symplectic nature of the transfer matrix and consequences

The $(2n \times 2n)$ transfer matrix $\boldsymbol{G}$ satisfies the relationship

$$\boldsymbol{G}^{\mathrm{T}} \boldsymbol{J}_{\mathrm{m}} \boldsymbol{G} = \boldsymbol{J}_{\mathrm{m}}, \tag{5.1}$$

where $\boldsymbol{J}_{\mathrm{m}}$ is the metric matrix

$$\boldsymbol{J}_{\mathrm{m}} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{I} \\ -\boldsymbol{I} & \boldsymbol{0} \end{bmatrix},$$

with $\boldsymbol{J}_{\mathrm{m}}^{\mathrm{T}} = \boldsymbol{J}_{\mathrm{m}}^{-1} = -\boldsymbol{J}_{\mathrm{m}}$, and $\boldsymbol{I}$ is the $(n \times n)$ identity matrix. This relationship depends solely on the symmetry of the stiffness matrix $\boldsymbol{K}$, and can be verified by direct substitution from equation (2.7). Moreover, the symplectic relationship, equation (5.1), requires that partitions of the transfer matrix satisfy the

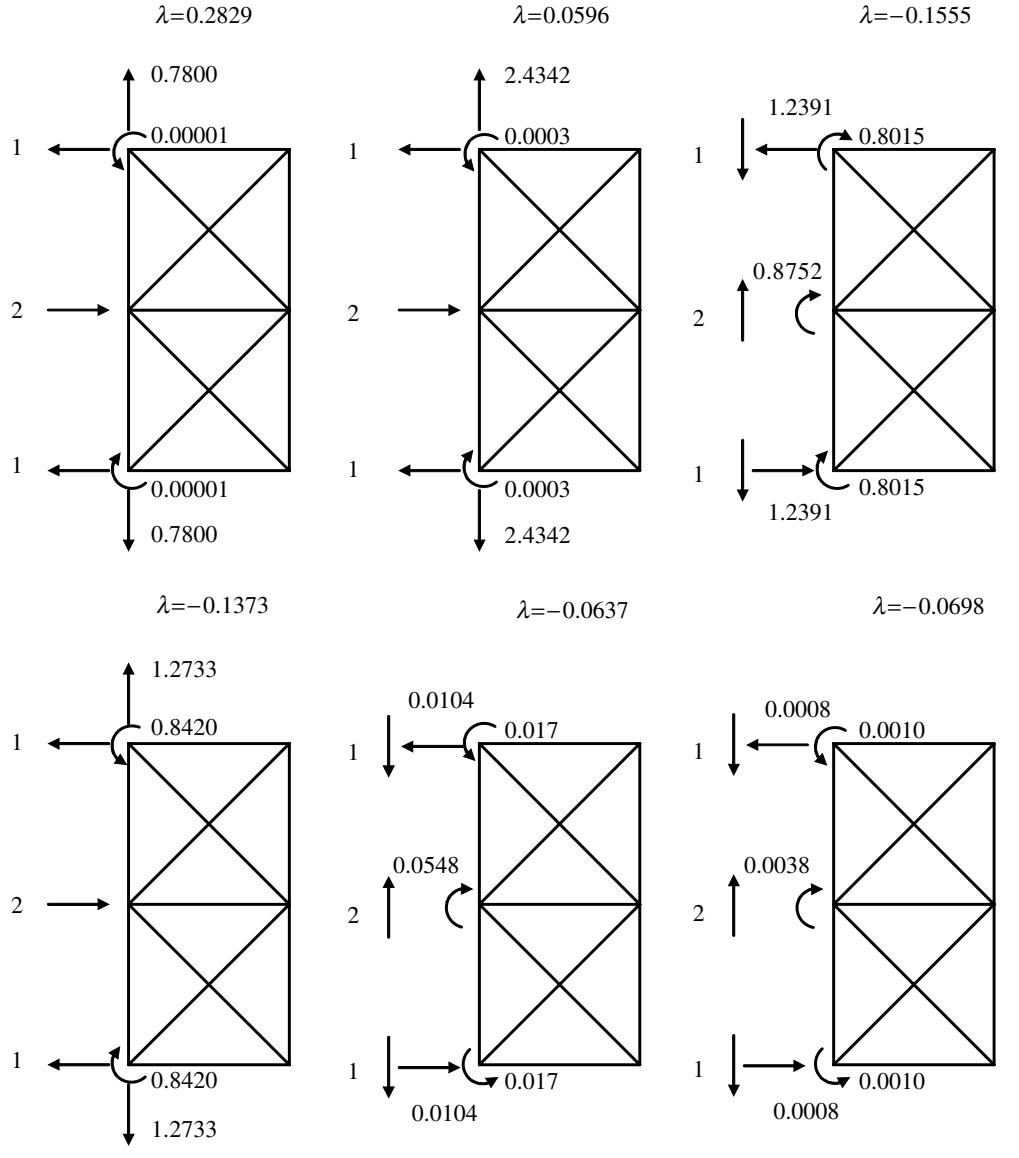

Figure 3. Nodal force and moment components for left-to-right decay in rigid-jointed cell.

relationships

$$\boldsymbol{G}_{dd}^{\mathrm{T}}\boldsymbol{G}_{pd} = (\boldsymbol{G}_{dd}^{\mathrm{T}}\boldsymbol{G}_{pd})^{\mathrm{T}}, \tag{5.2a}$$

$$\boldsymbol{G}_{dp}^{\mathrm{T}}\boldsymbol{G}_{pp} = (\boldsymbol{G}_{dp}^{\mathrm{T}}\boldsymbol{G}_{pp})^{\mathrm{T}}, \tag{5.2b}$$

$$\boldsymbol{G}_{dd}^{\mathrm{T}}\boldsymbol{G}_{pp} - \boldsymbol{G}_{pd}^{\mathrm{T}}\boldsymbol{G}_{dp} = \boldsymbol{I}. \tag{5.2c}$$

The symplectic relationship $\boldsymbol{G}^{\mathrm{T}}\boldsymbol{J}_{\mathrm{m}}\boldsymbol{G} = \boldsymbol{J}_{\mathrm{m}}$ can be rearranged to give $\boldsymbol{J}_{\mathrm{m}}^{-1}\boldsymbol{G}^{\mathrm{T}}\boldsymbol{J}_{\mathrm{m}} = \boldsymbol{G}^{-1}$; thus, the inverse of $\boldsymbol{G}$ is similar to the transpose of $\boldsymbol{G}$, which in turn has the same eigenvalues as $\boldsymbol{G}$. Thus, the eigenvalues occur as reciprocals. Alternatively, one may employ a theorem by Taussky & Zassenhaus (1959): *for*

Table 3. Effect of nodal joining on equivalent continuum stiffness properties.

equivalent property	rigid-joint	pin-joint	difference (%)
cross-sectional area, A (m^2)	3.52246×10^{-4}	3.52241×10^{-4}	$+0.0015$
second moment of area, I (m^4)	2.1306154×10^{-4}	2.1306019×10^{-4}	$+0.0006$
Poisson's ratio, ν	0.261188	0.2612039	-0.006
shear coefficient, κ	0.4957	0.4956	$+0.02$
shear modulus, G ($\mathrm{N\,m}^{-2}$)	79.29031×10^{9}	79.28932×10^{9}	$+0.00125$

every real square matrix $\boldsymbol{G}$*, there exists a (non-unique) non-singular real symmetric matrix* $\boldsymbol{S}$*, such that* $\boldsymbol{G}^{\mathrm{T}}=\boldsymbol{S}^{-1}\boldsymbol{G}\boldsymbol{S}$*; that is,* $\boldsymbol{G}$ *is similar to its transpose.* Combining the above two relationships yields

$$(\boldsymbol{S}\boldsymbol{J}_{\mathrm{m}})^{-1}\boldsymbol{G}(\boldsymbol{S}\boldsymbol{J}_{\mathrm{m}})=\boldsymbol{G}^{-1}; \tag{5.3}$$

thus, the inverse of $\boldsymbol{G}$ is similar to $\boldsymbol{G}$, and not only do the eigenvalues occur as reciprocals, but $\boldsymbol{G}$ and the inverse of $\boldsymbol{G}$ also have the same JCF. Thus, the multiplicity of an eigenvalue and its inverse must be the same, and any non-trivial Jordan block structure must be the same for an eigenvalue and its inverse. Moreover, from equation (2.8) one has $\boldsymbol{G}=\boldsymbol{V}\boldsymbol{J}\boldsymbol{V}^{-1}$ and its inverse $\boldsymbol{G}^{-1}=\boldsymbol{V}^{-1}\boldsymbol{J}^{-1}\boldsymbol{V}$, and substituting into equation (5.3) yields

$$(\boldsymbol{V}^{-1}\boldsymbol{S}\boldsymbol{J}_{\mathrm{m}}\boldsymbol{V})^{-1}\boldsymbol{J}(\boldsymbol{V}^{-1}\boldsymbol{S}\boldsymbol{J}_{\mathrm{m}}\boldsymbol{V})=\boldsymbol{J}^{-1}. \tag{5.4}$$

Thus, the JCF and its inverse are similar, and one may conclude that the JCF of $\boldsymbol{J}^{-1}$ is nothing other than $\boldsymbol{J}$. The structure of the similarity matrix $\boldsymbol{B}\triangleq\boldsymbol{V}^{-1}\boldsymbol{S}\boldsymbol{J}_{\mathrm{m}}\boldsymbol{V}$ may be determined (Gantmacher 1959) as follows: first re-arrange equation (5.4) as $\boldsymbol{J}\boldsymbol{B}-\boldsymbol{B}\boldsymbol{J}^{-1}=0$, or in block form

$$\begin{bmatrix}\boldsymbol{J}_1 & \boldsymbol{0} & \cdots\\ \boldsymbol{0} & \boldsymbol{J}_2 & \\ \vdots & & \ddots\end{bmatrix}\begin{bmatrix}\boldsymbol{B}_{11} & \boldsymbol{B}_{12} & \cdots\\ \boldsymbol{B}_{21} & \boldsymbol{B}_{22} & \\ \vdots & & \ddots\end{bmatrix}-\begin{bmatrix}\boldsymbol{B}_{11} & \boldsymbol{B}_{12} & \cdots\\ \boldsymbol{B}_{21} & \boldsymbol{B}_{22} & \\ \vdots & & \ddots\end{bmatrix}\begin{bmatrix}\boldsymbol{J}_1^{-1} & \boldsymbol{0} & \cdots\\ \boldsymbol{0} & \boldsymbol{J}_2^{-1} & \\ \vdots & & \ddots\end{bmatrix}=\boldsymbol{0}, \tag{5.5}$$

where $\boldsymbol{B}_{ij}$ is a (not necessarily square) block decomposition of $\boldsymbol{B}$, compatible (conforming) with the size of the Jordan blocks. Each of the Jordan blocks may be written as $\boldsymbol{J}_i=\lambda_i\boldsymbol{I}+\boldsymbol{N}_i$, where $\boldsymbol{N}_i$ is nilpotent; the inverse of the block, according to equation (2.16), may be written as $\boldsymbol{J}_i^{-1}=\lambda_i^{-1}\boldsymbol{I}+\bar{\boldsymbol{N}}_i$, where $\bar{\boldsymbol{N}}_i$ is also nilpotent. Equation (5.5) is then broken up and rearranged as

$$(\lambda_i-\lambda_j^{-1})\boldsymbol{B}_{ij}=\boldsymbol{B}_{ij}\bar{\boldsymbol{N}}_j-\boldsymbol{N}_i\boldsymbol{B}_{ij}. \tag{5.6}$$

Now for $(\lambda_i-\lambda_j^{-1})\neq0$, one may multiply by $(\lambda_i-\lambda_j^{-1})$, and substitute from equation (5.6) to give

$$(\lambda_i-\lambda_j^{-1})^2\boldsymbol{B}_{ij}=\boldsymbol{B}_{ij}\bar{\boldsymbol{N}}_j^2-2\boldsymbol{N}_i\boldsymbol{B}_{ij}\bar{\boldsymbol{N}}_j+\boldsymbol{N}_i^2\boldsymbol{B}_{ij}. \tag{5.7}$$

This process may be repeated a sufficient number of times, when the right-hand side eventually becomes equal to zero because of the nilpotency of $\boldsymbol{N}_i$ and $\bar{\boldsymbol{N}}_j$. Thus, one concludes that $(\lambda_i-\lambda_j^{-1})\boldsymbol{B}_{ij}=\boldsymbol{0}$, and hence

$$\boldsymbol{B}_{ij}=\boldsymbol{0},\quad\text{for } (\lambda_i-\lambda_j^{-1})\neq0. \tag{5.8}$$

For $(\lambda_i - \lambda_j^{-1}) = 0$, or $\lambda_i \lambda_j = 1$, one has

$$\boldsymbol{B}_{ij}\bar{\boldsymbol{N}}_j - \boldsymbol{N}_i \boldsymbol{B}_{ij} = \boldsymbol{0} \tag{5.9}$$

for a non-trivial block. This may be solved according to the structure of the nilpotent matrices, and several of the elements of the $\mathbf{B}_{ij}$ will be arbitrary. The process is best illustrated by a simple example: assume a 6×6 JCF

$$\begin{bmatrix} \boldsymbol{J}_1 & & & \\ & \boldsymbol{J}_2 & & \\ & & \lambda_2 & \\ & & & \lambda_2^{-1} \end{bmatrix} = \begin{bmatrix} \lambda_1 & 1 & & & & \\ & \lambda_1 & & & & \\ & & \lambda_1^{-1} & 1 & & \\ & & & \lambda_1^{-1} & & \\ & & & & \lambda_2 & \\ & & & & & \lambda_2^{-1} \end{bmatrix}, \tag{5.10}$$

where the remaining elements are zero. We thus have two Jordan blocks and two trivial diagonal blocks. The block decomposition of $\boldsymbol{B}$ has four 2×2 blocks ($\boldsymbol{B}_{11}$, $\boldsymbol{B}_{12}$, $\boldsymbol{B}_{21}$ and $\boldsymbol{B}_{22}$), four 2×1 blocks ($\boldsymbol{B}_{13}$, $\boldsymbol{B}_{14}$, $\boldsymbol{B}_{23}$ and $\boldsymbol{B}_{24}$), four 1×2 blocks ($\boldsymbol{B}_{31}$, $\boldsymbol{B}_{32}$, $\boldsymbol{B}_{41}$ and $\boldsymbol{B}_{42}$); the remaining blocks are scalar. Immediately, on account of equation (5.8), the only non-zero blocks are $\boldsymbol{B}_{12}$, $\boldsymbol{B}_{21}$ and the scalar blocks $\boldsymbol{B}_{34}$ and $\boldsymbol{B}_{43}$ For a block such as $\boldsymbol{B}_{12}$, equation (5.9) becomes $\boldsymbol{B}_{12}\bar{\boldsymbol{N}}_2 - \boldsymbol{N}_1\boldsymbol{B}_{12} = \boldsymbol{0}$, or explicitly

$$\begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}\begin{bmatrix} 0 & -\lambda_1^{-2} \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} = \boldsymbol{0}; \tag{5.11}$$

expanding gives

$$\begin{bmatrix} -b_3 & -b_4 - \lambda_1^{-2} b_1 \\ 0 & -\lambda_1^{-2} b_3 \end{bmatrix} = \boldsymbol{0}, \tag{5.12}$$

from which one has $b_3 = 0$ and $b_4 = -\lambda_1^{-2} b_1$, and the block takes the form

$$\boldsymbol{B}_{12} = \begin{bmatrix} b_1 & b_2 \\ 0 & -\lambda_1^{-2} b_1 \end{bmatrix}. \tag{5.13}$$

The scalar blocks are arbitrary, and the similarity matrix takes the form

$$\boldsymbol{B} = \begin{bmatrix} 0 & 0 & b_1 & b_2 & 0 & 0 \\ 0 & 0 & 0 & -\lambda_1^{-2} b_1 & 0 & 0 \\ b_3 & b_4 & 0 & 0 & 0 & 0 \\ 0 & -\lambda_1^2 b_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b_5 \\ 0 & 0 & 0 & 0 & b_6 & 0 \end{bmatrix}. \tag{5.14}$$

Since the repeating eigenvalue λ_1 and the (non-zero) elements $b_{1,2,\ldots,6}$ in this example are quite arbitrary, one may conclude from this example that similarity of a JCF to its inverse does not, in general, impose any restriction on the eigenvalues which can give rise to a non-trivial Jordan block.

6. Bi- and symplectic orthogonality and the reciprocal theorem

Since the transfer matrix $\boldsymbol{G}$ is not symmetric, one would normally employ *bi-orthogonality* as the means of modal decomposition of an arbitrary state vector. Let $\boldsymbol{X}_i$ be a (right) eigenvector of $\boldsymbol{G}$ associated with the eigenvalue λ_i, such that

$$\boldsymbol{G}\boldsymbol{X}_i = \lambda_i \boldsymbol{X}_i; \tag{6.1}$$

let $\boldsymbol{Y}_j$ be a (right) eigenvector of $\boldsymbol{G}^{\mathrm{T}}$ associated with the eigenvalue λ_j, such that

$$\boldsymbol{G}^{\mathrm{T}}\boldsymbol{Y}_j = \lambda_j \boldsymbol{Y}_j. \tag{6.2}$$

Pre-multiply equation (6.1) by $\boldsymbol{Y}_j^{\mathrm{T}}$ to give

$$\boldsymbol{Y}_j^{\mathrm{T}}\boldsymbol{G}\boldsymbol{X}_i = \lambda_i \boldsymbol{Y}_j^{\mathrm{T}}\boldsymbol{X}_i; \tag{6.3}$$

transpose equation (6.2), and post-multiply by $\boldsymbol{X}_i$ to give

$$\boldsymbol{Y}_j^{\mathrm{T}}\boldsymbol{G}\boldsymbol{X}_i = \lambda_j \boldsymbol{Y}_j^{\mathrm{T}}\boldsymbol{X}_i. \tag{6.4}$$

Subtraction gives

$$(\lambda_i - \lambda_j)\mathbf{Y}_j^{\mathrm{T}}\mathbf{X}_i = 0, \tag{6.5}$$

and the bi-orthogonality relationship

$$\boldsymbol{Y}_j^{\mathrm{T}}\boldsymbol{X}_i = 0, \quad \text{for } \lambda_i \neq \lambda_j. \tag{6.6}$$

In principle, the disadvantage of this approach is the need to perform a second eigenanalysis of the transpose of $\boldsymbol{G}$, although in practice one needs only to compute the eigen- and principal vectors, as the eigenvalues of $\boldsymbol{G}^{\mathrm{T}}$ are the same as those of $\boldsymbol{G}$. Instead, symplectic orthogonality is determined as follows: transpose equation (6.1) to give

$$\boldsymbol{X}_i^{\mathrm{T}}\boldsymbol{G}^{\mathrm{T}} = \lambda_i \boldsymbol{X}_i^{\mathrm{T}}. \tag{6.7}$$

Post-multiply by $\boldsymbol{J}_{\mathrm{m}}\boldsymbol{G}\boldsymbol{X}_j$ to give

$$\boldsymbol{X}_i^{\mathrm{T}}\boldsymbol{G}^{\mathrm{T}}\boldsymbol{J}_{\mathrm{m}}\boldsymbol{G}\boldsymbol{X}_j = \lambda_i \boldsymbol{X}_i^{\mathrm{T}}\boldsymbol{J}_{\mathrm{m}}\boldsymbol{G}\boldsymbol{X}_j. \tag{6.8}$$

Now $\boldsymbol{G}^{\mathrm{T}}\boldsymbol{J}_{\mathrm{m}}\boldsymbol{G} = \boldsymbol{J}_{\mathrm{m}}$ and $\boldsymbol{G}\boldsymbol{X}_j = \lambda_j \boldsymbol{X}_j$, and substituting these expressions into (6.8) yields

$$(1 - \lambda_i\lambda_j)\boldsymbol{X}_i^{\mathrm{T}}\boldsymbol{J}_{\mathrm{m}}\boldsymbol{X}_j = 0. \tag{6.9}$$

Thus, an eigenvector is *symplectic adjoint orthogonal* to all vectors, including itself, but excluding the vector(s) associated with its reciprocal eigenvalue.

The two orthogonality relationships are clearly related, as follows: $\boldsymbol{Y}_j$ is the (right) eigenvector of $\boldsymbol{G}^{\mathrm{T}}$ having eigenvalue λ_j or, equivalently, $\boldsymbol{Y}_j^{\mathrm{T}}$ is the left eigenvector of $\boldsymbol{G}$ having eigenvalue λ_j. On the other hand, from equation (6.1), one has $\boldsymbol{G}^{-1}\boldsymbol{X}_j = \lambda_j^{-1}\boldsymbol{X}_j$, but $\boldsymbol{G}^{-1} = \boldsymbol{J}_m^{-1}\boldsymbol{G}^{\mathrm{T}}\boldsymbol{J}_m$, leading to $\boldsymbol{J}_{\mathrm{m}}^{-1}\boldsymbol{G}^{\mathrm{T}}\boldsymbol{J}_{\mathrm{m}}\boldsymbol{X}_j = \lambda_j^{-1}\boldsymbol{X}_j$; finally, pre-multiply by $\boldsymbol{J}_{\mathrm{m}}$ and transpose to give $(\boldsymbol{J}_{\mathrm{m}}\boldsymbol{X}_j)^{\mathrm{T}}\boldsymbol{G} = \lambda_j^{-1}(\boldsymbol{J}_{\mathrm{m}}\boldsymbol{X}_j)^{\mathrm{T}}$, and one sees that $(\boldsymbol{J}_{\mathrm{m}}\boldsymbol{X}_j)^{\mathrm{T}}$ is a left eigenvector of $\boldsymbol{G}$ having eigenvalue λ_j^{-1}; effectively, the left eigenvectors do not have to be explicitly computed if the right eigenvectors are already known. Replace $\boldsymbol{Y}_j^{\mathrm{T}}$ and λ_j in the bi-orthogonality relationship (6.5) by $(\boldsymbol{J}_m\boldsymbol{X}_j)^{\mathrm{T}}$ and λ_j^{-1}, respectively, to give

$$(\lambda_i - \lambda_j^{-1})(\boldsymbol{J}_{\mathrm{m}}\boldsymbol{X}_j)^{\mathrm{T}}\boldsymbol{X}_i = 0; \tag{6.10}$$

but $(\boldsymbol{J}_{\mathrm{m}}\boldsymbol{X}_j)^{\mathrm{T}} = -\boldsymbol{X}_j^{\mathrm{T}}\boldsymbol{J}_{\mathrm{m}}$, and multiplying by λ_j gives the symplectic orthogonality relationship, equation (6.9).

Zhong & Williams (1993) have shown that the symplectic orthogonality relationship is a consequence of the reciprocal theorem of Betti–Maxwell; the

latter, in turn, underlies symmetry of the stiffness matrix $\boldsymbol{K}$. According to the reciprocal theorem, for two different load systems applied to the cell, denoted by subscripts 1 and 2, the work done by the forces $\boldsymbol{F}_1$ acting through the displacements $\boldsymbol{d}_2$ is equal to the work done by the forces $\boldsymbol{F}_2$ acting through the displacements $\boldsymbol{d}_1$. For the single cell, figure 2, this may be written as

$$\boldsymbol{d}_{\mathrm{L}1}^{\mathrm{T}}\boldsymbol{F}_{\mathrm{L}2}+\boldsymbol{d}_{\mathrm{R}1}^{\mathrm{T}}\boldsymbol{F}_{\mathrm{R}2}=\boldsymbol{d}_{\mathrm{L}2}^{\mathrm{T}}\boldsymbol{F}_{\mathrm{L}1}+\boldsymbol{d}_{\mathrm{R}2}^{\mathrm{T}}\boldsymbol{F}_{\mathrm{R}1}. \tag{6.11}$$

Express the right-hand side vectors, in terms of the left-hand side vectors according to

$$\begin{aligned}&\boldsymbol{d}_{\mathrm{R}1}=\lambda_1\boldsymbol{d}_{\mathrm{L}1},\quad \boldsymbol{d}_{\mathrm{R}2}=\lambda_2\boldsymbol{d}_{\mathrm{L}2},\quad \boldsymbol{F}_{\mathrm{R}1}=\boldsymbol{p}_{\mathrm{R}1}=\lambda_1\boldsymbol{p}_{\mathrm{L}1}=-\lambda_1\boldsymbol{F}_{\mathrm{L}1},\\&\boldsymbol{F}_{\mathrm{R}2}=\boldsymbol{p}_{\mathrm{R}2}=\lambda_2\boldsymbol{p}_{\mathrm{L}2}=-\lambda_2\boldsymbol{F}_{\mathrm{L}2},\end{aligned} \tag{6.12}$$

to give

$$-\boldsymbol{d}_{\mathrm{L}1}^{\mathrm{T}}\boldsymbol{p}_{\mathrm{L}2}+\lambda_1\lambda_2\boldsymbol{d}_{\mathrm{L}1}^{\mathrm{T}}\boldsymbol{p}_{\mathrm{L}2}=-\boldsymbol{d}_{\mathrm{L}2}^{\mathrm{T}}\boldsymbol{p}_{\mathrm{L}1}+\lambda_1\lambda_2\boldsymbol{d}_{\mathrm{L}2}^{\mathrm{T}}\boldsymbol{p}_{\mathrm{L}1}, \tag{6.13}$$

or

$$(1-\lambda_1\lambda_2)[\boldsymbol{d}_{\mathrm{L}1}^{\mathrm{T}}\boldsymbol{p}_{\mathrm{L}2}-\boldsymbol{d}_{\mathrm{L}2}^{\mathrm{T}}\boldsymbol{p}_{\mathrm{L}1}]=0. \tag{6.14}$$

But the term

$$[\boldsymbol{d}_{\mathrm{L}1}^{\mathrm{T}}\boldsymbol{p}_{\mathrm{L}2}-\boldsymbol{d}_{\mathrm{L}2}^{\mathrm{T}}\boldsymbol{p}_{\mathrm{L}1}]=\boldsymbol{X}_1^{\mathrm{T}}\boldsymbol{J}_{\mathrm{m}}\boldsymbol{X}_2, \tag{6.15}$$

which indicates that equation (6.14) is a re-expression of the symplectic orthogonality relationship (6.9).

7. Restrictions on complex unity eigenvalues and Jordan canonical form

Synge (1945) considered the *problem of Saint-Venant*, which is a beam subjected to end loading only, with the surface generators being free of traction, specifically for a prismatic homogenous elastic cylinder, and for the decay modes considered stress varying as

$$\mathrm{e}^{kx}f(y,z). \tag{7.1}$$

According to Synge, 'A purely imaginary k implies a periodic distribution of displacement and stress. Consider the energy in a length of cylinder equal to this period. It is equal to the work done by the terminal stress in passing from the natural state to the strained state. But from the periodicity, this is zero. Hence the energy of a strained state is zero, which is contrary to a basic postulate of elasticity. Hence there can be no purely imaginary eigenvalue k. It should be added that we cannot assert this if (Poisson's ratio) ν is arbitrary. It is necessarily only true if strain energy is positive definite, i.e. if $-1<\nu<1/2$', indicating that this simple and ingenious argument was originally put forward by Dougall (1913).

It is usual in such continuum problems to express the decay characteristic in terms of a greatest linear dimension ('gld') of the cross-section, when the decay characteristic can be expressed as $\mathrm{e}^{k'(x/\mathrm{gld})}$, where $k'=k\times\mathrm{gld}$ is now dimensionless. For a repetitive structure, the eigenvalue λ relates to the k' as $\lambda=\mathrm{e}^{k'}$. Thus, if an expression $k'=\mathrm{i}\theta$ is inadmissible, so too is an eigenvalue of the form $\lambda=\mathrm{e}^{k'}=\mathrm{e}^{\mathrm{i}\theta}=\cos\theta+\mathrm{i}\sin\theta$. On the other hand, such complex unity eigenvalues take the place of some of the real unity eigenvalues for structures having a pre-twisted form (Stephen & Zhang 2006), and also for a curved beam-like structure (Stephen & Ghosh 2005); for both cases, the transfer matrix expressed in a local

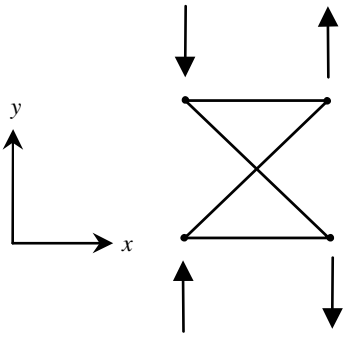

Figure 4. Single cell of pin-jointed cell having partial mechanism with displacements shown; force components are zero.

coordinate system rotating with the cross-section has such complex unity eigenvalues pertaining to the rigid body displacements, when obviously no energy is stored.

Elementary arguments based on the dimension of the transfer matrix of a repeating cell can lead to restrictions on the possible coupling of eigenvectors, and in turn the possible JCF. Consider the simple planar cell shown in figure 6: as there are two displacement degrees of freedom per node, so the stiffness matrix $\boldsymbol{K}$ and the transfer matrix $\boldsymbol{G}$ are both of size (8×8). As with any straight planar structure, there must be precisely six unity eigenvalues—rigid body displacement in the x-direction and tension are coupled within a (2×2) Jordan block, while rigid body displacement in the y-direction, cross-sectional rotation, bending moment and shear, are coupled within a (4×4) Jordan block. Thus, the Jordan block structure must take the form

$$\begin{bmatrix} \lambda & 0 & & & & & & \\ 0 & \lambda^{-1} & & & & & & \\ & & 1 & 1 & & & & \\ & & & 1 & & & & \\ & & & & 1 & 1 & & \\ & & & & & 1 & 1 & \\ & & & & & & 1 & 1 \\ & & & & & & & 1 \end{bmatrix}, \tag{7.2}$$

where the single unknown eigenvalue λ occurs as a pair with its inverse, and with a possible unity element replacing the zero to its right. Now a necessary, but not sufficient, condition for a principal vector to be coupled to an eigenvector, when one does have a unity on the superdiagonal in that block, is that an eigenvalue is repeated; for a matrix of this dimension this can only occur when $\lambda=\lambda^{-1}$, which in turn implies $\lambda=\pm1$. The repeated eigenvalue $\lambda=-1$, within a non-trivial block, does indeed occur for the cell, figure 4, which acts as a scissors-mechanism when subject to the self-equilibrated loading shown. A more general proof leading to the

same conclusion that the only possible repeating eigenvalues which can give rise to a non-trivial Jordan block are $\lambda = \pm 1$ is now developed.

(*a*) *Eigenvectors associated with* $\lambda = \pm 1$ *do no work*

If the left-hand side eigenvector is written as

$$\boldsymbol{e} = \begin{bmatrix} \boldsymbol{d}_{\mathrm{L}} \\ \boldsymbol{p}_{\mathrm{L}} \end{bmatrix}, \tag{7.3}$$

then the right-hand side vector is

$$\lambda \boldsymbol{e} = \begin{bmatrix} \lambda \boldsymbol{d}_{\mathrm{L}} \\ \lambda \boldsymbol{p}_{\mathrm{L}} \end{bmatrix}, \tag{7.4}$$

and the force and displacement vectors, according to the sign conventions of FEA, are

$$\boldsymbol{F} = \begin{bmatrix} \boldsymbol{F}_{\mathrm{L}} \\ \boldsymbol{F}_{\mathrm{R}} \end{bmatrix} = \begin{bmatrix} -\boldsymbol{p}_{\mathrm{L}} \\ \lambda \boldsymbol{p}_{\mathrm{L}} \end{bmatrix}, \quad \boldsymbol{d} = \begin{bmatrix} \boldsymbol{d}_{\mathrm{L}} \\ \lambda \boldsymbol{d}_{\mathrm{L}} \end{bmatrix}. \tag{7.5}$$

Now the strain energy stored within the cell is equal to the work done (WD), which is

$$\mathrm{WD} = \tfrac{1}{2}\boldsymbol{d}^{\mathrm{T}}\boldsymbol{F} = \tfrac{1}{2}[\boldsymbol{d}_{\mathrm{L}}^{\mathrm{T}} \quad \lambda \boldsymbol{d}_{\mathrm{L}}^{\mathrm{T}}] \begin{bmatrix} -\boldsymbol{p}_{\mathrm{L}} \\ \lambda \boldsymbol{p}_{\mathrm{L}} \end{bmatrix} = \tfrac{1}{2}(\lambda^2 - 1)\boldsymbol{d}_{\mathrm{L}}^{\mathrm{T}}\boldsymbol{p}_{\mathrm{L}} = \tfrac{1}{2}(1 - \lambda^2)\boldsymbol{d}_{\mathrm{L}}^{\mathrm{T}}\boldsymbol{F}_{\mathrm{L}}. \tag{7.6}$$

Clearly, the WD is zero for $\lambda = \pm 1$. This agrees with experience: the only known eigenvectors having $\lambda = 1$ are rigid body translations and rotations, while the only known eigenvectors having $\lambda = -1$ pertain to structures having a partial scissors-mechanism, as in figure 4.

(*b*) *Work associated with a principal vector*

If the left-hand side principal vector is written as

$$\boldsymbol{p} = \begin{bmatrix} \boldsymbol{d}_{\mathrm{L}}^{\mathrm{p}} \\ \boldsymbol{p}_{\mathrm{L}}^{\mathrm{p}} \end{bmatrix}, \tag{7.7}$$

where the superscript 'p' denotes principal, then the right-hand side vector is

$$\lambda \boldsymbol{p} + \boldsymbol{e} = \begin{bmatrix} \lambda \boldsymbol{d}_{\mathrm{L}}^{\mathrm{p}} + \boldsymbol{d}_{\mathrm{L}} \\ \lambda \boldsymbol{p}_{\mathrm{L}}^{\mathrm{p}} + \boldsymbol{p}_{\mathrm{L}} \end{bmatrix}, \tag{7.8}$$

and the force and displacement vectors are

$$\boldsymbol{F} = \begin{bmatrix} -\boldsymbol{p}_{\mathrm{L}}^{\mathrm{p}} \\ \lambda \boldsymbol{p}_{\mathrm{L}}^{\mathrm{p}} + \boldsymbol{p}_{\mathrm{L}} \end{bmatrix}, \quad \boldsymbol{d} = \begin{bmatrix} \boldsymbol{d}_{\mathrm{L}}^{\mathrm{p}} \\ \lambda \boldsymbol{d}_{\mathrm{L}}^{\mathrm{p}} + \boldsymbol{d}_{\mathrm{L}} \end{bmatrix}, \tag{7.9}$$

and the strain energy is

$$\mathrm{WD} = \tfrac{1}{2}\boldsymbol{d}^{\mathrm{T}}\boldsymbol{F} = \tfrac{1}{2}[(\lambda^2 - 1)\boldsymbol{d}_{\mathrm{L}}^{\mathrm{pT}}\boldsymbol{p}_{\mathrm{L}}^{\mathrm{p}} + \lambda(\boldsymbol{d}_{\mathrm{L}}^{\mathrm{pT}}\boldsymbol{p}_{\mathrm{L}} + \boldsymbol{d}_{\mathrm{L}}^{\mathrm{T}}\boldsymbol{p}_{\mathrm{L}}^{\mathrm{p}}) + \boldsymbol{d}_{\mathrm{L}}^{\mathrm{T}}\boldsymbol{p}_{\mathrm{L}}]. \tag{7.10}$$

Again, this expression agrees with experience: the only known repeating eigenvalues giving rise to a non-trivial Jordan block form are $\lambda = \pm 1$. Consider the coupling of tension with rigid body displacement in the x-direction: the first term in equation (7.10) is zero as $\lambda = 1$, the last term is zero as $\boldsymbol{p}_{\mathrm{L}} = \boldsymbol{0}$, that is the

rigid body displacement has no force components, and the second term reduces to

$$\mathrm{WD} = \tfrac{1}{2}\boldsymbol{d}_{\mathrm{L}}^{\mathrm{T}}\boldsymbol{p}_{\mathrm{L}}^{\mathrm{p}}, \tag{7.11}$$

which is one-half of the extension (the displacement components of the eigenvector) times the tensile force (the force components of the principal vector).

Now, it is known that principal vectors are not unique: one may add an arbitrary multiple, say β, of the generating eigenvector and it is still a principal vector, that is

$$\boldsymbol{G}\boldsymbol{p} = \lambda\boldsymbol{p} + \boldsymbol{e}, \tag{7.12}$$

and also

$$\boldsymbol{G}(\boldsymbol{p} + \beta\boldsymbol{e}) = \lambda(\boldsymbol{p} + \beta\boldsymbol{e}) + \boldsymbol{e}, \quad \text{since } \boldsymbol{G}\beta\boldsymbol{e} = \lambda\beta\boldsymbol{e}. \tag{7.13}$$

(On the other hand, one cannot arbitrarily add multiples of a principal vector of one grade to a principal vector of another grade.) Now re-calculate the work associated with the modified principal vector as follows: on the left-hand side one has

$$\boldsymbol{p} + \beta\boldsymbol{e} = \begin{bmatrix} \boldsymbol{d}_{\mathrm{L}}^{\mathrm{p}} + \beta\boldsymbol{d}_{\mathrm{L}} \\ \boldsymbol{p}_{\mathrm{L}}^{\mathrm{p}} + \beta\boldsymbol{p}_{\mathrm{L}} \end{bmatrix}, \tag{7.14}$$

while the right-hand side state vector is

$$\lambda(\boldsymbol{p} + \beta\boldsymbol{e}) + \boldsymbol{e} = \begin{bmatrix} \lambda\boldsymbol{d}_{\mathrm{L}}^{\mathrm{p}} + (\lambda\beta + 1)\boldsymbol{d}_{\mathrm{L}} \\ \lambda\boldsymbol{p}_{\mathrm{L}}^{\mathrm{p}} + (\lambda\beta + 1)\boldsymbol{p}_{\mathrm{L}} \end{bmatrix}, \tag{7.15}$$

with cell force and displacement vectors

$$\boldsymbol{F} = \begin{bmatrix} -\boldsymbol{p}_{\mathrm{L}}^{\mathrm{p}} - \beta\boldsymbol{p}_{\mathrm{L}} \\ \lambda\boldsymbol{p}_{\mathrm{L}}^{\mathrm{p}} + (\lambda\beta + 1)\boldsymbol{p}_{\mathrm{L}} \end{bmatrix}, \quad \boldsymbol{d} = \begin{bmatrix} \boldsymbol{d}_{\mathrm{L}}^{\mathrm{p}} + \beta\boldsymbol{d}_{\mathrm{L}} \\ \lambda\boldsymbol{d}_{\mathrm{L}}^{\mathrm{p}} + (\lambda\beta + 1)\boldsymbol{d}_{\mathrm{L}} \end{bmatrix}. \tag{7.16}$$

The strain energy is then

$$\begin{aligned}\mathrm{WD} = \tfrac{1}{2}[&(\lambda^2 - 1)\boldsymbol{d}_{\mathrm{L}}^{\mathrm{pT}}\boldsymbol{p}_{\mathrm{L}}^{\mathrm{p}} + ((\lambda^2 - 1)\beta + \lambda)(\boldsymbol{d}_{\mathrm{L}}^{\mathrm{pT}}\boldsymbol{p}_{\mathrm{L}} + \boldsymbol{d}_{\mathrm{L}}^{\mathrm{T}}\boldsymbol{p}_{\mathrm{L}}^{\mathrm{p}}) + ((\lambda^2 - 1)\beta^2 \\ &+ 2\lambda\beta + 1)\boldsymbol{d}_{\mathrm{L}}^{\mathrm{T}}\boldsymbol{p}_{\mathrm{L}}].\end{aligned} \tag{7.17}$$

Now make the assertion that the strain energy must be independent of arbitrary β, that is expressions (7.10) and (7.17) must be identical; this requires

$$A\beta^2 + B\beta = 0, \tag{7.18}$$

where

$$A = (\lambda^2 - 1)\boldsymbol{d}_{\mathrm{L}}^{\mathrm{T}}\boldsymbol{p}_{\mathrm{L}}, \quad B = (\lambda^2 - 1)(\boldsymbol{d}_{\mathrm{L}}^{\mathrm{pT}}\boldsymbol{p}_{\mathrm{L}} + \boldsymbol{d}_{\mathrm{L}}^{\mathrm{T}}\boldsymbol{p}_{\mathrm{L}}^{\mathrm{p}}) + 2\lambda\boldsymbol{d}_{\mathrm{L}}^{\mathrm{T}}\boldsymbol{p}_{\mathrm{L}}. \tag{7.19}$$

This assertion is equivalent to the demand that the strain energy should be independent of the choice of coordinate axes, since the eigenvectors are the rigid body displacements in the x- and y-directions. Now since β is arbitrary, equation (7.18) clearly requires that $A = B = 0$. The former requires either $(\lambda^2 - 1) = 0$, or $\boldsymbol{d}_{\mathrm{L}}^{\mathrm{T}}\boldsymbol{p}_{\mathrm{L}} = 0$, or both. If $(\lambda^2 - 1) = 0$, then the requirement $B = 0$ implies $\boldsymbol{d}_{\mathrm{L}}^{\mathrm{T}}\mathrm{p}_{\mathrm{L}} = 0$; if, from the requirement $A = 0$, one chooses $\boldsymbol{d}_{\mathrm{L}}^{\mathrm{T}}\boldsymbol{p}_{\mathrm{L}} = 0$, with $(\lambda^2 - 1) \neq 0$, the requirement $B = 0$ implies $\boldsymbol{d}_{\mathrm{L}}^{\mathrm{pT}}\boldsymbol{p}_{\mathrm{L}} + \boldsymbol{d}_{\mathrm{L}}^{\mathrm{T}}\boldsymbol{p}_{\mathrm{L}}^{\mathrm{p}} = 0$. Thus, one has the two possibilities:

(i) $(\lambda^2 - 1) = 0$ and $\boldsymbol{d}_{\mathrm{L}}^{\mathrm{T}}\boldsymbol{p}_{\mathrm{L}} = 0$, or
(ii) $(\lambda^2 - 1) \neq 0$, with $\boldsymbol{d}_{\mathrm{L}}^{\mathrm{T}}\boldsymbol{p}_{\mathrm{L}} = 0$ and $\boldsymbol{d}_{\mathrm{L}}^{\mathrm{pT}}\boldsymbol{p}_{\mathrm{L}} + \boldsymbol{d}_{\mathrm{L}}^{\mathrm{T}}\boldsymbol{p}_{\mathrm{L}}^{\mathrm{p}} = 0$.

Case (i) agrees with experience. However, the first scalar product of case (ii) implies the existence of an eigenvector associated with a non-unity decay eigenvalue, for which the strain energy in the cell and, indeed, the entire semi-infinite structure, is zero, which is inconceivable. Thus, one concludes that the only possible repeating eigenvalues that can give rise to a non-trivial Jordan block are $\lambda = \pm 1$. Karpov *et al.* (2002*a*) have recently classified possible modes in repetitive structures as 'exponential' (i.e. Saint-Venant decay according to a single non-unity eigenvalue), 'polynomial' (that is, transmission associated with multiple unity eigenvalues) and 'quasi-polynomial' (that is, degenerate multiple non-unity eigenvalues). It has been shown that these quasi-polynomial modes cannot exist.

The above arguments presume the eigenvalues and vectors to be real; should they be complex, then the transpose of the displacement vector is replaced by the Hermitian conjugate (complex conjugate transpose), and one concludes that $\lambda\lambda^* = 1$, where '$*$' denotes the conjugate, in which case the only possibilities are the complex unity eigenvalues. Again, this is in accord with experience in the eigenanalysis of pre-twisted (Stephen & Zhang 2006) and curved structures (Stephen & Ghosh 2005).

8. The effect of cell (a)symmetry

In this section, we note that eigenvalue degeneracy is intrinsic to translational symmetry, which is the same as the structure being repetitive. In particular, eigenanalysis of a tapered cell reveals splitting of two unity eigenvalues. On the other hand, for repetitive structures for which the cell has additional symmetries, additional constraints (relationships) may be derived for both the stiffness and transfer matrices; this is explored for a cell having left-to-right (reflective) symmetry. Symmetry is synonymous with constraint.

(*a*) *Tapered cell*

Consider the tapered, pin-jointed cell shown in figure 5, which is a modification of the cell shown in figure 2, in which the right-hand side vertical members have been increased in length to 1.5 m, and the diagonal members to suit. Clearly, such a cell cannot be part of a repetitive structure, since adjacent cells are obviously different; eigenanalysis is still a valid mathematical operation, but is not a practical proposition, as it would have to be repeated for each individual cell; nevertheless, some interesting observations arise. First, note that the eigenvalues still occur as reciprocal pairs

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3/2 \\ 2/3 \end{pmatrix}, \begin{pmatrix} 25.1235 \\ 0.0398 \end{pmatrix}, \begin{pmatrix} 2.7890 \\ 0.3585 \end{pmatrix}, \begin{pmatrix} -12.5159 \\ -0.0799 \end{pmatrix},$$

as they must, since the stiffness matrix is symmetric, and the transfer matrix will still be symplectic. More interestingly, the lack of nodal (translational) symmetry has the effect of eigenvalue splitting: of the original six unity eigenvalues for the cell of figure 1, two have split to become $2/3$ and $3/2$. These pertain to the (left-to-right) rotation vector, which from elementary kinematics must have $\lambda = 3/2$, while from consideration of equilibrium, the moment vector must have $\lambda = 2/3$. Moreover, the moment vector contains a self-equilibrated load whose nodal force components are of considerable magnitude—indeed, precisely what is necessary to

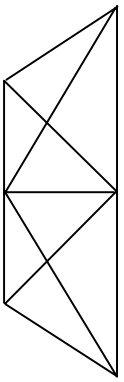

Figure 5. Tapered cell.

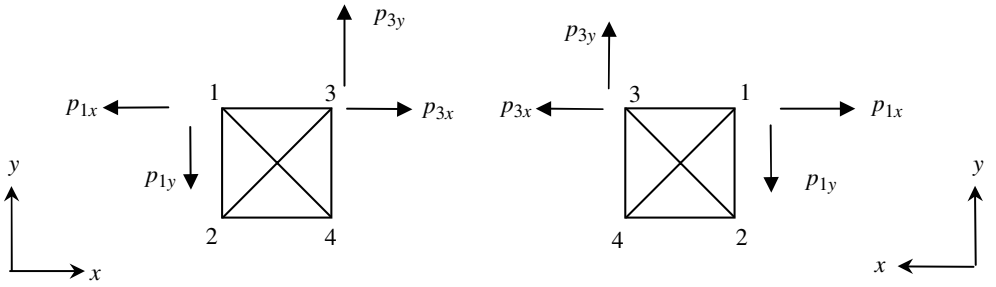

Figure 6. Symmetric cell showing left-to-right reflection.

ensure that the displacement components of the moment eigenvector should also transmit with $\lambda = 2/3$. Clearly, two of the unity eigenvalues must pertain to a rigid body displacement in the x-direction, coupled to a tensile force. The other two pertain to a rigid body displacement in the y-direction and a combined bending moment and shearing force vector. Shearing force and bending moment are no longer coupled within the same Jordan block.

(*b*) *Cells having left-to-right symmetry*

Consider the simple X-braced cell shown in figure 6, which clearly has left-to-right (denoted L⇔R) symmetry. In figure 6*a*, the nodal forces are shown according to the conventions of the *Theory of Elasticity*, and decay of a self-equilibrated loading applied to the left-hand nodes is assumed left-to-right, that is $|\lambda| < 1$. In reflecting the cell to figure 6*b*, the x-components of force are in the same direction, while the y-components have changed direction. Similarly, the components of displacement in the x-direction will change sign, while those in the y-direction remain unchanged. The eigenvector for the left-to-right decay, written in full, is

$$\boldsymbol{s}_{\mathrm{L}} = [d_{1x} \quad d_{1y} \quad d_{2x} \quad d_{2y} \quad p_{1x} \quad p_{1y} \quad p_{2x} \quad p_{2y}]^{\mathrm{T}}. \tag{8.1}$$

Now by virtue of the symmetry of the cell, the equivalent *right-to-left* eigenvector, figure 6*b*, also having a *right-to-left* decay of $|\lambda| < 1$, will be

$$[-d_{1x} \quad d_{1y} \quad -d_{2x} \quad d_{2y} \quad p_{1x} \quad -p_{1y} \quad p_{2x} \quad -p_{2y}]^{\mathrm{T}} = \boldsymbol{R}\boldsymbol{s}_{\mathrm{L}}, \tag{8.2}$$

where the reflection (or parity) matrix $\boldsymbol{R}$ may be written in general as

$$\boldsymbol{R} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{0} \\ \boldsymbol{0} & -\boldsymbol{A} \end{bmatrix}, \tag{8.3}$$

and for the particular cell shown

$$\boldsymbol{A} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{8.4}$$

For a planar structure, the essential pattern of ∓ 1 on the leading diagonal will extend according to the number of nodes on the cross-section. For a three-dimensional beam-like space structure, the z-components of displacement and force would be unchanged by the L$\Leftrightarrow$R reflection, in which case the structure of $\boldsymbol{R}$ would be unchanged, while matrix $\boldsymbol{A}$ would consist of the sequence $-1, 1, 1$ repeated on the leading diagonal.

Since $\boldsymbol{R}\boldsymbol{s}_{\mathrm{L}}$ is a right-to-left eigenvector having right-to-left decay with eigenvalue λ, it is also a left-to-right eigenvector having left-to-right decay eigenvalue λ^{-1}. Thus, we may write initially, $\boldsymbol{G}\boldsymbol{s}_{\mathrm{L}} = \lambda \boldsymbol{s}_{\mathrm{L}}$; pre-multiply by $\boldsymbol{R}\boldsymbol{G}^{-1}$ to give $\boldsymbol{R}\boldsymbol{G}^{-1}\boldsymbol{s}_{\mathrm{L}} = \lambda^{-1}\boldsymbol{R}\boldsymbol{s}_{\mathrm{L}}$. But from the symmetry argument above, one also has $\boldsymbol{G}\boldsymbol{R}\boldsymbol{s}_{\mathrm{L}} = \lambda^{-1}\boldsymbol{R}\boldsymbol{s}_{\mathrm{L}}$. Symmetry demands that the two should be indistinguishable, which requires $\boldsymbol{G}\boldsymbol{R} = \boldsymbol{R}\boldsymbol{G}^{-1}$, or $\boldsymbol{R}\boldsymbol{G}\boldsymbol{R} = \boldsymbol{G}^{-1}$. Note that the reflection matrix $\boldsymbol{R}$ is involutory, that is $\boldsymbol{R}^2 = \boldsymbol{I}$, as is matrix $\boldsymbol{A}$. These arguments are represented graphically in figure 7. A more rigorous approach is presented in §8*c*.

(*c*) *Four pole representation*

Following Easwaran *et al.* (1993), who considered a four pole representation with positive state variables according to figure 8, define *barred* positive force and displacement components according to figure 9. This representation is ideal for the L$\Leftrightarrow$R symmetry of the cell; force components are related to the FE sign convention according to

$$\bar{p}_{1x} = -F_{1x},\ \bar{p}_{1y} = F_{1y},\ \bar{p}_{2x} = -F_{2x},\ \bar{p}_{2y} = F_{2y},$$
$$\bar{p}_{3x} = F_{3x},\ \bar{p}_{3y} = F_{3y},\ \bar{p}_{4x} = F_{4x},\ \bar{p}_{4y} = F_{4y},$$

or

$$\begin{bmatrix} \bar{\boldsymbol{p}}_{\mathrm{L}} \\ \bar{\boldsymbol{p}}_{\mathrm{R}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{F}_{\mathrm{L}} \\ \boldsymbol{F}_{\mathrm{R}} \end{bmatrix}, \tag{8.5}$$

while displacement components are related according to

$$\bar{d}_{1x} = -d_{1x},\ \bar{d}_{1y} = d_{1y},\ \bar{d}_{2x} = -d_{2x},\ \bar{d}_{2y} = d_{2y},$$
$$\bar{d}_{3x} = d_{3x},\ \bar{d}_{3y} = d_{3y},\ \bar{d}_{4x} = d_{4x},\ \bar{d}_{4y} = d_{4y},$$

or

$$\begin{bmatrix} \bar{\boldsymbol{d}}_{\mathrm{L}} \\ \bar{\boldsymbol{d}}_{\mathrm{R}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{d}_{\mathrm{L}} \\ \boldsymbol{d}_{\mathrm{R}} \end{bmatrix}. \tag{8.6}$$

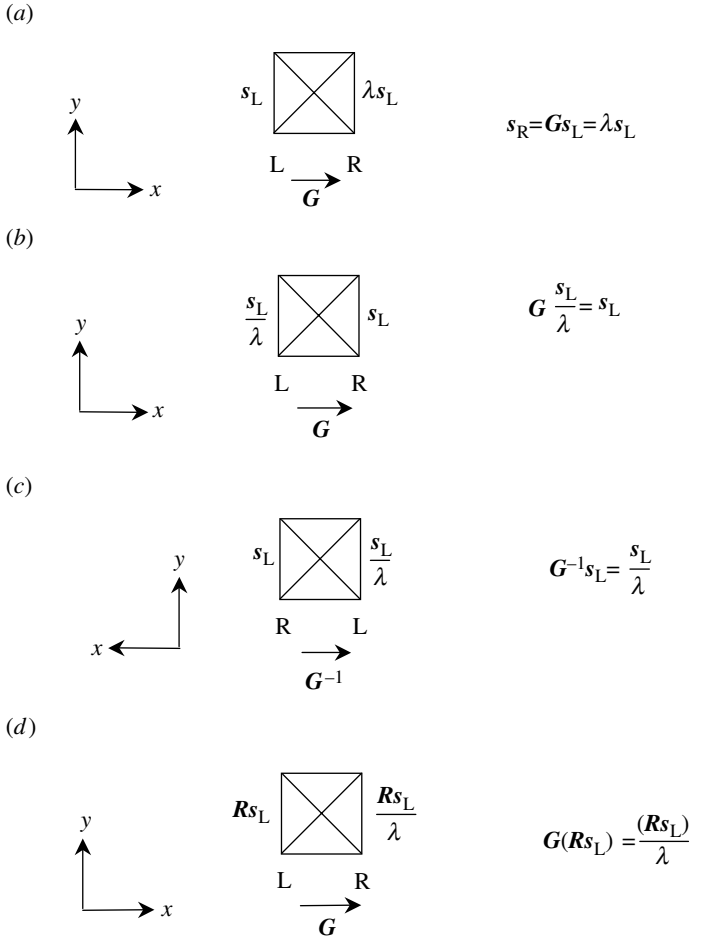

Figure 7. Symmetry arguments for reflection of symmetric cell. (*a*) Left-to-right decay, with eigenvalue $|\lambda|<1$. (*b*) Scale eigenvector: divide by λ on both sides of cell. (*c*) Reflect the cell, including the coordinate system. (*d*) Due to symmetry, this is indistinguishable from left-to-right decay; replace $\boldsymbol{G}^{-1}$ by $\boldsymbol{G}$, reflect the coordinate system, and pre-multiply state (eigen) vector by $\boldsymbol{R}$.

Now define the stiffness matrix in this representation as

$$\bar{\boldsymbol{p}} = \bar{\boldsymbol{K}}\bar{\boldsymbol{d}}, \tag{8.7}$$

or more fully

$$\begin{bmatrix} \bar{\boldsymbol{p}}_{\mathrm{L}} \\ \bar{\boldsymbol{p}}_{\mathrm{R}} \end{bmatrix} = \begin{bmatrix} \bar{\boldsymbol{K}}_{\mathrm{LL}} & \bar{\boldsymbol{K}}_{\mathrm{LR}} \\ \bar{\boldsymbol{K}}_{\mathrm{RL}} & \bar{\boldsymbol{K}}_{\mathrm{RR}} \end{bmatrix} \begin{bmatrix} \bar{\boldsymbol{d}}_{\mathrm{L}} \\ \bar{\boldsymbol{d}}_{\mathrm{R}} \end{bmatrix}, \tag{8.8}$$

which may be rearranged as

$$\begin{bmatrix} \bar{\boldsymbol{p}}_{\mathrm{R}} \\ \bar{\boldsymbol{p}}_{\mathrm{L}} \end{bmatrix} = \begin{bmatrix} \bar{\boldsymbol{K}}_{\mathrm{RR}} & \bar{\boldsymbol{K}}_{\mathrm{RL}} \\ \bar{\boldsymbol{K}}_{\mathrm{LR}} & \bar{\boldsymbol{K}}_{\mathrm{LL}} \end{bmatrix} \begin{bmatrix} \bar{\boldsymbol{d}}_{\mathrm{R}} \\ \bar{\boldsymbol{d}}_{\mathrm{L}} \end{bmatrix}. \tag{8.9}$$

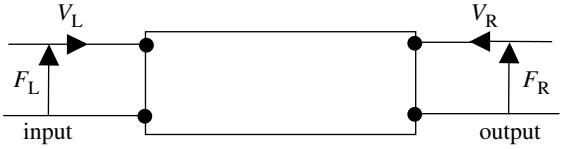

Figure 8. General four pole representation.

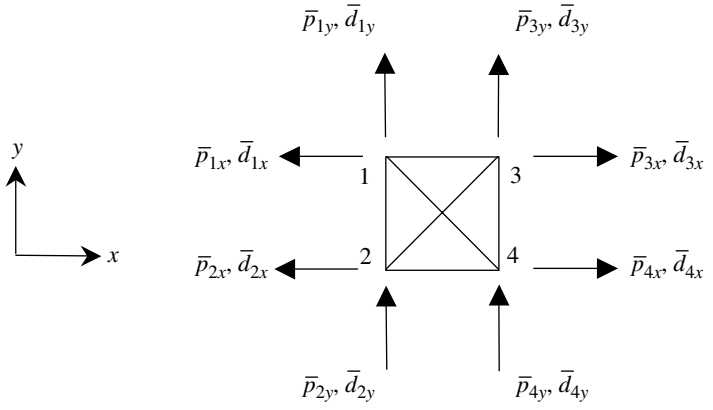

Figure 9. Definition of barred positive force and displacement components.

Now symmetry of the cell demands that equations (8.8) and (8.9) be indistinguishable, which requires

$$\bar{\boldsymbol{K}}_{\mathrm{LL}} = \bar{\boldsymbol{K}}_{\mathrm{RR}}, \tag{8.10a}$$

$$\bar{\boldsymbol{K}}_{\mathrm{LR}} = \bar{\boldsymbol{K}}_{\mathrm{RL}}. \tag{8.10b}$$

These *barred* stiffness matrix partitions can now be related to their conventional counterparts, as follows: first note

$$\bar{\boldsymbol{p}}_{\mathrm{L}} = \bar{\boldsymbol{K}}_{\mathrm{LL}}\bar{\boldsymbol{d}}_{\mathrm{L}} + \bar{\boldsymbol{K}}_{\mathrm{LR}}\bar{\boldsymbol{d}}_{\mathrm{R}} \tag{8.11a}$$

and

$$\boldsymbol{F}_{\mathrm{L}} = \boldsymbol{K}_{\mathrm{LL}}\boldsymbol{d}_{\mathrm{L}} + \boldsymbol{K}_{\mathrm{LR}}\boldsymbol{d}_{\mathrm{R}}. \tag{8.11b}$$

Pre-multiply the latter by $\boldsymbol{A}$, note that $\bar{\boldsymbol{p}}_{\mathrm{L}} = \boldsymbol{A}\boldsymbol{F}_{\mathrm{L}}$, $\bar{\boldsymbol{d}}_{\mathrm{L}} = \boldsymbol{A}\boldsymbol{d}_{\mathrm{L}}$, $\bar{\boldsymbol{d}}_{\mathrm{R}} = \boldsymbol{d}_{\mathrm{R}}$, and subtract the former to give

$$(\boldsymbol{A}\boldsymbol{K}_{\mathrm{LL}} - \bar{\boldsymbol{K}}_{\mathrm{LL}}\boldsymbol{A})\boldsymbol{d}_{\mathrm{L}} + (\boldsymbol{A}\boldsymbol{K}_{\mathrm{LR}} - \bar{\boldsymbol{K}}_{\mathrm{LR}})\boldsymbol{d}_{\mathrm{R}} = \boldsymbol{0}. \tag{8.12}$$

Now since the displacement vectors $\boldsymbol{d}_{\mathrm{L}}$ and $\boldsymbol{d}_{\mathrm{R}}$ are quite arbitrary, one must have

$$\bar{\boldsymbol{K}}_{\mathrm{LR}} = \boldsymbol{A}\boldsymbol{K}_{\mathrm{LR}} \tag{8.13a}$$

and

$$\bar{\boldsymbol{K}}_{\mathrm{LL}}\boldsymbol{A} = \boldsymbol{A}\boldsymbol{K}_{\mathrm{LL}} \tag{8.13b}$$

or

$$\bar{\boldsymbol{K}}_{\mathrm{LL}} = \boldsymbol{A}\boldsymbol{K}_{\mathrm{LL}}\boldsymbol{A}, \tag{8.13c}$$

since $\boldsymbol{A}^{-1} = \boldsymbol{A}$. Similarly, one has

$$\bar{\boldsymbol{p}}_{\mathrm{R}} = \bar{\boldsymbol{K}}_{\mathrm{RL}}\bar{\boldsymbol{d}}_{\mathrm{L}} + \bar{\boldsymbol{K}}_{\mathrm{RR}}\bar{\boldsymbol{d}}_{\mathrm{R}} \quad \text{and} \quad \boldsymbol{F}_{\mathrm{R}} = \boldsymbol{K}_{\mathrm{RL}}\boldsymbol{d}_{\mathrm{L}} + \boldsymbol{K}_{\mathrm{RR}}\boldsymbol{d}_{\mathrm{R}}; \tag{8.14}$$

substitute $\bar{\boldsymbol{p}}_{\mathrm{R}} = \boldsymbol{F}_{\mathrm{R}}$ in the latter, and subtract the former, again noting that $\bar{\boldsymbol{d}}_{\mathrm{L}} = \boldsymbol{A}\boldsymbol{d}_{\mathrm{L}}$, to give

$$(\boldsymbol{K}_{\mathrm{RL}} - \bar{\boldsymbol{K}}_{\mathrm{RL}}\boldsymbol{A})\boldsymbol{d}_{\mathrm{L}} + (\boldsymbol{K}_{\mathrm{RR}} - \bar{\boldsymbol{K}}_{\mathrm{RR}})\boldsymbol{d}_{\mathrm{R}} = \boldsymbol{0}, \tag{8.15}$$

and hence

$$\bar{\boldsymbol{K}}_{\mathrm{RR}} = \boldsymbol{K}_{\mathrm{RR}} \quad \text{and} \quad \bar{\boldsymbol{K}}_{\mathrm{RL}} = \boldsymbol{K}_{\mathrm{RL}}\boldsymbol{A}. \tag{8.16}$$

Thus, the symmetry requirements, equations (8.10*b*) and (8.13*a*), yield

$$\boldsymbol{A}\boldsymbol{K}_{\mathrm{LR}} = \boldsymbol{K}_{\mathrm{RL}}\boldsymbol{A} \quad \text{or} \quad \boldsymbol{K}_{\mathrm{LR}} = \boldsymbol{A}\boldsymbol{K}_{\mathrm{RL}}\boldsymbol{A} \tag{8.17a}$$

and

$$\boldsymbol{K}_{\mathrm{RR}} = \boldsymbol{A}\boldsymbol{K}_{\mathrm{LL}}\boldsymbol{A} \quad \text{or} \quad \boldsymbol{A}\boldsymbol{K}_{\mathrm{RR}} = \boldsymbol{K}_{\mathrm{LL}}\boldsymbol{A}. \tag{8.17b}$$

In addition, the stiffness matrix is symmetric, in which case one knows that $\boldsymbol{K}_{\mathrm{RL}} = \boldsymbol{K}_{\mathrm{LR}}^{\mathrm{T}}$; from equation (8.17*a*), this further implies that $\boldsymbol{A}\boldsymbol{K}_{\mathrm{RL}}^{\mathrm{T}} = \boldsymbol{K}_{\mathrm{RL}}\boldsymbol{A}$ and $\boldsymbol{K}_{\mathrm{LR}}^{\mathrm{T}}\boldsymbol{A} = \boldsymbol{A}\boldsymbol{K}_{\mathrm{LR}}$. In themselves, these relationships have no explicit implications for partitions of the transfer matrix $\boldsymbol{G}$.

The transfer matrix relationship may also be posed in terms of the barred state variables, as follows: from equations (8.5) and (8.6), note that

$$\begin{bmatrix} \boldsymbol{d}_{\mathrm{L}} \\ \boldsymbol{p}_{\mathrm{L}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{0} \\ \boldsymbol{0} & -\boldsymbol{A} \end{bmatrix} \begin{bmatrix} \bar{\boldsymbol{d}}_{\mathrm{L}} \\ \bar{\boldsymbol{p}}_{\mathrm{L}} \end{bmatrix}, \tag{8.18}$$

or more compactly

$$\boldsymbol{s}_{\mathrm{L}} = \boldsymbol{R}\bar{\boldsymbol{s}}_{\mathrm{L}}, \tag{8.19}$$

while

$$\boldsymbol{s}_{\mathrm{R}} = \bar{\boldsymbol{s}}_{\mathrm{R}}. \tag{8.20}$$

Substitute into equation (2.1), where the nth and $(n+1)$th sections are regarded as the left- and right-hand sides, respectively, to give

$$\bar{\boldsymbol{s}}_{\mathrm{R}} = \boldsymbol{G}\boldsymbol{R}\bar{\boldsymbol{s}}_{\mathrm{L}}. \tag{8.21}$$

Now, if the cell has left-to-right symmetry, then the subscripts 'L' and 'R' are interchangeable, that is

$$\bar{\boldsymbol{s}}_{\mathrm{L}} = \boldsymbol{G}\boldsymbol{R}\bar{\boldsymbol{s}}_{\mathrm{R}}. \tag{8.22}$$

Substitute (8.22) into (8.21) to give

$$\bar{\boldsymbol{s}}_{\mathrm{R}} = \boldsymbol{G}\boldsymbol{R}\boldsymbol{G}\boldsymbol{R}\bar{\boldsymbol{s}}_{\mathrm{R}}; \tag{8.23}$$

hence $\boldsymbol{G}\boldsymbol{R}$ is involutory, that is

$$(\boldsymbol{G}\boldsymbol{R})(\boldsymbol{G}\boldsymbol{R}) = \boldsymbol{I} \tag{8.24}$$

or

$$\boldsymbol{R}\boldsymbol{G}\boldsymbol{R} = \boldsymbol{G}^{-1}, \tag{8.25}$$

which is identical to the relationship derived by the previous, more heuristic, approach. A relationship can also be established for the JCF of such symmetric cells: substituting $\boldsymbol{G} = \boldsymbol{V}\boldsymbol{J}\boldsymbol{V}^{-1}$ and its inverse $\boldsymbol{G}^{-1} = \boldsymbol{V}\boldsymbol{J}^{-1}\boldsymbol{V}^{-1}$ into equation (8.25) gives

$$(\boldsymbol{J}\boldsymbol{B})(\boldsymbol{J}\boldsymbol{B}) = \boldsymbol{I}, \tag{8.26}$$

where $\boldsymbol{B} = \boldsymbol{V}^{-1}\boldsymbol{R}\boldsymbol{V}$; thus, $\boldsymbol{J}\boldsymbol{B}$ is involutory, as is $\boldsymbol{B}$ itself. Again one has relationship (5.3) showing that the JCF is similar to its inverse; for this left-to-right symmetric cell, the similarity matrix is further constrained to be involutory.

Expanding equation (8.24) into its square partitions leads to the relationships

$$\boldsymbol{G}_{dd}\boldsymbol{A}\boldsymbol{G}_{dd}\boldsymbol{A}-\boldsymbol{G}_{dp}\boldsymbol{A}\boldsymbol{G}_{pd}\boldsymbol{A}=\boldsymbol{I}, \tag{8.27a}$$

$$\boldsymbol{G}_{dp}\boldsymbol{A}\boldsymbol{G}_{pp}\boldsymbol{A}-\boldsymbol{G}_{dd}\boldsymbol{A}\boldsymbol{G}_{dp}\boldsymbol{A}=\boldsymbol{0}, \tag{8.27b}$$

$$\boldsymbol{G}_{pd}\boldsymbol{A}\boldsymbol{G}_{dd}\boldsymbol{A}-\boldsymbol{G}_{pp}\boldsymbol{A}\boldsymbol{G}_{pd}\boldsymbol{A}=\boldsymbol{0}, \tag{8.27c}$$

$$\boldsymbol{G}_{pp}\boldsymbol{A}\boldsymbol{G}_{pp}\boldsymbol{A}-\boldsymbol{G}_{pd}\boldsymbol{A}\boldsymbol{G}_{dp}\boldsymbol{A}=\boldsymbol{I}. \tag{8.27d}$$

9. Conclusions

A variety of results pertaining to the elastostatic transfer matrix analysis of repetitive structures has been presented. Results previously known relate to the reciprocal eigenvalue properties as a consequence of the symplectic nature of the transfer matrix, bi- and symplectic orthogonality and the impossibility of complex unity eigenvalues for prismatic repetitive structures. Multiple unity eigenvalues are a particular feature of the elastostatic eigenanalysis, and the Moore–Penrose pseudo-inverse is introduced as a rational approach to the computation of principal vectors. It is shown that only the eigenvalues $\lambda=\pm1$ can give rise to a non-trivial JCF, at least for the prismatic structure. An example of a structure for which the transfer matrix has repeating negative unity eigenvalue is one possessing a scissor-like mechanism. A planar structure, previously treated as pin-jointed, is reconsidered as rigid-jointed; the additional rotational nodal degrees of freedom give rise to new Saint-Venant decay modes—the number of transmission modes associated with unity eigenvalues is fixed—while the equivalent continuum properties are practically unaffected. This is in accord with the practice of treating real, rigid-jointed structures as pin-jointed, at least for small deflection elastic analysis. Symmetry implies restriction: a variety of relationships between partitions of both the stiffness and transfer matrices of a cell possessing left-to-right symmetry are derived; in contrast, one has splitting of unity eigenvalues for a tapered cell that lacks translational symmetry. The present elastostatic results may be seen as complementary to Langley's (1996) analysis of wave motion energetics using transfer matrices.

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