This paper examines the implications for eddy parameterisations of expressing them in terms of the quasi-Stokes velocity. Another definition of low-passed time averaged mean density (the modified mean) must be used, which is the inversion of the mean depth of a given isopycnal. This definition naturally yields lighter (denser) fluid at the surface (floor) than the Eulerian mean, since fluid with these densities occasionally occurs at these locations. The difference between the two means is second-order in perturbation amplitude, and so small, in the fluid interior (where formulae to connect the two exist). Near horizontal boundaries, the differences become first order, and so more severe. Existing formulae for quasi-Stokes velocities and streamfunction also break down here. It is shown that the low-passed time mean potential energy in a closed box is incorrectly computed from modified mean density, the error term involving averaged quadratic variability.

The layer in which the largest differences occur between the two mean densities is the vertical excursion of a mean isopycnal across a deformation radius, at most about 20 m thick. Most climate models would have difficulty in resolving such a layer. We show here that extant parameterisations appear to reproduce the Eulerian, and not modified mean, density field and so do not yield a narrow layer at surface and floor either. Both these features make the quasi-Stokes streamfunction appear to be non-zero right up to rigid boundaries. If this were not true, then more accurate results would be obtained by leaving the mean density non-zero at the boundary. The layer in which the largest differences occur between the two mean densities is the vertical excursion of a mean isopycnal across a deformation radius, at most about 20 m thick. Most climate models would have difficulty in resolving such a layer. We show here that extant parameterisations appear to reproduce the Eulerian, and not modified mean, density field and so do not yield a narrow layer at surface and floor either. Both these features make the quasi-Stokes streamfunction appear to be non-zero right up to rigid boundaries.

The linear Eady problem is used as a special case to investigate this, since terms can be explicitly computed. A variety of eddy parameterisations is employed for a channel problem, and

**Abstract**

This paper examines the implications for eddy parameterisations of expressing them in terms of the quasi-Stokes velocity. Another definition of low-passed time averaged mean density (the modified mean) must be used, which is the inversion of the mean depth of a given isopycnal. This definition naturally yields lighter (denser) fluid at the surface (floor) than the Eulerian mean, since fluid with these densities occasionally occurs at these locations. The difference between the two means is second-order in perturbation amplitude, and so small, in the fluid interior (where formulae to connect the two exist). Near horizontal boundaries, the differences become first order, and so more severe. Existing formulae for quasi-Stokes velocities and streamfunction also break down here. It is shown that the low-passed time mean potential energy in a closed box is incorrectly computed from modified mean density, the error term involving averaged quadratic variability.

The layer in which the largest differences occur between the two mean densities is the vertical excursion of a mean isopycnal across a deformation radius, at most about 20 m thick. Most climate models would have difficulty in resolving such a layer. We show here that extant parameterisations appear to reproduce the Eulerian, and not modified mean, density field and so do not yield a narrow layer at surface and floor either. Both these features make the quasi-Stokes streamfunction appear to be non-zero right up to rigid boundaries. If this were not true, then more accurate results would be obtained by leaving the mean density non-zero at the boundary. The layer in which the largest differences occur between the two mean densities is the vertical excursion of a mean isopycnal across a deformation radius, at most about 20 m thick. Most climate models would have difficulty in resolving such a layer. We show here that extant parameterisations appear to reproduce the Eulerian, and not modified mean, density field and so do not yield a narrow layer at surface and floor either. Both these features make the quasi-Stokes streamfunction appear to be non-zero right up to rigid boundaries.
the time-mean density is compared with that from an eddy-resolving calculation. Curiously, although most of the parameterisations employed are formally valid only in terms of the modified density, they all reproduce only the Eulerian mean density successfully. This is despite the existence of (numerical) delta-functions near the surface. The parameterisations were only successful if the vertical component of the quasi-Stokes velocity was required to vanish at top and bottom. A simple parameterisation of Eulerian density fluxes was, however, just as accurate and avoids delta-function behaviour completely.

1. Introduction

During the last decade, oceanographers have realised that coarse-resolution ocean models cannot adequately represent the ocean in a coupled climate model without some modifications to represent eddies. There has been a variety of schemes suggested to include eddy effects. These schemes divide into two categories. The first, which we shall be examining here, involves adding terms to represent the additional thickness flux by baroclinic eddies (Gent and McWilliams, 1990; Greatbatch and Lamb, 1990; Gent et al, 1995; Visbeck et al, 1997; Treguier et al, 1997; Killworth, 1997, 1998; Greatbatch, 1998). The second (Neptune) involves a representation of the statistical properties of eddies on the mean flow (Eby and Holloway, 1994; Merryfield and Holloway, 1997), and is not discussed here.

The effects of thickness flux can be written in a variety of ways which should formally be identical. One way is always a simple average of the product of two varying quantities. If isopycnal co-ordinates are employed, this term is the divergence of, where is the horizontal velocity and the thickness between two neighbouring isopycnals (proportional to, where is the height of an isopycnal and is the density). Analytically,

\[ \frac{\partial}{\partial t} \int \left( \frac{\partial u}{\partial z} h \right) dz = \nabla \cdot \nabla h \]

where the average is a low-pass time average on a density surface, and the suffix denotes purely horizontal terms. In (1.1) the thickness flux is written as an additional, horizontal 'bolus' velocity, which advects the mean thickness. An eddy parameterisation in an isopycnic model would supply a form for this term, which would vanish on vertical sidewalls.

If z-co-ordinates are employed, however, the situation is somewhat more awkward. The rough equivalent of thickness flux divergence becomes the divergence of:

\[ \nabla \cdot \left( \int \frac{\partial u}{\partial z} h \right) \]

where averages are Eulerian, and the divergence is fully three-dimensional. While \( \gamma \) can be parameterised, the more usual approach is to seek parameterisations for some equivalent of the bolus velocity.

One such parameterisation is suggested and tested later; in general the problems associated with diapycnal transport have caused researchers to avoid this approach.

The purpose of this paper is to be written in a variety of ways, which should formally be:

\[ \frac{\partial}{\partial t} \int \left( \frac{\partial u}{\partial z} h \right) dz = \nabla \cdot \nabla h \]

and be discussed here.

Proper soil and order to the mean flow (Eby and Holloway, 1994; Greatbatch and Holloway, 1994, 1997). The second (Neptune) involves a representation of the eddies. We consider the eddy-resolving formulation, by which we mean a model in which the eddies are resolved. This term is the divergence of the product of two varying quantities. If we denote the terms by and, we then have a simple expression for the product of two varying quantities. According to this formulation, the ocean in a coupled climate model, without some modifications to represent eddies, has been a major issue of concern for oceanographers interested in understanding the ocean's role in the carbon cycle. These expressions are derived from the parameterisations of ocean models, which have made the ocean's role in the carbon cycle less clear to many. Although the parameterisations developed for the ocean models have been successful in representing the ocean's role in the carbon cycle, it is important to note that these models can only represent the ocean's role in the carbon cycle if they are able to capture the physics of the ocean's role in the carbon cycle.
EMD, then the fluxes associated with the shaded fluid are ignored, producing an apparently non-
amounts because of the above discussion. The MMD is advected by the (Eulerian) mean flow and
for the total (mean plus eddy) flow is shown as a function of density. If 'density' is taken to be the
mean is. The densities lighter than are shown shaded. In Fig. 1b, the streamfunction
TRM theory is formally valid. However, the time derivatives of EMD and MMD differ by
for 'mean' density, and indicated schematically in Fig. 1 (Treguier, Held and Larichev, 1997 give
the surface (or bottom) density varies over the averaging period as shown in Fig. 1a. The time-
point (EMD for short), one should interpret density as being the inversion of the mean depth of a
number of technical issues relating to the intrinsic differences between averages on density surfaces
the quasi-Stokes streamfunction is given to second order in amplitude by

The quasi-Stokes vector streamfunction is given to second order in small quantities and is thus very small where the
intrinsically divergent, while the quasi-Stokes velocity is (by construction) non-divergent, and (b)
by the quasi-Stokes velocity:

1/2

The problems are best seen by considering recent direct eddy-resolving computations [that by
Rix and Willebrand (1996) did not discuss the shape of other peaks of quasi-Stokes vectors and
and on level surfaces (i.e., between pseudo-Lagrangian and Eulerian means). The most logical
number of technical issues relating to the intrinsic differences between averages on density surfaces

The quasi-Stokes vector streamfunction is given to second order in small quantities and is thus very small where the
intrinsically divergent, while the quasi-Stokes velocity is (by construction) non-divergent, and (b)
by the quasi-Stokes velocity:
There is, simply, nowhere to 'put' the extra fluxes in an Eulerian sense. The streamfunction is clearly zero at the minimum density: no fluid ever enters at lighter densities. Equally true is that the streamfunction is nonzero at the density. The question, which is far from just philosophical, is how to interpret mean density in a non-eddy-permitting model. Some readers may be surprised at this statement. After all, M has argued cogently for the definition to be \( \text{MMD} \). This causes both the total and quasi-Stokes streamfunctions to vanish at the surface and floor. For realistic finite amplitude fluctuations, however, the streamfunction changes rapidly very close to the surface, as we shall show. Most non-eddy-resolving models are unlikely to resolve the scale over which this changes, so that they would fail to reproduce the lightest density layers, and act as if the streamfunction had something approximating to a delta-function near-surface. If this layer is not resolved, the quasi-Stokes streamfunction cannot vanish at what is now the surface, inducing an apparent flux through the surface to represent the 'missing' flux on lighter density surfaces. In other words, there may well be a difference—which will be addressed in this paper—between the correct description, using \( \text{MMD} \), and the description in an under-resolved model or one using an eddy parameterisation, which may for numerical or physical reasons be using a density field truncated near surface and floor and so resembling the EMD.

In confirmation of this discussion, Killworth (1998) found it impossible to produce a streamfunction which vanished at top and bottom. Indeed, the streamfunction attained extreme values at the surface and floor. If other simple numerical inaccuracies were disguising a true zero value at surface and floor, or there were a damping down near surface and floor as suggested by M, one would expect a reduction in its value from the interior as the horizontal boundaries are approached; this is not seen.

Treguier (1999) used an extensive eddy-resolving channel computation to diagnose both eddy-induced velocities on density surfaces, as well as the quasi-geostrophic version of \( \text{MMD} \). The two sets of velocities were found to be very similar except near the surface, indirectly confirming M. However, Treguier's Fig. 7b shows clearly that reaches extreme values at the surface and floor, rather than vanishing. Gille and Davis (1999) ran channel models, both of the Eady problem and of a wind-forced problem, and diagnosed the eddy terms. In their Fig. 7, they show what is the majority of the TRM streamfunction, which again does not vanish at the surface (it is small at depth, so that no conclusions can be drawn from their figure as to whether the streamfunction vanishes at the floor). McIntosh and McDougall (1996) plotted overturning streamfunction, computed from FRAM, on \( \text{MMD} \) (their Fig. 4) and on \( \text{EMD} \) (their Fig. 5). It is clear that the latter case – albeit computed with M's interior formulae and so in error near-surface – does not capture the additional near-surface and floor fluxes which \( \text{MMD} \) does.

These direct calculations, then, show that the boundary conditions applied to quasi-Stokes vertical velocities in parameterisations, which historically are consistently those of zero flow at rigid surfaces, need investigation. Particularly, what differences are produced in simulations if the requirement of vanishing at surface and floor are relaxed? To reiterate, if the physics of the model being employed—e.g. some eddy parameterisation—fails to reproduce the fine density structure, it is not clear we would wish to enforce it. Following a discussion of the small amplitude theory used by M and MM, the behaviour of the \( \text{MMD} \) near horizontal boundaries is discussed (section 2). We show that the differences between \( \text{EMD} \) and \( \text{MMD} \) become large in certain situations. In regions where the \( \text{EMD} \) and \( \text{MMD} \) meet important boundaries are discussed. We show how the boundary conditions are redefined to capture the eddy phenomena in parameterisations, which historically have been based on small-amplitude theory.
Small-amplitude theory (section 3) is used to evaluate the relevant expressions making up either the flux divergence or the vector streamfunction. Small-amplitude theory has its disadvantages, but it is at least an exact solution to the equations of motion in the limit of vanishingly small perturbations; it is also accurate to precisely the same order as the M and MM theory. We show specifically that and the quasi-Stokes streamfunction do not vanish at surface or floor using the M formulae. En route, two equivalents of the isopycnal co-ordinate parameterisation of Killworth (1997), which had been restated as a-co-ordinate version in that paper without proof, are produced (section 4). Section 5 then briefly discusses these results, comparing them with the Killworth (1997), showing how the delta-functions at surface and floor present in that theory become precisely the vertical quasi-Stokes velocity computed at surface and floor from the second-order M formulae. Section 6 evaluates closed-form solutions for the Eady (1949) problem. We show more generally that mass and energy conservation holds for the EMD formulation, but energy conservation does not hold for the MMD formulation even if exact expressions are used through the entire water column, for reasons described earlier.

Section 8 asks the question: given that current climate models cannot resolve the differences between densities, can current eddy parameterisations? We revisit a test of parameterisations (Killworth, 1998), run both with and without the vertical quasi-Stokes velocity vanishing at the surface and floor in two parameterisation schemes. We find that the non-zero surface vertical quasi-Stokes velocity results are uniformly poor compared with zero values. However, an alternative parameterisation, using a direct estimate of the density flux divergence in co-ordinates, performs just as well, and would be relevant for an Eulerian definition of mean density. We conclude that parameterisations using quasi-Stokes formulations – which should formally reproduce the MMD – do apparently perform better with no advection through surface and floor than reproducing the EMD.

2. Eulerian and modified mean densities near a horizontal boundary
(a) Small amplitude

Both M and MM have derived formulae connecting Eulerian and isopycnic averages for the case when perturbations are of small amplitude. In particular, the MMD and EMD are connected by

\[ \varphi = \frac{\varphi'}{2 \bar{\sigma}} \]

where \( \bar{\sigma} \) is half the density variance:

\[ \bar{\sigma} = \frac{1}{2} \left( \sigma^2 \right) \]

Thus if \( \varphi \) is a representative amplitude of the small perturbations, although (examples will be given later) the rate of change of the two densities, being, can be quite different.

We assume without loss of generality. The use of density co-ordinates means that MMD can be calculated exactly. Define a low-pass average of some quantity \( \varphi \) on a density surface as

\[ \bar{\varphi} = \frac{1}{2} \left( \varphi + \varphi' \right) \]

This variation is produced by unspecified three-dimensional motions; the dependence on horizontal position is irrelevant for the current discussion.
There is apparently a choice whether to compute the average value of only during the time that density is present, 

\[ \text{...} + (i)^{(1)}(j) + \ldots + (n)^{(1)} \] 

... and so on.

In summary, when (the lightest fluid which never outcrops at the surface) 

\[ \text{...} + (i)G^{(1)} + (i)G^{(2)} + \ldots + (n)G^{(1)} + (n)G^{(2)} \times 1 \ldots + (n)G^{(m)} \] 

and so on. Now suppose \( z \) is within \( \mu \) of \( \delta \); we write.

\[ \mu (i)G^{(1)} + (i)G^{(2)} + \ldots + (n)G^{(m)} \] 

To proceed, we have.

\[ \frac{\delta}{\delta_0} \] 

where the range of densities where the integration is restricted is.

\[ \mu (i)G^{(1)} + (i)G^{(2)} + \ldots + (n)G^{(m)} \] 

and other terms.

\[ \mu (i)G^{(1)} + (i)G^{(2)} + \ldots + (n)G^{(m)} \] 

where we have used the definition of VMD in the notation. Since we have from that

\[ (\delta - \mu) \] 

and hence is apparent.

\[ \mu (i)G^{(1)} + (i)G^{(2)} + \ldots + (n)G^{(m)} \] 

and since is apparent.
As stated, the two densities differ at first, not second, order in small quantities. This is indicated schematically in Fig. 2, which also shows a specific example, for which and are very similar in the interior (for small amplitude) but differ much more strongly near surface and floor, in a manner similar to a delta-function.

Finite amplitude density fluctuations will equilibrate at about, where is the deformation radius and the horizontal gradient operator. This implies that the vertical scale is a familiar two-dimensional parameterization result, as suggested by M. It is the typical vertical excursion made when moving a short horizontal distance (b) above a mean isopycnal which moves significantly vertically only on the gyre scale.

This scale is rather small for the ocean, though not for the atmosphere. Even with fairly optimistic estimates, it is hard to produce a vertical scale much larger than 20 m. So the distance over which the MMD and EMD differ significantly is not resolved in most climate models, being concentrated in the last grid point. Thus the near-boundary differences between the two mean densities will probably appear to climate models as single grid-point effects, i.e. delta functions.

Figure 3 shows this effect clearly (one of Mckibbin and Macdonald, 1999b). On the horizontal gradient operator. The images which the model can resolve in this case are determined by the deformation radius of the eddies. The model is not sensitive to the horizontal length scale of variability, and as a result the differences between EMD and MMD become small.

The presence of a mixed layer (not treated here) makes no difference in this argument, since it is not the vertical excursion made when moving a short horizontal distance (c) above a mean isopycnal. The differences between the two densities have two important effects. The first is directly concerned with the interpretation of mean density. It is straightforward to see that the low-pass time filtered net mass in a water column, which is a uniquely defined value, is the same whether EMD or MMD is used:
parameter changes little over a scale, so that the equation

\[ H - 0 = \frac{dz}{\gamma} = \frac{\nu}{\gamma} \]

is a small quantity, where

\[ \frac{\gamma}{\nu} = \beta \]

Since parameters change, the horizontal variation for what follows, where is either smaller, or much smaller, than the horizontal variation of density may be less than or equal to the vertical variation in subpolar gyres, where we have retained the multiplicand at the surface for clarity, and higher order terms are neglected. In this section, we extend linear instability theory beyond the quasigeostrophic limit, using vertical co-ordinates. The same does not hold in the theory is not always a good predictor of the behaviour of a nonlinear eddying system, as Edmond et al. (1980) show clearly for the quasi-geostrophic limit. (Note that the TRM formulae of M and N (1975, 1976) for the low-pass filtered potential energy of a fluid column (a uniquely defined quantity) is straightforward, either from the M formulae or by direct evaluation, and is not given here. The difference between the two PE expressions lies in the variability, fundamentally a part of the low-pass filtering. The same does not hold in the quasigeostrophic theory, because the noncommutative operators on products of quantities. For small amplitude, the differences between EMD and MMD is a small quantity, where

\[ \frac{\gamma}{\nu} = \beta \]

and the second negative. The same does not hold in the quasigeostrophic theory, because the noncommutative operators on products of quantities. For small amplitude, the differences between EMD and MMD is a small quantity, where

\[ \frac{\gamma}{\nu} = \beta \]
The confirmation that the neglected terms remain small even after the vorticity equation is created is tedious and not particularly illuminating. We then find expressions for the quantities necessary for the expression of the potential vorticity normal to the direction of the stream. A combined expression involving the stream function and the pressure is:

\[
\frac{\partial}{\partial y} \left( \frac{\partial \psi}{\partial x} \right) - \frac{\partial}{\partial x} \left( \frac{\partial \psi}{\partial y} \right) = \frac{\partial}{\partial y} \left( \frac{\partial \psi}{\partial x} \right) - \frac{\partial}{\partial x} \left( \frac{\partial \psi}{\partial y} \right)
\]

where \( \psi \) is the stream function and \( \phi \) is the potential. The additional factor ensures that stretching of planetary vorticity does not dominate the problem. The potential vorticity normal to the direction of the stream is:

\[
\frac{\partial}{\partial y} \left( \frac{\partial \psi}{\partial x} \right) - \frac{\partial}{\partial x} \left( \frac{\partial \psi}{\partial y} \right) = \frac{\partial}{\partial y} \left( \frac{\partial \psi}{\partial x} \right) - \frac{\partial}{\partial x} \left( \frac{\partial \psi}{\partial y} \right)
\]

Hence, the expression for the potential vorticity normal to the direction of the stream is:

\[
\frac{\partial}{\partial y} \left( \frac{\partial \psi}{\partial x} \right) - \frac{\partial}{\partial x} \left( \frac{\partial \psi}{\partial y} \right) = \frac{\partial}{\partial y} \left( \frac{\partial \psi}{\partial x} \right) - \frac{\partial}{\partial x} \left( \frac{\partial \psi}{\partial y} \right)
\]

with boundary conditions of zero at zero and position. Here:

\[
0 = \frac{\partial \psi}{\partial y} + \left[ \frac{\partial}{\partial y} \left( \frac{\partial \psi}{\partial x} \right) - \frac{\partial}{\partial x} \left( \frac{\partial \psi}{\partial y} \right) \right] \frac{\partial \psi}{\partial x} \frac{\partial \psi}{\partial y}
\]

So the expression of the potential vorticity normal to the direction of the stream is:

\[
\frac{\partial}{\partial y} \left( \frac{\partial \psi}{\partial x} \right) - \frac{\partial}{\partial x} \left( \frac{\partial \psi}{\partial y} \right) = \frac{\partial}{\partial y} \left( \frac{\partial \psi}{\partial x} \right) - \frac{\partial}{\partial x} \left( \frac{\partial \psi}{\partial y} \right)
\]

Now, separate vorticity into two parts:

\[
\frac{\partial}{\partial y} \left( \frac{\partial \psi}{\partial x} \right) - \frac{\partial}{\partial x} \left( \frac{\partial \psi}{\partial y} \right) = \frac{\partial}{\partial y} \left( \frac{\partial \psi}{\partial x} \right) - \frac{\partial}{\partial x} \left( \frac{\partial \psi}{\partial y} \right)
\]

Cross-differential the momentum equations (neglecting terms containing small and not mass. Here we add the theorem of conservation of vorticity.
For the $u$-component, we need
\[ \int \frac{\partial^2 u}{\partial x \partial y} \, dy = \frac{\partial}{\partial y} \left( \frac{\partial u}{\partial y} \right) = \phi \]
so that
\[ \int \frac{\partial u}{\partial y} \, dy = \frac{\partial}{\partial y} u = \phi \]
parameterisation which performed well in a channel model simulation (Killworth, 1998). However, linear theory using density co-ordinates was used by Killworth (1997) to create an eddy

\[ \mathbf{v} = \mathbf{q} \cdot \mathbf{Dv} \]

For a channel problem, the quasi-Stokes streamfunction, using the M formulation for the interior of the fluid, does not vanish identically, and the surface and floor values of reduce to 16

\[ \frac{\partial \mathbf{v}}{\partial x} \]

Recall that the second terms in (3.13), (3.14) are usually small compared with the first terms, so

\[ \mathbf{v} + \mathbf{v} = \mathbf{v} \]

Similarly, we can compute from (3.1). Now, the product of the two terms in the divergence formulation is discussed in detail by Treguier (1997).

The scalings in Appendix A show that the main term acting to change the mean density is now potential vorticity is mixed (together with a possible rotation term), and not thickness.
The connection with possible co-ordinates

The second parameterisation uses the scalings and approximations to compute the quasi-Stokes formulae (3.5) to (3.7). This is intrinsically a scaling using EMD. Since the eddy terms can be evaluated everywhere, questions of boundary conditions do not enter the formulation: \( \alpha \) can be evaluated anywhere.

The first parameterisation simply evaluates directly, using these scalings and approximations to compute the eddy terms. Since the eddy terms are non-zero, the deformations and transformation of coordinates are needed. The transformation is given by

\[
\mathbf{\Psi} = \mathbf{\Psi}(\mathbf{\Phi})
\]

The eigenvector is given by

\[
\mathbf{e} = \mathbf{e}(\mathbf{\Phi})
\]

Section 7. Conclusions

In conclusion, the two parameterisations are different, but both provide good approximations to the problem, with the advantages of the first parameterisation being simplicity and ease of use, and the disadvantages of the second parameterisation being a more complex formulation.
additional contributions are negligible. Using the Gent and McWilliams (1990) formulation where parameterisations such as Killworth (1997), in which is well-behaved at surface and floor, the first terms cancel, leaving Killworth's (1997 eqn. 41a), namely

\[ \frac{\partial}{\partial z} \left( \frac{\partial \psi}{\partial z} \right) = \frac{1}{\sqrt{g}} \int \frac{f}{\sqrt{g}} \, dz \]

The first terms cancel, leaving Killworth's (1997) formulation

\[ \frac{\partial}{\partial z} \left( \frac{\partial \psi}{\partial z} \right) = \frac{1}{\sqrt{g}} \int \frac{f}{\sqrt{g}} \, dz \]

The last terms on both sides yields

\[ \frac{\partial}{\partial z} \left( \frac{\partial \psi}{\partial z} \right) = \frac{1}{\sqrt{g}} \int \frac{f}{\sqrt{g}} \, dz \]

We note from (3.13), (3.14) that, using interior formulae, the full range of approximations, so that the 'missing' fluxes, which belong to no available EMD, match precisely the values of the quasi-Stokes streamfunction evaluated using the M (interior) formulae. We write, more simply, still using the M formulation for small perturbations which holds

\[ \frac{\partial}{\partial z} \left( \frac{\partial \psi}{\partial z} \right) = \frac{1}{\sqrt{g}} \int \frac{f}{\sqrt{g}} \, dz \]

We now have (3.13), (3.14) that, using interior formulae, the full range of approximations, so that the 'missing' fluxes, which belong to no available EMD, match precisely the values of the quasi-Stokes streamfunction evaluated using the M (interior) formulae. We write, more simply, still using the M formulation for small perturbations which holds

\[ \frac{\partial}{\partial z} \left( \frac{\partial \psi}{\partial z} \right) = \frac{1}{\sqrt{g}} \int \frac{f}{\sqrt{g}} \, dz \]

We note from (3.13), (3.14) that, using interior formulae, the full range of approximations, so that the 'missing' fluxes, which belong to no available EMD, match precisely the values of the quasi-Stokes streamfunction evaluated using the M (interior) formulae. We write, more simply, still using the M formulation for small perturbations which holds

\[ \frac{\partial}{\partial z} \left( \frac{\partial \psi}{\partial z} \right) = \frac{1}{\sqrt{g}} \int \frac{f}{\sqrt{g}} \, dz \]
and the pseudo-vertical flux is

\[
\frac{\partial}{\partial V Y} = \varepsilon \delta \gamma = v^
\]

for the case (adjusted to density co-ordinates). The pseudo-vertical velocity is

\[
\frac{\partial}{\partial V Y} = \varepsilon \delta \gamma = v
\]

which is uniform in \(z\).

Next,

\[
\gamma = (\varepsilon \delta \gamma) V Y = \phi
\]

from which, if we compute all relevant quantities. We have

\[
\gamma = (\varepsilon \delta \gamma) V Y = \phi
\]

The formulation in density co-ordinates looks almost identical, with the replacement of \(z\) by \(p\) and of \(\delta\) by the pseudo-vertical velocity is

\[
\frac{\partial}{\partial V Y} = \varepsilon \delta \gamma = v
\]

which is uniform in \(z\). Thus

\[
\gamma = (\varepsilon \delta \gamma) V Y = \phi
\]

so we now compute all relevant quantities. We have

\[
\gamma = (\varepsilon \delta \gamma) V Y = \phi
\]

is the purely imaginary phase velocity. For the fastest growing mode,

\[
\gamma = (\varepsilon \delta \gamma) V Y = \phi
\]

where is the deformation radius, gives the familiar equation for the perturbation

\[
\gamma = (\varepsilon \delta \gamma) V Y = \phi
\]

Applying the boundary conditions (no seiche and no flow) gives the standard results that

\[
\gamma = (\varepsilon \delta \gamma) V Y = \phi
\]

Next.

\[
\gamma = (\varepsilon \delta \gamma) V Y = \phi
\]

so we now compute all relevant quantities. We have

\[
\gamma = (\varepsilon \delta \gamma) V Y = \phi
\]

is the purely imaginary phase velocity. For the fastest growing mode,

\[
\gamma = (\varepsilon \delta \gamma) V Y = \phi
\]

where is the deformation radius, gives the familiar equation for the perturbation

\[
\gamma = (\varepsilon \delta \gamma) V Y = \phi
\]

Applying the boundary conditions (no seiche and no flow) gives the standard results that

\[
\gamma = (\varepsilon \delta \gamma) V Y = \phi
\]
The solution is indicated in Fig. 6. Here we assume that the amplitude increases monotonically from zero at the southern boundary to a maximum in the centre of the channel, and locally-based eddy parameterisations. 

The solution here is indicated in Fig. 6. Here we assume that the amplitude increases monotonically from zero at the southern boundary to a maximum in the centre of the channel, and locally-based eddy parameterisations.

We now consider the temporal change terms. From these, we derive by (5.24) and substitution

(5.25)
Now we compute the equivalent using MMD. Again, this only yields sensible values if the exact
2
achieved by requiring \( \tau \) not to vanish at surface and floor. This balance, as noted, holds in general (see Appendix B).

For the Eady problem, we can compute potential energy changes directly. For the Eulerian case,
We consider first the change in the area-integrated density field in a channel geometry for linear
theory. Now this must be zero: integrating (1.2) across the channel area means that the divergences
of (2.1) do not hold if the
form.

Formally, there are only two approaches: to use EMD, with the eddy terms being; and to
use MMD, with the quasi-Stokes streamfunction evaluated correctly everywhere. However, (7.2) does not hold if the
theory. Now this must be zero: integrating (1.2) across the channel area means that the divergences
of (2.1) do not hold if the
form.

Each parameterisation (\( \rho_{\text{EMD}} \) and \( \rho_{\text{MMD}} \)) introduces an additional area integral, so that no mass can be gained or lost from the
system. We have seen that the column integral of the MMD is identical, so that (7.1) must hold
showing that a fuller representation of eddy effects will have to take nonlocal factors into account.
The difference between the and values is precisely equal to the rate of change computed using small perturbation Eady theory in density co-ordinates:

The comparisons are not ideal. Like other published work, they are of Eulerian means only, and over a period probably an order of magnitude too short for a good statistical comparison. However, the trends in Fig. 3 were visually unaltered by averaging over another period of similar length, so the statistics may be better than we suggest.

Comparisons cannot be sensibly made with two-dimensional calculations over the same time span, since the intermediate time behaviour of the full eddying simulation and the two-dimensional calculations is invariably different. Thus only steady state two-dimensional results can be compared with the long-time average.

The comparisons are shown in Table 1, and used both a direct correlation between the fields, which is of little discriminatory use, and a more stringent measure of explained variance due to Visbeck et al. (1997). As discussed by Killworth (1998), the two-dimensional runs have no depth-averaged field, so that only the baroclinic can be compared. The barotropic field, as noted by Killworth, plays a not inconsiderable role in the dynamics.
from numerical approximations. Note that direct attempts to parameterise the flux divergence
\[ \nabla \cdot \mathbf{V} \] (computing directly from small-amplitude formulae, also discussed below); the last two parameterisations for the EM2, directly evaluate either or
\[ \nabla \cdot \mathbf{V} + (\frac{\partial \psi}{\partial z}) \] of \( \psi \).
The former could contain some measure of the rotational flux, though the latter could not, apart
about \( \psi \), which was needed for a second-order accurate representation of the three-dimensional

the 2GM parameterisations used were (in order of appearance in Fig. 7):
1. \( K_s \) (Gent and McWilliams 1990, which has a constant diffusivity); Fig. 7d
2. \( 2K_97 \) (more properly, the depth co-ordinate version of Killworth, 1997, discussed earlier;
3. \( 3GM_90 \) (Gent and McWilliams 1990, which has a constant diffusivity); Fig. 7d
4. \( K_97 \) (Killworth 1997, adapted as discussed below); Fig. 7f

The other parameterisations used were (in order of appearance in Fig. 7):
1. \( 2K_97 \) (more properly, the depth co-ordinate version of Killworth, 1997, discussed earlier;
2. \( 3GM_90 \) (Gent and McWilliams 1990, which has a constant diffusivity); Fig. 7d

The fourth parameterisation, \( K_s \), attempted to do the same thing for \( K_97 \), which only specifies
floor; the delta-function changes are thus spread across the (relatively wide) top and bottom grid
points. The diffusivity is taken as a constant. The \( K_97 \) parameterisation is as discussed earlier,
"pushing forward" of isopycnals in the MMD, so that direct comparisons with it are not useful.
"pushing forward" of isopycnals in the MMD, so that direct comparisons with it are not useful.
the imposed vanishing of the vertical temperature gradient at surface and floor). A better yardstick

both measures exclude the forcing region. Both integration time and the area

Griffies 1998), which are to be compared with the averaged three-dimensional solution. Fig. 7a

'pushing forward' of isopycnals in the MMD, so that direct comparisons with it are not useful.
"pushing forward" of isopycnals in the MMD, so that direct comparisons with it are not useful.
the imposed vanishing of the vertical temperature gradient at surface and floor). A better yardstick

both measures exclude the forcing region. Both integration time and the area

Griffies 1998), which are to be compared with the averaged three-dimensional solution. Fig. 7a

'pushing forward' of isopycnals in the MMD, so that direct comparisons with it are not useful.
"pushing forward" of isopycnals in the MMD, so that direct comparisons with it are not useful.
the imposed vanishing of the vertical temperature gradient at surface and floor). A better yardstick

both measures exclude the forcing region. Both integration time and the area

Griffies 1998), which are to be compared with the averaged three-dimensional solution. Fig. 7a

'pushing forward' of isopycnals in the MMD, so that direct comparisons with it are not useful.
usually suffer from Veronis effects (Veronis, 1975); however, this approach does not, since the terms are derived from solutions to the equations of motion and so have the same conservation properties (for the flux terms) as the original system.

Table 1 shows the measures of fit for the solutions for each parameterisation, with the coefficient (diffusivity for Gent-McWilliams, the scaling factor for Killworth) adjusted to values which generate the best fit. Usually not all four fits can be optimised simultaneously, and the values cited are slightly subjective (small changes affecting the second significant figure).

The most accurate version of the GM90 parameterisation for this problem has a diffusion coefficient of $160 \, \text{m}^2 \text{s}^{-1}$, a little lower than that cited in Killworth (1998). The results for the GM90 (Fig. 7d) are very similar to those of pure diffusion (6c), although slightly less accurate than this in the field. The similarity is surprising since the GM90 includes the strong northward (southward) advection near the surface (floor) which is not present in the simple diffusive case.

The most accurate version of the K97 parameterisation (Fig. 7e) has, as used in Killworth (1998) for the same problem. As Fig. 7e shows, this parameterisation is the only one to produce the 'doming' of the 15.5° isotherm near the northern boundary with any accuracy. It is, as Table 1 shows, the most accurate of the parameterisations.

If it is not required to have an accurate solution at the surface and floor, then for this geometry the parameter values used hitherto are insufficient to reproduce the three-dimensional solution. This is because the high northward advection near-surface is now lacking. For the GM90 parameterisation (Fig. 7f), it was necessary to increase the diffusion coefficient an order of magnitude (to $1200 \, \text{m}^2 \text{s}^{-1}$) in order to reproduce an approximation to the three-dimensional fields. Although the temperature field looks reasonable, the corresponding velocity is poorly reproduced, due to the strong surface front near the southern boundary. A similar finding holds for the K97 parameterisation (Fig. 7g; recall that this could not be run with a sufficiently high value of diffusion coefficient for a proper parameterisation). Thus permitting non-zero quasi-Stokes streamfunctions at surface and floor has not achieved a higher accuracy than maintaining zero, for this problem and choice of parameterisations.

However, the final two parameterisations (VP, VPWP) do not use the (\alpha, \omega) formulation but simply insert a parameterisation for mixing directly. The results (Figs. 7h, 7i) are very similar, with VPWP being slightly superior; both yield an accurate reproduction of the three-dimensional result. In terms, then, of reproducing the Eulerian mean density, most schemes were successful, with the K97 and VPWP schemes marginally superior to the others, and schemes which permitted nonzero quasi-Stokes streamfunctions at surface and floor were quite inferior.

This paper has argued that within this narrow layer, quasi-Stokes streamfunctions, whether computed by inaccurate near-boundary second order formulae or exactly, possess a near-distribution function behaviour which cannot be well represented in numerical models. At finite amplitude, this layer would still be very thin and almost certainly unresolvable by most extant climate models. Thus it might well be that a better behaviour for parameterisations using the quasi-Stokes formulation would be to permit the streamfunction to be nonzero at surface and floor. Numerical experiments showed this not to be the case (the errors produced by nonzero surface streamfunctions were far larger than one would expect to be produced by the differences between the approximate formulae for the two mean densities).

The remainder of this paper exchanges the treatment of the two mean densities: the modified, and the mean density which we refer to as the Eulerian, mean density. The mean density is shown to be an alternative parameterisation of the dynamical equation, and of particular importance, that the Eulerian mean density is shown to be a better representation of the mean density than the modified mean density.
are represented correctly in ocean model parameterisations.

If the parameterisations are tuned adequately, it is thus important that the pseudo-vertical motions actually produced by pseudo-vertical motions are represented correctly in ocean model parameterisations. In terms of TRM motions as relaxing some originally stratified front, in co-ordinates the relaxation is

\[ \frac{\partial}{\partial t} \mathbf{v} + (\mathbf{u} \cdot \nabla) \mathbf{v} = \nabla \cdot \left( \mathbf{g} \mathbf{N} \right) - \mathbf{a} \]

where \( \mathbf{v} \) represents velocities, dominates over by the same amount. In other words, while we think of lateral

\[ \frac{\partial \mathbf{v}}{\partial t} = \mathbf{g} \mathbf{N} - \mathbf{a} \]

where \( \mathbf{a} \) represents .

Another result from the runs was that a direct parameterisation of the Eulerian mixing term yielded no solution resembling the MMD. This demonstrates Treguier & Gille’s (1997) arguments for the quasigeostrophic regime. In such cases (small), dominates over by an amount. However, in terms of TRM parameter ranges; those of Treguier (1999) and Gille and Davis (1999) would make a useful start.

Acknowledgments

My thanks to the two referees who patiently corrected my often careless representations of TRM, and to colleagues at SOC who both heard more about this problem than they ever wanted.

My thanks to the two referees who patiently corrected my often careless representations of TRM parameterisations.
\[
\{ \langle \varphi - v, \varphi - v \rangle \rangle \frac{\partial}{\partial x} (\varphi - v) + \frac{\partial}{\partial y} \langle \varphi - v, \varphi - v \rangle \rangle \frac{\partial}{\partial y} (\varphi - v) \rangle \rangle \frac{\partial}{\partial y} \frac{\partial}{\partial y} \frac{\partial}{\partial y} = \Delta \varphi
\]

Here we demonstrate that for the linear theory here; I am not aware of any proofs beyond the Galerkin procedure. For simplicity, define the mean follows: \[ \bar{\varphi} \]

\[ \text{Table 1} \]

<table>
<thead>
<tr>
<th>Parameterisation</th>
<th>Agreement measures for various two-dimensional parameterisations</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\varepsilon$</td>
<td>$\text{correlation}$</td>
</tr>
<tr>
<td>0.00</td>
<td>0.99</td>
</tr>
<tr>
<td>0.01</td>
<td>0.96</td>
</tr>
<tr>
<td>0.02</td>
<td>0.84</td>
</tr>
<tr>
<td>0.16</td>
<td>0.16</td>
</tr>
</tbody>
</table>

APPENDIX B: ENERGY CONVERSION

\[ \text{Similarly, again integrating only in the vertical, noting that the horizontal divergences give no} \]

\[ \text{contribution when integrated across the domain.} \]

\[ \text{Then we have, by the vertical} \]

\[ \text{Finally, we have confined the horizontal} \]

\[ \text{Then the boundary conditions on} \]

\[ \text{and the boundary condition becomes} \]

\[ \text{Here we demonstrate that} \]

\[ \text{for simplicity, define the mean follows:} \]

\[ \text{APPENDIX B: ENERGY CONVERSION} \]
References


1. (a) time variation of surface density (assumed sinusoidal). The shaded area shows densities which are lighter than the Eulerian mean. (b) Any eddy transport in density layers in this range does not appear if the streamfunction is plotted against Eulerian mean density (i.e. the shaded area is lost) so that the streamfunction is nonzero at the 'surface' density. If plotted against modified density, streamfunction values are correctly recorded and the streamfunction becomes zero at the surface.

2. The differences between Eulerian mean and modified density. The upper diagram shows that the densities are very close to each other in the fluid interior (differing by, where is the small amplitude of the fluctuations). In a zone of size near surface and floor, the two densities differ by a much larger amount,, as indicated in the exploded lower view (which is actually the exact solution for sinusoidal time variation and uniform interior density gradient).

3. The Eulerian and modified mean density for a 4-year and along-channel average of an eddy-permitting channel model discussed in the text. (The average over the previous 4 year period is almost identical.) The problem was chosen to provide a larger vertical range over which the EMD and MMD differ than would hold for the real ocean, so that the vertical resolution (10 m) was adequate. Also shown is a typical two-dimensional parameterisation steady-state result, in this case following Gent and McWilliams (1990), using an eddy diffusion of $2000 m^2 s^{-1}$. While the latter does not reproduce the EMD particularly well (true for a wide range of diffusivities), it does not reproduce the MMD at all where this differs from the EMD. This appears to hold for most extant parameterisations.

4. Tendency terms for the linear Eady problem. Shown are (assumed independent of the cross-stream direction for simplicity), , and the resulting .

5. Schematic of the quasi-Stokes streamfunction generated from linear Eady theory for a very wide channel in which the eddy amplitude is the same at all points across the channel (as would be produced by most parameterisation schemes). No flow is generated save for two delta-function vertical velocities at the vertical walls, and two more, this time horizontal, at surface and floor.

6. The correct solution of the linear Eady problem's quasi-Stokes velocity when the eddy amplitude varies smoothly across the channel. Broad pseudo-vertical velocities are produced with the signs as shown, acting to increase (decrease) the density of the light (heavy) water, and additional delta-function horizontal velocities induced by setting the quasi-Stokes streamfunction to zero at top and bottom.

7. Contours of temperature (°C; contour interval 0.5°C) and baroclinic velocity (m s\(^{-1}\); contour interval 0.004 m s\(^{-1}\) with negative contours dashed) for (a) the time- and along-channel-averaged three-dimensional eddy-resolving calculation. The remaining panels are all for two-dimensional parameterisations. These are: (b) simple advection and diffusion using the values used in the three-dimensional calculation; (c) as (b), but with a horizontal diffusivity of 200 m\(^2\) s\(^{-1}\); (d) the Gent and McWilliams (1990) parameterisation, using m\(^2\) s\(^{-1}\); (e) the Killworth (1997) parameterisation using; (f) the Gent and McWilliams parameterisation modified so that the streamfunction does not vanish at surface or floor, using m\(^2\) s\(^{-1}\); (g) the Killworth (1997) parameterisation, similarly modified, but for which is too small to reproduce the three-dimensional calculation accurately due to numerical instabilities; (h) parameterizing simply directly from linear theory, with; (i) parameterizing directly from linear theory, with .
Three-dimensional average fields

Temperature

Advection-diffusion (200 m s⁻¹)

Advection-diffusion (10 m s⁻¹)

Fig. 7a-c

Fig. 6
Fig. 7d-f

K97, modified surface condition, $u = 5$

VPWP ($u = 3$)

VP, $u = 3$. Modified surface condition ($u = 1200 \text{ m s}^{-1}$)

GM90, modified surface condition ($u = 160 \text{ m s}^{-1}$)

GM90, $u < 5$