A BEHAVIOURAL APPROACH TO THE POLE STRUCTURE OF 1D AND ND LINEAR SYSTEMS *

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Abstract. We use the tools of behavioural theory and commutative algebra to produce a new definition of a (finite) pole of a linear system. This definition agrees with the classical one, and allows a direct dynamical interpretation. It also generalizes immediately to the case of a multidimensional system. We make a natural division of the poles into controllable and uncontrollable poles. When the behaviour in question has latent variables, we make a further division into observable and unobservable poles. In the case of a 1D state-space model, the uncontrollable and unobservable poles correspond respectively to the input and output decoupling zeros, whereas the observable controllable poles are the transmission poles.

Most of these definitions can be interpreted dynamically in both the 1D and nD cases, and some can be connected to properties of kernel representations. We also examine the connections between poles, transfer matrices and their left and right MFDs. We find behavioural results which correspond to the concepts that a controllable system is precisely one with no input decoupling zeros, and an observable system is precisely one with no output decoupling zeros. We produce a decomposition of a behaviour as the sum of sub-behaviours associated with various poles. This is related to the integral representation theorem, which describes every system trajectory as a sum of integrals of polynomial exponential trajectories.

Key words. Pole, decoupling zero, behavioural approach, multidimensional systems, characteristic variety, polynomial exponential function, associated primes, primary decomposition, integral representation

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1. Introduction.

This paper is about the pole structure of 1D and nD linear systems. Although it is motivated by the need to establish a cohesive theory of poles for multidimensional (nD) systems, many of the results in the paper give a new perspective to the classical 1D theory. Furthermore, this paper places all of the classical results in the new and growing behavioural framework [39, 40, 41]. We consider only linear systems with constant coefficients; our results apply to both continuous and discrete cases.

The systematic theory of poles (and zeros) of 1D systems originated in the work of Rosenbrock [33] and others. Since then, a number of papers have reviewed, extended and refined the theory (e.g. [8, 23, 35]; see also the references below).

A multidimensional (nD) system is one in which information propagates in two or more independent directions. Such systems arise as the solutions of partial differential or difference equations. Shankar and Sule [36, 37] have defined poles for SISO nD systems, using the ring of stable causal rational functions. As yet, to our knowledge, no theory of pole structure or even a definition of a pole has been proposed for nD

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MIMO systems. The primary aim of this paper is to fill that gap. Our secondary aim is to extend the behavioural theory of both 1D and nD systems by introducing into it concepts concerned with pole structure. Thirdly, we intend to shed new light on the existing 1D theory, both through our use of behaviours and by identifying correspondences between system-theoretic concepts and algebraic ones.

In accordance with the behavioural paradigm, we define poles (decoupling zeros, etc) to be objects associated with a system's behaviour, rather than with any given representation of it. One advantage of such definitions is that they are all intrinsic, as is also the case with the recent definitions of Bourlès and Fliess [4] and of Pommaret and Quadrat [31]. For example, the input decoupling zeros are not defined as the zeros eliminated by a reduction procedure as in [33], but are instead defined directly.

The algebraic definitions of Bourlès and Fliess have been extended to the case of nD systems with variable coefficients by Pommaret and Quadrat [31]. Our definitions are equivalent to these in the 1D/ nD constant coefficients case, the relationship being given by module duality, which translates for example the definition of a pole given in terms of the finitely generated system module into a corresponding one in terms of the system behaviour. Our definitions are not directly related to those of Shankar and Sule.

One of our principal tools in this paper is the duality theory in [24] between finitely generated modules over a polynomial ring and nD behaviours described by linear partial differential or difference equations with constant coefficients. Consequently, we use a considerable amount of commutative algebra; see for example [3, 5, 7, 10] for background in this field.

The paper is structured as follows. We begin with an overview of relevant concepts from behavioural theory, with special emphasis on module duality (§2). Section 3 collects some results on nD behavioural theory which we need later in the paper.

In §4 we review the concepts of exponential trajectories and the characteristic variety. Exponential trajectories are central to any understanding of system poles, and the characteristic variety is a geometric object which captures the essential structure of a behaviour as regards its exponential trajectories. Based on these ideas, we define the poles of an nD system in §5. We also define the concepts of controllable and uncontrollable poles. The controllable poles have properties which relate strongly to those of the system's transfer matrix, whereas the uncontrollable poles play the role of input decoupling zeros.

Section 6 considers the case where the behaviour in question is a latent variable representation (e.g. a state-input-output behaviour). We divide the system poles into observable and unobservable poles, which have natural interpretations. This leads naturally to a further breakdown in terms of controllable observable poles, etc., all of which correspond naturally to known sets of poles in the classical 1D context.

Finally, in §7 we consider a decomposition which is dual to the standard algebraic concept of the primary decomposition of a module. This "polar decomposition" expresses the zero-input behaviour of a system as the finite sum of sub-behaviours associated with the distinct poles of the system. Using a similar decomposition we can write the system behaviour itself as the sum of the controllable part and a finite set of sub-behaviours associated with the system's uncontrollable poles (input decoupling zeros). This latter decomposition is a refinement of the well-known controllable–autonomous decomposition. Using the little-known but important integral representation theorem, it is possible to write any zero-input trajectory as a sum of integrals of polynomial exponential trajectories. In the case of a finite-dimensional behaviour, every zero-input trajectory can be written as a sum of polynomial-weighted exponential trajectories with frequencies corresponding to the system poles, as in the 1D case.

We leave a number of important issues untouched, for example the definition of multiplicity of a pole, the definition of an infinite pole, and the whole question of zeros. However, the basic structures which we uncover in this work should prove a good foundation for further developments of this type.

2. Behaviours, Modules and Duality. This section provides an overview of behavioural theory and the duality between finitely generated modules and behaviours given by partial differential/ difference equations.

2.1. The behavioural approach. The behavioural approach to 1D systems is due to Willems [39, 40, 41], and centres on the concept of the system behaviour, which is the set of associated trajectories. Formally, we define a *system* as a triple $(\mathcal{A}, q, \mathcal{B})$, where $\mathcal{A}$ is a set called the *signal space*, $q \in \mathbb{Z}^+$ is the number of components, and $\mathcal{B} \subseteq \mathcal{A}^q$ is the *behaviour*. The elements of $\mathcal{B}$ are called *trajectories*. We will make little distinction between a system and its behaviour. Note that the standard behavioural definition of a system as a triple $(T, W, \mathcal{B})$ (where T is the signal domain, W the signal value set and $\mathcal{B} \subseteq W^T$) does not cover the important case of distributions.

In all cases of interest (as listed below), $\mathcal{A}$ will have the structure of a module over a suitable ring of differential or difference operators; in particular, $\mathcal{A}$ is always a vector space over a field k, which is taken to be $\mathbb{R}$ or $\mathbb{C}$. Specifically, throughout this paper $\mathcal{A}$ denotes one of the following signal spaces, with module structure as described:

1. The discrete signal space $\mathcal{A} = k^{\mathbb{N}^n}$, $k = \mathbb{R}, \mathbb{C}$, which is a module over the polynomial ring $\mathcal{R} = k[\underline{\mathbf{z}}] = k[z_1, \ldots, z_n]$, where the action of z_i on a trajectory $w \in \mathcal{A}$ is taken to be the shift operator σ_i, defined by:

$$(\sigma_i w)(t_1, \ldots, t_n) \;:=\; w(t_1, \ldots, t_{i-1}, t_i + 1, t_{i+1}, \ldots, t_n) \tag{2.1}$$

By extension, any element of $\mathcal{R}$ has an action on $\mathcal{A}$:

$$\forall p \in k[\underline{\mathbf{z}}], w \in \mathcal{A} \qquad p(z_1, \ldots, z_n)w \;:=\; p(\sigma_1, \ldots, \sigma_n)(w) \tag{2.2}$$

2. The discrete signal space $\mathcal{A} = k^{\mathbb{Z}^n}$, $k = \mathbb{R}, \mathbb{C}$, which is a module over the Laurent polynomial ring $\mathcal{R} = k[\underline{\mathbf{z}}, \underline{\mathbf{z}}^{-1}] = k[z_1, \ldots, z_n, z_1^{-1}, \ldots, z_n^{-1}]$, where z_i acts as the shift operator σ_i, and z_i^{-1} as the inverse shift σ_i^{-1}.

3. The signal space $\mathcal{A} = \mathcal{C}^\infty(\mathbb{R}^n, k)$, $k = \mathbb{R}, \mathbb{C}$ of all k-valued $\mathcal{C}^\infty$ functions on $\mathbb{R}^n$, which is a module over $\mathcal{R} = k[\underline{\mathbf{z}}]$, where z_i acts as the partial derivative operator $\partial/\partial t_i$. This action is extended to $\mathcal{R}$ by:

$$\forall p \in k[\underline{\mathbf{z}}], w \in \mathcal{A} \qquad p(z_1, \ldots, z_n)w \;:=\; p(\partial/\partial t_1, \ldots, \partial/\partial t_n)(w) \tag{2.3}$$

4. The signal space $\mathcal{A} = \mathcal{D}'(\mathbb{R}^n, k)$, $k = \mathbb{R}, \mathbb{C}$ of all k-valued distributions on $\mathbb{R}^n$, which is a module over $\mathcal{R} = k[\underline{\mathbf{z}}]$, where z_i again acts as the partial derivative operator $\partial/\partial t_i$.

Throughout the paper, we consider the behaviour $\mathcal{B} \subseteq \mathcal{A}^q$ to be the solution space in $\mathcal{A}^q$ of a finite set of linear partial differential or difference equations in q dependent variables with constant coefficients. Such behaviours will be called *differential behaviours* and *difference behaviours* respectively. The term *autoregressive (AR) behaviour* has formerly been used, but is undesirable due to the inappropriate stochastic connotations.

The set of systems with differential/ difference behaviours covers all linear time-invariant systems dealt with by the classical 1D state-space framework and its nD analogues. Differential and difference behaviours are in particular linear and shift-invariant, and in the continuous case are also closed under partial differentiation. In each case any differential or difference behaviour $\mathcal{B}$ is therefore a submodule of $\mathcal{A}^q$; the ring action is the component-wise application of (2.2) or (2.3).

It is convenient to describe differential/ difference behaviours using polynomial matrices (Laurent polynomial matrices when $\mathcal{A} = k^{\mathbb{Z}^n}$). Thus let $E \in \mathcal{R}^{g,q}$; then the behaviour described by E is the submodule of $\mathcal{A}^q$ consisting of all w satisfying $Ew = 0$, where the meaning of Ew is given by the action of the ring $\mathcal{R}$ on $\mathcal{A}$. For example, the 3D difference behaviour $\mathcal{B}$ in 2 variables described by the equations

$$\begin{aligned} w_1(t_1,t_2,t_3) - w_1(t_1,t_2+1,t_3+1) + 2w_2(t_1+1,t_2,t_3) = 0 \\ w_1(t_1+2,t_2,t_3+1) - w_2(t_1,t_2,t_3) + 3w_2(t_1,t_2+1,t_3) = 0 \end{aligned}$$

is given by the polynomial matrix

$$E \;=\; \begin{pmatrix} 1 - z_2z_3 & 2z_1 \\ z_1^2z_3 & -1+3z_2 \end{pmatrix}$$

We say that E is a *kernel representation* of $\mathcal{B}$, and we write $\mathcal{B} = \ker_{\mathcal{A}} E$. For any matrix $E \in \mathcal{R}^{g,q}$, we also define the following modules[1]:

$$\mathrm{Ker}_{\mathcal{R}}\, E := \{v \in \mathcal{R}^{1,g} \;\mid\; vE = 0\}, \tag{2.4}$$

$$\mathrm{Im}_{\mathcal{R}}\, E := \{v \in \mathcal{R}^{1,q} \;\mid\; v = xE \text{ for some } x \in \mathcal{R}^{1,g}\}, \tag{2.5}$$

$$\mathrm{Coker}_{\mathcal{R}}\, E := \mathcal{R}^{1,q}/\,\mathrm{Im}_{\mathcal{R}}\, E \tag{2.6}$$

$$\mathrm{im}_{\mathcal{A}}\, E := \{w \in \mathcal{A}^g \;\mid\; w = El \text{ for some } l \in \mathcal{A}^q\} \tag{2.7}$$

Note the different subscripts used to denote different ring actions. Also, the modules $\mathrm{Ker}_{\mathcal{R}}\, E$, $\mathrm{Im}_{\mathcal{R}}\, E$, $\mathrm{Coker}_{\mathcal{R}}\, E$ are defined with respect to a left action of E, whereas $\ker_{\mathcal{A}} E$ and $\mathrm{im}_{\mathcal{A}} E$ are defined with respect to a right action.

2.2. Module duality. In this section we describe the duality between finitely generated $\mathcal{R}$-modules and differential/ difference behaviours [24]. For the purposes of the next definition and the discussion immediately following only, the space $\mathcal{A}$ can be thought of as an arbitrary module.

DEFINITION 2.1. *Let M be a finitely generated $\mathcal{R}$-module, and $\mathcal{A}$ an arbitrary $\mathcal{R}$-module. The* dual of M *with respect to $\mathcal{A}$, denoted $D(M)$, is defined by*

$$D(M) \;:=\; \mathrm{Hom}_{\mathcal{R}}(M,\mathcal{A}) \tag{2.8}$$

If $\phi : M \mapsto N$ is a morphism of finitely generated $\mathcal{R}$-modules, then the dual map $D(\phi) : D(N) \mapsto D(M)$ is given by:

$$\forall\, v \in D(N), \qquad (D(\phi))(v) \;:=\; v \circ \phi \tag{2.9}$$

The action of $\mathcal{R}$ on the dual module $D(M)$ is given by

$$\forall\, r \in \mathcal{R}, w \in D(M), \qquad (rw)(m) \;:=\; r \cdot w(m) \;=\; w(rm) \tag{2.10}$$

[1]Note that the modules $\mathrm{Ker}_{\mathcal{R}}\, E$, $\mathrm{Im}_{\mathcal{R}}\, E$, $\mathrm{Coker}_{\mathcal{R}}\, E$ are defined using row vectors, and do not therefore require transposition of E, contrary to the first author's previous practice.

for all $m \in M$.

Note for any $q \in \mathbb{Z}^+$ that $D(\mathcal{R}^{1,q}) \cong \mathcal{A}^q$. Furthermore, it turns out that:

THEOREM 2.2. [24, pp19–21] *Differential/ difference behaviours are precisely the dual modules of finitely generated $\mathcal{R}$-modules. Specifically, if $\mathcal{B} = \ker_{\mathcal{A}} E$ then $\mathcal{B} = D(M)$, where M is the finitely generated module* $\mathrm{Coker}_{\mathcal{R}}\, E$.

It is not hard to show [24, Cor. 2.57] that the dual action of a polynomial matrix map $E : \mathcal{R}^{1,g} \mapsto \mathcal{R}^{1,q}$, $v \mapsto vE$, is the map $E : \mathcal{A}^q \mapsto \mathcal{A}^g$, $w \mapsto Ew$ given by the ring action on $\mathcal{A}$ (equation (2.2) or (2.3) in our cases of interest).

DEFINITION 2.3. *A* complex *of modules is a set of maps $\phi_i : F_i \mapsto F_{i-1}$, $i \in \mathbb{Z}$, where $\phi_i \circ \phi_{i+1} = 0$ for all i. We represent a complex by a diagram of the form*

$$\cdots \xrightarrow{\phi_{i+2}} F_{i+1} \xrightarrow{\phi_{i+1}} F_i \xrightarrow{\phi_i} F_{i-1} \xrightarrow{\phi_{i-1}} \cdots$$

The complex is said to be exact, *and is called an* exact sequence, *if for every i we have that* $\ker\, \phi_i = \mathrm{im}\, \phi_{i+1}$.

Exact sequences are a compact way of expressing structural relationships. Exactness of the following two sequences express the injectivity and surjectivity respectively of the map ϕ:

$$0 \longrightarrow M \xrightarrow{\phi} N \qquad\qquad M \xrightarrow{\phi} N \longrightarrow 0$$

The next exact sequence expresses that K is (isomorphic to) the kernel of ϕ and L is (isomorphic to) the cokernel:

$$0 \longrightarrow K \xrightarrow{\iota} M \xrightarrow{\phi} N \xrightarrow{\rho} L \longrightarrow 0$$

Note that to any complex of modules, there is a corresponding dual complex, obtained by reversing the arrows and dualizing each module and each map, and conversely. The following crucial result shows that, in our cases of interest, the same can be said of exact sequences:

THEOREM 2.4. [24, Thm. 2.54, Cor. 2.43, 2.46, 2.47] *Each module $\mathcal{A}$ listed in §2.1 is an injective cogenerator of the category of $\mathcal{R}$-modules. This signifies that duality is contravariant and faithfully exact. In other words, given a complex of modules*

$$\cdots \xrightarrow{\phi_{i+2}} F_{i+1} \xrightarrow{\phi_{i+1}} F_i \xrightarrow{\phi_i} F_{i-1} \xrightarrow{\phi_{i-1}} \cdots \tag{2.11}$$

and its dual complex

$$\cdots \xrightarrow{D(\phi_{i-1})} D(F_{i-1}) \xrightarrow{D(\phi_i)} D(F_i) \xrightarrow{D(\phi_{i+1})} D(F_{i+1}) \xrightarrow{D(\phi_{i+2})} \cdots, \tag{2.12}$$

we have that (2.11) *is exact if and only if* (2.12) *is exact. In consequence, if $\mathcal{B} = \ker_{\mathcal{A}} E \subseteq \mathcal{A}^q$ then the* orthogonal module *$\mathcal{B}^{\perp} \subseteq \mathcal{R}^{1,q}$ (the module of all equations satisfied by the system $\mathcal{B}$) is equal to* $\mathrm{Im}_{\mathcal{R}}\, E$.

The module duality gives rise to a lattice duality $\mathcal{B} \mapsto \mathcal{B}^{\perp}$ between differential/ difference behaviours and submodules of $\mathcal{R}^{1,q}$. This duality takes summation to intersection and vice versa [24, 2.22], [42, Lemma 1]. Note also that $\mathcal{B} = D(M)$ where $M = \mathcal{R}^{1,q}/\mathcal{B}^{\perp}$. For any $\mathcal{R}$-matrix E it can easily be seen that $(\mathrm{im}_{\mathcal{A}}\, E)^{\perp} = \mathrm{Ker}_{\mathcal{R}}\, E$.

Now if $\mathcal{B} = \ker_{\mathcal{A}} E$ and $\mathcal{B}' = \ker_{\mathcal{A}} E$ are two behaviours with the same number of system variables, then $\mathcal{B}' \subseteq \mathcal{B}$ if and only if there exists an $\mathcal{R}$-matrix L with $E = LE'$ [24, 2.61]. Another important consequence of Theorem 2.4 is that the dual of a submodule is a factor module of the dual, and vice versa. More precisely, given a finitely generated $\mathcal{R}$-module M and a submodule N, we have an exact sequence

$$0 \longrightarrow N \longrightarrow M \longrightarrow M/N \longrightarrow 0 \tag{2.13}$$

This dualizes to an exact sequence

$$0 \longrightarrow D(M/N) \longrightarrow D(M) \longrightarrow D(N) \longrightarrow 0, \tag{2.14}$$

which we can interpret as meaning that $D(M/N)$ is a submodule of $D(M)$, with factor module $D(N)$. Indeed, we identify $D(M/N)$ with the set of functions of $D(M)$ which are zero on N.

An immediate application of the duality theorem is the following [24, pp24–25]:

LEMMA 2.5. *If $E \in \mathcal{R}^{g,q}$, then $\operatorname{im}_\mathcal{A} E = \mathcal{A}^g$ if and only if E has full row rank.*

Proof. The map $E : \mathcal{R}^{1,g} \mapsto \mathcal{R}^{1,q}$, $v \mapsto vE$, is monic if and only if the dual map $E : \mathcal{A}^q \mapsto \mathcal{A}^g$, $w \mapsto Ew$, is epic. □

We investigate some of the links between the theory of differential/ difference behaviours and commutative algebra resulting from this duality in §3.

Henceforth, $\mathcal{B} = \ker_\mathcal{A} E$ denotes a differential or difference behaviour with kernel representation E, which is contained in $\mathcal{A}^q$ for one of the signal spaces $\mathcal{A}$ listed in §2.1, and $M = \operatorname{Coker}_\mathcal{R} E$ denotes the finitely generated module to which $\mathcal{B}$ is dual:

$$E \in \mathcal{R}^{g,q}, \quad M = \operatorname{Coker}_\mathcal{R} E, \quad \mathcal{B} = D(M) = \ker_\mathcal{A} E \;\subseteq\; \mathcal{A}^q \tag{2.15}$$

A similar behavioural and/or module-theoretic approach can be taken for linear systems with variable coefficients, as shown in the 1D case for example in [12, 14, 16, 34], using non-commutative techniques. The algebraic approach has been extended to nD variable coefficient systems by Pommaret and Quadrat [30, 31]. Recent work has indicated that it may be possible to use behavioural tools in the analysis of non-linear systems [47].

Recall that the annihilator of a module M, denoted $\operatorname{ann} M$, is the ideal of all $r \in \mathcal{R}$ such that $rM = 0$. We denote the set of all elements of $\mathcal{B}$ annihilated by an ideal J (i.e. all elements w such that $rw = 0$ for all $r \in J$) by $[\mathcal{B} : J]$. The following properties of duality will be useful:

LEMMA 2.6. *Let J be an ideal of $\mathcal{R}$ and M a finitely generated $\mathcal{R}$-module with dual $\mathcal{B}$; then*

$$D(M/JM) \;=\; [\mathcal{B} : J] \tag{2.16}$$

Also, $\operatorname{ann} M = \operatorname{ann} \mathcal{B}$.

Proof. The equation (2.16) is given in [24, Cor. 2.101]. Considering the case $J = (r)$ for some $r \in \mathcal{R}\backslash 0$, we now find that $D(M/rM) = [\mathcal{B} : (r)]$. If r is in the annihilator of M then $M/rM = M$, so $[\mathcal{B} : (r)] = \mathcal{B}$, i.e. r is in the annihilator of $\mathcal{B}$. The converse is similar. □

3. Controllable, Observable and Autonomous Systems. In this section we recount relevant results from [24, 42, 46] and other sources, some of which are extended here from discrete to continuous systems. These results concern the characterization of controllable systems, observable systems and autonomous systems using algebraic tools.

3.1. Free variables. The following definition is adapted from [15]:

DEFINITION 3.1. *Let $\mathcal{B} \subseteq \mathcal{A}^q$. The set of variables $\{w_i \mid i \in \Phi\}$ for some subset Φ of $\{1, \dots, q\}$ is said to be a* set of free variables *if the mapping $\rho : \mathcal{A}^q \mapsto \mathcal{A}^\Phi$, which projects a trajectory onto the components of Φ, is epic when restricted to $\mathcal{B}$.*

The maximum size of a set of free variables is called the number of free variables *of $\mathcal{B}$, and denoted by $m(\mathcal{B})$.*

Thus a set of variables indexed by Φ is free if the corresponding components can take any value in $\mathcal{A}^\Phi$. Given a kernel representation $\mathcal{B} = \ker_{\mathcal{A}} E$ of $\mathcal{B}$, it can be shown that the number of free variables is equal to $q - \operatorname{rank}\, E$ [24, Thm. 2.69]. It can also be shown that the number of free variables is additive [42, remarks following Defn. 6], i.e. given an exact sequence

$$0 \longrightarrow \mathcal{B}_1 \longrightarrow \mathcal{B}_2 \longrightarrow \mathcal{B}_3 \longrightarrow 0$$

of differential/ difference behaviours, we have $m(\mathcal{B}_2) = m(\mathcal{B}_1) + m(\mathcal{B}_3)$. Consequently, if $\mathcal{B}_1, \mathcal{B}_2 \subseteq \mathcal{A}^q$ for some q, then we have

$$m(\mathcal{B}_1) + m(\mathcal{B}_2) \;=\; m(\mathcal{B}_1 + \mathcal{B}_2) + m(\mathcal{B}_1 \cap \mathcal{B}_2) \tag{3.1}$$

3.2. Autonomous systems. In the behavioural framework, an autonomous discrete system is defined as follows [15, 42]:

DEFINITION 3.2. *A difference behaviour $\mathcal{B}$ with signal domain T is called* autonomous *if there exists $T_1 \subseteq T$ such that any trajectory of $\mathcal{B}$ is determined by its values on T_1, and also $T \backslash T_1$ contains an n-dimensional cone.*

The main characterization of such behaviours [42, Thm. 2] follows; similar results or special cases have also appeared in [15, 46].

THEOREM 3.3. [15, 42, 46] *Let $\mathcal{B} = D(M)$ be a difference behaviour. The following are equivalent:*

1. *$\mathcal{B}$ is autonomous.*
2. *$\mathcal{B}$ has no free variables.*
3. *For any E with $\mathcal{B} = \ker_{\mathcal{A}} E$, E has full column rank.*
4. $\operatorname{ann} \mathcal{B} = \operatorname{ann} M \neq 0$*; equivalently, M is a torsion module.*

In the continuous case, Pillai and Shankar have defined an autonomous system as one with a characteristic variety which is not equal to $\mathbb{C}^n$ [29, Defn. 4.2]. We will discuss characteristic varieties in §4; for now, let us use the following definition, which is easily seen to be equivalent to this condition and to the conditions of Theorem 3.3.

DEFINITION 3.4. *A differential behaviour is called* autonomous *if* $\operatorname{ann} \mathcal{B} \neq 0$.

Recall that a behaviour is said to be *finite-dimensional* or *strongly autonomous* if and only if it is finite-dimensional as a vector space over k [15, 29]. Note that this has nothing to do with the traditional use of the term "finite-dimensional" in classical systems theory.

3.3. Minimal left annihilators and generalized factor primeness. We will also need the concepts of a minimal left annihilator [32, p24] and generalized factor primeness [24, Thm. 7.21], [46].

DEFINITION 3.5. *Let $\mathcal{R} = k[\underline{\mathbf{z}}]$ or $k[\underline{\mathbf{z}}, \underline{\mathbf{z}}^{-1}]$, and suppose that $E \in \mathcal{R}^{g,q}$. Then E is called a* minimal left annihilator *(of F) if there exists a matrix $F \in \mathcal{R}^{q,h}$ for some h for which the following conditions hold:*

1. *$EF = 0$.*
2. *For any E' satisfying $E'F = 0$, there exists L with $E' = LE$.*

The matrix E is said to be factor left prime in the generalized sense (GFLP), *if the existence of a $\mathcal{R}$-matrix factorization $E = LE_1$ (L not necessarily square) with* rank $E =$ rank E_1 *implies the existence of an $\mathcal{R}$-matrix L' with $E_1 = L'E$.*

It is shown in [24, Lem. 2.27], [42, Lem. 7] that E is a minimal left annihilator of F if and only if $\ker_{\mathcal{A}} E = \mathrm{im}_{\mathcal{A}} F$, or equivalently $\mathrm{Im}_{\mathcal{R}} E = \mathrm{Ker}_{\mathcal{R}} F$. Every matrix over $\mathcal{R}$ has a minimal left annihilator [24, Lem. 2.27], [42, Lem. 7]. A matrix over $\mathcal{R}$ is a minimal left annihilator if and only if it is generalized factor left prime [42, Lem. 10].

3.4. Controllable systems. Any differential/ difference behaviour admits certain input/ output structures. In this paper, as is traditional in control theory, the "inputs" are assumed to be a maximal set of free variables, and in consequence the outputs have no freedom once the inputs are determined. The more general situation, in which an input/ output structure is an arbitrary partitioning of the system variables, and which may be better suited to certain applications, is not considered in this paper. In [31], Pommaret and Quadrat extend some of the algebraic theory of this paper to this general case.

DEFINITION 3.6. [24, Thm. 2.69], [32, Defn. IV.8], [42, Defn. 12] *An* input/ output structure *on the behaviour $\mathcal{B}$ is a partitioning of the system variables $w = (u, y)$, such that the set of variables u is free and the* zero-input behaviour $\mathcal{B}_{0,y}$, *defined by*

$$\mathcal{B}_{0,y} \ = \ \{(u,y) \in \mathcal{B} \ \mid \ u = 0\} \tag{3.2}$$

is autonomous.

Equivalently, we can consider a partitioning $E = (-Q \ P)$ of any kernel representation E of $\mathcal{B}$ (to within a permutation), where the columns of Q correspond to the input variables u, and the columns of P to the output variables y, and we have the condition

$$\text{rank } E \ = \ \text{rank } P \ = \ \text{number of columns of } P$$

It is easy to show that the number of inputs is equal to m, the number of free variables. In particular, the number of inputs and number of outputs of a behaviour is independent of the input/ output structure.

If $\mathcal{B}$ has a kernel representation $(-Q \ P)$, where P is the submatrix corresponding to the output components, then $\mathcal{B}_{0,y}$ is (trivially) isomorphic to $\ker_{\mathcal{A}} P$. We usually make no distinction between these two behaviours.

For a given admissible input/ output structure, any behaviour $\mathcal{B}$ has a unique *transfer (function) matrix* $G \in k(\underline{\mathbf{z}})^{p,m}$ characterized by the equation $PG = Q$; see [24, Thm. 2.69] and also [32, p75] for the discrete case. Collecting together all behaviours with a given input/ output structure and the same transfer matrix with respect to that input/ output structure, we obtain a transfer class. The transfer class turns out to be independent of the i/o structure and resulting transfer matrix, and the transfer classes partition the set of behaviours. Furthermore, each transfer class has a unique element which is minimal with respect to set inclusion [24, Thm. 7.21], [32, p76]. Such a behaviour will be called the *minimal realization* of the transfer matrix. The minimal realization has properties which are closely linked to those of the transfer matrix, as we will see later in the paper.

The concept of a minimal realization is related to controllability, which in the behavioural context is defined as follows.

DEFINITION 3.7. [29, 32, 42, 44] *A differential behaviour $\mathcal{B}$ is* controllable *if, for any two open sets $T_1, T_2 \subseteq \mathbb{R}^n$ with disjoint closures, and for any two trajectories $w^{(1)}, w^{(2)} \in \mathcal{B}$, there exists a $w \in \mathcal{B}$ such that*

$$w(\underline{t}) \;=\; \begin{cases} w^{(1)}(\underline{t}) & \text{if } \underline{t} \in T_1 \\ w^{(2)}(\underline{t}) & \text{if } \underline{t} \in T_2 \end{cases} \tag{3.3}$$

A difference behaviour $\mathcal{B}$ with signal domain $T = \mathbb{Z}^n$ or $T = \mathbb{N}^n$ is controllable *if there exists a real number $\rho > 0$ such that for any sets $T_1, T_2 \subseteq T$ with $d(T_1, T_2) > \rho$, and for any $\underline{b}_1, \underline{b}_2 \in T$, and any two trajectories $w^{(1)}, w^{(2)} \in \mathcal{B}$, there exists a $w \in \mathcal{B}$ such that*

$$w(\underline{t}) \;=\; \begin{cases} w^{(1)}(\underline{t} - \underline{b}_1) & \text{if } \underline{t} \in T_1 \text{ and } \underline{t} - \underline{b}_1 \in T \\ w^{(2)}(\underline{t} - \underline{b}_2) & \text{if } \underline{t} \in T_2 \text{ and } \underline{t} - \underline{b}_2 \in T \end{cases} \tag{3.4}$$

In the discrete case, we can take $\underline{b}_1 = 0$ without loss of generality, and for $T = \mathbb{Z}^n$, we can also take $\underline{b}_2 = 0$. For both differential and difference behaviours, controllability expresses the idea of being able to join with a system trajectory any two system trajectories defined on regions which are sufficiently far apart.

The next result collects previous results in the literature on the characterization of nD behavioural controllability. In the 1D case, the equivalence of controllability and torsionfree-ness is first due to Fliess [13], following an observation of Pommaret. Special cases of the next result in the 1D and 2D discrete cases are due respectively to Willems [40] and to Rocha [32].

THEOREM 3.8. *Let $\mathcal{B} = D(M) = \ker_{\mathcal{A}} E$. The following are equivalent:*

1. *$\mathcal{B}$ is controllable.*
2. *$\mathcal{B}$ is minimal in its transfer class.*
3. *$\mathcal{B}$ has an image representation.*
4. *E is generalized factor left prime.*
5. *$\mathcal{B}$ is a divisible module.*
6. *M is a torsionfree module.*
7. *$\mathcal{B}$ has no proper differential/ difference sub-behaviours with the same number of free variables.*

Proof. The equivalence of 1 and 3 is given in [29, Prop. 3.4, Thm. 3.9] for differential behaviours, in [42, Thm. 5] for difference behaviours on $\mathbb{Z}^n$, and in [44] for general difference behaviours. The equivalence of 3 and 4 is given in [42, Lem. 10] and also [46]. The equivalence of 3 and 6 has been given in [30, Prop. VII.A.10], and the equivalence of 2,4 and 6 in [24, Thm. 7.21]. The equivalence of 5 and 6 can be proved easily using the exactness of duality. The condition 7 is a direct interpretation of the condition 4, as discussed following Defn. 10 in [42]. ∎

Note that any differential/ difference behaviour has a "controllable–autonomous decomposition" $\mathcal{B} = \mathcal{B}^c + \mathcal{B}^a$ [42, Thm. 7]; the argument given in that paper applies equally well to continuous systems. In this decomposition, the controllable behaviour $\mathcal{B}^c \subseteq \mathcal{B}$ is uniquely determined as the minimal element of the transfer class of $\mathcal{B}$; henceforth we call this behaviour the *controllable part* of $\mathcal{B}$. It is shown in [24, Thm. 7.21] that if $\mathcal{B} = D(M)$ then

$$\mathcal{B}^c \;=\; D(M/tM), \tag{3.5}$$

where tM denotes the torsion submodule of M. We will also find the behaviour $\mathcal{B}/\mathcal{B}^c$ interesting. This behaviour is dual to tM, as can be seen from equation (3.5) using basic duality principles (equations (2.13) and (2.14)). Given a representation E^c for $\mathcal{B}^c$, we can see by considering the restriction of the map E^c to $\mathcal{B}$ that the behaviour $E^c\mathcal{B}$ is naturally isomorphic to $\mathcal{B}/\mathcal{B}^c$.

The controllable–autonomous decomposition as defined above is in fact slightly more general (and less rich in structure) for a 1D system than as originally defined for 1D systems (see e.g. [41]). This is because we do not require that the sum should be direct.

The controllable part of $\mathcal{B}$ has the same input/ output structures as $\mathcal{B}$ itself (inevitably, since it can be defined with respect to any such i/o structure via the transfer matrix G). As we will see later in the paper, the zero-input behaviour of the controllable part, $(\mathcal{B}^c)_{0,y}$, has an interesting structure. It has already been shown in [24, Cor. 7.29] that the corresponding orthogonal module has the following form:

$$((\mathcal{B}^c)_{0,y})^\perp \;=\; \{v \in \mathcal{R}^{1,p} \mid vG \in \mathcal{R}^{1,m}\}, \tag{3.6}$$

where we treat $(\mathcal{B}^c)_{0,y}$ as a sub-behaviour of $\mathcal{A}^p$ rather than of $\mathcal{A}^q$, p being the number of outputs.

The next result has not been explicitly proved in the literature before.

Lemma 3.9. *Let $\mathcal{B}$ be a behaviour with zero-input behaviour $\mathcal{B}_{0,y}$ according to some i/o structure. Then $\mathcal{B}_{0,y}$ is an autonomous part of $\mathcal{B}$ in a controllable–autonomous decomposition.*

Proof. For any $(u,y) \in \mathcal{B}$, since the variables u are free in $\mathcal{B}^c$, there must exist a y^c with $(u,y^c) \in \mathcal{B}^c$. As $(0,y-y^c) \in \mathcal{B}_{0,y}$, we have the required decomposition $(u,y) = (u,y^c) + (0,y-y^c)$. □

Definition 3.10. *A matrix $E \in \mathcal{R}^{g,q}$ is called* zero left prime *if its gth order minors generate $\mathcal{R}$. A behaviour $\mathcal{B}$ is called* strongly controllable *if it has a zero left prime kernel representation.* It is a straightforward consequence that $\mathcal{B} = D(M)$ is strongly controllable if and only if M is free. Definition 3.10 is not the definition originally given for strong controllability [32], which has not yet been usefully extended from 2D to nD or to the continuous case.

3.5. Observable systems. We use the behavioural concept of observability [32, Defn. 19], [40]: given a behaviour $\mathcal{B}_{l,w}$ involving two sets of variables l and w, the variables l are said to be *observable from* the variables w if for any two trajectories $(l_1,w_1),(l_2,w_2) \in \mathcal{B}_{l,w}$, it holds that $w_1 = w_2$ implies $l_1 = l_2$. When the variables l are latent variables in a latent variable description of the behaviour $\mathcal{B}_w$ (see §6.1), we say that the behaviour $\mathcal{B}_{l,w}$ is *observable* if the latent variables l are observable from the manifest ones w. We have that $\mathcal{B}_{l,w}$ is observable if and only if $\mathcal{B}_{l,0} = 0$.

4. Exponential Trajectories and the Characteristic Variety. We commence our main discussion by explaining the notion of the characteristic variety of a behaviour, which is a geometric object describing the exponential trajectories contained in the behaviour. To motivate this analysis, we first review the classical interpretation of a pole of a 1D continuous system (see e.g. [8, 20]).

The point a is a pole of the system Σ if, when zero input $u(t)$ is fed to the system, there exists a non-zero initial condition $x(0)$ such that the resulting state trajectory has the form $x(t) = x(0)e^{at}$ [20]. This is sometimes rephrased by allowing $u(t)$ to be a finite sum of generalized delta functions which "kick the system" into the appropriate initial condition $x(0)$. Note that as $y(t)$ is determined linearly by $x(t)$ ($u(t)$ being zero), $y(t)$ must also be of the form $y(0)e^{at}$.

This *trajectory interpretation* of a pole can be formalized in behavioural theory as follows. Let $\mathcal{B}_{x,u,y}$ denote the state-input-output behaviour of our given state space representation of the system. Then a is a pole of the system (of $\mathcal{B}_{x,u,y}$) if there exists a non-zero trajectory of the form $v_0 e^{at}$ in $\mathcal{B}_{x,0,y}$, the set of zero-input trajectories. Exponential trajectories therefore play a crucial role in describing the poles of a system, indeed they are the motive for considering poles.

4.1. Polynomial exponential trajectories. The algebraic significance of a trajectory such as $w(t) = v_0 e^{at}$ is that it is killed by the differential operator $d/dt - a$. In the module formalism we are using, $w(t)$ is annihilated by the maximal ideal $(z-a)$. The characterization of trajectories annihilated by maximal ideals, or more generally by powers of maximal ideals, has been done in full detail in [25, (1.25–26), (5.26–29), (6.6), (6.10)]. Such trajectories turn out to be exponential trajectories, or more generally, polynomial exponential trajectories, in a sense appropriate to the underlying signal space. These are precisely the trajectories in which we are interested in our study of poles.

The description of exponential trajectories in [25] can be reformulated as in the next two results.

THEOREM AND DEFINITION 4.1. [25] *Consider a scalar trajectory $w(\underline{t})$ in $\mathcal{A}$, where $\mathcal{A}$ is one of the signal spaces listed in §2.1, which is a module over the ring $\mathcal{R} = k[\underline{\mathbf{z}}]$ or $k[\underline{\mathbf{z}}, \underline{\mathbf{z}}^{-1}]$. Let $\underline{a} \in \mathbb{C}^n$ or $(\mathbb{C}\backslash 0)^n$, respectively according to $\mathcal{R}$, and let $I(\underline{a})$ denote the maximal ideal of all elements of $\mathcal{R}$ which vanish at $\underline{a}$ (note that we may have $\underline{a} \notin k^n$, but $I(\underline{a})$ is still meaningful).*

Then $w(\underline{t})$ is annihilated by $I(\underline{a})$ if and only if it is of the following form (dependent on the signal space):

1. *For $\mathcal{A} = \mathbb{C}^{\mathbb{N}^n}$ or $\mathcal{A} = \mathbb{C}^{\mathbb{Z}^n}$,*

$$w(\underline{t}) \;=\; \alpha a_1^{t_1} \cdots a_n^{t_n}$$

for some $\alpha \in \mathbb{C}$.

2. *For $\mathcal{A} = \mathbb{R}^{\mathbb{N}^n}$, write $a_i = r_i e^{\imath\theta_i}$, $\theta_i := 0$ for $a_i = 0$, $i = 1, \ldots, n$;*

$$w(\underline{t}) \;=\; r_1^{t_1} \cdots r_n^{t_n} (\alpha \cos(\theta_1 t_1 + \cdots + \theta_n t_n) + \beta \sin(\theta_1 t_1 + \cdots + \theta_n t_n))$$

for some $\alpha, \beta \in \mathbb{R}$.

3. *For $\mathcal{A} = \mathbb{R}^{\mathbb{Z}^n}$, write $a_i = r_i e^{\imath\theta_i}$, $i = 1, \ldots, n$;*

$$w(\underline{t}) \;=\; r_1^{t_1} \cdots r_n^{t_n} (\alpha \cos(\theta_1 t_1 + \cdots + \theta_n t_n) + \beta \sin(\theta_1 t_1 + \cdots + \theta_n t_n)),$$

for some $\alpha, \beta \in \mathbb{R}$.

4. *For $\mathcal{A} = \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{C})$ or $\mathcal{A} = \mathcal{D}'(\mathbb{R}^n, \mathbb{C})$,*

$$w(\underline{t}) \;=\; \alpha e^{a_1 t_1 + \cdots + a_n t_n},$$

for some $\alpha \in \mathbb{C}$.

5. *For $\mathcal{A} = \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$ or $\mathcal{A} = \mathcal{D}'(\mathbb{R}^n, \mathbb{R})$, with $a_i = g_i + \imath h_i$, $i = 1, \ldots, n$;*

$$w(\underline{t}) \;=\; e^{g_1 t_1 + \cdots + g_n t_n} (\alpha \cos(h_1 t_1 + \cdots + h_n t_n) + \beta \sin(h_1 t_1 + \cdots + h_n t_n)),$$

for some $\alpha, \beta \in \mathbb{R}$.

A trajectory of $\mathcal{A}^q$ is annihilated by $I(\underline{a})$ if and only if every component is of the above form. Such a trajectory is called an exponential trajectory with frequency *$\underline{a}$.*

The full exponential-type properties of a behaviour can however only be captured by considering trajectories of a more general form, as described in the next result.

A trajectory $w \in \mathcal{A}$ is called *locally finite* if the submodule $\mathcal{R} \cdot w$ generated by w is finite-dimensional over k [25]. The $\mathcal{R}$-submodule of $\mathcal{A}$ consisting of all locally finite trajectories is denoted by $\mathcal{A}_{\mathrm{lf}}$.

THEOREM AND DEFINITION 4.2. [25] *Define $a, I(\underline{a})$ as above. A trajectory $w(\underline{t}) \in \mathcal{A}$ is annihilated by some power of $I(\underline{a})$ if and only if it is of the following form (dependent on the signal space):*

1. *For $\mathcal{A} = \mathbb{C}^{\mathbb{N}^n}$: write $S = \{i \in \{1, \ldots, n\} : a_i \neq 0\}$;*

$$w(\underline{t}) \;=\; p(\underline{t}) \prod_{i \in S} a_i^{t_i},$$

for some $p(\underline{t})$ which is the component-wise product of a polynomial function $p^{(1)}(\underline{t})$ of the parameters $t_i, i \in S$ only, and a finitely supported function $p^{(2)}(\underline{t})$ of the parameters $t_j, j \notin S$ only.

2. *For $\mathcal{A} = \mathbb{C}^{\mathbb{Z}^n}$:*

$$w(\underline{t}) \;=\; p(\underline{t}) a_1^{t_1} \cdots a_n^{t_n},$$

for some $p(\underline{t}) \in \mathbb{C}[\underline{t}]$.

3. *For $\mathcal{A} = \mathbb{R}^{\mathbb{N}^n}$, write $S = \{i \in \{1, \ldots, n\} : a_i \neq 0\}$, and, for $i \in S$, define $r_i, \theta_i \in \mathbb{R}$ by $a_i = r_i e^{\imath \theta_i}$. For $i \notin S$, set $\theta_i = 0$.*

$$w(\underline{t}) \;=\; p(\underline{t}) \left(\prod_{i \in S} r_i^{t_i} \right) (\alpha \cos(\theta_1 t_1 + \cdots + \theta_n t_n) + \beta \sin(\theta_1 t_1 + \cdots + \theta_n t_n)),$$

for some $\alpha, \beta \in \mathbb{R}$ and some $p(\underline{t})$ which is the component-wise product of a real polynomial function $p^{(1)}(\underline{t})$ of the parameters $t_i, i \in S$ only, and a finitely supported real function $p^{(2)}(\underline{t})$ of the parameters $t_j, j \notin S$ only.

4. *For $\mathcal{A} = \mathbb{R}^{\mathbb{Z}^n}$: for $i = 1, \ldots, n$, define $r_i, \theta_i \in \mathbb{R}$ by $a_i = r_i e^{\imath \theta_i}$, $i = 1 \ldots, n$;*

$$w(\underline{t}) \;=\; p(\underline{t}) r_1^{t_1} \cdots r_n^{t_n} (\alpha \cos(\theta_1 t_1 + \cdots + \theta_n t_n) + \beta \sin(\theta_1 t_1 + \cdots + \theta_n t_n)),$$

for some $\alpha, \beta \in \mathbb{R}$ and some $p(\underline{t}) \in \mathbb{R}[\underline{t}]$.

5. *For $\mathcal{A} = \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{C})$ or $\mathcal{A} = \mathcal{D}'(\mathbb{R}^n, \mathbb{C})$,*

$$w(\underline{t}) \;=\; p(\underline{t}) e^{a_1 t_1 + \cdots + a_n t_n},$$

for some $p(\underline{t}) \in \mathbb{C}[\underline{t}]$.

6. *For $\mathcal{A} = \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$ or $\mathcal{A} = \mathcal{D}'(\mathbb{R}^n, \mathbb{R})$: write $a_i = g_i + \imath h_i$, $g_i, h_i \in \mathbb{R}$, $i = 1, \ldots, n$;*

$$w(\underline{t}) \;=\; p(\underline{t}) e^{g_1 t_1 + \cdots + g_n t_n} (\alpha \cos(h_1 t_1 + \ldots + h_n t_n) + \beta \sin(h_1 t_1 + \ldots + h_n t_n)),$$

for some $p(\underline{t}) \in \mathbb{R}[\underline{t}]$.

Moreover,

$$\mathcal{A}_{\mathrm{lf}} \;=\; \bigoplus_{\underline{a}} \left\{ w \in \mathcal{A} \;\middle|\; I(\underline{a})^l \text{ annihilates } w \text{ for some } l \in \mathbb{Z}^+ \right\}, \tag{4.1}$$

where for $k = \mathbb{C}$ the direct sum runs over all $\underline{a}$ in either $\mathbb{C}^n$ or $(\mathbb{C} \backslash 0)^n$, according to $\mathcal{R}$; for $k = \mathbb{R}$ the direct sum runs over a set of representatives of the one- or two-element subsets $\{\underline{a}, \overline{\underline{a}}\}, \underline{a} \in \mathbb{C}^n$. Here $\overline{\underline{a}}$ denotes the component-wise complex conjugate of $\underline{a}$.

The locally finite trajectories in $\mathcal{A}_{\mathrm{lf}}^q$ *for any* q *are called* polynomial exponential functions/ trajectories.

Note that a finite-dimensional behaviour consists entirely of polynomial exponential trajectories, as it contains $\mathcal{R}w$ for each trajectory w, and so all trajectories are locally finite.

As discussed in [25], the space $\mathcal{A}_{\mathrm{lf}}$ is itself an injective cogenerator of the category of $\mathcal{R}$-modules, i.e. Theorem 2.4 applies with $D(M) = \mathrm{Hom}_{\mathcal{R}}(M, \mathcal{A})$ replaced by $D_{\mathrm{lf}}(M) = \mathrm{Hom}_{\mathcal{R}}(M, \mathcal{A}_{\mathrm{lf}})$. This enables the vast majority of the duality theory of [24] to be applied to differential/ difference behaviours in $\mathcal{A}_{\mathrm{lf}}^q$. In particular, the equation

$$(\mathcal{B} \cap \mathcal{A}_{\mathrm{lf}}^q)^{\perp} \;=\; \mathrm{Im}_{\mathcal{R}}\, E \;=\; \mathcal{B}^{\perp}$$

for any $\mathcal{B} = \ker_{\mathcal{A}} E$ shows that *the sub-behaviour* $\mathcal{B} \cap \mathcal{A}_{\mathrm{lf}}^q$ *of polynomial exponential trajectories determines* $\mathcal{B}$ *uniquely.* More specifically, in the case of a $\mathcal{C}^\infty$ differential behaviour, Hörmander gives the following result [18, Thm. 7.6.14]:

THEOREM 4.3. *Let* $\mathcal{B}$ *be a differential behaviour with signal space* $\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{C})$*. Then* $\mathcal{B}$ *is equal to the closure of the set of its polynomial exponential trajectories.*

Even in the general case, we have seen that a differential or difference behaviour is described fully by those trajectories annihilated by some power of an ideal $I(\underline{a})$, $\underline{a} \in \mathbb{C}^n$. As any differential/ difference behaviour containing such a trajectory must also contain a trajectory annihilated by $I(\underline{a})$ itself, from §4.2 to §6 we will discuss only exponential trajectories. In §7 we will look briefly at the integral representation theorem, which for a behaviour with signal space $\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{C})$ describes how any system trajectory can be written as an integral of polynomial exponential trajectories.

4.2. The characteristic variety. We now look at the concept of the characteristic variety, which contains the essential information on the exponential trajectories of a behaviour.

Let $J \subseteq k[\underline{\mathbf{z}}]$, $k = \mathbb{R}, \mathbb{C}$. Then we define the variety $V(J)$ as

$$V(J) \;:=\; \{\underline{a} \in \mathbb{C}^n \;\mid\; p(\underline{a}) = 0 \text{ for all } p \in J\} \tag{4.2}$$

Note that $V(J)$ is defined as a subset of $\mathbb{C}^n$ even when $k = \mathbb{R}$. In the case where the ring of interest is $\mathcal{R} = k[\underline{\mathbf{z}}, \underline{\mathbf{z}}^{-1}]$, the definition is the same, except that only points $\underline{a} \in (\mathbb{C}\backslash 0)^n$ are considered. Henceforth we write: $\underline{a} \in \mathbb{C}^n$ $((\mathbb{C}\backslash 0)^n)$, on the understanding that the former applies except when working with the signal space $\mathcal{A} = k^{\mathbb{Z}^n}$.

The next result and definition are fundamental to the paper, and come from the theory of pdes; see [2, p200, p340], and also [28, p138–139], where the term, *variety associated with the finitely generated module* M, is used. The characteristic variety is also investigated in [30, 31].

THEOREM AND DEFINITION 4.4. *Let* $\mathcal{B} = D(M)$ *be a behaviour with kernel representation* E*. Let* $\underline{a}$ *be a point in* $\mathbb{C}^n$ $((\mathbb{C}\backslash 0)^n)$*. The following are equivalent:*

1. $\underline{a} \in V(\mathrm{ann}\, \mathcal{B}) = V(\mathrm{ann}\, M)$.
2. $E(\underline{a})$ *has less than full column rank.*
3. $\mathcal{B}$ *contains a non-zero exponential trajectory* w *with frequency* $\underline{a}$*.*

If $\underline{a}$ *satisfies these conditions, then it is called a* characteristic point *of* $\mathcal{B}$ *or of* M*. The variety* $V(\mathrm{ann}\, \mathcal{B})$ *of all such points is called the* characteristic variety *of* $\mathcal{B}$ *or of* M*.*

Proof. Let g be the number of rows of E. By Lemma 2.6, duality preserves the annihilator, so $V(\mathrm{ann}\, \mathcal{B}) = V(\mathrm{ann}\, M)$. Write $I(\underline{a})$ for the set of all polynomials of $\mathcal{R}$

which vanish at the complex point $\underline{a}$. $I(\underline{a})$ is a maximal ideal of $\mathcal{R}$. The equivalence of the first two conditions follow since ann M and $\mathrm{Fitt}_0\, M$, the ideal of qth order minors of E, are known to have the same radical [10, Prop. 20.6] and therefore vanish at the same points. Now we have that $(\mathcal{R}/I(\underline{a}))^{1,q}/(\mathcal{R}/I(\underline{a}))^{1,g}E(\underline{a}) = M/I(\underline{a})M$, and by Lemma 2.6 we also have

$$D(M/I(\underline{a})M) \;=\; [\mathcal{B} : I(\underline{a})],$$

the set of trajectories of $\mathcal{B}$ annihilated by $I(\underline{a})$. Hence $E(\underline{a})$ has less than full column rank if and only if $D(M/I(\underline{a})M) \neq 0$, or if and only if $\mathcal{B}$ contains a non-zero trajectory annihilated by $I(a)$. Finally, apply Theorem 4.1. □

The first condition of Theorem 4.4 has the advantage of being succinct, algebraic, and representation-independent. The second condition, which considers the points where the kernel representation has less than full column rank, is familiar from the classical theory of poles. The third condition gives a dynamic (trajectory) interpretation of the point, the details of which depend upon the signal space. The equivalence of the first two conditions of Theorem 4.4 is well-known.

The characteristic variety is only interesting for autonomous behaviours, for it is different from $\mathbb{C}^n$ $((\mathbb{C}\backslash 0)^n)$ precisely when $\mathcal{B}$ is autonomous.

In fact, for a given characteristic point $\underline{a}$ of a behaviour $\mathcal{B} = \ker_{\mathcal{A}} E$, it is possible to construct explicitly a non-zero exponential trajectory with frequency $\underline{a}$ in $\mathcal{B}$. Such a trajectory is given for example in the continuous complex case by $v_0 e^{a_1 t_1 + \cdots + a_n t_n}$, where v_0 is any non-zero element of the kernel of $E(\underline{a})$, acting on $\mathbb{C}^q$ on the right. Conversely, over the field $\mathbb{C}$ if $v_0 e^{a_1 t_1 + \cdots + a_n t_n} \in \mathcal{B}$ then $E(\underline{a})v_0 = 0$, because when working over $\mathbb{C}$ we can easily show for any exponential trajectory w that $E(\underline{\mathbf{z}})w = E(\underline{a})w$.

Note that the characteristic points are defined over the algebraic closure $\mathbb{C}$ of k. In the case $k = \mathbb{R}$, the subset of characteristic points in $\mathbb{R}^n$ or $(\mathbb{R}\backslash 0)^n$ correspond to exponential trajectories of the form $v_0 e^{a_1 t_1 + \cdots + a_n t_n}$ (continuous case) or $v_0 a_1^{t_1} \cdots a_n^{t_n}$ (discrete case). However, the information contained in this subset of the characteristic variety is insufficient to describe the exponential phenomena which the system may exhibit.

Example Consider the 1D behaviour over $\mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$ given by the kernel representation:

$$\mathcal{B} \;=\; \ker_{\mathcal{A}} E, \qquad E \;=\; \begin{pmatrix} z-5 & -2 \\ z & z-1 \end{pmatrix}$$

Now E loses rank nowhere in $\mathbb{R}$, for its determinant is $z^2 - 4z + 5$, which has no roots in $\mathbb{R}$. Over $\mathbb{C}$, we have the roots $2 + \imath$, $2 - \imath$, and indeed, $\mathcal{B}$ admits the trajectory

$$w(t) \;=\; \begin{pmatrix} e^{2t}(3\sin t - \cos t) \\ e^{2t}(3\cos t - 4\sin t) \end{pmatrix},$$

which is exponential with frequency $2 + \imath$. However, $\mathcal{B}$ has no trajectories of the form $w(t) = \alpha e^{at}$, $a \in \mathbb{R}$, and the exponentially increasing/ decreasing trajectories can only be found by considering the characteristic points in $\mathbb{C}\backslash\mathbb{R}$.

4.3. Associated and coassociated primes. We will find it useful to consider objects which describe the structure of a behaviour $\mathcal{B}$ in a similar but slightly more detailed manner than ann $\mathcal{B}$ and its corresponding variety.

Recall that an $\mathcal{R}$-module M and a prime ideal J give rise to the *localization* $M_J = \{\frac{m}{s} \mid m \in M, s \in \mathcal{R}\backslash J\}$, which is a module over the local ring $R_J = \{\frac{r}{s} \mid r \in \mathcal{R}, s \in \mathcal{R}\backslash J\}$.

DEFINITION 4.5. *Let M be an $\mathcal{R}$-module. The* support supp M *of M is the set of prime ideals J such that $M_J \neq 0$. The set* Ass M *of* associated primes *of M, is the set of prime ideals of $\mathcal{R}$ which are annihilators of elements of M. These will also be called the* coassociated primes *of $\mathcal{B} = D(M)$.*

The following standard result can be found in e.g. [3, Prop. II.4.17, Thm. IV.1.2] or [10, Cor. 2.7, Thm. 3.1]:

THEOREM 4.6. *Let M be a finitely generated non-zero $\mathcal{R}$-module. Then the support of M is the set of all prime ideals containing* ann M. *The set* Ass M *is finite, nonempty, contained in* supp M, *and contains all the minimal prime divisors of* ann M.

From the first claim of Theorem 4.6, a point $\underline{a}$ is a characteristic point of $\mathcal{B} = D(M)$ if and only if $\operatorname{ann} M \subseteq I(\underline{a})$, or equivalently if and only if $M_{I(\underline{a})} \neq 0$. Indeed, much of the theory in this paper can be expressed formally in the language of localization; we will however use it sparingly.

From the last claim of Theorem 4.6, we can write the characteristic variety of $\mathcal{B}$ as the union of the varieties $V(J)$, J a coassociated prime of $\mathcal{B}$. There is redundancy in this decomposition, as it is also possible to write the characteristic variety as the union of the varieties corresponding to the minimal prime divisors of $\operatorname{ann} M$, called the *minimal primes* of M, only. The remaining associated primes of M, called *embedded primes*, correspond to proper subvarieties of those corresponding to the minimal primes. It is however necessary to consider all associated primes, since this makes available the primary decomposition theorem, which will be applied in §7. In general, the coassociated primes of a behaviour contain more information than the characteristic variety.

The next standard result will prove invaluable in breaking down the set of poles into subsets (e.g. controllable poles) with certain properties.

LEMMA 4.7. *Let M, M', M'' denote finitely generated $\mathcal{R}$-modules which form an exact sequence*

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

Then $V(\operatorname{ann} M) = V(\operatorname{ann} M') \cup V(\operatorname{ann} M'')$, *and* $\operatorname{Ass} M' \subseteq \operatorname{Ass} M \subseteq (\operatorname{Ass} M') \cup (\operatorname{Ass} M'')$.

Proof. The first claim is found in [3, Ch. II, §4.4], and the second in [3, Ch. IV, §1.2] or [10, p. 92]. □

5. Controllable and Uncontrollable Poles. In this section we present our new definition of a pole of an nD system, and provide a decomposition of the set of poles into *controllable* and *uncontrollable* poles. The controllable poles are poles of the controllable part of the system, and are associated with the transfer matrix. The uncontrollable poles play the role of input-decoupling zeros.

5.1. Controllable and uncontrollable poles. In our motivational discussion at the beginning of §4, we discussed the classical interpretation of a pole, as a frequency $\underline{a}$ for which the zero-input behaviour contains a non-zero exponential trajectory. We then discussed the characterization of such points $\underline{a}$ using the notion of the characteristic variety. We now use this notion to define a pole (point) in the obvious way. However, we do not yet consider states; these are introduced under the guise of latent variables in §6. Note also the distinction between "poles", which are prime ideals, and "pole points", which are points in real or complex space.

DEFINITION 5.1. *Let $\mathcal{B} = D(M)$ be a behaviour with a given input/ output structure and controllable part $\mathcal{B}^c$.*

1. *The* pole variety, pole points *and* poles *of $\mathcal{B}$ are defined to be the characteristic variety, characteristic points and coassociated primes respectively of the zero-input behaviour $\mathcal{B}_{0,y}$.*
2. *The* controllable pole variety, controllable pole points *and* controllable poles *of $\mathcal{B}$ are defined to be the characteristic variety, characteristic points and coassociated primes respectively of the behaviour $(\mathcal{B}^c)_{0,y}$.*
3. *The* uncontrollable pole variety, uncontrollable pole points *and* uncontrollable poles *of $\mathcal{B}$ are defined to be the characteristic variety, characteristic points and coassociated primes respectively of the behaviour $\mathcal{B}/\mathcal{B}^c$.*

These definitions are all new, and are not trivially equivalent to any previously given definitions which we are aware of. The behaviours concerned in Definition 5.1, and the corresponding finitely generated modules, can be captured in a dual pair of commutative exact diagrams (5.1)–(5.2); compare with the diagrams on p14 of [31]).

$$
\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & F \arrow[r, equal] \arrow[d] & F \arrow[d] & \\
0 \arrow[r] & tM \arrow[r] \arrow[d, equal] & M \arrow[r] \arrow[d] & M/tM \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & tM \arrow[r] & M/F \arrow[r] \arrow[d] & M/(F \oplus tM) \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}
\tag{5.1}
$$

In the diagram (5.1), F denotes a maximal free submodule of M, for example that generated by the elements $e_i + \mathcal{B}^\perp$, where e_i is the ith natural basis vector in $\mathcal{R}^{1,q}$, for each index i corresponding to an input component. The dual of M/F is $\mathcal{B}_{0,y}$, the dual of M/tM is $\mathcal{B}^c$, as already discussed, and the dual of $M/(tM \oplus F)$ is $\mathcal{B}^c \cap \mathcal{B}_{0,y} = (\mathcal{B}^c)_{0,y}$. Diagram (5.1) dualizes to (5.2).

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & (\mathcal{B}^c)_{0,y} \arrow[r] \arrow[d] & \mathcal{B}_{0,y} \arrow[r] \arrow[d] & \mathcal{B}/\mathcal{B}^c \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{B}^c \arrow[r] \arrow[d] & \mathcal{B} \arrow[r] \arrow[d] & \mathcal{B}/\mathcal{B}^c \arrow[r] & 0 \\
 & \mathcal{A}^m \arrow[r, equal] \arrow[d] & \mathcal{A}^m \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}
\tag{5.2}
$$

Note: it is important not to confuse the zero-input behaviour $(\mathcal{B}^c)_{0,y}$ with the behaviour $(\mathcal{B}_{0,y})^c$. The latter is uninteresting as it is always zero, whereas the former in general is not, and we will henceforth write $\mathcal{B}^c_{0,y}$ for $(\mathcal{B}^c)_{0,y}$.

It is possible to regard the behaviour $\mathcal{B}_{0,y}$ as the "pole module" of the system, though we feel that it would be more appropriate to give this name to the finitely generated module M/F to which $\mathcal{B}_{0,y}$ is dual. This terminology ties in loosely with that of Kalman, Conte, Perdon, Wyman and Sain [6, 21, 45], and we could extend it to an "input decoupling zero module" etc, though there does not seem to be a strong link between the two sets of definitions. The significance of a "pole module" is that the pole points themselves are the points where the annihilator of the pole module vanishes. This interpretation is also possible in the earlier work of Kalman and others. The term "pole module" (etc) has also been used by Bourlès and Fliess [4], where it is equivalent (for 1D systems) to the definition we have suggested; see also Pommaret and Quadrat [31].

One important consequence of our definitions is that the number of poles (and also the number of controllable poles and the number of uncontrollable poles) is finite. This follows from Theorem 4.6. Also, for any pole J, $V(J)$ is contained in the pole variety, and indeed the pole variety is the union of such $V(J)$'s; similarly for controllable and uncontrollable poles.

The following characterization of pole points is immediate from Theorem 4.4.

COROLLARY 5.2. *Let $\mathcal{B}$ be a behaviour with a given input/ output structure, and let $\underline{a}$ be a point in $\mathbb{C}^n$ $((\mathbb{C}\backslash 0)^n)$. Let $(-Q\ P)$ be a kernel representation of $\mathcal{B}$, where the submatrix P corresponds to the output variables. Then the following are equivalent:*

1. *$\underline{a}$ is a pole point of $\mathcal{B}$.*
2. *$P(\underline{a})$ has less than full column rank.*
3. *$\mathcal{B}_{0,y}$ contains a non-zero exponential trajectory of frequency $\underline{a}$.*

5.2. Controllable poles. The controllable poles (pole points, etc) of $\mathcal{B}$ are poles (pole points, etc) of the controllable part $\mathcal{B}^c$ of $\mathcal{B}$. Hence for any controllable pole point $\underline{a}$, there is an exponential trajectory $w(\underline{t}) \in \mathcal{B}^c_{0,y} \subseteq \mathcal{B}_{0,y}$ which can be concatenated with the zero trajectory in the sense of behavioural controllability, i.e. such a $w(\underline{t})$ can be controlled.

Recall that $\mathcal{B}^c$ is the unique minimal element of the transfer class of $\mathcal{B}$. Since there is a strong relationship between $\mathcal{B}^c$ and the transfer matrix G of $\mathcal{B}$, it is not surprising that the controllable pole points of a behaviour are captured in the structure of its transfer matrix.

Not only the variety $V(\operatorname{ann}\mathcal{B}^c_{0,y})$ but also the ideal $\operatorname{ann}\mathcal{B}^c_{0,y}$ can be described via the transfer matrix, and this gives $\operatorname{ann}\mathcal{B}^c_{0,y}$ and also the corresponding coassociated primes a special structure.

THEOREM 5.3. *Let $\mathcal{B}$ be a behaviour with a given input/ output structure and transfer matrix G. Then we have*

$$\operatorname{ann}\mathcal{B}^c_{0,y} \;=\; \{r \in \mathcal{R} \mid rG \textit{ is a polynomial matrix}\} \tag{5.3}$$

In particular, $\operatorname{ann}\mathcal{B}^c_{0,y}$ *is principal, and indeed generated by the least common denominator of the entries of G. The controllable poles of $\mathcal{B}$ are also principal.*

Proof. Recall equation (3.6):

$$(\mathcal{B}^c_{0,y})^\perp \;=\; \{v \in \mathcal{R}^{1,p} \mid vG \in \mathcal{R}^{1,m}\}$$

For any element v of $\mathcal{R}^{1,p}$, we therefore have

$$\operatorname{ann}(v + (\mathcal{B}^c_{0,y})^\perp) \;=\; \{r \in \mathcal{R} \mid (rv)G \in \mathcal{R}^{1,m}\} \tag{5.4}$$

Taking the intersection of these sets for all $v \in \mathcal{R}^{1,p}$ gives us $\operatorname{ann} \mathcal{R}^{1,p}/(\mathcal{B}_{0,y}^c)^\perp = \operatorname{ann} \mathcal{B}_{0,y}^c$, and thus we obtain (5.3). Furthermore, equation (5.4) also tells us that the annihilator of any $v + (\mathcal{B}_{0,y}^c)^\perp$ is equal to (d), where d is the least common denominator of the entries of vG. In particular, each such annihilator is principal, including the coassociated primes of $\mathcal{B}_{0,y}^c$, which are the controllable poles. □

COROLLARY 5.4. *Let $\mathcal{B}$ be a behaviour with a given input/ output structure and transfer matrix G. Let $(-Q^c\ P^c)$ be a generalized factor left prime matrix such that $P^cG = Q^c$, e.g. P^c is square and $(P^c)^{-1}Q^c = G$ is a minor left coprime matrix fraction description of G, if such exists. Let $\underline{a} \in \mathbb{C}^n$ $((\mathbb{C}\backslash 0)^n)$. Then the following are equivalent:*

1. *$\underline{a}$ is a controllable pole point of $\mathcal{B}$.*
2. *$P^c(\underline{a})$ has less than full column rank.*
3. *The denominator of some entry of G vanishes at $\underline{a}$.*
4. *$\mathcal{B}_{0,y}^c$ contains a non-zero exponential trajectory with frequency $\underline{a}$.*

Proof. If $(-Q^c\ P^c)$ is generalized factor left prime and $P^cG = Q^c$ then $(-Q^c\ P^c)$ is a kernel representation of a controllable behaviour with transfer matrix G, which is therefore the controllable part of $\mathcal{B}$. From [43, Cor. 7.9], any minor left prime matrix is generalized factor left prime, and therefore this situation particularly applies when $(P^c)^{-1}Q^c$ is a minor left coprime matrix fraction description of G.

P^c is then a kernel representation of $\mathcal{B}_{0,y}^c$ (up to trivial isomorphism), and the equivalence of 1,2 and 4 now follows from Theorem 4.4. Condition 3 is immediate from equation (5.3) in Theorem 5.3. □

Note that the condition "the denominator of some entry of G vanishes at $\underline{a}$" can be expressed formally by the localization condition: $G \notin \mathcal{R}_{I(\underline{a})}^{p,m}$, where $I(\underline{a})$ is the maximal ideal corresponding to the point $\underline{a}$.

Corollary 5.4 also characterizes controllable pole points in terms of left matrix fraction descriptions (MFDs) of G; these must be minor left coprime for the characterization to apply, and not every transfer matrix has a minor left coprime MFD. The same characterization can be applied to right MFDs, since it is known that if $P^{-1}Q = \overline{Q}\,\overline{P}^{-1}$ are minor left coprime and minor right coprime MFDs of a transfer matrix, then $|P| = |\overline{P}|$ to within a unit [19, Thm. 2.12].

This correspondence between the controllable pole points and the transfer matrix is similar to the classical 1D characterization of transmission poles. However, transmission poles are defined with respect to a state-space representation, and so it is not accurate to identify controllable pole points with transmission poles. The nD behavioural equivalent of the classical transmission poles will be discussed in §6.2.

5.3. Uncontrollable poles. The uncontrollable poles *are defined independently of any input/ output structure*, i.e. a system has the same uncontrollable poles regardless of the input/ output structure imposed. This is only interesting if we have a proper interpretation of an uncontrollable pole, but we will see that it corresponds in appropriate cases to the idea of an input decoupling zero.

The uncontrollable poles can be characterized as follows:

LEMMA 5.5. *Let $\mathcal{B} = D(M)$ be a behaviour with a given input/ output structure. The uncontrollable poles are the coassociated primes of*

$$\mathcal{B}/\mathcal{B}^c \;\cong\; \mathcal{B}_{0,y}/\mathcal{B}_{0,y}^c \;\cong\; D(tM). \tag{5.5}$$

The uncontrollable poles of $\mathcal{B}$ are precisely the non-zero coassociated primes of $\mathcal{B}$. Furthermore, a behaviour is controllable if and only if it has no uncontrollable poles.

Proof. The duality $\mathcal{B}/\mathcal{B}^c \cong D(tM)$ is known (see equation (3.5) and the discussion following), and the isomorphism $\mathcal{B}/\mathcal{B}^c \cong \mathcal{B}_{0,y}/\mathcal{B}^c_{0,y}$ comes from the standard module isomorphism theorems; see also diagram (5.2). The uncontrollable poles of $\mathcal{B}$ are the associated primes of tM, and it is easy to see that these are the non-zero associated primes of M, i.e. the non-zero coassociated primes of $\mathcal{B}$. Finally, $\mathcal{B}$ has no uncontrollable poles if and only if tM has no associated primes, i.e. if and only if $tM = 0$, i.e. $\mathcal{B} = \mathcal{B}^c$. □

The last result of Lemma 5.5 is reminiscent of the result that a controllable state-space representation of a 1D system is precisely one with no input decoupling zeros. However it is not appropriate to identify the uncontrollable poles with input decoupling zeros, since the concept of an input decoupling zero only has meaning when inputs, outputs and states are involved; we will develop this connection further in §6.2.

We now consider the characterization of uncontrollable pole points in terms of properties of $\mathcal{B}$ and its representations.

THEOREM 5.6. *Let $\mathcal{B} = D(M)$ be a behaviour with kernel representation E. Then:*

1. *$\mathcal{B}$ is controllable if and only if it has no uncontrollable pole points.*
2. *Let $\underline{a} \in \mathbb{C}^n$ $((\mathbb{C}\backslash 0)^n)$, and write $I(\underline{a}) \subseteq \mathcal{R}$ for the ideal of polynomials vanishing at $\underline{a}$. We have that:*

$\mathcal{B}\backslash\mathcal{B}^c$ contains a non-zero exponential trajectory with frequency $\underline{a}$
$\Rightarrow$ *$\underline{a}$ is an uncontrollable pole point of $\mathcal{B}$*
$\Rightarrow$ *E loses rank at $\underline{a}$, i.e.* rank $E(\underline{a}) \lneq$ rank E, *or equivalently, the localization $M_{I(\underline{a})}$ is not free.*

3. *The statements in claim 2 are equivalent when $\mathcal{B}^c$ is strongly controllable, and in particular for a 1D system. More generally, if E^c is a kernel representation of $\mathcal{B}^c$, then the statements are equivalent for points $\underline{a}$ at which E^c does not lose rank. In particular, these statements are equivalent for points $\underline{a}$ which are not controllable pole points.*
4. *Suppose that $\underline{a}$ is not an uncontrollable pole point of $\mathcal{B}$. Then E loses rank at $\underline{a}$ if and only if E^c loses rank at $\underline{a}$.*

Proof.

1. $\mathcal{B}$ has no uncontrollable pole points if and only if $\operatorname{ann} \mathcal{B}/\mathcal{B}^c = \mathcal{R}$, i.e. $\mathcal{B} = \mathcal{B}^c$.
2. Write $J = I(\underline{a})$, and suppose that $\mathcal{B}\backslash\mathcal{B}^c$ contains a non-zero exponential trajectory w with frequency $\underline{a}$. Then $Jw = 0 \in \mathcal{B}^c$, so $w + \mathcal{B}^c$ is an element of $\mathcal{B}/\mathcal{B}^c$ which is annihilated by J. By Theorem 4.4, $\underline{a}$ is a characteristic point of $\mathcal{B}/\mathcal{B}^c$, i.e. an uncontrollable pole point of $\mathcal{B}$. From this it follows that $\operatorname{ann} tM \subseteq J$, or equivalently by Theorem 4.6 that $t(M_J) = (tM)_J \neq 0$, i.e. M_J is not torsionfree. Hence M_J is not free, which by [24, Thm. 7.69] means that E loses rank at $\underline{a}$.
3. Now suppose that E loses rank at $\underline{a}$ but that E^c does not. Equivalently, $(M/tM)_J$ is free but M_J is not. Due to the exact sequence

$$0 \longrightarrow (tM)_J \longrightarrow M_J \longrightarrow (M/tM)_J \longrightarrow 0, \tag{5.6}$$

we must have $(tM)_J \neq 0$, and so by Theorem 4.6 $\operatorname{ann} tM \subseteq J$, from which $\underline{a}$ is an uncontrollable pole point of $\mathcal{B}$. Next, if $\underline{a}$ is such a point, and furthermore E^c does not lose rank at $\underline{a}$, we again have the exact sequence (5.6). Since again $(M/tM)_J$ is free, (5.6) splits, and so we can tensor with $R_J/J_J = R/J$ to obtain the exact sequence

$$0 \longrightarrow tM/JtM \longrightarrow M/JM \longrightarrow M^c/JM^c \longrightarrow 0,$$

where M^c denotes M/tM, and its dual

$$0 \longrightarrow [\mathcal{B}^c : J] \longrightarrow [\mathcal{B} : J] \xrightarrow{\rho} [\mathcal{B}/\mathcal{B}^c : J] \longrightarrow 0 \tag{5.7}$$

Since $\underline{a}$ is an uncontrollable pole point of $\mathcal{B}$, by Theorem 4.4 there exists a non-zero element v in $[\mathcal{B}/\mathcal{B}^c : J]$. From the exactness of (5.7), there must be an element $w \in [\mathcal{B} : J]$ with $\rho(w) = w + \mathcal{B}^c = v$. Since $v \neq 0$, we must have $w \notin \mathcal{B}^c$, and as J annihilates w, it is an exponential trajectory with frequency $\underline{a}$, as required. This proves the equivalence of the three conditions in claim 2 for points $\underline{a}$ at which E^c does not lose rank. If $E^c = (-Q^c\ P^c)$ is a decomposition of E^c with respect to an input/ output structure, so that P^c is a kernel representation of $\mathcal{B}^c_{0,y}$, then we know that rank E^c = rank P^c. Now if $\underline{a}$ is not a controllable pole point of $\mathcal{B}$, so that by Corollary 5.4 P^c does not lose rank at $\underline{a}$, then E^c does not lose rank at $\underline{a}$, and the equivalence of the given statements applies.
If $\mathcal{B}^c$ is strongly controllable, then E^c is zero left prime, and so E^c loses rank nowhere. For a 1D system, $\mathcal{B}^c$ is always strongly controllable.

4. Again we have an exact sequence

$$0 \longrightarrow (tM)_J \longrightarrow M_J \longrightarrow (M/tM)_J \longrightarrow 0$$

As in the proof of the previous claim, the condition that $\underline{a}$ is not an uncontrollable pole point of $\mathcal{B}$ means that $(tM)_J = 0$, from which $M_J \cong (M/tM)_J$. Hence M_J is free if and only if $(M/tM)_J$ is free. By [24, Thm. 7.69], an equivalent statement is that E loses rank at $\underline{a}$ if and only if E^c loses rank at $\underline{a}$.

□

It is not true in general that an uncontrollable pole point of $\mathcal{B} = \ker_{\mathcal{A}} E$ is precisely a point where E loses rank. Consider the example:

$$E = \begin{pmatrix} z_1 & 0 & z_2 \\ 0 & z_2 & z_3 \end{pmatrix}$$

and the corresponding behaviour $\mathcal{B}$ with signal space $\mathcal{A} = \mathcal{C}^\infty(\mathbb{R}^3, \mathbb{C})$. The matrix E has a rank-loss point $(0, 0, a_3)$ for any $a_3 \in \mathbb{C}$, but it is minor left prime (i.e. the second order minors have no common factor), and therefore GFLP by [43, Cor. 7.9]. $\mathcal{B}$ is therefore controllable by Theorem 3.8, and so by Theorem 5.6 it has no uncontrollable pole points.

Uncontrollable pole points can also be characterized in terms of other behaviours and representations:

THEOREM 5.7. *Let $\mathcal{B} = \ker_{\mathcal{A}} E$ be any behaviour with controllable part $\mathcal{B}^c = \ker_{\mathcal{A}} E^c$. Let K be any polynomial matrix with $E = KE^c$, and let C be a minimal left annihilator of E^c; write*

$$L = \begin{pmatrix} K \\ C \end{pmatrix}.$$

Then L is a kernel representation of $E^c\mathcal{B} \cong \mathcal{B}/\mathcal{B}^c$, and the following are equivalent for any point $\underline{a} \in \mathbb{C}^n$ $((\mathbb{C}\backslash 0)^n)$:

1. *$\underline{a}$ is an uncontrollable pole point of $\mathcal{B}$.*
2. *$L(\underline{a})$ has less than full column rank.*
3. *$E^c\mathcal{B}$ contains a non-zero exponential trajectory with frequency $\underline{a}$.*

Proof. Given E, E^c, K, C and L as described, it is easy to see that LE^c is a kernel representation of $\mathcal{B}$, and we also have $\ker_{\mathcal{A}} L \subseteq \operatorname{im}_{\mathcal{A}} E^c$. Now $w \in E^c\mathcal{B}$ implies $Lw = 0$, and conversely $w \in \ker_{\mathcal{A}} L$ implies that $w = E^c v$ for some v, so $(LE^c)v = 0$, i.e. $v \in \mathcal{B}$. Hence $\ker_{\mathcal{A}} L = E^c\mathcal{B}$. As $E^c\mathcal{B} \cong \mathcal{B}/\mathcal{B}^c$, the uncontrollable pole points are the characteristic points of $E^c\mathcal{B}$, and the remaining assertions follow on applying Theorem 4.4. □

5.4. The uncontrollable–controllable decomposition. As implied by the nomenclature, poles are either uncontrollable poles or controllable poles (or possibly both), and similarly for pole points:

THEOREM 5.8.

1. *The union of the uncontrollable pole variety and the controllable pole variety is the pole variety.*
2. *The union of the set of uncontrollable poles and the set of controllable poles is the set of poles.*

Proof. From the commutative exact diagram (5.2), we have an exact sequence

$$0 \longrightarrow (\mathcal{B}^c)_{0,y} \longrightarrow \mathcal{B}_{0,y} \longrightarrow \mathcal{B} \ / \ \mathcal{B}^c \longrightarrow 0,$$

which is dual to a corresponding sequence of finitely generated modules in reverse order in the diagram (5.1). Applying Lemma 4.7 gives us the first claim concerning pole points immediately, and also the following:

$$\{\text{uncontrollable poles}\} \ \subseteq \ \{\text{poles}\} \ \subseteq \ \{\text{uncontrollable poles, controllable poles}\}$$

It remains to show that every controllable pole is a pole. Write $M_{0,y}$ and $M^c_{0,y}$ respectively for the modules to which $\mathcal{B}_{0,y}$ and $\mathcal{B}^c_{0,y}$ are dual. We now have the following:

$$\operatorname{Ass} M^c_{0,y} \ \subseteq \ \operatorname{supp} \ M^c_{0,y} \ \subseteq \ \operatorname{supp} \ M_{0,y}$$

As $\operatorname{ann} M_{0,y} \neq 0$ (its dual is autonomous), $0 \notin \operatorname{supp} \ M_{0,y}$. Now by Theorem 5.3, every element of $\operatorname{Ass} M^c_{0,y}$ is principal and therefore has height 1. Each such element must therefore be minimal in $\operatorname{supp} \ M_{0,y}$, and so by Theorem 4.6 is therefore in $\operatorname{Ass} M_{0,y}$. This proves that $\operatorname{Ass} M^c_{0,y} \subseteq \operatorname{Ass} M_{0,y}$, i.e. the controllable poles are also poles, as required. □

In the proof above, the fact that every controllable pole is a pole relies on the fact that every controllable pole is principal (from Theorem 5.3). In §6 we will encounter decompositions of sets of poles where such an observation does not apply, and therefore only partial results of this nature can be obtained.

Note that, contrary to the implication of the nomenclature, it is possible for a pole to be both a controllable pole and an uncontrollable pole, and similarly for pole points! A complete partitioning of poles into controllable and uncontrollable ones would require a suitable notion of multiplicity, which is beyond the scope of this paper.

5.5. The dimension of the characteristic variety. Another important question for an nD system is how "large" the poles are, in a geometric sense. For a 1D system, the pole variety is just a set of isolated points. For a 2D system, the pole variety may be a set of isolated points, or it may include curves. For a 3D system, there is the additional possibility that the pole variety may contain a two-dimensional surface. The dimensions of the pole variety, controllable pole variety and uncontrollable pole variety, or more generally of the characteristic variety, are therefore of interest.

LEMMA 5.9. *Let $\mathcal{B} = \ker_{\mathcal{A}} E$. Then:*

1. *For $\mathcal{B} \neq 0$, the dimension of the characteristic variety of $\mathcal{B}$ is n minus the "right primeness degree" [43] of E.*

2. *$\mathcal{B}$ is finite-dimensional if and only if it has a finite number of characteristic points.*

3. *The controllable pole variety has dimension either -1 (it is empty) or $n-1$. The former case occurs if and only if the transfer matrix is a polynomial matrix. In the latter case, each maximal irreducible subvariety of the controllable pole variety has dimension $n-1$.*

4. *If the transfer matrix is not a polynomial matrix, the pole variety has dimension $n-1$.*

Proof. The first claim is immediate from Theorem 4.4 and the definition of the right primeness degree. The second claim is given in [26, 29].

From Theorem 5.3, $\operatorname{ann} \mathcal{B}^c_{0,y}$ is principal, and it must therefore be either $\mathcal{R}$ or an ideal of height 1. Hence $V(\operatorname{ann} \mathcal{B}^c_{0,y})$ is either empty or of dimension $n-1$. Again by Theorem 5.3, the former occurs precisely when the transfer matrix is polynomial. In the other case, Theorem 5.3 says further that every coassociated prime of $\mathcal{B}^c_{0,y}$, including the minimal prime divisors of $\operatorname{ann} \mathcal{B}^c_{0,y}$, has height 1, so the maximal irreducible subvarieties of $V(\operatorname{ann} \mathcal{B}^c_{0,y})$ have dimension $n-1$.

Claim 4 follows from claim 3, as the pole variety contains the controllable pole variety. □

The dimension of the characteristic variety (pole variety, etc), can be found algorithmically, as discussed in [43]. For a system with signal space $k^{\mathbb{N}^n}$, this quantity can also be interpreted as a measure of the order of magnitude of the system's initial condition set, the "autonomy degree" [43]. The interpretation for systems with other signal spaces is still open.

When the transfer matrix is not a polynomial matrix, Lemma 5.9 states that every maximal irreducible subvariety of the controllable pole variety has dimension $n-1$. In consequence, every maximal irreducible subvariety of the pole variety having dimension less than $n-1$ must also be a maximal irreducible subvariety of the uncontrollable pole variety. This is also trivially true when the transfer matrix is a polynomial matrix.

6. Observable and Unobservable Poles.

In this section we will look at the additional structure which occurs when the behaviour is a latent variable representation of some other behaviour. The principal example of this is when the latent variables are "state variables", e.g. in the context of the classical 1D theory. When the behaviour is defined through a Rosenbrock system matrix (or "polynomial matrix description"), this additional structure allows us to relate certain submatrices to various sets of poles.

6.1. Observable and Unobservable poles.

Consider now a behaviour $\mathcal{B}_{l,w}$ which is a latent variable description of some other behaviour $\mathcal{B}_w$. The behaviour $\mathcal{B}_w$ is called the *manifest behaviour* of the *full behaviour* $\mathcal{B}_{l,w}$, the variables of w are called *manifest variables* and the variables of l are called *latent variables.* These ideas are described in [40] for the 1D case and [22, 32] for the 2D case. Formally, we have the following for some polynomial matrices E, F:

$$\mathcal{B}_{l,w} = \left\{ \begin{pmatrix} l \\ w \end{pmatrix} \in \mathcal{A}^{d+q} \;\middle|\; Ew = Fl \right\} \tag{6.1}$$

$$\mathcal{B}_w = \left\{ w \in \mathcal{A}^q \;\middle|\; \exists\, l \in \mathcal{A}^d \ s.t. \begin{pmatrix} l \\ w \end{pmatrix} \in \mathcal{B}_{l,w} \right\} \tag{6.2}$$

It is easy to derive a kernel representation of $\mathcal{B}_w$: we construct a minimal left annihilator L of F, and we have $\mathcal{B}_w = \ker_{\mathcal{A}}(LE)$ [22], [24, p27]. There is also the behaviour $\mathcal{B}_{l,0}$, which we will call the *unobservable behaviour*, consisting of all elements of $\mathcal{B}_{l,w}$ with $w = 0$. $\mathcal{B}_{l,0}$ is isomorphic to $\ker_{\mathcal{A}} F$, and vanishes precisely when the latent variables are observable (in the behavioural sense) from the manifest variables. We can show that $\mathcal{B}_w$ is the factor module of $\mathcal{B}_{l,w}$ by $\mathcal{B}_{l,0}$, so we have an exact sequence

$$0 \longrightarrow \mathcal{B}_{l,0} \longrightarrow \mathcal{B}_{l,w} \longrightarrow \mathcal{B}_w \longrightarrow 0 \tag{6.3}$$

Now consider the pole structure of these behaviours. Take an input/ output structure on $\mathcal{B}_{l,w}$ for which the number of inputs which are in the variables w is maximized (when the latent variables are state variables, all the input variables can of course be taken to be manifest variables). We will refer to such an input/ output structure on $\mathcal{B}_{l,w}$ as a *maximally manifest input/ output structure*. Divide the components w and l into input and output components, so $w = (w_1^T, w_2^T)^T$ and $l = (l_1^T, l_2^T)^T$, where $(l_1^T, w_1^T)^T$ is a complete input vector for $\mathcal{B}_{l,w}$. It follows that (w_1, w_2) is an input/ output structure on $\mathcal{B}_w$, and (l_1, l_2) is one on $\mathcal{B}_{l,0}$ (it is for this reason that we restrict our consideration to i/o structures which are maximally manifest). These i/o structures are called the *induced input/ output structures* on $\mathcal{B}_w$ and $\mathcal{B}_{l,0}$.

The poles of $\mathcal{B}_{l,w}$ fall naturally into two categories. The first is the set of poles which are observable from the behaviour $\mathcal{B}_w$, and the second is the set of poles which cannot be determined from the manifest behaviour (they are poles of the unobservable behaviour).

DEFINITION 6.1. *Let $\mathcal{B}_{l,w}$ be a behaviour with latent variables l and a maximally manifest i/o structure with input variables (w_1, l_1) and output variables (w_2, l_2). Take the induced i/o structures on $\mathcal{B}_w$ and $\mathcal{B}_{l,0}$.*

1. *The* unobservable pole variety, unobservable pole points *and* unobservable poles *of $\mathcal{B}_{l,w}$ are defined to be the pole variety, pole points and poles of the unobservable behaviour $\mathcal{B}_{l,0}$.*
2. *The* observable pole variety, observable pole points *and* observable poles *of $\mathcal{B}_{l,w}$ are defined to be the pole variety, pole points and poles of the manifest behaviour $\mathcal{B}_w$.*

In effect we have defined the unobservable poles as the coassociated primes of the behaviour $\mathcal{B}_{l_1=0,l_2,w_1=0,w_2=0}$ (notation interpreted in the obvious way), and the observable poles as those of the behaviour $\mathcal{B}_{w_1=0,w_2} \subseteq \mathcal{B}_w$. The poles of $\mathcal{B}_{l,w}$ itself are the coassociated primes of $\mathcal{B}_{l_1=0,l_2,w_1=0,w_2}$ and $\mathcal{B}_{w_1=0,w_2}$ is the factor module of $\mathcal{B}_{l_1=0,l_2,w_1=0,w_2}$ by $\mathcal{B}_{l_1=0,l_2,w_1=0,w_2=0}$, so we have an exact sequence

$$0 \longrightarrow \mathcal{B}_{l_1=0,l_2,w_1=0,w_2=0} \longrightarrow \mathcal{B}_{l_1=0,l_2,w_1=0,w_2} \longrightarrow \mathcal{B}_{w_1=0,w_2} \longrightarrow 0 \tag{6.4}$$

THEOREM 6.2. *Let $\mathcal{B}_{l,w}$ be a behaviour with latent variables l and a maximally manifest i/o structure. Then the poles/ pole points of the full behaviour $\mathcal{B}_{l,w}$ have the following properties:*

1. *A point $\underline{a}$ is an observable pole point if and only if there exists $(l, w) \in \mathcal{B}_{l,w}$ such that $w_1 = 0$ and w_2 is non-zero and exponential with frequency $\underline{a}$.*
2. *A point $\underline{a}$ is an unobservable pole point if and only if there exists $(l, w) \in \mathcal{B}_{l,w}$ such that $w = 0, l_1 = 0$ and l_2 is non-zero and exponential with frequency $\underline{a}$.*
3. *Suppose that the latent variables l contain no free variables, i.e. $\mathcal{B}_{l,0}$ is autonomous. Then $\mathcal{B}_{l,w}$ is observable if and only if it has no unobservable poles, or equivalently no unobservable pole points.*

4. *The union of the unobservable pole variety and the observable pole variety is the pole variety.*

5. *Every observable pole is a pole. Every pole is either an unobservable pole or an observable pole (or both).*

Proof. The first two claims follow on applying Theorem 4.4, and the last two are immediate from Lemma 4.7 together with the exact sequence (6.4). For claim 3, the condition that $\mathcal{B}_{l,w}$ is observable is equivalent to the condition $\mathcal{B}_{l,0} = 0$, or alternatively to the vanishing of the module to which $\mathcal{B}_{l,0}$ is dual. The result follows since a finitely generated module M is 0 precisely when it has no associated primes (Theorem 4.6), or equivalently when $V(\text{ann}\, M) = \emptyset$. □

We do not claim that every unobservable pole is a pole. Claim 3 of Theorem 6.2 is similar to the classical 1D result that a system in state-space form is observable if and only if it has no output decoupling zeros. Note that the condition on this last result is satisfied if the latent variables l are interpreted as states.

6.2. Complete classification of poles. We can combine the partitioning of pole points into unobservable and observable pole points with the partitioning into uncontrollable and controllable pole points. This involves the controllable part of $\mathcal{B}_{l,w}$, and its associated unobservable and manifest behaviours. We denote by $\mathcal{B}^c_{l,w}$ the controllable part of $\mathcal{B}_{l,w}$, by $\mathcal{B}^c_{l,0}$ the unobservable behaviour of $\mathcal{B}^c_{l,w}$, by $\mathcal{B}^c_w$ the manifest behaviour of $\mathcal{B}^c_{l,w}$, and so on. Care must be taken with this notation, as it is not generally the case that $\mathcal{B}^c_{l,0} = (\mathcal{B}_{l,0})^c$.

Our final refinement of the pole structure is defined as follows:

DEFINITION 6.3. *Let $\mathcal{B}_{l,w}$ be a behaviour with latent variables l and a maximally manifest i/o structure. Then:*

1. *The* unobservable uncontrollable pole variety, unobservable uncontrollable pole points *and* unobservable uncontrollable poles *are the characteristic variety, characteristic points and coassociated primes of $\mathcal{B}_{l,0} \,/\, \mathcal{B}^c_{l,0}$.*

2. *The* unobservable controllable pole variety, unobservable controllable poles points *and* unobservable controllable poles *are the pole variety, pole points and poles of $\mathcal{B}^c_{l,0}$.*

3. *The* observable uncontrollable pole variety, observable uncontrollable pole points *and* observable uncontrollable poles *are the characteristic variety, characteristic points and coassociated primes of $\mathcal{B}_w \,/\, \mathcal{B}^c_w$.*

4. *The* observable controllable pole variety, observable controllable pole points *and* observable controllable poles *are the pole variety, pole points and poles of $\mathcal{B}^c_w$.*

To show that the unobservable uncontrollable poles are meaningfully defined, we have to show that $(\mathcal{B}^c)_{l,0}$ is a sub-behaviour of $\mathcal{B}_{l,0}$, which is evident from the fact that $(\mathcal{B}^c)_{l,0} = \mathcal{B}_{l,0} \cap \mathcal{B}^c$. To show that the observable uncontrollable poles are meaningfully defined, we have to show that $(\mathcal{B}^c)_w$ is a sub-behaviour of $\mathcal{B}_w$, which follows by definition. In fact, a stronger claim is possible:

THEOREM 6.4. *Let $\mathcal{B}_{l,w}$ be a behaviour with latent variables l. Then the manifest behaviour of the controllable part of $\mathcal{B}_{l,w}$ is equal to the controllable part of the manifest behaviour of $\mathcal{B}_{l,w}$.*

Proof. We write $\mathcal{B} = \mathcal{B}_{l,w}$ as usual. Clearly $(\mathcal{B}^c)_w$ is a sub-behaviour of $\mathcal{B}_w$. It is also clear from the definition of controllability that as $\mathcal{B}^c$ is controllable, so is $(\mathcal{B}^c)_w$. Now write E for any kernel representation of $\mathcal{B}$, and E^c for any kernel representation of $\mathcal{B}^c$. Since $\mathcal{B}$ and $\mathcal{B}^c$ have the same input/ output structures, any submatrices comprised of corresponding columns of E and of E^c must have the same rank. In particular, $m(\mathcal{B}_{l,0}) = m((\mathcal{B}^c)_{l,0})$. Now we use the additivity of the number of free variables:

$$\begin{aligned} m((\mathcal{B}^c)_w) &= m((\mathcal{B}^c)_{l,w}) - m((\mathcal{B}^c)_{l,0}) \\ &= m(\mathcal{B}_{l,w}) - m(\mathcal{B}_{l,0}) \\ &= m(\mathcal{B}_w) \end{aligned}$$

Summarizing, $(\mathcal{B}^c)_w$ is a sub-behaviour of $\mathcal{B}_w$ which has the same number of free variables and is controllable. This suffices to prove that $(\mathcal{B}^c)_w$ is the controllable part of $\mathcal{B}_w$. $\square$

Theorem 6.4 proves that the observable controllable poles are equal to the controllable poles of the manifest behaviour (if you like, the "controllable observable poles"). This result also shows that we can use the notation $\mathcal{B}^c_w$ without fear of ambiguity.

Currently we have a pair of exact sequences of autonomous behaviours whose coassociated primes correspond to various sets of poles:

$$0 \longrightarrow \mathcal{B}_{l_1=0,l_2,w_1=0,w_2=0} \longrightarrow \mathcal{B}_{l_1=0,l_2,w_1=0,w_2} \longrightarrow \mathcal{B}_{w_1=0,w_2} \longrightarrow 0 \tag{6.5}$$

$$0 \longrightarrow \mathcal{B}^c_{l_1=0,l_2,w_1=0,w_2=0} \longrightarrow \mathcal{B}^c_{l_1=0,l_2,w_1=0,w_2} \longrightarrow \mathcal{B}^c_{w_1=0,w_2} \longrightarrow 0 \tag{6.6}$$

and each module in the sequence (6.6) is a submodule of the corresponding module in the upper sequence (6.5). Let us look at the corresponding factor modules. Since $\mathcal{B}^c_{l_1=0,l_2,w_1=0,w_2=0} = \mathcal{B}_{l_1=0,l_2,w_1=0,w_2=0} \cap \mathcal{B}^c_{l_1,l_2,w_1,w_2}$, the factor module is

$$\frac{\mathcal{B}_{l_1=0,l_2,w_1=0,w_2=0}}{\mathcal{B}_{l_1=0,l_2,w_1=0,w_2=0} \cap \mathcal{B}^c_{l_1,l_2,w_1,w_2}} \cong \frac{\mathcal{B}_{l_1=0,l_2,w_1=0,w_2=0} + \mathcal{B}^c_{l_1,l_2,w_1,w_2}}{\mathcal{B}^c_{l_1,l_2,w_1,w_2}} \tag{6.7}$$

This is obviously a submodule of $\mathcal{B}_{l_1,l_2,w_1,w_2} \;/\; \mathcal{B}^c_{l_1,l_2,w_1,w_2}$, which by equation (5.5) is isomorphic to $\mathcal{B}_{l_1=0,l_2,w_1=0,w_2}/\mathcal{B}^c_{l_1=0,l_2,w_1=0,w_2}$. These behaviours are all shown in Figure 6.1. By the Snake Lemma (e.g. [10, Ex. A3.10]), we also have an isomorphism

$$\frac{\mathcal{B}_{l_1,l_2,w_1,w_2}}{\mathcal{B}_{l_1=0,l_2,w_1=0,w_2=0} + \mathcal{B}^c_{l_1,l_2,w_1,w_2}} \cong \frac{\mathcal{B}_{w_1=0,w_2}}{\mathcal{B}^c_{w_1=0,w_2}}, \tag{6.8}$$

and the diagram in Figure 6.1 is commutative and exact.

Reading the non-trivial behaviours in Figure 6.1 from left to right and from top to bottom, we see that the coassociated primes are respectively the following: the unobservable uncontrollable poles, the uncontrollable poles, the observable uncontrollable poles, the unobservable poles, the poles, the observable poles, the unobservable controllable poles, the controllable poles, and the observable controllable poles.

We now apply Lemma 4.7 again. This gives us the basic structure of the poles of a behaviour given by a latent variable description.

1. The union of the unobservable uncontrollable pole variety and the observable uncontrollable pole variety is the uncontrollable pole variety. Every observable uncontrollable pole is an uncontrollable pole. Every uncontrollable pole is either an unobservable uncontrollable pole or an observable uncontrollable pole.

2. The union of the unobservable pole variety and the observable pole variety is the pole variety. Every observable pole is a pole. Every pole is either an unobservable pole or an observable pole.

3. The union of the unobservable controllable pole variety and the observable controllable pole variety is the controllable pole variety. Every observable controllable pole is a controllable pole. Every controllable pole is either an unobservable controllable pole or an observable controllable pole.

$$\begin{tikzcd}
& 0 & 0 & 0 & \\
0 \arrow[r] & \frac{\mathcal{B}_{l_1=0,l_2,w=0}+\mathcal{B}^c_{l,w}}{\mathcal{B}^c_{l,w}} \arrow[r] \arrow[u] & \frac{\mathcal{B}_{l,w}}{\mathcal{B}^c_{l,w}} \arrow[r] \arrow[u] & \frac{\mathcal{B}_{l,w}}{\mathcal{B}_{l_1=0,l_2,w=0}+\mathcal{B}^c_{l,w}} \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \mathcal{B}_{l_1=0,l_2,w=0} \arrow[r] \arrow[u] & \mathcal{B}_{l_1=0,l_2,w_1=0,w_2} \arrow[r] \arrow[u] & \mathcal{B}_{w_1=0,w_2} \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \mathcal{B}^c_{l_1=0,l_2,w=0} \arrow[r] \arrow[u] & \mathcal{B}^c_{l_1=0,l_2,w_1=0,w_2} \arrow[r] \arrow[u] & \mathcal{B}^c_{w_1=0,w_2} \arrow[r] \arrow[u] & 0 \\
& 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

FIG. 6.1. *Exact commutative diagram for complete classification of poles. Note that the columns of this diagram are read upwards, contrary to convention; this is to achieve symmetry in Figure 6.2.*

4. The union of the unobservable uncontrollable pole variety and the unobservable controllable pole variety is the unobservable pole variety. Every unobservable uncontrollable pole is an unobservable pole. Every unobservable pole is either an unobservable controllable pole or an unobservable uncontrollable pole.

5. The union of the uncontrollable pole variety and the controllable pole variety is the pole variety. The union of the set of uncontrollable poles and the set of controllable poles is the set of poles.

6. The union of the observable uncontrollable pole variety and the observable controllable pole variety is the observable pole variety. The union of the set of observable uncontrollable poles and the set of observable controllable poles is the set of observable poles.

In general, we do not expect the inclusions of sets of poles missing from the list above to hold (e.g. we do not expect every unobservable uncontrollable pole to be an uncontrollable pole). Note however that we are able to make a full decomposition in the last case of observable poles. This is possible because the observable controllable poles are a subset of the controllable poles, and are therefore principal by Theorem 5.3. We can therefore apply the same argument as in the proof of Theorem 5.8 to deduce that every observable controllable pole is an observable pole. This reasoning cannot be applied to cases 1–4.

We summarize the relationships between the various sets of poles in a conceptual exact diagram (Figure 6.2). This diagram should be interpreted as follows: if $0 \longrightarrow$ Set 1 $\longrightarrow$ Set 2 $\longrightarrow$ Set 3 $\longrightarrow 0$ is a row or column, then the union of the pole variety of Set 1 and the pole variety of Set 3 is the pole variety of Set 2. The corresponding statements for the poles themselves is slightly weaker, except for the middle and right-hand columns, for which an analogous law holds.

Example. Take the 2D behaviour given by the following polynomial matrix:

$$\mathcal{B}_{l,u,y} = \ker_{\mathcal{A}} E_1,$$
$$E_1 = \left(\begin{array}{c|cc|c} (z_1^2 - z_2)(z_1 + z_2)(z_1 - 1)z_2 & 0 & z_2^2(z_1^2 - z_2) & -(z_1^2 + z_2^2 - 1) \\ (z_1^2 - z_2)(z_1 + z_2)(z_1 - 1)z_1 & (z_1 + z_2)^2 & z_1 z_2(z_1^2 - z_2) & (z_1^2 + z_2^2 - 1) \end{array}\right)$$

$$
\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & \text{unobs. unctrl. poles} \arrow[r] \arrow[u] & \text{unctrl. poles} \arrow[r] \arrow[u] & \text{obs. unctrl. poles} \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \text{unobs. poles} \arrow[r] \arrow[u] & \text{poles} \arrow[r] \arrow[u] & \text{obs. poles} \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \text{unobs. ctrl. poles} \arrow[r] \arrow[u] & \text{ctrl. poles} \arrow[r] \arrow[u] & \text{obs. ctrl. poles} \arrow[r] \arrow[u] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}
$$

FIG. 6.2. *Relationships between sets of poles.*

The signal space $\mathcal{A}$ can be taken as $k^{\mathbb{N}^n}$, $\mathcal{C}^\infty(\mathbb{R}^n, k)$ or $\mathcal{D}'(\mathbb{R}^n, k)$ for $k = \mathbb{R}$ or $\mathbb{C}$; the ring $\mathcal{R} = k[\underline{\mathbf{z}}]$. The behaviour $\mathcal{B}_{l,u,y}$ has 2 inputs, 1 output and 1 latent variable. We will now find the 9 pole varieties of this behaviour. This requires constructing representations for 9 related behaviours — those appearing in Figure 6.1. To begin with, we have the zero-input and unobservable behaviours:

$$
\begin{aligned}
\mathcal{B}_{l,u=0,y} \cong \ker_{\mathcal{A}} E_2, \quad E_2 &= \left(\begin{array}{c|c} (z_1^2 - z_2)(z_1 + z_2)(z_1 - 1)z_2 & -(z_1^2 + z_2^2 - 1) \\ (z_1^2 - z_2)(z_1 + z_2)(z_1 - 1)z_1 & (z_1^2 + z_2^2 - 1) \end{array}\right) \\
\mathcal{B}_{l,u=0,y=0} \cong \ker_{\mathcal{A}} E_3, \quad E_3 &= \begin{pmatrix} (z_1^2 - z_2)(z_1 + z_2)(z_1 - 1)z_2 \\ (z_1^2 - z_2)(z_1 + z_2)(z_1 - 1)z_1 \end{pmatrix}
\end{aligned}
$$

Next, observe that we can write $E_1 = LE_4$, where L and E_4 are given by:

$$
\begin{aligned}
L &= \begin{pmatrix} z_2 & -1 \\ z_1 & 1 \end{pmatrix}, \\
E_4 &= \left(\begin{array}{c|cc|c} (z_1^2 - z_2)(z_1 + z_2)(z_1 - 1) & (z_1 + z_2) & z_2(z_1^2 - z_2) & 0 \\ 0 & (z_1 + z_2)z_2 & 0 & (z_1^2 + z_2^2 - 1) \end{array}\right)
\end{aligned}
$$

Since E_4 is minor left prime, $\ker_{\mathcal{A}} E_4 = \mathcal{B}^c_{l,u,y}$. Hence we have the following representations:

$$
\begin{aligned}
\mathcal{B}^c_{l,u=0,y} \cong \ker_{\mathcal{A}} E_5, \quad E_5 &= \left(\begin{array}{c|c} (z_1^2 - z_2)(z_1 + z_2)(z_1 - 1) & 0 \\ 0 & (z_1^2 + z_2^2 - 1) \end{array}\right) \\
\mathcal{B}^c_{l,u=0,y=0} \cong \ker_{\mathcal{A}} E_6, \quad E_6 &= \begin{pmatrix} (z_1^2 - z_2)(z_1 + z_2)(z_1 - 1) \\ 0 \end{pmatrix}
\end{aligned}
$$

For the manifest behaviours of $\mathcal{B}_{l,u,y}$ and $\mathcal{B}^c_{l,u,y}$, we need minimal left annihilators of E_3 and E_6, and these are easily found to be the matrices $E_7 = (z_1 \;\; -z_2)$ and $E_8 = (0 \;\; 1)$ respectively. The zero-input manifest behaviours are now found by taking the appropriate sub-matrices of $E_7 E_1$ and $E_8 E_4$ respectively:

$$
\begin{aligned}
\mathcal{B}_{u=0,y} \cong \ker_{\mathcal{A}} E_9, \quad E_9 &= (-(z_1^2 + z_2^2 - 1)(z_1 + z_2)) \\
\mathcal{B}^c_{u=0,y} \cong \ker_{\mathcal{A}} E_{10}, \quad E_{10} &= (z_1^2 + z_2^2 - 1)
\end{aligned}
$$

Since E_4 has full row rank, Theorem 5.7 tells us that a kernel representation of $E_4\mathcal{B}_{l,u,y} \cong \mathcal{B}_{l,u,y}/\mathcal{B}^c_{l,u,y}$ is the matrix L. Similarly, the representation found for $\mathcal{B}^c_{u,y}$

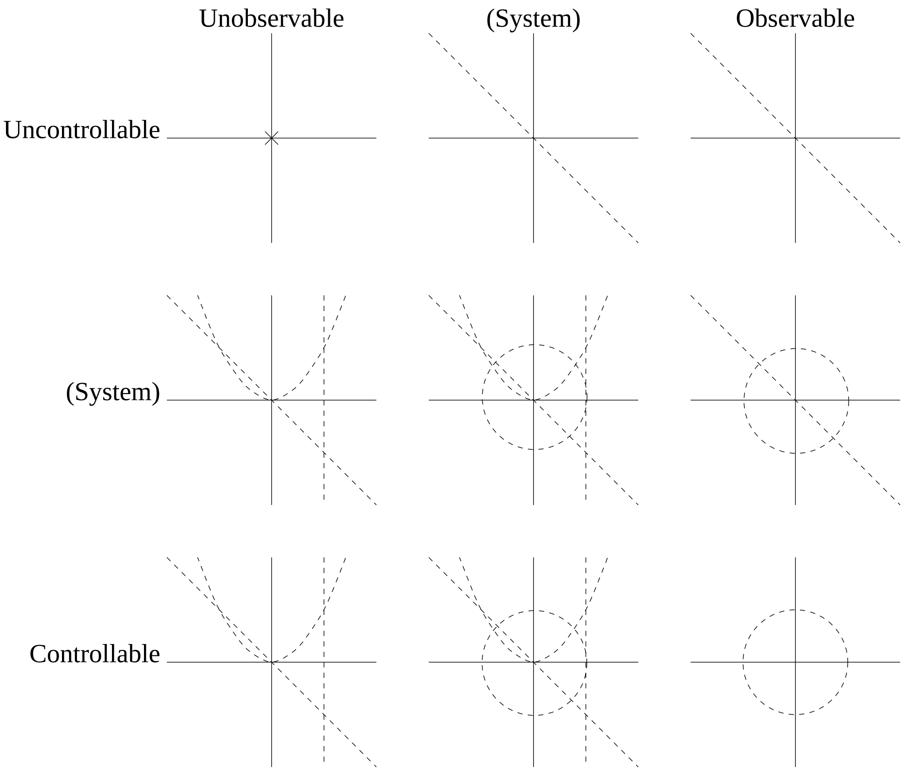

FIG. 6.3. *Example of pole varieties*

also has full row rank, and so a kernel representation for $\mathcal{B}_{u,y}/\mathcal{B}^c_{u,y}$ is the matrix $E_{11} = (-(z_1+z_2))$. Using a similar trick, a kernel representation for $\mathcal{B}_{l,u=0,y=0}/\mathcal{B}^c_{l,u=0,y=0}$ is

$$E_{12} = \begin{pmatrix} z_2 \\ z_1 \end{pmatrix}.$$

Now the 9 sets of pole points, 9 behaviours of relevance and 9 kernel representations are listed as follows:

Pole point set	Behaviour	Rep. matrix
Unobservable uncontrollable pole points	$\mathcal{B}_{l,u=0,y=0}/\mathcal{B}^c_{l,u=0,y=0}$	E_{12}
Uncontrollable pole points	$\mathcal{B}_{l,u,y}/\mathcal{B}^c_{l,u,y}$	L
Observable uncontrollable pole points	$\mathcal{B}_{u,y}/\mathcal{B}^c_{u,y}$	E_{11}
Unobservable pole points	$\mathcal{B}_{l,u=0,y=0}$	E_3
(System) pole points	$\mathcal{B}_{l,u=0,y}$	E_2
Observable pole points	$\mathcal{B}_{u=0,y}$	E_9
Unobservable controllable pole points	$\mathcal{B}^c_{l,u=0,y=0}$	E_6
Controllable pole points	$\mathcal{B}^c_{l,u=0,y}$	E_5
Observable controllable pole points	$\mathcal{B}^c_{u=0,y}$	E_{10}

For this example, it is now easy to plot the various pole varieties as the rank-loss varieties of the corresponding matrices (however in general we suspect that computation of the annihilator is a more efficient method). These varieties are shown in Figure 6.3.

Note that the behaviour $\mathcal{B}_{l,u=0,y=0}/\mathcal{B}^c_{l,u=0,y=0}$ giving the unobservable uncontrollable pole points has a single pole (z_1, z_2). We have conjectured that not every unobservable uncontrollable pole is an uncontrollable pole. We can confirm this using the current example; we have to show that (z_1, z_2) is not an associated prime of $\mathrm{Coker}_{\mathcal{R}}\, L$, where L is the matrix given above. Equivalently, we show that no element of $\mathcal{R}^{1,2} \backslash \mathrm{Im}_{\mathcal{R}}\, L$ is taken into $\mathrm{Im}_{\mathcal{R}}\, L$ by the actions of z_1 and z_2. Let N denote the set of all elements x of $\mathcal{R}^{1,2}$ such that $z_1 x \in \mathrm{Im}_{\mathcal{R}}\, L$ and $z_2 x \in \mathrm{Im}_{\mathcal{R}}\, L$. Then N is the set of polynomial vectors corresponding to the first two components of the kernel (natural left action) of the matrix

$$K = \left(\begin{array}{cc|cccc} z_1 & 0 & z_2 & z_1 & 0 & 0 \\ 0 & z_1 & -1 & 1 & 0 & 0 \\ z_2 & 0 & 0 & 0 & z_2 & z_1 \\ 0 & z_2 & 0 & 0 & -1 & 1 \end{array} \right)^T$$

However, it is possible to show that the kernel of this matrix is equal to the image of the matrix

$$\left(\begin{array}{cc|cccc} z_2 & -1 & -z_1 & 0 & -z_2 & 0 \\ z_1 & 1 & 0 & -z_1 & 0 & -z_2 \end{array} \right)$$

and hence $N = \mathrm{Im}_{\mathcal{R}} L$. This proves that (z_1, z_2) is not an associated prime of $\mathrm{Coker}_{\mathcal{R}} L$, and hence is not an uncontrollable pole of the behaviour, although it is an unobservable uncontrollable pole.

6.3. Rosenbrock system matrices. One important class of representations of 1D and nD systems is the class of Rosenbrock system matrices, also called polynomial matrix descriptions; see e.g. [33] for the 1D case and [19] for the nD case.

We write here x for the latent variables, u for the (manifest) free variables, and y for the remaining manifest variables. We can think of u as the input, x as the state, and y as the output. Since x is to be interpreted as a vector of state variables, it is assumed to contain no free variables, i.e. the behaviour with zero input and zero output is assumed to be autonomous. We consider state-input-output behaviours given by the following classical system of equations:

$$T(\underline{\mathbf{z}})x(\underline{t}) = U(\underline{\mathbf{z}})u(\underline{t}) \tag{6.9}$$

$$y(\underline{t}) = V(\underline{\mathbf{z}})x(\underline{t}) + W(\underline{\mathbf{z}})u(\underline{t}) \tag{6.10}$$

We can write the resulting behaviour in kernel form as follows:

$$\mathcal{B}_{x,u,y} = \ker_{\mathcal{A}} \begin{pmatrix} T & -U & 0 \\ V & W & -I \end{pmatrix} \tag{6.11}$$

The submatrix

$$\begin{pmatrix} T & -U \\ V & W \end{pmatrix}$$

is called a *Rosenbrock system matrix*. We define the *pole points of the Rosenbrock system matrix* to be the points where T fails to have full column rank. This is the definition previously given in the 1D case (e.g. in [20]).

Note that the standard input/ output structure on $\mathcal{B}_{x,u,y}$ is trivially a maximally manifest input/ output structure, with the states being treated as additional outputs in the behavioural sense.

LEMMA 6.5. *Let $\mathcal{B}_{x,u,y}$ be a differential or difference behaviour given by a Rosenbrock system matrix* (6.9–6.10). *Then:*

1. *The pole points of $\mathcal{B}_{x,u,y}$ are precisely the pole points of the Rosenbrock system matrix.*
2. *The unobservable pole points of $\mathcal{B}_{x,u,y}$ are precisely the points where the matrix $(T^T \ V^T)^T$ has less than full column rank.*
3. *Each uncontrollable pole point of $\mathcal{B}_{x,u,y}$ is a point where $(T \ -U)$ has less than its normal rank; the converse holds for 1D systems.*
4. *The observable controllable pole points of $\mathcal{B}_{x,u,y}$ are precisely the points $\underline{a}$ such that the denominator of some entry of the input-to-output transfer matrix of the Rosenbrock system matrix vanishes at $\underline{a}$.*

Proof.

1. A kernel representation of $\mathcal{B}_{x,0,y}$ is

$$\begin{pmatrix} T & 0 \\ V & -I \end{pmatrix}$$

(to within isomorphism). By Corollary 5.2, the pole points are the points where this matrix, or equivalently T, fails to have full column rank.

2. The unobservable pole points are the characteristic points of

$$\mathcal{B}_{x,0,0} \cong \ker_{\mathcal{A}} \begin{pmatrix} T \\ V \end{pmatrix}.$$

By Theorem 4.4, the unobservable pole points are the points where this matrix fails to have full column rank.

3. By Theorem 5.6, in the 1D case the uncontrollable pole points are the points where the whole matrix

$$\begin{pmatrix} T & -U & 0 \\ V & W & -I \end{pmatrix}$$

has less than its normal rank. These are equal to the rank-loss points of $(T \ -U)$.

4. By Theorem 6.4, the observable controllable pole points are the controllable pole points of $\mathcal{B}_{u,y}$, the transfer matrix of which is the input-to-output transfer matrix of the PMD. The claim now follows from Corollary 5.4.

□

Lemma 6.5 is enough to prove that, in the 1D case, the pole points, unobservable pole points, uncontrollable pole points, and observable controllable pole points of $\mathcal{B}_{x,0,y}$ correspond respectively to the poles, output decoupling zeros, input decoupling zeros and transmission poles of the system, as defined classically. Note also that, in the general nD case, we have proved in Theorem 5.6 and Theorem 6.2 that a behaviour is controllable (resp. observable) if and only if it has no uncontrollable pole points (resp. no unobservable pole points).

We can readily apply our theory to characterize the pole points of special classes of nD systems, for example systems governed by the Roesser or Fornasini–Marchesini state space models. In these cases we find that the pole points are the points where the "characteristic polynomial" vanishes, as has been anticipated by many other authors. Such an analysis is however beyond the scope of this paper.

7. Polar Decomposition and Integral Representation. In this section we discuss a decomposition of a behaviour that emerges from the theory of poles, and also the related integral representation theorem.

7.1. Polar and decoupling zero decompositions. In a 1D system, any trajectory with zero input can be written as a sum of polynomial exponential trajectories with frequencies corresponding to the system poles. This results from a decomposition of the behaviour as a sum of certain sub-behaviours, a decomposition which is dual to the classical algebraic "primary decomposition". We can apply the same principles in the nD case.

Consider the ring $\mathcal{R} = k[\underline{\mathbf{z}}]$ or $\mathcal{R} = k[\underline{\mathbf{z}}, \underline{\mathbf{z}}^{-1}]$ (or more generally any Noetherian ring) and a finitely generated $\mathcal{R}$-module L. Given a proper submodule N of L, N has a *primary decomposition* (e.g. [3, 5, 10]) into submodules of L containing N:

$$N = \bigcap_{i=1}^{l} N_i, \tag{7.1}$$

where each quotient L/N_i is J_i-coprimary for some prime J_i, i.e. Ass (L/N_i) consists of the single prime J_i, and the J_i's include all the associated primes of L/N. The intersection can be taken to be such that the J_i's are precisely the associated primes of L/N, each occurring precisely once, in which case it is called a *minimal primary decomposition*. Such a decomposition is in general non-unique.

The dual of (7.1) allows us to express any behaviour $\mathcal{B}$ as the sum of sub-behaviours $\mathcal{B}_i$, where each $\mathcal{B}_i$ has a single coassociated prime, and these primes are precisely the coassociated primes of $\mathcal{B}$. This idea has two interesting applications:

THEOREM 7.1. *Let $\mathcal{B}$ be a differential or difference behaviour. Then the zero-input behaviour has a decomposition:*

$$\mathcal{B}_{0,y} = \sum_{i=1}^{l} \mathcal{B}_i, \tag{7.2}$$

where each behaviour $\mathcal{B}_i$ has only a single coassociated prime J_i, and the J_i's are precisely the poles of $\mathcal{B}$. We call this decomposition a polar decomposition *of $\mathcal{B}_{0,y}$.*

The behaviour $\mathcal{B}$ itself also has a decomposition:

$$\mathcal{B} = \mathcal{B}^c + \sum_{l=1}^{r} \mathcal{B}'_l, \tag{7.3}$$

where each behaviour $\mathcal{B}'_l$ has only a single coassociated prime J'_l, and the J'_l's are precisely the uncontrollable poles of $\mathcal{B}$.

Proof. The first claim follows immediately by application of the dual version of (7.1) to the zero-input behaviour $\mathcal{B}_{0,y}$. For the second part, assume firstly that $\mathcal{B}$ is not autonomous, since otherwise $\mathcal{B}^c = 0$, the uncontrollable poles are the coassociated primes of $\mathcal{B}$, and the result is immediate by the dual of (7.1). Hence by Theorem 3.3 M is not a torsion module and so 0 is an associated prime. Thus $\mathcal{B}^\perp$ has a minimal primary decomposition: $\mathcal{B}^\perp = \bigcap_{l=0}^{r} N_l$, where in particular N_0 is the module such that Ass $(\mathcal{R}^{1,q}/N_0) = \{0\}$. Define $\mathcal{B}'_l = (N_l)^\perp$ for $l = 0, 1, \ldots, r$. This gives us:

$$\mathcal{B} = \sum_{l=0}^{r} \mathcal{B}'_l \tag{7.4}$$

Now by Lemma 5.5, the uncontrollable poles of $\mathcal{B} = D(M)$ are precisely the coassociated primes of the behaviours $\mathcal{B}'_l$ for $l = 1, \ldots, r$. Finally, $\mathcal{R}^{1,q}/N_0$ is 0-primary, or equivalently torsionfree (e.g. on applying [10, Prop. 3.4]), and so by Theorem 3.8 $\mathcal{B}_0'$ is controllable. The sum $\sum_{l=1}^{r} \mathcal{B}_l'$ is autonomous as the sum of autonomous behaviours, and so by the controllable–autonomous decomposition $\mathcal{B}_0'$ is the controllable part of $\mathcal{B}$. This completes the proof. □

Note that the polar decomposition also leads to a decomposition of $\mathcal{B}$ itself, by combination with the controllable–autonomous decomposition. However such a decomposition of $\mathcal{B}$ must be redundant, since by (7.3) it is possible to do this using only the uncontrollable poles. Indeed, the uncontrollable pole decomposition of $\mathcal{B}$ is a refinement of the controllable–autonomous decomposition. The 2D polar decomposition is also similar to a decomposition discussed by Valcher [38], in which any 2D autonomous behaviour can be decomposed into the sum of a finite-dimensional behaviour and a behaviour with a square kernel representation.

The annihilator of a finitely generated module M can be computed using Gröbner basis techniques; an algorithm has been presented in [43]. This is in general the closest we can get to computing the characteristic variety of a behaviour. The decompositions in Theorem 7.1 can be computed via the primary decomposition, algorithms for which have been given in the algebraic literature, for example [1, 11, 17]. The construction of a primary decomposition necessarily includes an identification of the associated primes, i.e. the poles or uncontrollable poles.

7.2. The integral representation theorem. The integral representation theorem was originally formulated by Ehrenpreis in 1961 and then proven in full generality by Malgrange and Palamodov [28, Thm. VI.4.1]. A simplified special case was presented by Björk in [2, Thm. 8.1.3]. Roughly, the integral representation theorem gives an explicit form for any $\mathcal{C}^\infty$ behaviour, in which each trajectory is expressed as a sum of integrals of polynomial exponential functions which also lie in the behaviour. In the following, we write $\langle \underline{a}, \underline{t} \rangle$ for the inner product of two vectors $\underline{a}, \underline{t}$. Using the language of the behavioural approach, Björk's statement of the theorem reads as follows:

THEOREM 7.2. (Integral Representation Theorem)

Let $\mathcal{B}$ be a differential behaviour with signal space $\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{C})$. Then there exist polynomial vectors $p_1, \ldots, p_L$ in $2n$ variables with the following property. For any compact and convex subset $\mathcal{H}$ of $\mathbb{R}^n$ with non-empty interior $\mathcal{K}$, and for any trajectory $w \in \mathcal{B}$, there exists measures $\mu_1, \ldots, \mu_L$ on $\mathbb{C}^n$ for which

$$w(\underline{t}) \;=\; \sum_{i=1}^{L} \int_{\mathbb{C}^n} p_i(\underline{a}, \underline{t}) e^{\langle \underline{a}, \underline{t} \rangle} \; d\mu_i(\underline{a}) \tag{7.5}$$

for all $\underline{t} \in \mathcal{H}$, and for which the following conditions also hold:

1. *Each function $p_i(\underline{a}, \underline{t}) e^{\langle \underline{a}, \underline{t} \rangle}$ is a polynomial exponential trajectory of $\mathcal{B}$ for any $\underline{a}$ in the support of the measure μ_i. In particular the support of μ_i is contained in the characteristic variety of $\mathcal{B}$.*
2. *We have*

$$\int_{\mathbb{C}^n} e^{\mathcal{H}(\underline{a})} (1 + ||\underline{a}||^c) \; d|\mu_i|(\underline{a}) \;<\; \infty$$

for all c, where $\mathcal{H}(\underline{a}) = \sup_{\underline{t} \in \mathcal{H}} (\langle \Re e(\underline{a}), \underline{t} \rangle)$. In particular, the integrals on the right of (7.5) are absolutely convergent and can be differentiated with respect to $\underline{t}$ under the integral sign for $\underline{t} \in \mathcal{K}$.

The integral representation (7.5) can be broken down by first applying the decomposition (7.3) in Theorem 7.1 in order to write $\mathcal{B}$ as a sum of sub-behaviours $\mathcal{B}_i$ each with a single coassociated prime. For each $\mathcal{B}_i$, we then obtain an integral representation, and when summed these give a representation of $\mathcal{B}$.

Assume therefore that $\mathcal{B}$ has a single coassociated prime J, or equivalently that $\mathcal{R}^{1,q}/\mathcal{B}^\perp$ is a J-coprimary. Then according to [28, Cor. IV.4.2] or [2, Thm. 8.4.5], there is a $q \times L$ matrix $A(\underline{\mathbf{z}}, \partial/\partial\underline{\mathbf{z}})$ of differential operators with polynomial coefficients such that

$$\mathcal{B}^\perp \;=\; \{v \in \mathcal{R}^{1,q} \mid A^T(\underline{a}, \partial/\partial\underline{a})v^T(\underline{a}) = 0 \text{ for all } \underline{a} \in V\},$$

where V denotes the characteristic variety of $\mathcal{B}$. Such a matrix A is called a *Noetherian operator* for $\mathcal{B}^\perp \subseteq \mathcal{R}^{1,q}$, and we can show that, for all $\underline{a} \in V$,

$$E(\partial/\partial\underline{t})(A(\underline{a},\underline{t})e^{\langle \underline{a},\underline{t}\rangle}) \;=\; (A^T(\underline{a}, \partial/\partial\underline{a})E^T(\underline{a})e^{\langle \underline{a},\underline{t}\rangle})^T \;=\; 0,$$

i.e. that $A(\underline{a},\underline{t})e^{\langle \underline{a},\underline{t}\rangle}$ is a trajectory of $\mathcal{B}$. The columns of $A(\underline{a},\underline{t})$ then become the polynomial vectors $p_i(\underline{a},\underline{t})$ in the right-hand side of (7.5); see [2, proof of Thm. 8.1.3].

It is easy to construct a Noetherian operator in the special case when we know in addition that $\operatorname{ann} \mathcal{R}^{1,q}/\mathcal{B}^\perp$ is a radical ideal, i.e. when the annihilator of $\mathcal{R}^{1,q}/\mathcal{B}^\perp$ is equal to its single associated prime. In this case, $M = \mathcal{R}^{1,q}/\mathcal{B}^\perp$ is a torsionfree $\mathcal{R}/J$-module. We write $\overline{\mathcal{R}} = \mathcal{R}/J$, and for any matrix Q over $\mathcal{R}$ we use the notation $\overline{Q}$ for the matrix over $\overline{\mathcal{R}}$ obtained by projection of each entry. Now consider a kernel representation $E \in \mathcal{R}^{g,q}$ of $\mathcal{B}$, and construct an $A \in \mathcal{R}^{q,L}$ such that

$$\operatorname{Im}_{\overline{\mathcal{R}}} \overline{E} \;=\; \operatorname{Ker}_{\overline{\mathcal{R}}} \overline{A},$$

which is possible as $\operatorname{Coker}_{\overline{\mathcal{R}}} \overline{E}$ is torsionfree over $\overline{\mathcal{R}}$. We now have

$$\begin{aligned}\mathcal{B}^\perp &= \operatorname{Im}_{\mathcal{R}} E \\ &= \{v \in \mathcal{R}^{1,q} \mid \overline{vA} = \overline{v}\overline{A} = 0 \in \overline{\mathcal{R}}^{1,L}\} \\ &= \{v \in \mathcal{R}^{1,q} \mid A^T(\underline{a})v^T(\underline{a}) = 0 \text{ for all } \underline{a} \in V(\operatorname{ann}\mathcal{B})\}\end{aligned}$$

This proves that $A(\underline{z}, \partial/\partial\underline{z}) = A(\underline{z})$ is a Noetherian operator for M. The polynomial vectors in the integral representation are then extracted as the columns of A.

The preceding argument can be applied in the case where $\mathcal{B}$ is controllable (and with no other conditions). For it then holds that $\mathcal{B}$ has the single coassociated prime $\{0\}$, which is equal to $\operatorname{ann}\mathcal{B}$. The ring $\overline{\mathcal{R}}$ is simply equal to $\mathcal{R}$, and so the Noetherian operator A is just any matrix such that $\operatorname{Im}_{\mathcal{R}} E = \operatorname{Ker}_{\mathcal{R}} A$ or equivalently $\ker_{\mathcal{A}} E = \operatorname{im}_{\mathcal{A}} A$, i.e. A is any image representation of $\mathcal{B}$. Thus in the controllable case we obtain:

COROLLARY 7.3.
Let $\mathcal{B}$ be a controllable differential behaviour with signal space $\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{C})$ and an image representation F. Let $F_1, \dots, F_L$ denote the columns of F. Then for any $\underline{a} \in \mathbb{C}^n$, $F_i(\underline{a})e^{\langle \underline{a},\underline{t}\rangle}$ is an exponential trajectory of $\mathcal{B}$, and every trajectory w of $\mathcal{B}$ can be written

$$w(\underline{t}) \;=\; \sum_{i=1}^{L} \int_{\mathbb{C}^n} F_i(\underline{a})e^{\langle \underline{a},\underline{t}\rangle} \; d\mu_i(\underline{a}), \tag{7.6}$$

for suitable measures $\mu_1, \dots, \mu_L$ as in Theorem 7.2.

COROLLARY 7.4.
Let $u \in \mathcal{A}^m$, $y \in \mathcal{A}^p$ satisfy

$$P(\underline{\mathbf{z}})y \ = \ u,$$

where $\mathcal{A} = \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{C})$ and P is a square non-singular matrix. Then there are measures $\mu_1, \ldots, \mu_p$ on $\mathbb{C}^n$ such that

$$y_j(\underline{t}) = \int_{\mathbb{C}^n} e^{\langle \underline{a}, \underline{t} \rangle} \, d\mu_j(\underline{a}), \qquad j = 1, \ldots, p$$

$$\text{and} \qquad u(\underline{t}) = \sum_{i=1}^{p} \int_{\mathbb{C}^n} P_i(\underline{a}) e^{\langle \underline{a}, \underline{t} \rangle} \, d\mu_i(\underline{a})$$

as in Theorem 7.2, where P_i denotes the ith column of P.

Proof. This result is immediate from Corollary 7.3, since $(U^T \ I_p)^T$ is an image representation of $\ker_{\mathcal{A}} (-I_p \ U)$, the given controllable behaviour. $\square$

In the case of a finite-dimensional behaviour $\mathcal{B}$, the characteristic variety is finite (see Lemma 5.9), and each integral in the summation of (7.5) can be taken as an evaluation of the integrand at a given characteristic point, multiplied by a suitable constant. In this situation we can recover the result that every trajectory is a polynomial exponential trajectory [26], which generalizes the well-known decomposition of trajectories of 1D systems. Results for the case of a zero-dimensional characteristic variety are also given in [2, p365]. The general construction of Noetherian operators is addressed in [27].

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