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Optimizing Stability Bounds of Finite-Precision PID Controller Structures

S. Chen, J. Wu, R. H. Istepanian, and J. Chu

***Abstract*— This paper investigates a recently derived lower bound stability measure for sampled-data controller structures subject to finite-wordlength (FWL) constraints. The optimal realization of the digital PID controller with FWL considerations is formulated as a nonlinear optimization problem, and an efficient strategy based on adaptive simulated annealing (ASA) is adopted to solve this complex optimization problem. A numerical example of optimizing the finite-precision PID controller structure for a simulated steel rolling mill system is presented to illustrate the effectiveness of the proposed strategy.**

***Index Terms*—Finite wordlength, optimization, sampled data system, stability.**

I. INTRODUCTION

Controller implementations with fixed-point arithmetic offer the advantages of speed, memory space, cost, and simplicity over floating-point arithmetic. However, a designed stable closed-loop system may become unstable when the infinite-precision controller is implemented using a fixed-point processor due to finite-wordlength (FWL) effects. In recent years, many studies have addressed FWL issues [1]–[6]. It remains an unsolved problem to compute the exact stability robustness measure with FWL considerations for sampled-data control systems [1]. To overcome this problem, a tractable lower bound stability measure has been developed [6]. Recently, a new lower bound stability measure [4], [5] has been derived, which provides a better estimate of stability robustness than the one given in [6].

In this paper, we discuss the generic optimal realization problem of the digital PID controller with FWL considerations based on this new closed-loop lower bound stability measure. We prove that this problem can be solved as an unconstrained nonlinear optimization problem. The optimization criteria in this case are, however, nonsmooth and nonconvex functions, and conventional optimization methods may fail to obtain an optimum solution. To overcome this difficulty, an efficient global optimization method, known as the ASA [7]–[10], is employed to search for the true optimal PID controller realization. The effectiveness of the proposed optimization strategy is illustrated by the numerical example of a simulated steel rolling mill control problem.

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II. STABILITY ROBUSTNESS MEASURES WITH FWL CONSIDERATIONS

Consider the sampled-data system depicted in Fig. 1, where $P(s)$ is strictly proper. The plant $P(z) = S_h P(s) H_h$ has a realization $(A_z \in \mathcal{R}^{m\times m}, B_z \in \mathcal{R}^{m\times l}, C_z \in \mathcal{R}^{q\times m}, 0)$. The controller $C(z)$ has a realization $(A_c \in \mathcal{R}^{n\times n}, B_c \in \mathcal{R}^{n\times q}, C_c \in \mathcal{R}^{l\times n}, D_c \in \mathcal{R}^{l\times q})$. The realizations of $C(z)$ are not unique. If (A_c, B_c, C_c, D_c) is a realization of $C(z)$, so is $(\mathcal{T}^{-1}A_c\mathcal{T}, \mathcal{T}^{-1}B_c, C_c\mathcal{T}, D_c)$ for any similarity transformation $\mathcal{T} \in \mathcal{R}^{n\times n}$. The corresponding realization $(\overline{A}, \overline{B}, \overline{C}, \overline{D})$ of the closed-loop system is given by

$$\begin{aligned}\overline{A} &= \begin{bmatrix} A_z + B_z D_c C_z & B_z C_c \\ B_c C_z & A_c \end{bmatrix} \\ &= \begin{bmatrix} A_z & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} B_z & 0 \\ 0 & I_n \end{bmatrix} \begin{bmatrix} D_c & C_c \\ B_c & A_c \end{bmatrix} \begin{bmatrix} C_z & 0 \\ 0 & I_n \end{bmatrix} \\ &= M_0 + M_1 X M_2 = \overline{A}(X) \\ \overline{B} &= \begin{bmatrix} B_z \\ 0 \end{bmatrix} \\ \overline{C} &= [C_z \quad 0] \\ \overline{D} &= 0 \end{aligned} \tag{1}$$

where all 0's are zero matrices of proper dimensions, I_n is the $n \times n$ identity matrix, and

$$X = \begin{bmatrix} D_c & C_c \\ B_c & A_c \end{bmatrix} = \begin{bmatrix} p_1 & p_2 & \cdots & p_{q+n} \\ p_{q+n+1} & p_{q+n+2} & \cdots & p_{2(q+n)} \\ \vdots & \vdots & \cdots & \vdots \\ p_{(l+n-1)(q+n)+1} & p_{(l+n-1)(q+n)+2} & \cdots & p_{(l+n)(q+n)} \end{bmatrix} \tag{2}$$

is the controller matrix. Let $C(z)$ be chosen to make the feedback system stable. Then all of the eigenvalues $\{\lambda_i, 1 \le i \le m+n\}$ of $\overline{A}(X)$ are in the interior of the unit circle.

When the realization (A_c, B_c, C_c, D_c) is implemented with a digital control processor, X is perturbed into $X + \Delta X$ due to the

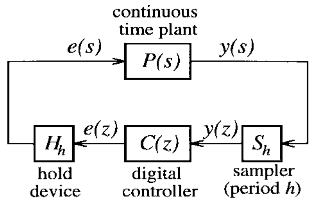


Fig. 1. Sampled-data system with digital controller realization.

FWL effects, shown in (3) at the bottom of the page. Each element of ΔX is bounded by $\epsilon/2$, that is,

$$\mu(\Delta X) \triangleq \max_{1\le i\le N} |\Delta p_i| \le \frac{\epsilon}{2} \tag{4}$$

where $N = (l+n)(q+n)$. For a fixed-point processor of B_s bits, $\epsilon = 2^{-(B_s - B_X)}$, and 2^{B_X} is a normalization factor. With the perturbation ΔX, λ_i may be moved to $\tilde{\lambda}_i$. The sampled-data system is unstable if and only if there exists $|\tilde{\lambda}_i| \ge 1$. Define

$$\mu_0(X) \triangleq \inf\{\mu(\Delta X): \overline{A}(X) + M_1 \Delta X M_2 \text{ is unstable}\}. \tag{5}$$

It describes the stability robustness of the realization X to the FWL effects [1]. However, explicitly computing the value of $\mu_0(X)$ is still an unsolved open problem.

To overcome the difficulty in the computation of $\mu_0(X)$, a lower bound of $\mu_0(X)$ has recently been derived [4]. Define

$$\mu_1(X) \triangleq \min_{1\le i\le m+n} \frac{1-|\lambda_i|}{\sum_{j=1}^{N} \left|\frac{\partial \lambda_i}{\partial p_j}\bigg|_X\right|}. \tag{6}$$

We have the following theorem from [4].

Theorem 1: $\overline{A}(X) + M_1 \Delta X M_2$ is stable when $\mu(\Delta X) < \mu_1(X)$.

Comparing (6) with (5), it is easily seen that $\mu_1(X)$ is a lower bound of $\mu_0(X)$. The following lemma from [4] shows that $\mu_1(X)$ can be computed easily.

Lemma 1: Let $\{\lambda_i, 1 \le i \le m+n\}$ be the eigenvalues of $\overline{A}(X)$, and let $\boldsymbol{x}_i$ and $\boldsymbol{y}_i$ be the right eigenvector and reciprocal left eigenvector corresponding to λ_i, respectively. Then, see (7) at the bottom of this page, where T denotes the transpose operation and * is the conjugate operation.

Let $B_s^{\min}$ be the smallest word length that can guarantee the closed-loop stability. When X is implemented with a digital control processor of B_s bits, it is easily seen that the closed-loop sampled-data system is stable if

$$\mu_1(X) > \frac{2^{-(B_s - B_X)}}{2}. \tag{8}$$

Define $\hat{B}_{s1}^{\min}$ as the smallest integer that is not less than $-\log_2 \mu_1(X) - 1 + B_X$. $\hat{B}_{s1}^{\min}$ can be used as an estimate of $B_s^{\min}$. Another tractable stability robustness measure with FWL considerations given in [6] is defined as

$$\mu_2(X) \triangleq \min_{1\le i\le m+n} \frac{1-|\lambda_i|}{\sqrt{N \sum_{j=1}^{N} \left|\frac{\partial \lambda_i}{\partial p_j}\bigg|_X\right|^2}}. \tag{9}$$

It is also a lower bound of $\mu_0(X)$. Similarly, an estimate $\hat{B}_{s2}^{\min}$ of $B_s^{\min}$ can be computed based on $\mu_2(X)$. Since

$$\left(\sum_{j=1}^{N} \left|\frac{\partial \lambda_i}{\partial p_j}\bigg|_X\right|\right)^2 \le N \sum_{j=1}^{N} \left|\frac{\partial \lambda_i}{\partial p_j}\bigg|_X\right|^2 \tag{10}$$

we have $\mu_2(X) \le \mu_1(X) \le \mu_0(X)$. It is clear that $\mu_1(X)$, which is closer to $\mu_0(X)$, is a better stability robustness measure, and can provide a better estimate of $B_s^{\min}$.

III. Optimal Realization of PID Controller Structures with FWL Considerations

Since there are different realizations for a given $C(z)$ and the stability robustness measure $\mu_1(X)$ is a function of the realization, it is of practical importance to find a realization such that $\mu_1(X)$ is maximized. Such a realization is optimal in the sense that it has maximum stability robustness to FWL effects. The digital controller implemented with an optimal realization means that the stability of the closed-loop system is guaranteed with a minimum hardware requirement in terms of word length. In this section, we specifically discuss the optimal realization problem of digital PID controllers.

The digital PID controller $C(z)$ is an order $n = 2$ system. We will assume that $C(z)$ is "single-input single-output," that is, $l = q = 1$. Let $(A_c^0 \in \mathcal{R}^{2\times 2}, B_c^0 \in \mathcal{R}^{2\times 1}, C_c^0 \in \mathcal{R}^{1\times 2}, D_c^0 \in \mathcal{R})$ be a controllable canonical realization of $C(z)$, and let $(A_z \in \mathcal{R}^{m\times m}, B_z \in \mathcal{R}^{m\times 1}, C_z \in \mathcal{R}^{1\times m})$ be a realization of the plant $P(z)$. Then the initial control matrix is

$$X_0 = \begin{bmatrix} D_c^0 & C_c^0 \\ B_c^0 & A_c^0 \end{bmatrix} \in \mathcal{R}^{3\times 3}. \tag{11}$$

Any realization of $C(z)$ can be represented as $(\mathcal{T}^{-1}A_c^0\mathcal{T}, \mathcal{T}^{-1}B_c^0, C_c^0\mathcal{T}, D_c^0)$ or

$$X_{\mathcal{T}} \triangleq \begin{bmatrix} 1 & 0 \\ 0 & \mathcal{T}^{-1} \end{bmatrix} X_0 \begin{bmatrix} 1 & 0 \\ 0 & \mathcal{T} \end{bmatrix} \tag{12}$$

$$\Delta X = \begin{bmatrix} \Delta p_1 & \Delta p_2 & \cdots & \Delta p_{q+n} \\ \Delta p_{q+n+1} & \Delta p_{q+n+2} & \cdots & \Delta p_{2(q+n)} \\ \vdots & \vdots & \cdots & \vdots \\ \Delta p_{(l+n-1)(q+n)+1} & \Delta p_{(l+n-1)(q+n)+2} & \cdots & \Delta p_{(l+n)(q+n)} \end{bmatrix} \tag{3}$$

$$\frac{\partial \lambda_i}{\partial X} = \begin{bmatrix} \frac{\partial \lambda_i}{\partial p_1} & \frac{\partial \lambda_i}{\partial p_2} & \cdots & \frac{\partial \lambda_i}{\partial p_{q+n}} \\ \frac{\partial \lambda_i}{\partial p_{q+n+1}} & \frac{\partial \lambda_i}{\partial p_{q+n+2}} & \cdots & \frac{\partial \lambda_i}{\partial p_{2(q+n)}} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial \lambda_i}{\partial p_{(l+n-1)(q+n)+1}} & \frac{\partial \lambda_i}{\partial p_{(l+n-1)(q+n)+2}} & \cdots & \frac{\partial \lambda_i}{\partial p_{(l+n)(q+n)}} \end{bmatrix} = M_1^T \boldsymbol{y}_i^* \boldsymbol{x}_i^T M_2^T \tag{7}$$

where $\mathcal{T} \in \mathcal{R}^{2\times 2}$ and $\det(\mathcal{T}) \neq 0$. The transition matrix of the closed-loop system is

$$\begin{aligned}\overline{A}(X_\mathcal{T}) &= \begin{bmatrix} A_z & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} B_z & 0 \\ 0 & I_n \end{bmatrix} X_\mathcal{T} \begin{bmatrix} C_z & 0 \\ 0 & I_n \end{bmatrix} \\ &= \begin{bmatrix} I_m & 0 \\ 0 & \mathcal{T}^{-1} \end{bmatrix} \overline{A}(X_0) \begin{bmatrix} I_m & 0 \\ 0 & \mathcal{T} \end{bmatrix}. \end{aligned} \tag{13}$$

Let λ_i^0 be the ith eigenvalue of $\overline{A}(X_0)$. From (13), applying Lemma 1, we have

$$\left.\frac{\partial \lambda_i}{\partial X}\right|_{X=X_\mathcal{T}} = \begin{bmatrix} 1 & 0 \\ 0 & \mathcal{T}^T \end{bmatrix} \left.\frac{\partial \lambda_i}{\partial X}\right|_{X=X_0} \begin{bmatrix} 1 & 0 \\ 0 & \mathcal{T}^{-T} \end{bmatrix}. \tag{14}$$

From (6), (7), and (14), we can define the optimal realization problem of digital PID controllers as the following optimization problem:

$$\nu \triangleq \max_{X_\mathcal{T}} \mu_1(X_\mathcal{T}) = \max_{X_\mathcal{T}} \min_{1\le i\le m+2} \frac{1 - |\lambda_i^0|}{\sum_{j=1}^{N} \left| \left.\frac{\partial \lambda_i}{\partial p_j}\right|_{X=X_\mathcal{T}} \right|}. \tag{15}$$

For the complex-valued matrix $M \in \mathcal{C}^{m\times n}$ with elements $M_{i,j}$, define

$$\|M\|_s \triangleq \sum_{i=1}^{m} \sum_{j=1}^{n} |M_{i,j}|. \tag{16}$$

The optimization problem (15) is equivalent to

$$\begin{aligned} \nu &\triangleq \min_{X_\mathcal{T}} \max_{1\le i\le m+2} \frac{\left\| \left.\frac{\partial \lambda_i}{\partial X}\right|_{X=X_\mathcal{T}} \right\|_s}{1 - |\lambda_i^0|} \\ &= \min_{\substack{\mathcal{T}\in\mathcal{R}^{2\times 2} \\ \det(\mathcal{T})\neq 0}} \max_{1\le i\le m+2} \left\| \begin{bmatrix} 1 & 0 \\ 0 & \mathcal{T}^T \end{bmatrix} \Phi_i \begin{bmatrix} 1 & 0 \\ 0 & \mathcal{T}^{-T} \end{bmatrix} \right\|_s \end{aligned} \tag{17}$$

where

$$\Phi_i = \frac{\left.\frac{\partial \lambda_i}{\partial X}\right|_{X=X_0}}{1 - |\lambda_i^0|} \tag{18}$$

are eigenvalue sensitivity matrices. It is difficult to handle the constraint $\det(\mathcal{T}) \neq 0$ directly in numerical optimization. The following theorem shows that the optimization problem (17) can be solved by solving the two "simpler" problems. First, we define

$$\begin{aligned} &f_1(x, y, w) \\ &\quad = \max_{1\le i\le m+2} \left\| \begin{bmatrix} w & 0 & 0 \\ 0 & x & 0 \\ 0 & y & 1/x \end{bmatrix} \Phi_i \begin{bmatrix} 1/w & 0 & 0 \\ 0 & 1/x & 0 \\ 0 & -y & x \end{bmatrix} \right\|_s \end{aligned} \tag{19}$$

and

$$\begin{aligned} f_2(x, y, u, w) = \max_{1\le i\le m+2} &\left\| \begin{bmatrix} w & 0 & 0 \\ 0 & x & u \\ 0 & (xy-1)/u & y \end{bmatrix} \right. \\ &\left. \cdot \Phi_i \begin{bmatrix} 1/w & 0 & 0 \\ 0 & y & -u \\ 0 & (1-xy)/u & x \end{bmatrix} \right\|_s. \end{aligned} \tag{20}$$

Theorem 2: Let

$$\nu_1 = \min_{\substack{x\in(0,+\infty) \\ y\in(-\infty,+\infty) \\ w\in(0,+\infty)}} f_1(x, y, w) \tag{21}$$

and

$$\nu_2 = \min_{\substack{x\in(-\infty,+\infty) \\ y\in(-\infty,+\infty) \\ u\in(0,+\infty) \\ w\in(0,+\infty)}} f_2(x, y, u, w). \tag{22}$$

Then

$$\nu = \min\{\nu_1, \nu_2\}. \tag{23}$$

If $\nu = \nu_1$ and $(x_{\text{opt}1}, y_{\text{opt}1}, w_{\text{opt}1})$ is the optimal solution of (21), the optimal solution of (17) is given as

$$\mathcal{T}_{\text{opt}} = \frac{1}{w_{\text{opt}1}} \begin{bmatrix} x_{\text{opt}1} & y_{\text{opt}1} \\ 0 & 1/x_{\text{opt}1} \end{bmatrix}. \tag{24}$$

If $\nu = \nu_2$ and $(x_{\text{opt}2}, y_{\text{opt}2}, u_{\text{opt}2}, w_{\text{opt}2})$ is the optimal solution of (22), the optimal solution of (17) is given as

$$\mathcal{T}_{\text{opt}} = \frac{1}{w_{\text{opt}2}} \begin{bmatrix} x_{\text{opt}2} & (x_{\text{opt}2} y_{\text{opt}2} - 1)/u_{\text{opt}2} \\ u_{\text{opt}2} & y_{\text{opt}2} \end{bmatrix}. \tag{25}$$

The proof of Theorem 2 is given in the the Appendix. Because $f_1(x, y, w)$ and $f_2(x, y, u, w)$ are nonsmooth and nonconvex functions, it is very difficult for a conventional optimization method to obtain a global minimum solution. To overcome this difficulty, we adopt an efficient global optimization strategy based on the ASA [7]–[10]. Space limitation precludes a detailed description of the ASA algorithm here.

IV. Application Example

We consider the implementation of a finite-precision PID controller for a steel rolling mill system. The continuous-time plant model $P(s)$ was developed in [11]. The entire digital PID control system is simulated. Discretizing $P(s)$ with the sampling period $h = 0.001$ yields $P(z)$:

$$\begin{aligned} A_z &= \begin{bmatrix} 0.9951 & -9.7260 & 0.0049 \\ 0.0010 & 0.9884 & -0.0010 \\ 0.0067 & 13.3732 & 0.9933 \end{bmatrix} \\ B_z &= \begin{bmatrix} 0.2486 \\ 0.0001 \\ 0.0006 \end{bmatrix} \\ C_z &= [1 \quad 0 \quad 0] \\ D_z &= [0]. \end{aligned} \tag{26}$$

A stabilized PID controller for vibration suppression and disturbance rejection is designed:

$$\frac{0.00269s}{0.001s + 1} - 0.435 - \frac{14.26}{s}. \tag{27}$$

Inserting the bilinear transformation into (27) gives rise to the digital PID controller:

$$C(z) = -\frac{0.01426}{z - 1} - \frac{1.1956}{z - 0.3333} + 1.3512. \tag{28}$$

The initial realization of $C(z)$ is set to the controllable canonical realization:

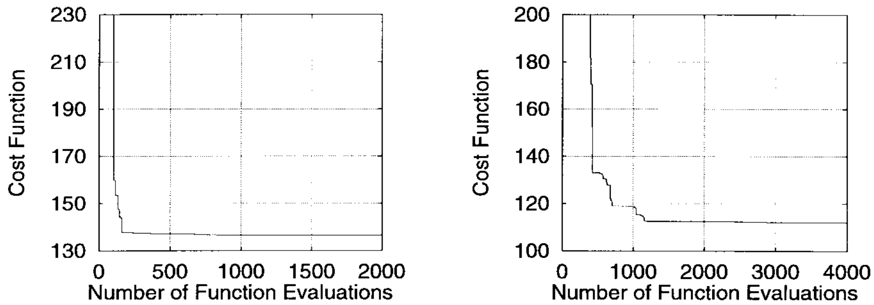
$$\begin{aligned} A_c^0 &= \begin{bmatrix} 1 & 0 \\ 0 & 0.3333 \end{bmatrix} \\ B_c^0 &= \begin{bmatrix} -1 \\ -1 \end{bmatrix} \\ C_c^0 &= [0.01426 \quad 1.1956] \\ D_c^0 &= [1.3512]. \end{aligned} \tag{29}$$

Notice that the data given above are shown to only four significant digits in fractional part.

From $\overline{A}(X_0)$, the poles of the ideal closed-loop system are computed and given as

$$\begin{bmatrix} \lambda_{1,2} \\ \lambda_{3,4} \\ \lambda_5 \end{bmatrix} = \begin{bmatrix} 0.9089 \pm 0.2371i \\ 0.9431 \pm 0.0725i \\ 0.9422 \end{bmatrix} \tag{30}$$

where $i = \sqrt{-1}$. The corresponding eigenvalue sensitivity matrices are shown in (31) at the bottom of the next page.



(a) (b)

Fig. 2. Typical convergence performance of the ASA in optimizing: (a) cost function $f_1(x, y, w)$ with initial $(x, y, w) = (1.0, 0.0, 1.0)$ and (b) cost function $f_2(x, y, u, w)$ with initial $(x, y, u, w) = (1.0, 1.0, 1.0, 1.0)$.

For the optimization problem (21), the ASA algorithm always converged to the solution

$$(x_{\rm opt1}, y_{\rm opt1}, w_{\rm opt1}) = (2.3704, 3.3598, 0.2004) \quad \text{with } \nu_1 = 136.5897. \tag{32}$$

The realization corresponding to $(x_{\rm opt1}, y_{\rm opt1}, w_{\rm opt1})$ is

$$X_{\rm opt1} = \begin{bmatrix} 1.3512 & 0.1687 & 2.7560 \\ 0.5888 & 1 & 0.9450 \\ -0.4750 & 0 & 0.3333 \end{bmatrix}. \tag{33}$$

The cost function $f_1(x, y, w)$ in a typical run is shown in Fig. 2(a). It is worth pointing out that, in the previous study [5], a conventional optimization method, the Rosenbrock algorithm, failed to achieve this global optimum. For the optimization problem (22), two solutions were found by the ASA, and they are

$$\left(x_{\rm opt2}^{(1)}, y_{\rm opt2}^{(1)}, u_{\rm opt2}^{(1)}, w_{\rm opt2}^{(1)}\right) = (2.7967, 0.1540, 0.3512, 0.2565) \quad \text{with } \nu_2 = 111.9901 \tag{34}$$

$$\left(x_{\rm opt2}^{(2)}, y_{\rm opt2}^{(2)}, u_{\rm opt2}^{(2)}, w_{\rm opt2}^{(2)}\right) = (-3.0481, -0.1868, 0.2895, 0.4824) \quad \text{with } \nu_2 = 111.9899. \tag{35}$$

The corresponding realizations are

$$X_{\rm opt2}^{(1)} = \begin{bmatrix} 1.3512 & 1.7925 & 0.6277 \\ -0.4553 & 0.6204 & -0.1664 \\ -0.6273 & -0.6548 & 0.7129 \end{bmatrix} \tag{36}$$

$$X_{\rm opt2}^{(2)} = \begin{bmatrix} 1.3512 & 0.6274 & -0.5069 \\ -0.6274 & 0.7129 & 0.1852 \\ 1.6101 & 0.5883 & 0.6204 \end{bmatrix}. \tag{37}$$

The cost function $f_2(x, y, u, w)$ in a typical run is shown in Fig. 2(b). Since $\nu = \min\{\nu_1, \nu_2\} = \nu_2$, the optimal PID controller realization is either $X_{\rm opt2}^{(1)}$ or $X_{\rm opt2}^{(2)}$.

Table I summarizes the stability lower bound measures, estimated, and true minimal bit lengths that can ensure closed-loop stability for different controller realizations. The results given in Table I show that $\mu_1(X)$ is a better measure of stability robustness, as it provides a larger stability bound and a better estimate of $B_s^{\min}$ compared with $\mu_2(X)$. The ASA optimization strategy is very effective, and the optimization process converges fast, as confirmed in Fig. 2. From Table I, two realizations $X_{\rm opt2}^{(1)}$ and $X_{\rm opt2}^{(2)}$ have the same stability lower bound measure and the same estimate of minimum word length. The largest absolute parameter value is 1.6101 for $X_{\rm opt2}^{(2)}$ and 1.7925 for $X_{\rm opt2}^{(1)}$. For practical implementation, therefore, $X_{\rm opt2}^{(2)}$ is preferred.

TABLE I
LOWER STABILITY BOUNDS, ESTIMATED MINIMAL BIT LENGTHS, AND TRUE MINIMAL BIT LENGTHS FOR DIFFERENT CONTROLLER REALIZATIONS

Realization	μ_1	$\hat{B}_{s1}^{\min}$	μ_2	$\hat{B}_{s2}^{\min}$	$B_s^{\min}$
X_0	0.001900	10	0.001100	10	7
$X_{\rm opt1}$	0.007321	9	0.004706	9	4
$X_{\rm opt2}^{(1)}$	0.008929	7	0.004896	8	4
$X_{\rm opt2}^{(2)}$	0.008929	7	0.004896	8	4

V. CONCLUSIONS

Based on a new lower bound measuring stability robustness of sampled-data systems with FWL considerations, the optimal realization of an FWL PID controller can be interpreted as a nonlinear optimization problem. An efficient global optimization strategy based on the ASA has been adopted to solve this FWL optimal realization problem. The theoretical results have been verified using the numerical example of a simulated steel rolling mill system. This method can be extended to other finite-precision controller realizations. In this work, the main emphasis has been focused on the stability issues of digital controller structures subject to FWL constraints. Ongoing work will explore the integration of the proposed ASA optimization procedure with the closed-loop performance and sparseness consideration of controller realizations.

APPENDIX

Define the $n \times n$ diagonal matrix set: $\xi_n \triangleq \{U = \text{diag}(u_1, u_2, \cdots, u_n)$: $u_i \in \{-1, 1\}, 1 \le i \le n\}$. From the definition (16), we have the following.

$$\Phi_{1,2} = \begin{bmatrix} 5.3222 \pm 2.4117i & 16.3783 \mp 16.1509i & -6.4294 \mp 6.8389i \\ -0.2336 \pm 0.2303i & 0.5165 \pm 1.1835i & 0.4878 \mp 0.1992i \\ 7.6870 \pm 8.1766i & 40.9014 \mp 16.6993i & -6.4145 \mp 16.8486i \end{bmatrix}$$

$$\Phi_{3,4} = \begin{bmatrix} 0.6130 \pm 6.0505i & 55.7394 \pm 35.2729i & 0.1727 \mp 9.9017i \\ -0.7948 \mp 0.5030i & -9.6135 \pm 3.4161i & 1.1885 \pm 0.9662i \\ -0.2065 \pm 11.8384i & 99.6482 \pm 81.0121i & 2.6112 \mp 19.1031i \end{bmatrix}$$

$$\Phi_5 = \begin{bmatrix} -8.0215 & -138.6951 & 13.1745 \\ 1.9778 & 34.1969 & -3.2483 \\ -15.7514 & -272.3494 & 25.8702 \end{bmatrix} \tag{31}$$

Lemma 2: $\forall M \in \mathcal{C}^{m\times n}$, $U_1 \in \xi_m$, and $U_2 \in \xi_n$,

$$\|U_1 M\|_s = \|M\|_s \quad \text{and} \quad \|MU_2\|_s = \|M\|_s.$$

Proof of Theorem 2: Define the sets

$$\begin{aligned}
\zeta_0 &\triangleq \left\{\mathcal{T} = \begin{bmatrix} t_1 & t_2 \\ t_3 & t_4 \end{bmatrix}: t_1 \in \mathcal{R},\, t_2 \in \mathcal{R},\, t_3 \in \mathcal{R},\right.\\
&\qquad \left. t_4 \in \mathcal{R},\, t_1 t_4 - t_2 t_3 \neq 0\right\}\\
\zeta_1 &\triangleq \left\{\mathcal{T} = \begin{bmatrix} t_1 & t_2 \\ 0 & t_4 \end{bmatrix}: t_1 \in \mathcal{R},\, t_2 \in \mathcal{R},\, t_4 \in \mathcal{R},\right.\\
&\qquad \left. t_1 t_4 \neq 0\right\}\\
\zeta_2 &\triangleq \left\{\mathcal{T} = \begin{bmatrix} t_1 & t_2 \\ t_3 & t_4 \end{bmatrix}: t_1 \in \mathcal{R},\, t_2 \in \mathcal{R},\, t_3 \in \mathcal{R},\right.\\
&\qquad \left. t_4 \in \mathcal{R},\, t_3 \neq 0,\, t_1 t_4 - t_2 t_3 \neq 0\right\}.
\end{aligned} \tag{38}$$

Construct the optimization problems

$$\nu_1 \triangleq \min_{\mathcal{T}\in\zeta_1} \max_{1\le i\le m+2} \left\| \begin{bmatrix} 1 & 0 \\ 0 & \mathcal{T}^T \end{bmatrix} \Phi_i \begin{bmatrix} 1 & 0 \\ 0 & \mathcal{T}^{-T} \end{bmatrix} \right\|_s \tag{39}$$

and

$$\nu_2 \triangleq \min_{\mathcal{T}\in\zeta_2} \max_{1\le i\le m+2} \left\| \begin{bmatrix} 1 & 0 \\ 0 & \mathcal{T}^T \end{bmatrix} \Phi_i \begin{bmatrix} 1 & 0 \\ 0 & \mathcal{T}^{-T} \end{bmatrix} \right\|_s. \tag{40}$$

Obviously, $\zeta_0 = \zeta_1 \cup \zeta_2$, and therefore, $\nu = \min\{\nu_1, \nu_2\}$. Define the function sgn$(\cdot)$: sgn$(x) = 1$ for $x \ge 0$ and sgn$(x) = -1$ for $x < 0$. Consider the optimization problem (39). Utilizing Lemma 2, $\forall \mathcal{T} \in \zeta_1$ and $\forall i \in \{1, \cdots, m+2\}$ we have

$$\begin{aligned}
&\left\| \begin{bmatrix} 1 & 0 \\ 0 & \mathcal{T}^T \end{bmatrix} \Phi_i \begin{bmatrix} 1 & 0 \\ 0 & \mathcal{T}^{-T} \end{bmatrix} \right\|_s\\
&= \left\| \begin{bmatrix} 1 & 0 & 0 \\ 0 & t_1 & 0 \\ 0 & t_2 & t_4 \end{bmatrix} \Phi_i \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/t_1 & 0 \\ 0 & -t_2/(t_1 t_4) & 1/t_4 \end{bmatrix} \right\|_s\\
&= \left\| \begin{bmatrix} 1/\sqrt{|t_1 t_4|} & 0 & 0 \\ 0 & \sqrt{|t_1/t_4|} & 0 \\ 0 & \text{sgn}(t_4) t_2/\sqrt{|t_1 t_4|} & \sqrt{|t_4/t_1|} \end{bmatrix} \Phi_i \right.\\
&\quad \left. \cdot \begin{bmatrix} \sqrt{|t_1 t_4|} & 0 & 0 \\ 0 & \sqrt{|t_4/t_1|} & 0 \\ 0 & -\text{sgn}(t_4) t_2/\sqrt{|t_1 t_4|} & \sqrt{|t_1/t_4|} \end{bmatrix} \right\|_s.
\end{aligned} \tag{41}$$

Define

$$\begin{aligned}
x &= \sqrt{\frac{|t_1|}{|t_4|}} \in (0, +\infty)\\
y &= \text{sgn}(t_4) \frac{t_2}{\sqrt{|t_1 t_4|}} \in (-\infty, +\infty)\\
w &= \frac{1}{\sqrt{|t_1 t_4|}} \in (0, +\infty).
\end{aligned} \tag{42}$$

Then

$$\begin{aligned}
&f_1(x, y, w)\\
&\triangleq \max_{1\le i\le m+2} \left\| \begin{bmatrix} w & 0 & 0 \\ 0 & x & 0 \\ 0 & y & 1/x \end{bmatrix} \Phi_i \begin{bmatrix} 1/w & 0 & 0 \\ 0 & 1/x & 0 \\ 0 & -y & x \end{bmatrix} \right\|_s\\
&= \max_{1\le i\le m+2} \left\| \begin{bmatrix} 1 & 0 \\ 0 & \mathcal{T}^T \end{bmatrix} \Phi_i \begin{bmatrix} 1 & 0 \\ 0 & \mathcal{T}^{-T} \end{bmatrix} \right\|_s
\end{aligned} \tag{43}$$

and

$$\begin{aligned}
\nu_1 &\triangleq \min_{\mathcal{T}\in\zeta_1} \max_{1\le i\le m+2} \left\| \begin{bmatrix} 1 & 0 \\ 0 & \mathcal{T}^T \end{bmatrix} \Phi_i \begin{bmatrix} 1 & 0 \\ 0 & \mathcal{T}^{-T} \end{bmatrix} \right\|_s\\
&= \min_{\substack{x\in(0,+\infty)\\ y\in(-\infty,+\infty)\\ w\in(0,+\infty)}} f_1(x, y, w).
\end{aligned} \tag{44}$$

If $\nu = \nu_1$ and $(x_{\text{opt1}}, y_{\text{opt1}}, w_{\text{opt1}})$ is the solution of the optimization problem (44)

$$\begin{aligned}
\nu &= \nu_1\\
&= \max_{1\le i\le m+2} \left\| \begin{bmatrix} w_{\text{opt1}} & 0 & 0 \\ 0 & x_{\text{opt1}} & 0 \\ 0 & y_{\text{opt1}} & 1/x_{\text{opt1}} \end{bmatrix} \right.\\
&\quad \left. \cdot\, \Phi_i \begin{bmatrix} 1/w_{\text{opt1}} & 0 & 0 \\ 0 & 1/x_{\text{opt1}} & 0 \\ 0 & -y_{\text{opt1}} & x_{\text{opt1}} \end{bmatrix} \right\|_s\\
&= \max_{1\le i\le m+2} \left\| \frac{1}{w_{\text{opt1}}} \begin{bmatrix} w_{\text{opt1}} & 0 & 0 \\ 0 & x_{\text{opt1}} & 0 \\ 0 & y_{\text{opt1}} & 1/x_{\text{opt1}} \end{bmatrix} \right.\\
&\quad \left. \cdot\, \Phi_i \begin{bmatrix} 1/w_{\text{opt1}} & 0 & 0 \\ 0 & 1/x_{\text{opt1}} & 0 \\ 0 & -y_{\text{opt1}} & x_{\text{opt1}} \end{bmatrix} w_{\text{opt1}} \right\|_s
\end{aligned} \tag{45}$$

which means that

$$\mathcal{T}_{\text{opt}} = \frac{1}{w_{\text{opt1}}} \begin{bmatrix} x_{\text{opt1}} & y_{\text{opt1}} \\ 0 & 1/x_{\text{opt1}} \end{bmatrix} \tag{46}$$

is the optimal solution of the problem (17).

By considering (40) in a similar way, we can prove the rest of Theorem 2.

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