Design of Near Perfect Reconstruction Oversampled Filter Banks for Subband Adaptive Filters

Moritz Harteneck, Stephan Weiss, and Robert W. Stewart

Abstract— **In this paper a design algorithm for real valued and complex valued oversampled filter banks which yield a low level of inband alias and enable simple subband adaptive structures is presented. The filter banks are either based on complex modulation of a real valued low pass prototype or on the direct or modulated set-ups of real valued filter banks. If real valued filter banks are required then the different channels will have different subsampling ratios in order that the bandpass sampling theorem is not violated. The paper also presents design examples of real valued and complex valued filter banks.**

Keywords— **Oversampled near perfect reconstruction filter banks, filter bank design, adaptive filtering.**

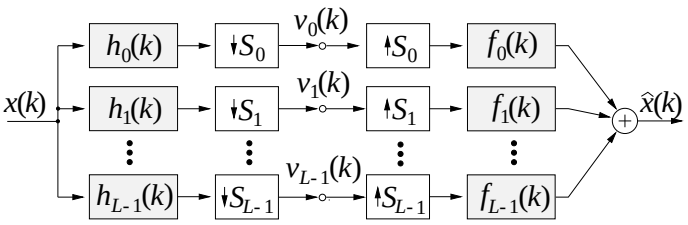

Fig. 1. General filter bank structure consisting of analysis and synthesis stages.

I. Introduction

Over the last two decades, adaptive filtering has been of considerable interest for many applications such as acoustic echo cancelation [1], noise reduction, equalization, and beamforming [2]. Often the adaptive system has to "model" a long duration impulse response, as in the case of, for example, the identification of room acoustics [3], [4]. Hence with the high number of adaptive filter weights required, popular adaptive algorithms based on least mean squares and least squares techniques [5] become very computationally complex and exhibit a slow convergence.

One possibility to combat these problems is the use of adaptive algorithms together with multirate techniques to split the fullband problem into smaller subband problems. If the well researched critically sampled perfect reconstruction filter banks are used to decompose the input signals, the subband signals are contaminated by aliasing which requires adaptive filters in between adjacent bands to compensate for this distortion [6]. Thus multichannel adaptive algorithms are being used in the subbands which exhibit a higher computational complexity and a slower convergence for correlated input signals. If the perfect reconstruction condition is dropped, critically sampled structures are obtainable which do not need cross adaptive filters, as spectral gaps ensure that aliasing in the subbands does not occur [7], [8]. This spectral loss however prohibits exact system identification and can significantly distort signals passing through the filter bank. Applications in data communications or high quality audio can be very sensitive to such errors.

Another possibility to achieve the decomposition of the input signals is to use oversampled filter banks (OSFBs), i.e. filter banks which generate redundancy in the subbands. One class of OSFBs is obtained via complex modulation from a real valued lowpass prototype filter. If the prototype filter possesses a high stopband attenuation, then aliasing in the subbands is sufficiently suppressed. The subband signals in this class of oversampled filter banks are all subsampled by the same subsampling ratio S. This filter bank can be modified to perform a single-side-band (SSB) modulation-like filter bank yielding real valued subband signals [9], [10], [11]. The trick is to modulate the bandpass signals into the baseband prior to decimation, thus avoiding problems with the bandpass sampling [12]. Since the prototype design is identical to the design for complex modulated filter banks it is not further discussed in this paper. Another OSFB class consists of real valued non-uniform filter banks [13] which have different subsampling ratios S_l in each channel to satisfy the bandpass sampling theorem and avoid inband aliasing. In this case the analysis filters posses different bandwidths which might be advantageous in, for example, psychoacoustic coding [14].

In this paper we present a fast converging and efficient design algorithm for OSFBs with near perfect reconstruction property which suppresses aliasing in the subbands, and is thus well suited for subband adaptive filtering. Therefore, Sec. II, will introduce two classes of real and complex valued OSFBs. Sec. III discusses an iterative least squares design algorithm, for which examples will be shown in Sec. IV. Our notation uses a normalized sampling frequency $f_s = 2$. Lower case italic letters denote scalar values, lower case boldface and upper case boldface letters represent vector and matrix quantities, respectively, and k is the discrete time index.

II. Types of Filter Banks

Fig. 1 shows the concatenation of an analysis and a synthesis bank. On the analysis side, the input signal $x(k)$ is split into L frequency bands by analysis filters $h_l(k)$ and decimated by a factor of S_l in the lth branch giving the subband signal $v_l(k)$. On the synthesis side, each corresponding branch, consisting of an upsampler and a synthesis filter $f_l(k)$, restores the original fullband sampling rate and interpolates the subband signal. Summing over all branches then forms the output signal $\hat{x}(k)$. Such a filter bank possesses perfect or near perfect reconstruction property, if the output signal $\hat{x}(k)$ is essentially a copy of the input signal $x(k)$ delayed by Δ samples, i.e. $\hat{x}(k) \approx x(k-\Delta)$. Writing the transfer function of the filter bank shown in Fig.1 in the frequency domain gives

$$\hat{X}(z) = \sum_{l=0}^{L-1} \sum_{m=0}^{S_l-1} \underbrace{\frac{1}{S_l} F_l(z) H_l(zW_{S_l}^m)}_{T_{l,m}(z)} X(zW_{S_l}^m), \qquad (1)$$

where $\hat{X}(z)$, $H_l(z)$, $F_l(z)$ and $X(z)$ are the z-transforms of $\hat{x}(k)$, $h_l(k)$, $f_l(k)$ and $x(k)$, respectively, and W_S is a modulation factor $e^{-j\frac{2\pi}{S}}$. The transfer functions $T_{l,m}(z)$ describe the transfer of the different aliasing terms m of each channel l. For the

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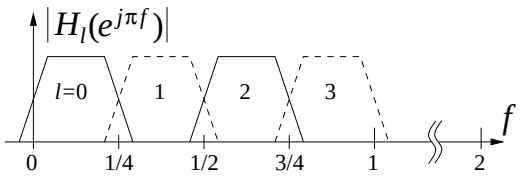


Fig. 2. Analysis filter set-up for a 4 channel complex valued OSFB.

adaptive filters to work in the subbands without the need for information from adjacent subbands [6], the aliasing terms for $m > 0$ have to be approximately zero, i.e.

$$T_{l,m}(z) \approx 0, \quad \forall l = 0, \ldots, L-1, \; m = 1, \ldots, S_l. \tag{2}$$

Note that if (2) is satisfied, (1) simplifies to

$$\hat{X}(z) = \sum_{l=0}^{L-1} \frac{1}{S_l} F_l(z) H_l(z) X(z), \tag{3}$$

i.e. perfect reconstruction and power complementarity become equivalent claims. To minimize the alias level $T_{l,m}(z)$ in the subband signals, complex valued and real valued set-ups are possible. These set-ups are outlined in Sec. II-A and II-B, respectively.

A. Complex Valued Filter Banks

One way to implement an OSFB where inband aliasing is reduced to the level of the stop band attenuation of the analysis filters is to use a complex valued set of analysis (see Fig. 2) and synthesis filters, whereby the filters are derived from a real valued linear phase prototype $p(k)$ by a generalized discrete Fourier transform (GDFT) [10]:

$$h_l(k) = p(k) \cdot e^{j\frac{2\pi}{L}(l+\frac{1}{2})(k+k_0)} \quad . \tag{4}$$

A time offset $k_0 = -\frac{l_p-1}{2}$ in (4) ensures linear phase analysis and synthesis filters if the prototype filter $p(k)$ is linear phase; its length l_p is assumed to be even throughout this paper. Furthermore, this choice of k_0 causes real and imaginary part of the analysis filters to separately fulfill linear phase conditions. With the usual selection as complex conjugate and time reversed versions of the analysis filter, the synthesis filter take on a particularly simple form:

$$f_l(k) = \tilde{h}_l(k) = h_l(k) \quad . \tag{5}$$

The offset $\frac{1}{2}$ on the band index l affects the position of the passbands of the $H_l(z)$ [10] and assures that the analysis filters are lined up equally in the frequency interval $f \in [0; 1]$ as shown in Fig. 2. For real valued input $x(k)$ only the frequency bands from 0 to 1 have to be covered by the analysis bank since the remaining subbands are complex conjugate copies and therefore redundant. The oversampling ratio (OSR) for this filter bank equates to $2L/S$, where S is the common subsampling ratio for all channels. Note that to avoid aliasing in the subbands, the prototype filter $p(k)$ has to be a real valued lowpass filter strictly limited to the frequency range $[-\frac{1}{S}; \frac{1}{S}]$. This type of OSFB approximates an implementation of Weyl-Heisenberg frames which are studied in detail in [15].

Collecting the coefficients of the prototype $p(k)$, and the analysis and synthesis filters $h_l(k)$ and $f_l(k)$ in vectors $\mathbf{p}$, $\mathbf{h}_l$ and $\mathbf{f}_l$

$$\mathbf{p} = [p(0)\ p(1)\ \ldots\ p(l_p-1)]^T \tag{6a}$$

$$\mathbf{h}_l = \mathbf{f}_l = [h_l(0)\ h_l(1)\ \ldots\ h_l(l_p-1)]^T, \tag{6b}$$

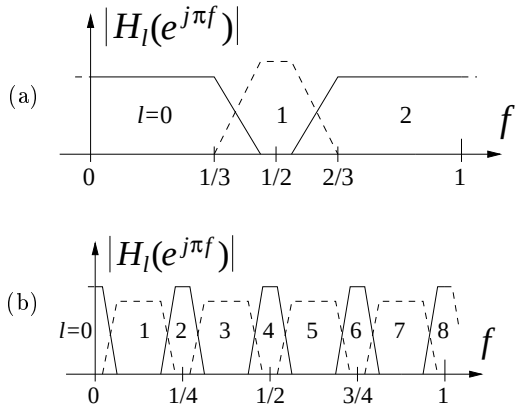


Fig. 3. Analysis filter set-up for real valued OSFBs using (a) 3 channels; and (b) 9 channels.

TABLE I
MODULATION RULES.

band l	analysis filter $h_l(k)$
0	$\sqrt{S_0} \cdot p_1(k)$
$1, 3, \ldots, 2L_0 - 1$	$2\sqrt{S_l} \cdot p_0(k) \cdot \cos\left(k \cdot \frac{l}{2} \cdot \frac{\pi}{L_0}\right)$
$2, 4, \ldots, 2L_0 - 2$	$2\sqrt{S_l} \cdot p_1(k) \cdot \cos\left(k \cdot \frac{l}{2} \cdot \frac{\pi}{L_0}\right)$
$2L_0$	$\sqrt{S_{2L_0}} \cdot p_1(k) \cdot \cos(k \cdot \pi)$

where superscript T denotes transpose, the modulation in (4) can be expressed by a convenient matrix-vector notation,

$$\mathbf{h}_l \;=\; \mathbf{f}_l \;=\; \mathbf{M}_l \cdot \mathbf{p} \quad . \tag{7}$$

The matrices $\mathbf{M}_l$ are diagonal matrices with modulation factors (c.f. (4)) on their main diagonal. For further details on this type of filter bank and efficient polyphase implementations, refer to [10], [16], [11].

B. Real Valued Filter Banks

Another way to obtain a filter bank satisfying (2) is to use a real valued non-uniform OSFB and choose a set-up where the bandpass sampling theorem is not violated [12], [13]. For a real valued signal in the l^{th} channel to be undistorted due to decimation by a factor S_l, an interpretation of the bandpass sampling theorem requires the undecimated subband signal's band to lie entirely between any two consecutive frequency points among $f = i \cdot \frac{1}{S_l}, i = 1, 2, \ldots, S_l - 1$.

Extending this to OSFBs means that every analysis filter $h_l(k)$ has to be placed in the frequency domain such that the resulting signal does not violate the bandpass sampling when subsampled by a factor S_l and that the analysis filters have to be placed such that all frequencies are covered by at least one filter in order to allow reconstruction.

Fig. 3(a) shows the simplest set-up of analysis filters of a non-uniform real valued OSFB. In this filter bank, the low pass and high pass channels, produced by $H_0(z)$ and $H_2(z)$, are subsequently decimated by 2 and therefore are alias-free. The bandpass channel $H_1(z)$ is subsampled by a factor of 3 and also remains undistorted. Hence, the OSR for this filter bank equates to $\sum_{l=0}^{2} \frac{1}{S_l} = 133\%$.

Another possibility to obtain real valued OSFBs is to start from a cosine modulated filter bank with L_0 channels, where the bandwidth of the analysis filters is reduced such that no aliasing occurs during decimation, and to fill the resulting spectral gaps

by additional analysis filters which are derived from a second prototype to allow reconstruction. Filling the gaps with additional channels leads to a filter bank with a total of $L = 2L_0 + 1$ channels. All filters are derived from the two prototypes $p_0(k)$ and $p_1(k)$ as shown in Tab. I.

Fig. 3(b) shows, as an example, a 9 channel filter bank which is modulated according to Tab. I. The channels 1, 3, 5 and 7 are derived from the prototype $p_0(k)$ and can be subsampled by 4. To avoid aliasing, the prototype filter $p_0(k)$ has to have a bandwidth of less than $\frac{1}{8}$, i.e. be limited to the frequency range $f \in [-\frac{1}{8}; \frac{1}{8}]$. The channels 0, 2, 4, 6 and 8, however, are derived from the prototype $p_1(k)$ and are subsampled by 8, 6, 5, 6 and 8, respectively. These decimation ratios are chosen such that the allowed bandwidth of the prototype $p_1(k)$ is as large as possible and results in this example in the passband of the prototype to be restricted to the region $[-0.0833; 0.0833]$. The OSR of this filter bank is 178%.

Analogous to Sec. II-A, the modulation can be expressed in matrix notation. More information on this type of filter bank can be found in [13], [17], [18], [19].

If a filter bank with a higher number of channels is required, the presented cosine modulation approach can be used, starting with a higher L_0, or different set-ups can be found which satisfy the bandpass sampling theorem. Alternatively, a previously designed filter bank can be iterated, which, however, leads in general to suboptimal solutions.

III. Design Algorithm

A common choice in the design of filter banks is to choose the synthesis filters to be time-reversed and complex conjugate versions of the analysis filters. This yields a reduction in the free design parameters. According to the theory of frames [20] this also leads to very close approximation of signal decompositions with a fixed energy translation between time and subband domain, regardless of the input signal.

To design OSFBs as presented in Sec. II, an iterative least squares algorithm [22], [23] is used. The performance criterion to be minimized by the algorithm is a combination of the the filter bank reconstruction error ξ_1 and the stop band energy ξ_2 of the analysis filters $h_l(k)$,

$$\xi = \xi_1 + \gamma \cdot \xi_2, \tag{8}$$

where γ is a positive weighting factor which trades off between the importance of the reconstruction error and the stop band attenuation. In the following we separately discuss formulations for both ξ_1 and ξ_2, and finally present the design algorithm in Sec. III-C.

A. Reconstruction Error

If inband aliasing is sufficiently suppressed, the impulse response $t(k)$ of the overall filter bank as given in Fig. 1 is the time domain formulation of (3), and can be written as a convolution of the analysis and the synthesis filters,

$$t(k) = \sum_{l=0}^{L-1} \frac{1}{S_l} (h_l(k) * f_l(k)), \tag{9}$$

where "$*$" denotes convolution and the factor $1/S_l$ takes account of the power loss in the channels due to decimation. Using matrix notation, the convolution in the lth branch can be calculated by [24]

$$\mathbf{t}_l = \begin{bmatrix} t_l(0) \\ t_l(1) \\ t_l(2) \\ \vdots \\ t_l(2l_p - 2) \end{bmatrix} = \begin{bmatrix} h_l(0) & 0 & \cdots & 0 \\ h_l(1) & h_l(0) & \cdots & 0 \\ h_l(2) & h_l(1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & h_l(l_p-1) \end{bmatrix} \begin{bmatrix} f_l(0) \\ f_l(1) \\ f_l(2) \\ \vdots \\ f_l(l_p-1) \end{bmatrix} = \mathbf{H}_l(1\mathbf{f}_l) \tag{10}$$

Evaluating the summation of (9), the impulse response of the whole filter bank can be expressed as

$$\mathbf{t} = \begin{bmatrix} \frac{1}{S_0}\mathbf{H}_0 & \frac{1}{S_1}\mathbf{H}_1 & \cdots & \frac{1}{S_{L-1}}\mathbf{H}_{L-1} \end{bmatrix} \begin{bmatrix} \mathbf{f}_0 \\ \mathbf{f}_1 \\ \vdots \\ \mathbf{f}_{L-1} \end{bmatrix} = \mathbf{H} \cdot \mathbf{f}. \tag{11}$$

To get a measure of the reconstruction error, the Euclidean distance between the impulse response $\mathbf{t}$ and a prefect delay (represented by the vector $\mathbf{v}$) is evaluated:

$$\xi_1 = \|\mathbf{t} - \mathbf{v}\|^2 = \|\mathbf{H} \cdot \mathbf{f} - \mathbf{v}\|^2 \quad . \tag{12}$$

B. Stop Band Energy

As described in [25], [22], a measure for the energy contained in the stop band of a linear phase analysis filter $h_l(k)$ can be calculated by using a dense grid of frequency points $\{\omega_0, \omega_1, \ldots, \omega_N\}$ covering the whole stop band and calculating

$$\xi_{2,l} = \left\| \begin{bmatrix} 1 & \cos(\omega_0 \cdot 1) & \cdots & \cos(\omega_0 \cdot (l_p-1)) \\ 1 & \cos(\omega_1 \cdot 1) & \cdots & \cos(\omega_1 \cdot (l_p-1)) \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \cos(\omega_N \cdot 1) & \cdots & \cos(\omega_N \cdot (l_p-1)) \end{bmatrix} \begin{bmatrix} h_l(0) \\ h_l(1) \\ \vdots \\ h_l(l_p-1) \end{bmatrix} \right\|^2 = \|\mathbf{P}_l \cdot \mathbf{h}_l\|^2 \quad . \tag{13}$$

Here the matrix $\mathbf{P}_l$ describes the required specifications on the stop bands of the lth analysis filter.

C. Minimization Algorithm

Using the above definitions and rewriting (8) in a matrix vector notation for a general L channel filter bank, and assuming analysis and synthesis filters to be the same such that a tight frame is achieved in very close approximation, the performance criterion can be written as

$$\xi = \left\| \begin{bmatrix} \mathbf{H} \\ \gamma\mathbf{P} \end{bmatrix} \mathbf{h} - \begin{bmatrix} \mathbf{v} \\ \mathbf{0} \end{bmatrix} \right\|^2, \tag{14}$$

where $\mathbf{P} = \mathrm{diag}(\mathbf{P}_0, \mathbf{P}_1, \ldots, \mathbf{P}_{L-1})$ is a matrix describing the frequency specifications on all analysis filters and $\mathbf{h} = [\mathbf{h}_0^T \ \mathbf{h}_1^T \ \ldots \ \mathbf{h}_{L-1}^T]^T$ is a collection of the analysis filters. As defined in (11) the matrix $\mathbf{H}$ consists of the matrices $\mathbf{H}_l$ and as these matrices are dependent on the analysis filters $\mathbf{h}_l$. Therefore the performance criterion (14) is to the power of four dependent on the design parameters.

To enforce linear phase filters in the design algorithm, the axial symmetry of the filter impulse responses can be exploited.

TABLE II
ITERATIVE LEAST SQUARES DESIGN ALGORITHM.

Step	Instruction
1:	initialize filters $\mathbf{h}(i)$, set $i = 1$
2:	build $\mathbf{H}_l(i-1)$ from $\mathbf{h}_l(i-1)$
3:	minimize with respect to $\mathbf{h}(i)$
	$\xi = \left\| \begin{bmatrix} \mathbf{H}(i-1) \\ \gamma\mathbf{P} \end{bmatrix} \mathbf{h}(i) - \begin{bmatrix} \mathbf{v} \\ \mathbf{0} \end{bmatrix} \right\|^2$
4:	apply relaxation
	$\mathbf{h}(i) = \tau\mathbf{h}(i) + (1-\tau)\mathbf{h}(i-1)$
5:	check stop condition
	if $\|\mathbf{h}(i) - \mathbf{h}(i-1)\| < \epsilon$ then stop
6:	$i = i+1$, goto 2:

As an example, the prototype filter $p(k)$ is completely characterized by its first half of the impulse response. In matrix notation, this relationship can be expressed as

$$\mathbf{p} = \mathbf{L} \cdot \bar{\mathbf{p}} \quad , \tag{15}$$

where $\mathbf{L}$ is an $l_p \times \frac{l_p}{2}$ matrix defined by $\mathbf{L} = [\mathbf{I}_{l_p/2}\ \mathbf{J}_{l_p/2}]^T$, $\mathbf{I}$ being an identity and $\mathbf{J}$ an inverse identity matrix. Incorporating (15) into the design effectively halfs the degrees of freedom.

Thus, for the three channel real valued OSFB discussed in Sec. II-B, the performance criterion can now be written as

$$\xi = \left\| \begin{bmatrix} \frac{1}{S_0}\mathbf{H}_0\mathbf{L} & \frac{1}{S_1}\mathbf{H}_1\mathbf{L} & \frac{1}{S_2}\mathbf{H}_2\mathbf{L} \\ \gamma\mathbf{P}_0\mathbf{L} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \gamma\mathbf{P}_1\mathbf{L} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \gamma\mathbf{P}_2\mathbf{L} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{h}}_0 \\ \bar{\mathbf{h}}_1 \\ \bar{\mathbf{h}}_2 \end{bmatrix} - \begin{bmatrix} \mathbf{v} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \right\|^2 , \tag{16}$$

where the matrices $\mathbf{P}_l$ describes the stop band requirements to eliminate the aliasing caused by the subsampling process.

For the nine channel real valued OSFB as presented in Sec. II-B, which is derived from two prototypes $\mathbf{p}_0$ and $\mathbf{p}_1$, the performance criterion simplifies to

$$\xi = \left\| \begin{bmatrix} \sum_{l\in\mathbb{S}_0} \frac{1}{S_l}\mathbf{H}_l\mathbf{M}_l\mathbf{L} & \sum_{l\in\mathbb{S}_1} \frac{1}{S_l}\mathbf{H}_l\mathbf{M}_l\mathbf{L} \\ \gamma\mathbf{P}_0\mathbf{L} & \mathbf{0} \\ \mathbf{0} & \gamma\mathbf{P}_1\mathbf{L} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{p}}_0 \\ \bar{\mathbf{p}}_1 \end{bmatrix} - \begin{bmatrix} \mathbf{v} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \right\|^2 , \tag{17}$$

where the matrices $\mathbf{P}_0$ and $\mathbf{P}_1$ describe the frequency requirements on the prototypes $\mathbf{p}_0$ and $\mathbf{p}_1$, respectively, and the set $\mathbb{S}_l$ denotes the set of filters derived from the prototype $\mathbf{p}_l$.

The performance criterion for the complex valued OSFB presented in Sec. II-A, which is derived from only one real valued prototype $\mathbf{p}$, further simplifies to

$$\xi = \left\| \begin{bmatrix} \frac{1}{S}\sum_{l=0}^{L-1} \mathbf{H}_l\mathbf{M}_l\mathbf{L} \\ \gamma\mathbf{P}\mathbf{L} \end{bmatrix} \bar{\mathbf{p}} - \begin{bmatrix} \mathbf{v} \\ \mathbf{0} \end{bmatrix} \right\|^2 , \tag{18}$$

where S is the common subsampling ratio and $\mathbf{P}$ describes the frequency requirements on the prototype.

To perform the minimization of (14), an iterative least squares technique [22], [26], [23] is used where the matrix $\mathbf{H}$, which is being built from the analysis filters, is substituted by the matrix being built from an old set of analysis filters. This linearisation gives a quadratic approximation of the performance surface. To ensure convergence, a relaxation step is added to the algorithm. The design algorithm can now be formulated iteratively as shown in Table II, where τ is the relaxation parameter which has to be chosen between 0 and 1 and ϵ quantifies the stop condition. The minimization of Step 2 can be conveniently performed by a QR decomposition [27].

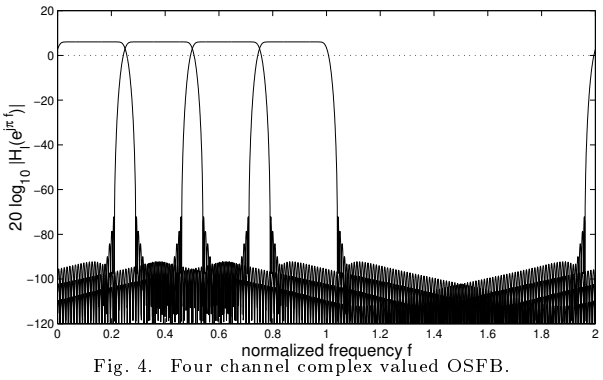
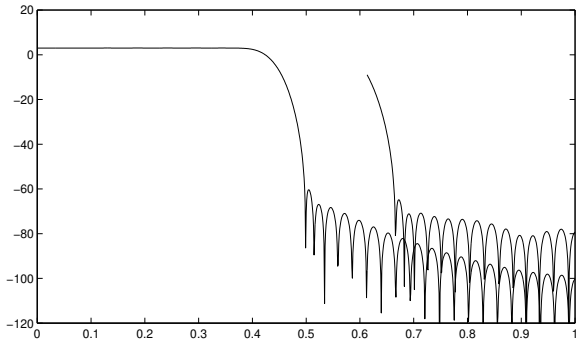


Fig. 4. Four channel complex valued OSFB.

(a)