Higher order discretisation methods for a class of 2-D continuous-discrete linear systems

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Abstract: Differential linear repetitive processes are a distinct class of two-dimensional linear systems which can be used, for example, to model industrial processes such as long-wall coal cutting operations. Also, they can be used to study key properties of classes of linear iterative learning schemes. The key feature of interest in the paper is the fact that information propagation in one of the two separate directions in such processes evolves continuously over a finite fixed duration and in the other direction it is, in effect, discrete. The paper develops discrete approximations for the dynamics of these processes and examines the effects of the approximation techniques used on two key systems-related properties. These are stability and the structure of the resulting discrete state-space models. Some ongoing work and areas for further development are also briefly noted.

1 Introduction

The essential unique characteristic of a repetitive, or multi-pass, process can be illustrated by considering machining operations where the material or workpiece involved is processed by a sequence of passes of the processing tool. Assuming that the pass length $\alpha < +\infty$ (i.e. the duration of a pass of the processing tool) is constant, the output vector, or pass profile, $y_k(t)$, $0 \le t \le \alpha$, $k \ge 0$, (t being the independent spatial or temporal variable) acts as a forcing function on the next pass and hence contributes to the dynamics of the new pass profile $y_{k+1}(t)$, $0 \le t \le \alpha$, $k \ge 0$.

Industrial examples of repetitive processes include long-wall coal cutting and metal rolling operations [13]. In recent years, problem areas have also arisen where adopting a repetitive process perspective has major advantages over alternatives, so-called algorithmic examples. This is especially true for classes of iterative learning control schemes [1].

The basic unique control problem for repetitive processes is that the sequence of pass profiles generated can contain oscillations that increase in amplitude in the pass to pass direction (i.e. the k direction in the notation used here). Such behaviour is easily generated in simulation studies and in experiments on scaled models of industrial examples, such as long-wall coal cutting (see [13] for a detailed treatment). Early approaches to stability analysis for (linear single-input/single-output) repetitive processes was based on first converting the system into an infinite-length single pass process to enable the application of standard techniques (e.g. those based on the inverse Nyquist criterion). It was soon realised, however, that this approach to stability analysis would, except in a few very restrictive special cases, lead to incorrect conclusions [9].

The basic reason for this situation is that the infinite-length single pass model effectively neglects their inherent two-dimensional (2-D) systems structure, i.e. information propagation along a given pass (t direction) and from pass to pass (k direction). In particular, differential linear repetitive processes are a distinct class of continuous-discrete 2-D linear systems where, in contrast to other classes of such systems (see, for example, [6]), information is propagated in one direction as a function of a continuous variable over the finite and constant (by definition) pass length and as a function of a discrete variable in the other direction. Also, the infinite-length single pass model ignores the fact that the initial conditions are reset before the start of each new pass, where it is known that this is a critical feature of the overall process dynamics.

To remove these deficiencies, a rigorous stability theory has been developed [9] based on an abstract model of the dynamics in a Banach space setting which includes all processes with linear dynamics and a constant pass length as special cases. Also the results of applying this theory to a range of subclasses, including those considered in this paper, have been reported [13, 9].

Of particular interest in terms of applications are the subclasses of so-called differential and discrete linear repetitive processes, respectively. Discrete linear repetitive processes have clear structural links with 2-D discrete linear systems described by the well known Roesser [8] state space model (or equivalents). As noted above, however, a key difference in all cases is the fact that in a repetitive process information propagation along a pass (i.e. one of the two separate directions) only occurs over a finite duration, namely, the pass length.

The only essential difference between differential and discrete linear repetitive processes is that in the former the dynamics along a given pass evolve as a function of a continuous variable defined over the pass length, as opposed to a discrete variable in the latter. Suppose therefore that a

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filter/controller has been designed for a differential linear repetitive process. Then an obvious means of implementing this filter/controller is by the construction of an appropriate discrete approximation. One possible approach here would be to use numerical integration techniques, such as the Euler or trapezoidal rule, which for standard, or one-dimensional (1-D), linear systems is equivalent to the well known bilinear transform in the frequency domain.

Use of such numerical integration techniques in the case of 1-D differential linear systems results in dynamics described by a 1-D discrete linear systems state space model. Here it is argued that, for differential linear repetitive processes, the dynamics of the resulting discrete approximation should be described by a so-called discrete linear repetitive process state space model. In particular, this makes the well established links to 2-D linear systems theory available for further analysis.

This paper shows that, although the trapezoidal rule is applicable here, it is by no means as powerful as in the 1-D case. To remove difficulties which this may cause, such as 'poor' approximation errors, this paper investigates the use of higher order single step approximations and shows that these are particularly appropriate for the discretisation of differential linear repetitive processes. Some ongoing work and areas for further development are also briefly noted.

2 Background

The state space model of the differential linear repetitive processes considered here have the following form over $0 \leq t \leq \alpha$, $k \geq 0$:

$$\begin{aligned} \dot{\boldsymbol{x}}_{k+1}(t) &= \boldsymbol{A}\boldsymbol{x}_{k+1}(t) + \boldsymbol{B}\boldsymbol{u}_{k+1}(t) + \boldsymbol{B}_0\boldsymbol{y}_k(t) \\ \boldsymbol{y}_{k+1}(t) &= \boldsymbol{C}\boldsymbol{x}_{k+1}(t) + \boldsymbol{D}_0\boldsymbol{y}_k(t) \\ \boldsymbol{x}_{k+1}(0) &= \boldsymbol{0} \end{aligned} \qquad (1)$$

where on pass k, $\boldsymbol{x}_k(t)$ is the $n \times 1$ state vector, $\boldsymbol{y}_k(t)$ is the $m \times 1$ vector pass profile, $\boldsymbol{u}_k(t)$ is the $l \times 1$ vector of control inputs.

The stability theory [9] for linear constant pass length repetitive processes is based on the following abstract model of the underlying dynamics, where $\boldsymbol{E}_\alpha$ is a suitably chosen Banach space (a complete normed linear space) with norm $\|.\|$ and $\boldsymbol{W}_\alpha$ is a linear subspace of $\boldsymbol{E}_\alpha$:

$$\boldsymbol{y}_{k+1} = \boldsymbol{L}_\alpha \boldsymbol{y}_k + \boldsymbol{b}_{k+1} \qquad k \geq 0 \qquad (2)$$

In this model, $\boldsymbol{y}_k \in \boldsymbol{E}_\alpha$ is the pass profile on pass k, $\boldsymbol{b}_{k+1} \in \boldsymbol{W}_\alpha$, and $\boldsymbol{L}_\alpha$ is a bounded linear operator mapping $\boldsymbol{E}_E$ into itself. The term $\boldsymbol{L}_\alpha \boldsymbol{y}_k$ represents the contribution from pass k to pass $k+1$ and $\boldsymbol{b}_{k+1}$ represents known initial conditions, disturbances and input effects on the current pass. In this paper the model is denoted by $\boldsymbol{S}$.

In the case of eqn. 1, the authors choose $\boldsymbol{E}_\alpha = \boldsymbol{L}_2^m[0, \alpha] \cap \boldsymbol{L}_\infty[0, \alpha]$ and it is routine to show that

$$(\boldsymbol{L}_\alpha \boldsymbol{y})(t) = \boldsymbol{C} \int_0^t e^{\boldsymbol{A}(t-\tau)} \boldsymbol{B}_0 \boldsymbol{y}(\tau) d\tau + \boldsymbol{D}_0 \boldsymbol{y}(t) \qquad 0 \leq t \leq \alpha \qquad (3)$$

and

$$\boldsymbol{b}_{k+1} = \boldsymbol{C} \int_0^t e^{\boldsymbol{A}(t-\tau)} \boldsymbol{B}\boldsymbol{u}_{k+1}(\tau) d\tau \qquad 0 \leq t \leq \alpha \qquad (4)$$

The stability theory for linear repetitive processes consists of the distinct concepts of asymptotic stability and stability along the pass, respectively. Asymptotic stability can be interpreted (in terms of the norm on the underlying function space) as demanding that bounded input sequences (inputs plus initial conditions plus disturbances on the current pass) produce bounded sequences of pass profiles over the finite and fixed process pass length. It is defined in terms of the abstract model $\boldsymbol{S}$ (for complete details, including proofs, see [9]) as follows:

Definition 1: The linear repetitive process $\boldsymbol{S}$ is said to be asymptotically stable if there exists a real scalar $\delta > 0$ such that, given any initial profile y_0 and any disturbance sequence $\{\boldsymbol{b}_k\}_{k\geq 1} \subset \boldsymbol{W}_\alpha$ bounded in norm (i.e. $\|\boldsymbol{b}_k\| \leq c_1$ for some constant $c_1 \geq \boldsymbol{0}$ and for all $k \geq 1$) the output sequence generated by the perturbed process

$$\boldsymbol{y}_{k+1} = (\boldsymbol{L}_\alpha + \gamma)\boldsymbol{y}_k + \boldsymbol{b}_{k+1} \qquad k \geq 0 \qquad (5)$$

is bounded in norm whenever $\|\gamma\| \leq \delta$.

The necessary and sufficient condition for this property to hold is that

$$r(\boldsymbol{L}_\alpha) < 1 \qquad (6)$$

where $r(.)$ denotes the spectral radius (in effect, the modulus of the largest eigenvalue) of its argument. *Note:* This definition demands that asymptotic stability is retained if the model is perturbed slightly due to modelling errors or simulation approximations.

In the case of processes described by eqn. 1 it is routine to show that asymptotic stability holds if, and only if,

$$r(\boldsymbol{D}_0) < 1 \qquad (7)$$

At first sight, this result is counter-intuitive in the sense that asymptotic stability is largely independent of the processes dynamics and, in particular, of the eigenvalues of $\boldsymbol{A}$ which clearly govern the dynamics produced along any pass. This situation is due entirely to the fact that the pass length is finite (over which duration even an unstable 1-D system can only produce a bounded output) and will change drastically when an arbitrary pass length ($\alpha \to +\infty$) is considered.

The limit profile is used to characterise transient behaviour in the pass to pass (i.e. k) direction. Suppose that the abstract process $\boldsymbol{S}$ is asymptotically stable and is subjected to a sequence $\{\boldsymbol{b}_k\}_{k\geq 1}$ which converges strongly to $\boldsymbol{b}_\infty$. Then the strong limit

$$\boldsymbol{y}_\infty := \lim_{k\to\infty} \boldsymbol{y}_k \qquad (8)$$

is termed the limit profile corresponding to this input sequence. In the special case of eqn. 1, the limit profile is described (see [9] for the details) over $0 \leq t \leq \alpha$ by

$$\begin{aligned} \dot{\boldsymbol{x}}_\infty(t) &= \left(\boldsymbol{A} + \boldsymbol{B}_0(\boldsymbol{I}_m - \boldsymbol{D}_0)^{-1}\boldsymbol{C}\right) \boldsymbol{x}_\infty(t) + \boldsymbol{B}\boldsymbol{u}_\infty(t) \\ \boldsymbol{y}_\infty(t) &= (\boldsymbol{I}_m - \boldsymbol{D}_0)^{-1} \boldsymbol{C}\boldsymbol{x}_\infty(t) \end{aligned} \qquad (9)$$

This is simply a 1-D linear time-invariant systems state space model and, hence, after a 'sufficiently large' number of passes, the dynamics of an asymptotically stable differential linear repetitive process can be replaced by those of a 1-D differential linear time-invariant system.

Given that the pass length is finite by definition, asymptotic stability cannot guarantee that the resulting limit profile has 'acceptable' along the pass dynamics. In particular, it cannot guarantee that the limit profile is stable as a 1-D linear system, i.e. all eigenvalues of the matrix $\boldsymbol{A} + \boldsymbol{B}_0(\boldsymbol{I}_m - \boldsymbol{D}_0)^{-1}\boldsymbol{C}$ have strictly negative real parts. Applications do exist where asymptotic stability is all that can be achieved or that is required, but, in general, it is the stronger requirement of stability along the pass which will be required.

Stability along the pass demands that the bounded-input bounded-output property of asymptotic stability holds uniformly, i.e. independent of the pass length, and is defined formally as follows:

Definition 2: The abstract model $\boldsymbol{S}$ is said to be stable along the pass if, and only if, there exist finite real scalars $M_\infty > 0$ and $\lambda_\infty \in (0, 1)$ which are independent of α and satisfy for all $\alpha > 0$ and $k \geq 0$

$$\|\boldsymbol{L}_\alpha^k\| \leq M_\infty \lambda_\infty^k \qquad (10)$$

where $\|.\|$ is also used to denote the induced operator norm.

Necessary and sufficient conditions for this property are

$$r_\infty := \sup_{\alpha>0} r(L_\alpha) < 1 \qquad (11)$$

and

$$M_0 := \sup_{\alpha>0} \sup_{|z|\geq\lambda} \|(zI - L_\alpha)^{-1}\| < +\infty \qquad (12)$$

for some real number $\lambda \in (r_\infty, 1)$.

The first of these two conditions states that asymptotic stability for all possible values of the pass length is a necessary condition for stability along the pass, which in the case of differential linear repetitive processes of the form considered here reduces to eqn. 7. The extra conditions imposed by stability along the pass are available in several equivalent forms but in this work the authors will make use of the following set (for a proof see [11]):

Theorem 1: Suppose that the pair $\{\boldsymbol{A}, \boldsymbol{B}_0\}$ is controllable and the pair $\{\boldsymbol{C}, \boldsymbol{A}\}$ is observable. Then differential linear repetitive processes described by the state space model eqn. 1 are stable along the pass if, and only if,

(*a*) all eigenvalues of $\boldsymbol{D}_0$ have modulus strictly less than unity

(*b*) all eigenvalues of $\boldsymbol{A}$ have strictly negative real parts

(*c*) the two variable polynomial

$$p_c(s,z) := \begin{vmatrix} s\boldsymbol{I}_n - \boldsymbol{A} & -\boldsymbol{B}_0 \\ -\boldsymbol{C} & z\boldsymbol{I}_m - \boldsymbol{D}_0 \end{vmatrix} \qquad (13)$$

satisfies

$$p_c(s,z) \neq 0 \qquad \mathrm{Re}(s) \geq 0 \qquad |z| \geq 1 \qquad (14)$$

The natural discrete analogue of a differential linear repetitive process is a so-called discrete linear repetitive process. Hence the discretisation of a process described by eqn. 1 should result in a discrete linear repetitive process state space model. Note also that a number of the key systems theoretic problems solved to date for discrete linear repetitive processes, such as so-called local reachability and controllability [4], have made, as appropriate, extensive use of the fact that these processes can be transformed into the well known 2-D linear systems state space model due to Roesser [8] (or equivalents). (Note that not all systems theoretic questions for discrete linear repetitive processes can be solved by this route.) This is an added incentive in aiming to obtain a discrete linear repetitive process state space model as the result of discretising eqn. 1, i.e. the well established systems theory for these 2-D linear systems is (potentially) available to study the effects of discretisation on, say, the controllability properties of eqn. 1.

3 Trapezoidal discretisation

By analogy with the 1-D linear systems case, the natural starting point to derive discrete approximations to the dynamics of eqn. 1 is to consider the use of the trapezoidal method of numerical integration or, equivalently, the bilinear transform. This method preserves stability in the 1-D case but can lead to unacceptably low accuracy. The analysis of this Section shows that this method applied to eqn. 1 again preserves stability (asymptotic and hence stability along the pass). It is noted that an attempt to increase the accuracy by increasing the number of steps in the approximation procedure at any point also leads to very undesirable features which mean that the resulting approximations cannot be usefully employed in subsequent analysis and design.

In the 1-D case, application of the trapezoidal rule assumes that the elements in the control input vector, say, $\boldsymbol{u}(t)$ are stepwise, i.e. $\boldsymbol{u}(iT + \Delta) \approx \boldsymbol{u}(iT)$, $\Delta \leq T$ where T denotes the sampling period. The trapezoidal rule, expressed in terms of a signal $w(t)$ with index $i \geq 0$ and sampling period T is given by

$$\boldsymbol{w}(iT) = \boldsymbol{w}((i-1)T) + \frac{T}{2}(\dot{\boldsymbol{w}}(iT) + \dot{\boldsymbol{w}}((i-1)T)) \quad (15)$$

Alternatively, the trapezoidal rule can be expressed in terms of the well known bilinear transform between the continuous and discrete domains, i.e.

$$s = \frac{2}{T}\left[\frac{z-1}{z+1}\right] \qquad (16)$$

which preserves stability, i.e. if the original continuous time system is stable then so is the resulting discrete system under this rule. Suppose, therefore, that the entries in both the control input and pass profile vectors of eqn. 1 are stepwise, i.e. for $\Delta \leq T$,

$$\begin{aligned} \boldsymbol{u}_k(iT + \Delta) &\approx \boldsymbol{u}_k(iT) \\ \boldsymbol{y}_k(iT + \Delta) &\approx \boldsymbol{y}_k(iT) \end{aligned} \qquad (17)$$

Then it is straightforward to show that, under the trapezoidal rule with sampling period T, the dynamics of eqn. 1 can be approximated by those of a discrete linear repetitive process of the form

$$\begin{aligned} \boldsymbol{x}_{k+1}(i+1) &= \hat{\boldsymbol{A}}\boldsymbol{x}_{k+1}(i) + \hat{\boldsymbol{B}}\boldsymbol{u}_{k+1}(i) + \hat{\boldsymbol{B}}_0\boldsymbol{y}_k(i) \\ \boldsymbol{y}_{k+1}(i) &= \hat{\boldsymbol{C}}\boldsymbol{x}_{k+1}(i) + \hat{\boldsymbol{D}}_0\boldsymbol{y}_k(i) \end{aligned} \qquad (18)$$

where, for ease of notation, we have suppressed the explicit dependence on T and

$$\begin{aligned} \hat{\boldsymbol{A}} &= \left[\boldsymbol{I}_n - \left(\frac{T}{2}\right)\boldsymbol{A}\right]^{-1}\left[\boldsymbol{I}_n + \left(\frac{T}{2}\right)\boldsymbol{A}\right] \\ \hat{\boldsymbol{B}} &= T\left[\boldsymbol{I}_n - \left(\frac{T}{2}\right)\boldsymbol{A}\right]^{-1}\boldsymbol{B} \\ \hat{\boldsymbol{B}}_0 &= T\left[\boldsymbol{I}_n - \left(\frac{T}{2}\right)\boldsymbol{A}\right]^{-1}\boldsymbol{B}_0 \\ \hat{\boldsymbol{C}} &= \boldsymbol{C} \\ \hat{\boldsymbol{D}}_0 &= \boldsymbol{D}_0 \end{aligned} \qquad (19)$$

The discrete linear repetitive process described by eqn. 18 is asymptotically stable if, and only if, $r(\hat{D}_0) = r(D_0) < 1$ and the following result (for a proof see [10]) gives a set of necessary and sufficient conditions for stability along the pass:

Theorem 2: Suppose that the pair $\{\hat{\boldsymbol{A}}, \hat{\boldsymbol{B}}_0\}$ is controllable and the pair $\{\hat{\boldsymbol{C}}, \hat{\boldsymbol{A}}\}$ is observable. Then discrete linear repetitive processes described by the state space model eqn. 18 are stable along the pass if, and only if,

(*a*) all eigenvalues of the matrix $\hat{\boldsymbol{D}}_0$ have modulus strictly less than unity (*b*) all eigenvalues of the matrix $\hat{\boldsymbol{A}}$ have modulus strictly less than unity

(*c*) the two variable polynomial

$$\boldsymbol{p}_d(z_1, z) := \begin{vmatrix} z_1\boldsymbol{I}_n - \hat{\boldsymbol{A}} & -\hat{\boldsymbol{B}}_0 \\ -\hat{\boldsymbol{C}} & z\boldsymbol{I}_m - \hat{\boldsymbol{D}}_0 \end{vmatrix} \tag{20}$$

satisfies

$$\boldsymbol{p}_d(z_1, z) \neq 0 \qquad |z_1| \geq 1 \qquad |z| \geq 1 \tag{21}$$

Now consider the stability properties (asymptotic and along the pass) of eqn. 18 as the result of applying the trapezoidal rule to eqn. 1. Then the fact that (*a*) and (*b*) of theorem 1 are preserved is obvious. Also the following relation holds between the polynomials $\boldsymbol{p}_c(s, z)$ and $\boldsymbol{p}_d(z_1, z)$

$$\boldsymbol{p}_d(z_1, z) = \boldsymbol{p}_c(s, z)_{\left(\frac{2}{T}\left[\frac{z_1-1}{z_1+1}\right], z\right)} \tag{22}$$

Hence, as the transformation

$$(s, z) = \left(\frac{2}{T}\left[\frac{z_1 - 1}{z_1 + 1}\right], z\right) \tag{23}$$

is bilinear, it follows immediately that the stability along the pass is also preserved. This result is stated formally as follows:

Theorem 3: Suppose that the differential linear repetitive process decribed by eqn. 1 is numerically approximated by application of the trapezoidal rule under the assumptions of eqn. 17 to yield the discrete linear repetitive process state space model of eqn. 18. Then the resulting discrete linear repetitive process state space model is stable along the pass.

To assess the accuracy of this method it is necessary to consider its approximation errors (which is, of course, well studied in the numerical analysis literature). Consider, therefore, a function, say, $f(\zeta)$ and let $f^{[j]}(\zeta)$ denote its jth derivative. Then the local truncation error with the trapezoidal rule is given by

$$E(\zeta, T) = \left[\frac{2T^3}{12}\right] f^{[3]}(\zeta) \qquad t_i = iT < \zeta < (i+1)T \tag{24}$$

Suppose now that the truncation error induced by the trapezoidal rule is unacceptably high. Then an obvious step is to consider a higher order method where in the 1-D case the higher-order Adams-Moulton methods, which belong to the general class of multistep methods for numerical integration (see, for example, [12]), are an obvious choice. The key point to note here, however, is that the use of higher order methods to reduce the truncation error in the discretisation of a differential linear repetitive process would result (see [3] for the details) in a model structure with 'very poor transparency'. In particular, it is very difficult, if not impossible, to determine what happens to key systems theoretic properties of differential linear repetitive processes under approximation. Hence, it is not a suitable basis for further analysis and design studies. The remainder of this paper shows that this basic requirement, i.e. 'transparency' with reduced error relative to the trapezoidal rule, can be achieved using the well known higher order single step numerical integration methods.

4 Single step higher order discretisation

The intrinsic feature of these methods is that higher order derivatives are employed. The simplest of these is the 4th-order method (see, for example, [7]) which (in the same notation as eqn. 15) takes the form

$$\begin{aligned} \boldsymbol{w}((i+1)T) = \boldsymbol{w}(iT) + \frac{T}{2}[\dot{\boldsymbol{w}}(iT) + \dot{\boldsymbol{w}}((i+1)T)] \\ + \frac{T^2}{12}[\ddot{\boldsymbol{w}}(iT) - \ddot{\boldsymbol{w}}((i+1)T)] \end{aligned} \tag{25}$$

The associated s to z plane mapping is given by

$$\frac{a}{s} + bs = \frac{z+1}{z-1} \qquad a = \frac{2}{T} \quad b = \frac{T}{6} \tag{26}$$

This mapping function is well known in filter theory (and also robust control) where it is termed an all-pass transformation (see, for example, [14]). In particular, the left-hand side (defined in terms of s only) is used to join band-pass analogue filters with low-pass equivalents and the right-hand side (defined in terms of z only) is an inverse bilinear transform. The following result shows that this method preserves stability properties in the 1-D case (and, hence, its basic feasibility for use in the discretisation of dynamic systems).

Theorem 4: The transformation given by eqn. 26 maps the left half of the complex s-plane, i.e. $s = \sigma + i\omega$, $s \neq 0$, $\sigma \leq 0$ into the closed unit circle $|z| \leq 1$ of the complex z-plane.

Proof: The transformation eqn. 26 can be rewritten as

$$z = \frac{bs^2 + s + a}{bs^2 - s + a} \qquad s \neq 0 \tag{27}$$

and setting $s = i\omega$, it follows that the imaginary axis in the s plane is transformed to the set

$$z = \frac{((a - b\omega^2)^2 - \omega^2) + i2(a - b\omega^2)\omega}{(a - b\omega^2)^2 + \omega^2} \tag{28}$$

Hence $|z| = 1$ and the associated argument is $\phi = 2\arctan(\omega/(a - b\omega^2))$, and $\omega \in \{0, \sqrt{(a/b)}\} \Rightarrow \phi \in \{0, \pi\}$, $\omega \in \{\sqrt{(a/b)}\} \Rightarrow \phi \in \{\pi, 2\pi\}$. From this it follows that this mapping is onto the unit circle in the z-plane. Finally, for an arbitrary σ,

$$z = \frac{(b\sigma^2 - b\omega^2 + \sigma + a) + i(\omega + 2b\sigma\omega)}{(b\sigma^2 - b\omega^2 - \sigma + a) + i(-\omega + 2b\sigma\omega)} \tag{29}$$

and it is easy to see that $\sigma \leq 0 \Rightarrow |z| \leq 1$.

The local truncation error for this method (defined in the same terms as eqn. 24) is

$$E(\zeta, T) = \frac{T^5}{720} f^{[5]}(\zeta) \qquad t_i = iT < \zeta < t_{i+1} \tag{30}$$

Note that the presence of second derivative terms in eqn. 30 means that the first derivatives of the entries in the input vector of eqn. 1 are required for its application. Hence, the stepwise assumption used previously is no longer appropriate because it would clearly introduce additional error. Instead, a simple argument leads to the requirement that the first derivatives of the entries in the input vector are such that $\dot{\boldsymbol{u}}_k(iT + \Delta) \approx \dot{\boldsymbol{u}}_k(iT)$, $\Delta \leq T$, i.e. the entries are approximated between samples by straight line segments where at each sample instant the left and right limits of the approximations are equal. This is termed a piecewise linear approximation here.

Assuming that the entries in the input vector are piecewise linear and applying eqn. 25 to eqn. 1 yields the following state space model:

$$\begin{aligned} \boldsymbol{x}_{k+1}(i+1) = \boldsymbol{A}_3\boldsymbol{x}_{k+1}(i) + \boldsymbol{B}_3\boldsymbol{u}_{k+1}(i+1) \\ + \boldsymbol{B}_4\boldsymbol{u}_{k+1}(i) + \boldsymbol{B}_5\boldsymbol{y}_k(i+1) + \boldsymbol{B}_6\boldsymbol{y}_k(i) \\ \boldsymbol{y}_{k+1}(i) = \boldsymbol{C}\boldsymbol{x}_{k+1}(i) + \boldsymbol{D}_0\boldsymbol{y}_k(i) \end{aligned} \tag{31}$$

where

$$A_3 = \left[I_n - \frac{T}{2}A + \frac{T^2}{12}A^2\right]^{-1}\left[I_n + \frac{T}{2}A + \frac{T^2}{12}A^2\right]$$
$$B_3 = \left[I_n - \frac{T}{2}A + \frac{T^2}{12}A^2\right]^{-1}\left[\frac{T}{2}B - \frac{T^2}{12}AB\right]$$
$$B_4 = \left[I_n - \frac{T}{2}A + \frac{T^2}{12}A^2\right]^{-1}\left[\frac{T}{2}B + \frac{T^2}{12}AB\right]$$
$$B_5 = \left[I_n - \frac{T}{2}A + \frac{T^2}{12}A^2\right]^{-1}\left[\frac{T}{2}B_0 - \frac{T^2}{12}AB_0\right]$$
$$B_6 = \left[I_n - \frac{T}{2}A + \frac{T^2}{12}A^2\right]^{-1}\left[\frac{T}{2}B_0 + \frac{T^2}{12}AB_0\right] \tag{32}$$

It is also easy to show that eqn. 31 have strong structural links with the Roesser model. All that is required is a simple 'forward transformation' of the pass profile vector followed by a change of variable in the pass number. These are given by

$$y_{k-1}(i) \equiv Y_k(i) \qquad 0 \le i < \alpha,\ k \ge 0$$
$$l := k+1 \tag{33}$$

and introducing them into eqn. 31 yields the state space model

$$x_l(i+1) = A_3 x_l(i) + B_3 u_l(i+1) + B_4 u_l(i) + B_5 Y_l(i+1) + B_6 Y_l(i)$$
$$Y_{l+1}(i) = C x_l(i) + D_0 Y_l(i) \tag{34}$$

Now introduce the state vector transformation

$$X_l(i) := x_l(i) - B_3 u_l(i) - B_5 Y_l(i) \tag{35}$$

into eqn. 34 to yield

$$X_l(i+1) = A_3 X_l(i) + (B_4 + A_3 B_3) u_l(i) + (B_6 + A_3 B_5) Y_l(i)$$
$$Y_{l+1}(i) = C X_l(i) + C B_3 u_l(i) + (D_0 + C B_5) Y_l(i) \tag{36}$$

which is precisely the Roesser model without advanced or retarded arguments. Note that, even if the original differential process does not have an inertial term, i.e. of the form $Du_{k+1}(t)$, in the equation defining the pass profile vector, the resulting discretisation has such an inertial term unless $CB_3 = 0$. Numerical examples which demonstrate the advantages of the discretisation method developed in this Section can be found in [5].

5 1-D model

It is known from previous work [4] that a number of key systems theoretic properties for discrete linear repetitive processes cannot be completely characterised using 2-D Roesser model (or equivalent) representations. One such property is so-called complete pass controllability which, in effect, demands the existence of an admissible (in a well defined sense) input sequence to drive the process to a pre-specified pass profile on a pre-specified pass number. In [4] this problem was solved by developing an equivalent 1-D discrete linear time invariant systems state space model of the process dynamics. (Compare this to the 1-D representations of other classes of 2-D linear systems developed by [2] where the matrices and vectors involved increase in dimension as the process evolves, a feature which means that these representations are of extremely limited use in actual problem solving.) This Section develops the 1-D model representation of the dynamics of the discrete approximation eqn. 31 to the dynamics of differential linear repetitive processes of the form of eqn. 1 and considers its (potential) role in onward analysis.

First introduce the so-called pass profile, state and input vectors for eqn. 31 as

$$Y(l) := \left[y_l^T(0), \ldots, y_l^T(\alpha-1)\right]^T$$
$$X(l) := \left[x_l^T(1), \ldots, x_l^T(\alpha)\right]^T$$
$$U(l) := \left[u_l^T(0), \ldots, u_l^T(\alpha-1)\right]^T \tag{37}$$

Then it follows from some extensive, but routine, manipulations that the 1-D discrete linear time invariant state space representation for the dynamics of eqn. 31 has the form

$$Y(l+1) = \Phi Y(l) + \Delta U(l)$$
$$X(l) = \Gamma Y(l) + \Sigma U(l) \tag{38}$$

where

$$\Phi = \begin{bmatrix} D_0 & 0 & 0 & \cdots & 0 \\ CB_6 & \overline{D}_0 & 0 & \cdots & 0 \\ CA_3B_6 & C\overline{B}_6 & \overline{D}_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ CA_3^{\alpha-2}B_6 & CA_3^{\alpha-3}\overline{B}_6 & CA_3^{\alpha-4}\overline{B}_6 & \cdots & \overline{D}_0 \end{bmatrix} \tag{39}$$

$$\Delta = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ CB_4 & \overline{D} & 0 & \cdots & 0 \\ CA_3B_4 & C\overline{B}_4 & \overline{D} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ CA_3^{\alpha-2}B_4 & CA_3^{\alpha-3}\overline{B}_4 & CA_3^{\alpha-4}\overline{B}_4 & \cdots & \overline{D} \end{bmatrix} \tag{40}$$

$$\Gamma = \begin{bmatrix} B_6 & B_5 & 0 & \cdots & 0 & 0 \\ A_3B_6 & \overline{B}_6 & B_5 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ A_3^{\alpha-1}B_6 & A_3^{\alpha-2}\overline{B}_6 & A_3^{\alpha-3}\overline{B}_6 & \cdots & \overline{B}_6 & B_5 \end{bmatrix} \tag{41}$$

$$\Sigma = \begin{bmatrix} B_4 & B_3 & 0 & \cdots & 0 & 0 \\ A_3B_4 & \overline{B}_4 & B_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ A_3^{\alpha-1}B_4 & A_3^{\alpha-2}\overline{B}_4 & A_3^{\alpha-3}\overline{B}_4 & \cdots & \overline{B}_4 & B_3 \end{bmatrix} \tag{42}$$

where

$$\overline{D}_0 = D_0 + CB_5 \qquad \overline{D} = CB_3$$
$$\overline{B}_4 = B_4 + A_3B_3 \qquad \overline{B}_6 = B_6 + A_3B_5 \tag{43}$$

Consider now the discrete counterpart of eqn. 1, a 'true' discrete system. Then it has been shown [4] that stability of a process in the normal sense of its 1-D systems representation is equivalent to asymptotic stability of the process from which it was constructed. The 1-D model, eqn. 38, is, however, stable in the normal sense, and hence asymptotically stable if, and only if, all eigenvalues of the matrix $\boldsymbol{D}_0$ and those of $\bar{\boldsymbol{D}}_0$ have modulus strictly less than unity. The role of the 1-D representation in the construction and analysis of discrete approximations to the dynamics of eqn. 1 is currently being investigated. In addition to the controllability aspects this includes the formulation and solution of optimal control problems and related controller design issues.

6 Conclusions

This paper has considered the problem of developing discrete approximations to the dynamics of differential linear repetitive processes. It has been shown that the trapezoidal rule is not as powerful as in the 1-D linear systems case. In particular, attempting to improve the accuracy of the trapezoidal rule by using more steps produces discrete approximations which are not 'very transparent' in the sense that it is very difficult, if not impossible, to determine what happens to key systems theoretic properties of the differential process under approximation.

To remove these difficulties, it has been shown that higher order single step methods have the basic transparency required, i.e. 'higher accuracy' combined with a resulting discrete linear systems state space model which enables stability properties of the approximation to be immediately established. Also, it is feasible via the existence of Roesser and alternative 2-D linear systems state space model interpretations of the resulting discrete process to study the effects of the approximation on key systems theoretic properties such as reachability and controllability. Similarly, the 1-D representation is available to assist in the investigation of the effects of the approximation on pass controllability and related properties. These and related aspects are currently under investigation and will be reported on in due course.

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