Nonlinear Iterative Learning by an Adaptive Lyapunov Technique

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Abstract

We consider the iterative learning control problem from an adaptive control viewpoint. It is shown that some standard Lyapunov adaptive designs can be modified in a straightforward manner to give a solution to either the feedback or feedforward ILC problem. Some of the common assumptions of nonlinear iterative learning control are relaxed: eg. we relax the common linear growth asssumption on the nonlinearities and handle systems of arbitrary relative degree. It is shown that generally a linear rate of convergence of the MSE can be achieved, and a simple robustness analysis is given. For linear plants we show that a linear rate of MSE convergence can be achieved for non-minimum phase plants.

1 Introduction

The purpose of this paper is to consider the Iterative Learning Control (ILC) problem from an adaptive control viewpoint. The aim is to demonstrate how standard adaptive control designs may be used for iterative learning, provided some simple conditions are observed. We highlight the differences between adaptive and iterative problems, and examine the related issues in some depth.

We take the defining feature of iterative learning control to be the learning of controls to achieve a given task, where the learning is achieved by a sequence of trials

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(or passes) executed on a plant. The plant is assumed to be identical for each pass, although it may be subject to disturbances. A practical ILC design should be capable of learning good controls under limited a-priori knowledge of the plant.

Contrasting to identification experiments, the learning experiments should also simultaneously produce increasingly desirable outputs. The iterative learning scenario is similar to repetitive control, where the system is assumed to be periodic and the objective is to design an asymptotically 'good' control. The difference lies in the ILC assumption that the plant is identical for each experiment: this requires that the internal state of the plant is reset to an identical (nominal) value at the start of each experiment, contrasting to the repetitive control scenario where the state at the start of each period is equal (by definition) to it's value at the end of the preceeding period. A final feature of a good ILC design which we incoporate into our analysis, is the requirement that ultimately (after a finite but suitably large number of trials) the learning can be 'switched off' – this is primarily driven by a technological requirement since the 'learnt' controller is likely to be cheaper to implement than the 'learning' controller which in general will be computationally far more expensive. We formalise the definitions of Iterative Learning in Section 2.

In this paper the algorithms proposed are essentially adaptive control designs but used in the Iterative Learning environment where state resets occur periodically. The main difference between these adaptive ILC designs and their conventional ILC counterparts (see [8] for a recent survey) is that the 'learning' is achieved during the passes, as opposed to the standard designs which only allow learning computations to occur at the end of each pass [8]. This feature distinguishes the contribution in this paper from other work in adaptive ILC such as [10]. Although some researchers consider that learning should only occur at the end of each pass, we do not feel this should be a defining feature of an ILC design. It is true that online learning increases the complexity of the control implementation; which may be unacceptable in some applications; but in many applications this extra real-time computational load will be feasible. Furthermore, as conventional ILC designs are extended from feedforward to feedback, from linear to nonlinear, from state feedback to output feedback etc. the on line computational load on the designs is increasing anyhow, see eg. [2].

The most obvious advantage is that the designs are robust to global parameter variations. There is also a distinct advantage in memory costs of the proposed adaptive feedback design: the memory requirement is small; instead of requiring the storage of the trajectories of the system, we will only require storage of some estimated parameters. In contrast to many other nonlinear ILC designs, we have no need to make restrictive global Lipschitz assumptions [14], [4], nor do we restrict attention to systems with matched uncertainty (ie. those whose uncertainty lies within that span of the control distribution).

We consider a wide variety of systems (eg. linear, matched, strict feedback, output feedback) and show that by simple modifications to wide classes of adaptive designs, we can utilize the adaptive designs to solve the ILC problems. The essential idea is to use a standard adaptive design, and to set the initial parameter estimates of the controller to equal the final parameter estimates of the controller on the preceeding pass. In the linear context, this intuitive idea has been observed by other authors such as R. Longman [6] and K. Moore [7]. Longman largely dismisses this solution due to the inability of adaptive control to handle non-minimum phase plants. However, in this paper we show that the corresponding adaptive ILC designs can handle non-minimum phase plants, in contrast to their adaptive infinite horizon counterparts. Further, in this paper we consider carefully the contributions adaptive control has to make to ILC, eg. in establishing convergence rates and considering generalisation problems. Other work which has considered adaptive control for ILC include [7] who considered the discrete linear case under a persitancy of excitation assumption, and the recent work [13] which considers systems with matched uncertainty, but with non-parametric uncertainties.

Although the adaptive ILC designs are essentially identical to the adaptive design on which they are based, there are a number of features of the ILC problem which do not have counterparts in the standard theory of adaptive control. The most significant of these concerns the effect of the key assumption of ILC, namely that the state is reset at the start of each pass.[1] Although ILC is essentially a 2D process ([11], [1]) by viewing the ILC situation as a 1D dynamics with a standard adaptive design operating, *in some circumstances the periodic resetting of the state can have a destabilising effect on the parameter estimators.* This is not an obvious phenomena, but we establish it's existance in Section 4.1. For the gradient style backstepping designs considered in this paper, this instability occurs only when the state is reset to an non-equilibrium position, and hence does not occur in the linear problems.

We also examine the robustness of the designs, and show that in a natural manner, that the robust adaptive theory can be used to solve ILC problems in the prescence of disturbances, either of eg. the initial conditions, or due to exogenous disturbances. It will be seen that l^1 disturbances of the initial conditions do not destabilise or prevent convergence of the adaptive ILC designs, but merely affect the transient performance in a quantifiable manner. Disturbances to the initial conditions in l^∞ can destabilise, this is illustrated and it is shown how standard robust adaptive theory can overcome this problem.

The standard adaptive theory can then be brought in to study transient performance. In general we establish linear rates of mean square error (MSE) convergence.

A significant application of these adaptive designs is to the linear ILC problem. Due to the stabilising effect of the state resetting, it is shown that the ubiquitous minimum phase assumption of adaptive control is not required for the ILC analogue. Furthermore we establish a linear rate of convergence for this algorithm.

Throughout the paper we do not aim for maximum generality, rather we aim to demonstrate the tools required to establish the adaptive ILC results, which are illustrated on a number of significant plant structures. We develop the matched cases rather further than the strict feedback or output feedback case, but observe that most of the matched results are easily generalised via backstepping ideas.

2 Notation and Problem Definition

The classic feedforward ILC problem is as follows. Let $\mathcal{U}$, the inputs, denote a collection of signals $u: \mathbb{R}_+ \to \mathbb{R}^m$, and let $\mathcal{Y}$, the outputs, denote a second set of signals $y: \mathbb{R}_+ \to \mathbb{R}^n$. We let $\mathcal{C}(\mathcal{U}, \mathcal{O})$ denote all the causal maps from $\mathcal{U}$ to $\mathcal{O}$. Suppose $H_k \in \mathcal{C}(\mathcal{U}, \mathcal{O})$ is

[1]It should be noted that a periodic resetting of the state is not one of the generic peturbations considered in robust adaptive control [9].

an operator which describes the input output behaviour of a plant on pass k:

$$y_k = H_k u_k, \quad k \geq 1. \tag{1}$$

A sequence of inputs $\{u_k\}_{k\geq 1}$ are applied to the plant and the outputs $\{y_k\}_{k\geq 1}$ over the time interval $[0, T]$ are observed.

It is neccessary to model two different types of uncertainty which can occur within ILC problems. The first is uncertainty due to disturbances which act on the system and which vary from pass to pass. Adopting a worst case perspective, we assume that disturbances are $l^p/\mathcal{D}$ bounded, ie that if:

$$d = (d_1|d_2|\ldots|d_k|\ldots), \quad d_k \in \mathcal{D},\ k \geq 1 \tag{2}$$

where the $|$-notation denotes concatenation, ie:

$$d(t) = d_i(t) \quad \text{if } t \in [(i-1)T, iT) \tag{3}$$

and $\mathcal{D}$ is a normed space of admissable signals defined on $[0, T]$, then d is such that:

$$\|d\|_{l^p/\mathcal{D}} = \left(\sum_{k=1}^{\infty} \|d_k\|_{\mathcal{D}}^p\right)^{1/p} \leq \delta_d^{l^p/\mathcal{D}} = \delta_d < \infty. \tag{4}$$

The operator H_k is of the form $H_k = G(d_k)$, so the only variation in the input/output behaviour of the plant is due to the disturbances. Such disturbances are used to model exogenous influences on the system, or disturbances to the initial conditions due to inaccurate state resetting. The second type of uncertainty reflects the a-priori knowledge of the system. This is simply the knowledge available to the control design about G at the start of the first pass. It is denoted by Δ, and corresponds to the knowledge of the set membership $G \in \Delta$.

The feedforward ILC control task is as follows. Given a constant reference signal sequence

$$y_{\mathrm{ref}\,k} = y_{\mathrm{ref}} \in \mathcal{Y}^{\mathrm{ref}} \subset C^n, \quad k \geq 1 \tag{5}$$

where $y_{\mathrm{ref}}\colon \mathbb{R}_+ \to \mathbb{R}$, or equivalently a reference signal $Y^{\mathrm{ref}}\colon \mathbb{R}_+ \to \mathbb{R}^n$ where $Y^{\mathrm{ref}} = (y_{\mathrm{ref}}, {y_{\mathrm{ref}}}^{(1)}, \ldots, {y_{\mathrm{ref}}}^{(n-1)})$, find a sequence of inputs $\{u_k\}_{k\geq 1}$, dependant only on Δ, m_k where m_k is the memory of the passes $1, \ldots, k$, and will be defined subsequently, and causally dependant on y_k, such that the corresponding outputs

$$y_k = H_k u_k \tag{6}$$

converge to a closed ball $B(y_{\mathrm{ref}}, \epsilon)$ of fixed radius ϵ centred on the specified y_{ref}. It is usual to consider perfect tracking ie. $\epsilon = 0$, but in the sequel we will consider problems which preclude perfect tracking. The convergence is usually understood in the topology of $\mathcal{O}$, restricted to the time interval $[0, T]$ although it may be desirable to also consider other convergence measures such as eg. pointwise convergence. We let $\mathcal{P}_k \geq 0$ denote a cost which reflects this convergence, eg.

$$\mathcal{P}_k = \|y_k - B(y_{\mathrm{ref}}, \epsilon)\|_{L^2[0,T]}^2. \tag{7}$$

There is essentially one further constraint placed to obtain a sensible ILC set-up. Specifically we want to eliminate the possibility of high gain solutions. One possibility is to include a penalty on u_k in the cost, and seek to minimise the cost; but in the nonlinear case this leads to a generally intractable optimal control problem. The simplest strategy therefore is simply to assume the reference signals are uniformly bounded, ie. that there exists $\gamma_1, \gamma_2 > 0$ such that:

$$\mathcal{Y}^{\text{ref}} \subset \{Y^{\text{ref}} \in C(\mathbb{R}^+, \mathbb{R}^n) \mid \|Y^{\text{ref}}\| \leq \gamma_1,\ |{y_{\text{ref}}}^{(n)}| \leq \gamma_2\}, \tag{8}$$

and then to demand that all pass signals are uniformly bounded over the domain $\mathbb{N}\times[0,T]$, ie uniformly pointwise bounded for all passes ($k \geq 1$) along the pass intervals $[0,T]$. The pass signals on pass k, $\mathcal{K}_k$, are defined to be all the inputs u_k, outputs y_k and internal states x_k of the plant and of the controller (which here we denote by $\hat{\theta}_k$), ie.

$$\mathcal{K}_k = (u_k, y_k, x_k, \hat{\theta}_k). \tag{9}$$

The uniform boundedness criterion is then that

$$\|\mathcal{K}(\cdot,\cdot)\|_{L^\infty(\mathbb{N}\times[0,T])} < \infty \tag{10}$$

where

$$\mathcal{K}(k,t) = \mathcal{K}_k(t). \tag{11}$$

To propogate information up through the trials, we define the memory on pass k as

$$m_k = F(k, m_{k-1}, \mathcal{K}_k) \in \mathcal{M} \tag{12}$$

for some suitably defined operator

$$F\colon \mathbb{N} \times \mathcal{M} \times \mathcal{K}_k \to \mathcal{M}, \tag{13}$$

where typically $\mathcal{M}$ is the space of all compactly supported sequences (finite memory), or Euclidean space (uniformly bounded memory), or even

$$m_k = (\mathcal{K}_1, \mathcal{K}_2, \ldots \mathcal{K}_k) \tag{14}$$

(complete memory). Note that standard feedforward ILC designs typically store the entire error trajectory in the memory, ie. $\mathcal{M}_k = (y_k - {y_{\text{ref}}}_k)$: this continuous signal is typically sampled to achieve a finite but large memory requirement. The designs considered in this paper will store parameter estimates only, which has a much lower memory requirement.

The feedforward ILC problem can then be formalised as follows.

Definition 2.1. *Given*

$$y_{\text{ref}k} = y_{\text{ref}} \in \mathcal{Y}^{\text{ref}},\ d_k \in \mathcal{D}, \quad k \geq 1, \tag{15}$$

the feedforward ILC problem is to find an operator $\mathcal{B}\colon \mathcal{M} \to \mathcal{U}$ such that for all $G \in \Delta$:

$$\mathcal{P}_k \to 0 \quad \text{as } k \to \infty, \quad \|\mathcal{K}\|_{L^\infty(\mathbb{N}\times[0,T])} < \infty \tag{16}$$

where H_k is of the form $H_k = G(d_k)$, and $u_k = \mathcal{B}m_k$.

The feedback ILC problem is formalised as follows.

Definition 2.2. *Given*
$$y_{\mathrm{ref}_k} \in \mathcal{Y}^{\mathrm{ref}},\ d_k \in \mathcal{D}, \quad k \geq 1, \tag{17}$$
the feedback ILC problem is to find an operator $\mathcal{A}\colon \mathcal{M} \times \mathcal{Y}^{\mathrm{ref}} \to \mathcal{C}(\mathcal{O},\mathcal{U})$, such that for all $G \in \Delta$:
$$\mathcal{P}_k \to 0 \quad \text{as } k \to \infty, \quad \|\mathcal{K}\|_{L^\infty(\mathbb{N}\times[0,T])} < \infty \tag{18}$$
where H_k, Ξ_k are of the form $H_k = G(d_k)$, $\Xi_k(m_k, y_{\mathrm{ref}_k}) = \mathcal{A}(m_k, y_{\mathrm{ref}_k})$ and satisfy the closed loop equations:

$$\begin{aligned}(H_k, \Xi_k(m_k, y_{\mathrm{ref}_k}))\ :\ y_k &= H_k u_k \\ u_k &= \Xi_k(m_k, y_{\mathrm{ref}_k})(y_k).\end{aligned} \tag{19}$$

3 Key Technical Results

The key technical proposition for establishing existence, boundedness and convergence of solutions is given by Proposition 3.1.

Proposition 3.1. *Let $f, h \in C^1(\mathbb{R}^n, \mathbb{R}_+)$, $g \in C^1(\mathbb{R}^p, \mathbb{R}_+)$ and suppose f, g, h are positive definite. Let $[0, \tau_k)$ be the maximal interval of existence for $\mathcal{K}_k = (u_k, y_k, \xi_k, \pi_k)$ for signals $\xi_k \in C^1([0, \tau_k), \mathbb{R}^m)$, $\pi_k \in C^1([0, \tau_k), \mathbb{R}^p)$, for $k \geq 1$. Suppose additionally that $\zeta_k \in C[0, T]$ for $k \geq 1$.*

Suppose $V\colon \mathbb{R}^{n+p} \to \mathbb{R}_+$ satisfies:

1. *$V(\xi_k, \pi_k) = f(\xi_k) + g(\pi_k)$,*
2. *$|\mathcal{K}_k(t)| \leq \nu(V(\xi_k(t), \pi_k(t)))$, $\quad \forall t \in [0, \tau_k)$ for some continuous function $\nu\colon \mathbb{R}_+ \to \mathbb{R}_+$,*
3. *$\dot{V}(\xi_k(t), \pi_k(t)) \leq -h(\xi_k(t)) + \zeta_k(t)$, $\forall t \in [0, \tau_k)$,*
4. *There exists $\lambda > 0$ such that*
$$\lambda \int_0^T h(\xi_k(t))\, dt \geq \mathcal{P}_k, \tag{20}$$
whenever the LHS is defined.

If $\xi_k(0) = 0$, $\pi_k(0) = \pi_{k-1}(T)$, then the pass signals $\mathcal{K}$ are defined on $\mathbb{N} \times [0, T]$ and for all $K \geq 1$, $\forall t \in [0, T]$

$$V(\xi_K(t), \pi_K(t)) \leq V(\xi_1(0), \pi_1(0)) + \sum_{k=1}^{K-1} \int_0^T \zeta_k(t)dt + \int_0^t \zeta_K(t)dt \tag{21}$$

$$\sum_{k=1}^{K} \mathcal{P}_k \leq \lambda g(\pi_1(0)) + \lambda \sum_{k=1}^{K} \int_0^T \zeta_k(t)dt. \tag{22}$$

In particular if $\sum_{k=1}^\infty \int_0^T \zeta_k(t)dt < \infty$, then $\sum_{k=1}^\infty \mathcal{P}_k < \infty$, and $\|\mathcal{K}\|_{L^\infty(\mathbb{N}\times[0,T])} < \infty$.

NB. A-priori we do not assume that solutions exist, hence all the assumptions of the Proposition are in terms of $[0, \tau_k)$, the maximal interval of existence of the solutions on pass k. For an ILC application, we are only interested in the behaviour of the system along each pass over the interval $[0, T]$ so it suffices to check that $\tau_k > T$.

The requirement of condition 2. is typically verified by bounding $|\xi_k|, |\pi_k|$ by V and, and then in turn bounding $|u_k|, |y_k|$ in terms of ξ_k, π_k and y_{ref}.

Proof. To prove that all pass signals are defined on $\mathbb{N} \times [0, T]$ it suffices to show that $\tau_k > T \ \forall k \geq 1$. By induction, assume $\tau_1, \ldots, \tau_{k-1} > T$. Then for $t \in [0, \tau_k)$ we have: $V(\xi_k(0), \pi_k(0)) = V(0, \pi_{k-1}(T)) < \infty$, $\int_0^t |\zeta_k(t)| \, dt \leq \int_0^T |\zeta_k(t)| \, dt < \infty$ as $\zeta_k \in C[0, T]$, and $\int_0^t -h(\xi(t)) \, dt \leq 0$ by positive definiteness of h. Hence,

$$\begin{aligned}
V(\xi_k(t), \pi_k(t)) &= V(\xi_k(0), \pi_k(0)) + \int_0^t \dot{V} \, dt \\
&\leq V(0, \pi_{k-1}(T)) + \int_0^t -h(\xi_k(t)) \, dt + \int_0^t \zeta_k(t) \, dt \\
&\leq V(0, \pi_{k-1}(T)) + \int_0^T -h(\xi_k(t)) \, dt + \int_0^T |\zeta_k(t)| \, dt \\
&< \infty. \qquad (23)
\end{aligned}$$

Since we have shown that $V(\xi_k(t), \pi_k(t))$ is bounded independantly of τ_k, and since by 2, the boundedness of V implies the boundedness of $\mathcal{K}_k(t)$, it follows that $\tau_k > T$ as required. Observe that $V(0, \pi_{k-1}(T)) \leq V(\xi(T), \pi_{k-1}(T))$ and hence that:

$$\begin{aligned}
V(0, \pi_{K-1}(T)) &\leq V(\xi(T), \pi_{K-1}(T)) \\
&\leq V(0, \pi_{K-2}(T)) + \int_0^T \zeta_{K-1}(t) \, dt \\
&\vdots \\
&\leq V(0, \pi_1(T)) + \sum_{k=2}^{K-1} \int_0^T \zeta_{k-1}(t) \, dt \\
&\leq V(0, \pi_1(0)) + \sum_{k=1}^{K-1} \int_0^T \zeta_{k-1}(t) \, dt \\
&= V(\xi_1(0), \pi_1(0)) + \sum_{k=1}^{K-1} \int_0^T \zeta_{k-1}(t) \, dt. \qquad (24)
\end{aligned}$$

Then inequality 21 follows from inequality 23, and inequality 24. The inequality 22 is established as follows:

$$\begin{aligned}
\sum_{k=1}^{K} \mathcal{P}_k &\leq \sum_{k=1}^{K} \lambda \int_0^T h(\xi_k(t)) \, dt \\
&\leq \lambda \sum_{k=1}^{K} \int_0^T -\dot{V}(\xi_k(t), \pi_k(t)) + \zeta_k(t) \, dt \\
&\leq \lambda \sum_{k=1}^{K} \left(V(\xi_k(0), \pi_k(0)) - V(\xi_k(T), \pi_k(T)) + \int_0^T \zeta_k(t) \, dt \right)
\end{aligned}$$

$$
\begin{aligned}
&\leq \lambda V(\xi_1(0), \pi_1(0)) + \lambda \sum_{k=1}^{K} \int_0^T \zeta_k(t)\, dt \\
&\leq \lambda g(\pi_1(0)) + \lambda \sum_{k=1}^{K} \int_0^T \zeta_k(t)\, dt. \qquad (25)
\end{aligned}
$$

This completes the proof. □

We illustrate the applications of this result to the simple case of a chain of intergrators with matched uncertainty. Define:

$$
\begin{aligned}
\Delta = \{G(d)\colon \mathcal{U} \to \mathcal{Y} \quad | \quad & G(d)(u(t)) = x(t),\ \theta \in \mathbb{R}^p,\ d = (d', d'')^T \\
& \dot{x} = Ax + B(\theta^T \phi(x) + u + d') \quad x(0) = d''\} \qquad (26)
\end{aligned}
$$

where the nonlinearity $\phi\colon \mathbb{R}^n \to \mathbb{R}^p$ is known and where

$$
A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}. \qquad (27)
$$

We first consider the case of no disturbances, ie. $d = (d', d'') = 0$. The standard adaptive tracking design is given as follows. Suppose $y_{\text{ref}}\colon \mathbb{R}_+ \to \mathbb{R}$ is a given reference trajectory, and let

$$
Y^{\text{ref}} = (y_{\text{ref}}, {y_{\text{ref}}}^{(1)}, \dots, {y_{\text{ref}}}^{(n-1)})^T. \qquad (28)
$$

Let a be a vector such that $A_* = A - Ba^T$ is Hurwitz, and suppose Q is a positive definite, symmetric matrix. Then let P be the (positive definite, symmetric) solution to the Lyapunov equation $A_*^T P + P A_* = -Q$. Take $b = (P^T + P)B$, and let the adaption gain α be positive.

We define the control to be:

$$
\begin{aligned}
\Xi(\hat{\theta}_0, \alpha)(y_{\text{ref}}) \ : \ u \ &= \ -\hat{\theta}^T \phi(x) - a^T (x - Y^{\text{ref}}) + y_{\text{ref}}^{(n)} \qquad (29) \\
\dot{\hat{\theta}} \ &= \ \alpha (x - Y^{\text{ref}})^T b \phi(x) \quad \hat{\theta}(0) = \hat{\theta}_0. \qquad (30)
\end{aligned}
$$

The closed loop is described by the following equations:

$$
\begin{aligned}
\frac{d}{dt}(x - Y^{\text{ref}}) &= A_*(x - Y^{\text{ref}}) + B((\theta - \hat{\theta})^T \phi(x), \quad x(0) = x_0 \\
\dot{\hat{\theta}} &= \alpha (x - Y^{\text{ref}})^T b \phi(x), \quad \hat{\theta}(0) = \hat{\theta}_0. \qquad (31)
\end{aligned}
$$

It is easily shown in a standard manner that the Lyapunov function

$$
V(x, \hat{\theta}) = (x - Y^{\text{ref}})^T P (x - Y^{\text{ref}}) + \frac{1}{2\alpha} (\theta - \hat{\theta})^T (\theta - \hat{\theta}) \qquad (32)
$$

satisfies

$$
\dot{V} = -(x - Y^{\text{ref}})^T Q (x - Y^{\text{ref}}), \qquad (33)
$$

and hence by La-Salles theorem $x - Y^{\mathrm{ref}} \to 0$ as $t \to \infty$.

We now show this control design is easily applied for feedback ILC. Take $\mathcal{M} = \mathbb{R}^p$, and define the ILC design by:

$$\begin{aligned} m_k &= \theta_{k-1}(T), \quad m_1 = 0, \\ \mathcal{A}m &= \Xi(m, \alpha)(\cdot). \end{aligned} \tag{34}$$

The cost is taken to be:

$$\mathcal{P}_k = \int_0^T (x_k - Y^{\mathrm{ref}})^T Q (x_k - Y^{\mathrm{ref}})\, dt \geq \sqrt{\underline{\lambda}(Q)} \| x_{k1} - y_{\mathrm{ref}k} \|^2_{L^2[0,T]}, \tag{35}$$

where $\underline{\lambda}(Q)$ denotes the minimum eigenvalue of Q.

We make one further assumption, which stands throughout the remainder of the paper. Namely, we assume that all reference signals are compatiable with the initial condition, ie.

$${Y^{\mathrm{ref}}}_k(0) = 0, \quad k \geq 1, \tag{36}$$

since we are assuming that $x_0 = 0$ – later we will perturb x_0, but it's nominal value will still be 0.

We can then establish:

Proposition 3.2. *Let Δ be given by equation (26), with $d = (d^{'}, d^{''}) = 0$. Then ILC design defined by equation (34) based on the adaptive controller defined by equations (29), (30) solves the feedback ILC problem with the cost given by equation (35).*

Proof. This is a straightforward application of Proposition 3.1, with the identifications: $\xi_k = x_k - {Y^{\mathrm{ref}}}_k$, $\pi_k = \theta - \hat{\theta}_k$, $f(\xi_k) = \xi_k^T P \xi_k$, $g(\pi_k) = \frac{1}{2\alpha}\pi_k^T \pi_k$, $h(\xi_k) = \xi_k^T Q \xi_k$, $\lambda = 1$ and $\zeta_k = 0$. Taking $\mathcal{K}_k = (x_k, y_k, u_k, \hat{\theta}_k)$, condition 2 follows since the boundedness of V and $\mathcal{Y}_{ref}$ implies the boundedness of x_k, $\hat{\theta}_k$, and hence of u_k, y_k. □

Let us now consider the effect of disturbances on the closed loop. The first application of the above results is to the case where the initial condition reset is not perfect and is subject to l^1 perturbations, ie when:

$$\|d''\|_{l^1} = \sum_{k=1}^{\infty} \|x_k(0)\|_2 < \infty. \tag{37}$$

Proposition 3.3. *Let Δ be given by equation 26, with $d = (0, d^{''})$ such that $\|d^{''}\|_{l^1} \leq \delta_{d''}$. Suppose further that $\|{Y^{\mathrm{ref}}}_k\|$ is uniformly bounded for $k \geq 1$. Then the ILC design given by equation 34 based on the adaptive controller given by equations 29, 30 solves the feedback ILC problem with cost given by equation 35.*

Proof. Take the same identifications as in Proposition 3.2, Then along pass k,

$$\dot{V} = -x^T Q x \leq -\xi_k^T Q \xi_k, \tag{38}$$

We firstly establish the uniform boundedness of V. By the properties of $\{\pi_k\}_{k\geq 1}$, $\{\xi_k\}_{k\geq 1}$ and V, we can establish that

$$\begin{aligned} V(\xi_k(t), \pi_k(t)) &\leq V(\xi_k(0), \pi_k(0)) \\ &= V(\xi_k(0), \pi_{k-1}(T)) \\ &= V(\xi_{k-1}(T), \pi_{k-1}(T)) + (V(\xi_k(0), \pi_{k-1}(T)) - V(\xi_{k-1}(T), \pi_{k-1}(T))) \\ &\leq V(\xi_{k-1}(T), \pi_{k-1}(T)) + (f(\xi_k(0)) - f(\xi_{k-1}(T))) \\ &\leq V(\xi_{k-1}(T), \pi_{k-1}(T)) + f(\xi_k(0)), \end{aligned} \tag{39}$$

and hence that:

$$\begin{aligned} V(\xi_k(t), \pi_k(t)) &\leq V(\xi_1(T), \pi_1(T)) + \sum_{j=2}^{k} f(\xi_j(0)) \\ &\leq V(\xi_1(0), \pi_1(0)) + \sum_{j=1}^{k} f(\xi_j(0)) \\ &\leq V(\xi_1(0), \pi_1(0)) + \overline{\lambda}(P)\|d^{''}\|_{l^2} \\ &= W. \end{aligned} \tag{40}$$

Similarly to Proposition 3.2, we have thus established the uniform boundedness of $\mathcal{K}$ (by a function of W), and result follows similarly to Propositions 3.1, 3.2. □

Perturbations which are non-decaying require modifications to the basic design. For example, if the inital condition is perturbed in l^∞, the following example shows that unstable behaviour can arise. This unstable behaviour manifests itself as a progressively higher gain control.

Example 3.4. *We take $n = p = 1$, $T = 1$, $\phi(x) = x$, $\theta = 0$, $y_{\text{ref}} = 0$, and $x_0 = 1$. This initial condition corresponds to $x_k(0) = d_k^{''} = 1$, where clearly $d^{''} \in l^\infty$. The control design is defined by $a = 1$, $\alpha = 1$. This gives closed loop equations of the form:*

$$\begin{aligned} \dot{x}_k &= u_k, \quad x_k(0) = 1 \\ u_k &= -\hat{\theta}_k x_k - x_k \\ \dot{\hat{\theta}}_k &= x_k^2, \quad \hat{\theta}_k(0) = \hat{\theta}_{k-1}(T), \hat{\theta}_1(0) = 0. \end{aligned} \tag{41}$$

Let us consider two cases.
Case i). *Suppose $\int_0^T x_k^2 \, dt \to 0$ as $k \to \infty$. Then*

$$T_k = \inf\{t \in [0,T] \mid x_k(t) = 1/2\}, \tag{42}$$

is defined for large k. Then by Cauchy Schwarz:

$$\left(\int_0^{T_k} x_k^2 \, dt\right)\left(\int_0^{T_k} u_k^2 \, dt\right) \geq \left(\int_0^{T_k} x_k u_k \, dt\right)^2 = \left(\int_0^{T_k} x_k \dot{x}_k \, dt\right)^2 = \frac{3}{8}. \tag{43}$$

Hence it follows that $\int_0^T u_k^2 \, dt \to \infty$ as $k \to \infty$, and thus in particular that

$$\|u_k\|_{L^\infty[0,T]} \to \infty \text{ as } k \to \infty. \tag{44}$$

Case ii). *Suppose $\int_0^T x_k^2 \, dt \not\to 0$ as $k \to \infty$. Then $\exists \epsilon > 0$ and a sequence $k_1, \ldots, k_n, \ldots$ such that $\int_0^T x_{k_i}^2 \, dt \geq \epsilon$ for $i \geq 1$. Now we have the inequalities:*

$$\hat{\theta}_{k_i+1}(0) = \hat{\theta}_{k_i}(T) = \hat{\theta}_{k_i}(0) + \int_0^T x_{k_i}^2 \, dt \geq \hat{\theta}_{k_i}(0) + \epsilon \geq \hat{\theta}_1(0) + (i-1) \to \infty \quad \text{as } i \to \infty. \tag{45}$$

Consequently

$$\|u_{k_i}\|_{L^\infty[0,T]} \geq |u_{k_i}(0)| = |\hat{\theta}_{k_i}(0) x_{k_i}(0) - 1| \to \infty \text{ as } i \to \infty. \tag{46}$$

Hence in either case the L^∞ norm of the control signal will grow without bound.

The simplest modification to an adaptive design to ensure stability in the face of l^∞/L^∞ disturbances is to consider a dead-zone modification. Here we simply replace Equation 30 with

$$\dot{\hat{\theta}} = \alpha D(\Omega_0, x - Y^{\mathrm{ref}})(x - Y^{\mathrm{ref}})^T b\phi(x) \quad \hat{\theta}(0) = \hat{\theta}_0, \tag{47}$$

where $D(\Omega_0, e) = 0, 1$ if $e \in \Omega_0$, $e \notin \Omega_0$ respectively. Here Ω_0 is a neighbourhood of zero of the form:

$$\Omega_0 = \{e \in \mathbb{R}^n \mid e^T P e \leq \eta_0\}. \tag{48}$$

As the adaptive design achieves convergence of $x - Y^{\mathrm{ref}} \to \Omega_0$ only, we measure the performance in the corresponding ILC by the cost

$$\mathcal{P}_k = \int_{T_k} (x_k - {Y^{\mathrm{ref}}}_k)^T Q (x_k - {Y^{\mathrm{ref}}}_k)\, dt, \tag{49}$$

where T_k, $k \geq 1$ is defined to be the time set:

$$T_k = \{t \in [0, T] \mid (x_k - {Y^{\mathrm{ref}}}_k)(t) \notin \Omega_0\}. \tag{50}$$

Proposition 3.5. *Let Δ be given by equation 26, with $d = (d', 0)$ such that $\|d'\|_{l^\infty/L^\infty} \leq \delta_{d'}$. Then for sufficiently small $\delta_{d'}$, the ILC design given by equation 34 based on the adaptive controller given by equations 29, 47 solves the feedback ILC problem with the cost given by equation 49.*

Proof. The proof is omitted for brevity. ☐

3.1 Convergence Rates

A general feature of these adaptive ILC designs is that a linear rate of convergence for the MSE costs can be achieved. Establishing good rates of convergence has stimulated other work in ILC such as [1]. We guarantee that the best controller in the first k passes has a performance proportional to $1/k$. Therefore if only a finite number of passes can be executed before learning must switched off, simply selecting the best control signal used to date will guarantee a performance inversely proportional to the number of 'training' passes. We refer to this as the stopping feedforward problem.

The general result required is as follows:

Lemma 3.6. *Under the condition of Proposition 3.1, suppose $\zeta_k = 0\ \forall k \geq 1$. Then*

$$\inf_{1 \leq j \leq k} \mathcal{P}_j \leq \lambda \frac{g(\pi_1(0))}{k}. \tag{51}$$

Proof. Define

$$\epsilon_k = \lambda \frac{g(\pi_1(0))}{k}. \tag{52}$$

Claim that $\inf_{1 \leq j \leq k} \mathcal{P}_j \leq \epsilon_k$. For suppose not, then $\mathcal{P}_j > \epsilon_k$ for $1 \leq j \leq k$. Since this implies:

$$k\epsilon_k < \sum_{j=1}^{k} \mathcal{P}_j \ \leq \ \lambda \sum_{j=1}^{k} g(\pi_j(0)) - g(\pi_j(T))$$

$$
\begin{aligned}
&\leq \lambda \sum_{j=1}^{k} g(\pi_j(0)) - g(\pi_{j+1}(0)) \\
&\leq \lambda \left(g(\pi_1(0)) - g(\pi_{k+1}(0))\right) \\
&\leq k\epsilon_k, \qquad (53)
\end{aligned}
$$

we have a contradiction. □

The basic result on the convergence rate is then as follows:

Proposition 3.7. *Let Δ be given by Equation 26, with $d = 0$. The ILC design 34 based on the adaptive controller 29, 30, solves the stopping feedforward ILC problem with a linear convergence rate when the costs are given by Equations 35.*

4 Strict Feedback Systems

We have applied the results in the previous section to the simplest scenario in nonlinear control, namely when the uncertainty is matched. This situation is particularly simple because the Lyapunov analysis is completed in the same coordinate system as the original state-space equations. Of course, this is not the generic situation in nonlinear control: we typically must consider a coordinate transformation before a Lyapunov analysis can be completed. This coordinate transformation complicates the treatment of the initial condition resetting: whilst the initial state may be reset identically in the original state space, it does not follow that this property automatically holds in the transformed state space, and this requires careful consideration. An example of how this can cause unstable behaviour is given in Section 4.1. Thereafter we make an assumption that the state reset is to an equilibrium point and demonstrate how under this assumption the coordinate transformation may be handled in the exemplar case of a tuning function design for strict feedback systems: however the ideas utilized are generic and can be used for many other designs.

4.1 An unstable ILC controller

Example 4.1. *Consider the following (strict feedback) system:*

$$
\begin{aligned}
\dot{x}_1 &= x_2 + \phi_1(x_1) \quad x(0) = 0 \\
\dot{x}_2 &= u + \theta\phi_2(x_1, x_2) \quad \theta \in \mathbb{R} \\
y &= x_1 \qquad (54)
\end{aligned}
$$

The control task is that of stabilisation, namely to track the reference $y_{\mathrm{ref}} = 0$.

Let us develop a standard adaptive backstepping design [5] for this problem assuming that ϕ_1, ϕ_2 are known, but θ is an a-priori unknown scalar. We define:

$$
\begin{aligned}
z_1 &= x_1 \\
z_2 &= x_2 - \alpha_1 \qquad (55)
\end{aligned}
$$

where we define

$$\alpha_1 = -z_1 - \phi_1(x_1). \qquad (56)$$

The control is taken to be

$$\begin{aligned} u &= \alpha_2 = -z_1 - z_2 - \hat{\theta}\phi_2 - \left(1 + \frac{\partial \phi_1}{\partial x_1}\right)(x_2 + \phi_1) \\ \dot{\hat{\theta}} &= z_2\phi_2. \end{aligned} \tag{57}$$

Writing the system in the z coordinates we have:

$$\begin{aligned} \dot{z}_1 &= -z_1 + z_2 \\ \dot{z}_2 &= -z_1 - z_2 + (\theta - \hat{\theta})\phi_2 \\ \dot{\hat{\theta}} &= z_2\phi_2. \end{aligned} \tag{58}$$

Using the Lyapunov function

$$V(z, \hat{\theta}) = \frac{1}{2}(z_1^2 + z_2^2) + \frac{1}{2}(\theta - \hat{\theta})^2, \tag{59}$$

it is routine to establish that

$$\dot{V} \le -(z_1^2 + z_2^2), \tag{60}$$

and hence that $y \to 0$ as $t \to \infty$.

Now we consider the analagous ILC design, ie. we index all variables by their pass number and use the controller state initialisation:

$$\hat{\theta}_1(0) = 0, \quad \hat{\theta}_k(0) = \hat{\theta}_{k-1}(T) \quad \text{for } k \ge 2. \tag{61}$$

We will now show that for a particular choice of ϕ_1, ϕ_2 we have unstable control signal behaviour, namely that

$$\|u_k\|_{L^\infty} \to \infty \quad \text{as } k \to \infty. \tag{62}$$

We take $\phi_1(0) > 0$ and define:

$$\phi_2 = z_2 = x_2 + x_1 + \phi(x_1). \tag{63}$$

The critical observation is that the z term in the definition of V is not reset to 0 at the start of each pass, contrasting to earlier sections (the reset of x does not imply $z = 0$).

The claim is established as follows.

Case i. *Suppose $\int_0^T z_{k2}^2; dt \not\to 0$ as $k \to \infty$, then*

$$\hat{\theta}_k(0) = \sum_{i=1}^{k-1} \int_0^T \dot{\hat{\theta}}_i \, dt = \sum_{i=1}^{k-1} \int_0^T z_{i2}^2 \, dt \to \infty \quad \text{as } k \to \infty. \tag{64}$$

Then

$$\begin{aligned} \|u_k\|_{L^\infty} &\ge |u_k(0)| \\ &= \left| \phi_1(0) - \hat{\theta}_k\phi_2(0) - \left(1 + \frac{\partial \phi_1}{\partial x_1}\Big|_{x_1=0}\right)(\phi_1(0)) \right| \\ &\ge \hat{\theta}_k(0)\phi_2(0) - \lambda \to \infty \quad \text{as } k \to \infty, \end{aligned} \tag{65}$$

for some $\lambda \in \mathbb{R}$ since $\phi_2(0) = \phi_1(0) > 0$.

Case ii. *Suppose $\int_0^T z_{k2}^2 \, dt \to 0$ as $k \to \infty$, then since $z_{k2}(0) = \phi_1(0) > 0$ it follows that $\dot{z}_{k2}(0) \to \infty$ as $k \to \infty$. Now*

$$\dot{z}_{k2} = u_k(0) + \theta\phi_2(0) + \left(1 + \frac{\partial \phi_1}{\partial x_1}\Big|_{x_1=0}\right)(x_2(0) + \phi_1(0)), \tag{66}$$

so

$$|\dot{z}_{k2}| = |u_k(0)| + \left|\theta\phi_2(0) + \left(1 + \frac{\partial \phi_1}{\partial x_1}\Big|_{x_1=0}\right)(x_2(0) + \phi_1(0))\right|, \tag{67}$$

and since the final term in inequality 67 is constant for all $k \geq 1$, it follows that $|u_k(0)| \to \infty$ as $k \to \infty$, hence $\|u_k\|_{L^\infty} \to \infty$ as $k \to \infty$ as required.

This example clearly demonstrates that adaptive control design must be used with care in ILC environments – the state reset *can* have a destabilising effect, perhaps contary to expectation!

4.2 The adaptive design

The a-priori uncertainty Δ is described by the state space equations:

$$\begin{aligned}
\Delta = \{H\colon \mathcal{U} \to \mathcal{Y} \quad | \quad & H(u(t)) = x(t), \ \theta \in \mathbb{R}^p \\
& \dot{x}_i = x_{i+1} + \phi_i(x_1, \ldots, x_i)^T\theta \quad x(0) = 0, \quad 1 \leq i \leq n-1 \\
& \dot{x}_n = u + \phi_n(x_1, \ldots, x_n)^T\theta \\
& \phi_1(0) = \ldots = \phi_n(0) = 0\}
\end{aligned} \tag{68}$$

Note that whilst by a coordinate shift we assume without loss of generality that $x_0 = 0$ throughout this paper, here we are imposing the stronger requirement than in the preceeding section, namely that the origin is also an equilibrium point.[2] This is to avoid in instability phenomenon demonstrated in example 4.1 which arises due to the coordinate transformation considered in the control design. It is expected that it is possible to dispense with this assumption using the concept of trajectory initialisation [5] when the reference signal sequence is persistantly exciting, but we do not consider this refinement in this paper.

The tuning functions design [5] is reproduced below for convenience (for notational convenience we define $x_{n+1} = u$, $\alpha_0 = 0$):

$$z_i = x_i - {y_{\mathrm{ref}}}^{(i-1)} - \alpha_{i-1} \tag{69}$$

$$\begin{aligned}
\alpha_i = \; & -z_{i-1} - c_i z_i - w_i^T\hat{\theta} + \sum_{k=1}^{i-1}\left(\frac{\partial \alpha_{i-1}}{\partial x_k}x_{k+1} + \frac{\partial \alpha_{i-1}}{\partial {y_{\mathrm{ref}}}^{(k-1)}}{y_{\mathrm{ref}}}^{(k-1)}\right) \\
& -\kappa_i|w_i|^2 z_i + \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}}\Gamma\tau_i + \sum_{k=2}^{i-1}\frac{\partial \alpha_{k-1}}{\partial \hat{\theta}}\Gamma w_i z_k
\end{aligned} \tag{70}$$

$$\tau_i = \tau_{i-1} + w_i z_i \tag{71}$$

$$w_i = \phi_i - \sum_{k=1}^{i-1}\frac{\partial \alpha_{i-1}}{\partial x_k}\phi_k \tag{72}$$

[2]Note that there is no loss of generality in assuming that $\phi_1(0) = \ldots = \phi_n(0) = 0$ if $x = 0$ is an equilibrium point.

where we assume that $c_1, \dots c_n > 0$. The adaptive controller is then taken to be:

$$\begin{aligned} \Xi(\hat{\theta}_0, \Gamma)(y_{\text{ref}}) \; : \; u &= \alpha_n + {y_{\text{ref}}}^{(n)} \\ \dot{\hat{\theta}} &= \Gamma \tau_n \quad \hat{\theta}(0) = \hat{\theta}_0 \end{aligned} \tag{73}$$

We define a coordinate transformation $T\colon \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n \times \mathbb{R}^p$ by $T(x, \hat{\theta}) = (z, \hat{\theta})$. Due to the equilibrium point at the origin, this transformation has the convenient properties: i) $T(0, \hat{\theta}) = (0, \hat{\theta}) \; \forall \hat{\theta} \in \mathbb{R}^p$ as we assume ${Y^{\text{ref}}}_k(0) = x(0) = 0$, and ii) $z_1 = y - y_{\text{ref}}$, (as $\alpha_0 = 0$).

The key standard fact about this adaptive design is as follows:

Proposition 4.2. *Consider the adaptive design $\Xi(y_{\text{ref}}, \hat{\theta}_0)$ defined above, applied to any system $G \in \Delta$ where Δ is given by Equation 68. If*

$$V(z, \hat{\theta}) = \frac{1}{2} z^T z + \frac{1}{2} (\theta - \hat{\theta})^T \Gamma (\theta - \hat{\theta}), \tag{74}$$

then

$$\dot{V} \leq -\sum_{i=1}^{n} c_i z_i^2. \tag{75}$$

Despite the similarity between the quadratic nature of the Lyapunov function and the Lyapunov function for the matched case, it is not trivial in general to establish the convergence of the corresponding ILC design. This is due to the fact that resetting the state x, does not correspond to resetting the transformed state z, so we cannot directly apply eg. Proposition 3.1. However we have made an additional simplifying assumption namely that 0 is an equilibrium point, under which assumption it follows that $x = 0$ implies that $z = 0$, and hence Proposition 3.1 applies (the implication $x = 0 \Rightarrow z = 0$ was not valid in example 4.1).

4.3 The ILC Design

The ILC design is taken to be:

$$\begin{aligned} m_k &= \theta_{k-1}(T), \quad m_1 = 0, \\ \mathcal{A} m &= \Xi(m, \Gamma)(\cdot), \end{aligned} \tag{76}$$

and we consider the L^2 cost:

$$\mathcal{P}_k = \|z_k\|^2_{L^2[0,T]} \geq c_1 \|x_{1k} - {y_{\text{ref}}}_k\|^2_{L^2[0,T]} \tag{77}$$

As throughout the paper, we assume that ${Y^{\text{ref}}}_k(0) = x(0) = 0$.

Proposition 4.3. *Suppose Δ is defined by 68. Then the ILC design 76 based on the adaptive controller 73 solves the feedback ILC problem with the L^2 cost 77.*

Proof. We take $\xi_k = z_k$, $\pi = \theta - \hat{\theta}_k$, $f(\xi_k) = \xi_k^T P \xi_k$, $g(\pi_k) = \frac{1}{2\alpha} \pi_k^T \pi_k$, $h(\zeta_k) = \sum_{i=1}^n c_i z_{ki}^2$, $\lambda = c_1$, and $\zeta_k = 0$. The result then follows from Proposition 3.1, using the facts that $T(0, \hat{\theta}) = (0, \hat{\theta})$ and that $\dot{V} \leq -\sum_{i=1}^n c_i z_{ki}^2$. ☐

The corresponding solution to the stopping feedforward ILC problem, has a linear MSE convergence rate by Lemma 3.6.

5 Output Feedback ILC

5.1 The adaptive design

The a-priori uncertainty Δ is described by the state space equations:

$$\begin{aligned}\Delta = \{H{:}\mathcal{U} \to \mathcal{Y} \quad | \quad & H(u(t)) = y(t),\ \theta \in \mathbb{R}^n \\ & \dot{x} = Ax + \phi(y) + \Phi(y)a + B\sigma(y)u \quad x(0) = 0, \\ & y = e_1^T x \\ & \phi(0) = 0,\ \Phi(0) = 0\}\end{aligned} \tag{78}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}, B = \begin{bmatrix} 0_{(n-m-1)\times 1} \\ b_m \\ \vdots \\ b_1 \\ b_0 \end{bmatrix}. \tag{79}$$

$$\phi(y) = \begin{bmatrix} \varphi_{0,1}(y) \\ \vdots \\ \varphi_{0,n}(y) \end{bmatrix}, \quad \Phi(y) = \begin{bmatrix} \varphi_{1,1}(y) & \dots & \varphi_{q,1}(y) \\ \vdots & & \vdots \\ \varphi_{1,n}(y) & \dots & \varphi_{q,n}(y) \end{bmatrix}. \tag{80}$$

It is important to observe that we are *not* imposing a minimum phase assumption (ie. that b is Hurwitz) as is typical for the adaptive design (we do however assume that the sign of b_m is known). Consequently there can be components of the state which diverge over the infinite horizon. Over the finite horizon considered in ILC, these components will remain bounded: particularly an explosive divergence in finite time is not possible.

As in the previous section we have assumed an equilibrium point at the origin. The discussion in that section is relevant here also.

We do not reproduce the tuning functions design here (see p341 [5]). The key properties of this design are as follows. It is a constructive procedure which yields a quadratic Lyapunov function

$$V = \frac{1}{2}z^T z + \frac{1}{2}\tilde{\theta}^T \Gamma^{-1}\tilde{\theta} + \frac{|b_m|}{2\gamma}\tilde{\varrho}^2 + \sum_{i=1}^{\rho} \frac{1}{4d_i}\varepsilon^T P \varepsilon \tag{81}$$

of the filter error ε, parameter estimates errors $((\tilde{\theta}, \tilde{\rho}) = (\theta - \hat{\theta}, 1/b_m - \hat{\varrho})$, and a transformed state z. Initialising all filters (eg. equations 8.28–8.30 p331 of [5]) to 0 at the start of each pass, we have $\varepsilon = x(0) - Y^{\text{ref}}(0) = 0$, by the standing assumption on the initial state of the reference signal, so that if a coordinate transformation T is defined by: $T(z, \hat{\theta}, \hat{\varrho}, \varepsilon) = (x, \hat{\theta}, \hat{\varrho}, \varepsilon)$ then we have the property that $T(0, \hat{\theta}, \hat{\varrho}, 0) = (0, \hat{\theta}, \hat{\varrho}, 0)$. Furthermore we have the property that $z_{k1} = y_k - {y_{\text{ref}}}_k$.

Defining $\hat{\Theta} = (\hat{\theta}, \hat{\varrho})$, we denote the tuning functions controller (p341 [5]) by

$$\Xi(\hat{\Theta}_0, \Gamma)(y_{\text{ref}}), \tag{82}$$

where Γ is the adaption gain matrix, where the parameter estimates initialised at $\hat{\Theta}_0$, and filters initialised at 0.

The key standard fact about this adaptive design is as follows:

Proposition 5.1. *Consider the adaptive design $\Xi(y_{\mathrm{ref}}, \hat{\theta}_0)$ defined as in (p341 of [5]), applied to any system $G \in \Delta$ where Δ is given by Equation 78. Then*

$$\dot{V} \leq -\sum_{i=1}^{\rho} c_i z_i^2. \tag{83}$$

The ILC design is defined analagously to 76. For brevity, we only establish the simplest ILC result.

Theorem 5.2. *Suppose Δ is defined by 78. Then the ILC design 76 based on the adaptive controller $\Xi(y_{\mathrm{ref}}, \hat{\theta}_0)$ defined as in (p341 of [5]) solves the feedback ILC problem with the L^2 cost 77.*

Proof. A slight modification of the wording of the existence proof of [5] p342, shows that the signals are bounded over any finite time interval as an increasing function of V, even if the system is not minimum phase. We can then apply 3.1 with the identifications: $\xi_k = (z_k, \varepsilon_k)$, $\pi_k = (\tilde{\theta}_k, \tilde{\rho}_k)$,

$$f(z_k, \varepsilon_k) = \frac{1}{2} z_k^T z_k + \sum_{i=1}^{n-m} \frac{1}{4d_i} \varepsilon_k^T P \varepsilon_k \tag{84}$$

$$g(\pi_k) = \frac{1}{2\alpha} \tilde{\theta}_k^T \tilde{\theta}_k + \frac{|b_m|}{2\gamma} \tilde{\rho}_k^2, \tag{85}$$

$$h(z_k, \varepsilon_k) = \sum_{i=1}^{n-m} -c_i z_{ki}^2 \tag{86}$$

$\lambda = c_1$ and $\zeta_k = 0$. □

The corresponding solution to the stopping feedforward ILC problem, has a linear convergence rate by Lemma 3.6.

5.2 Linear adaptive ILC

It is well known that any linear system satisfying the standard assumptions for adaptive linear control, namely that i) the system is minimum phase, ii) the sign of the high frequency gain is known, iii) the relative degree is known, iv) an upper bound on the order is known, can be written in observer canonical form. The observer canonical form is in the output feedback form, hence it is well known therefore that the above adaptive design, (and hence the ILC design) is valid. We have already observed that we have weakened the standard assumption that b is Hurwitz – this is the minimum phase assumption.

We define Δ as follows. Let Δ be defined:

$$\begin{aligned} \Delta = \{H{:}\,\mathcal{U} \to \mathcal{Y} \quad | \quad & H(u(t)) = y(t), \\ & \dot{x} = Ax + Bu \quad x(0) = 0, \\ & y = Cx \\ & \text{Conditions ii)} - \text{iv) hold.}\} \end{aligned} \tag{87}$$

We emphasise that we are not requiring condition i), ie. the system can be non-minimum-phase.

Then by defining the ILC design by Equations 76, and by applying the results of the previous section, we can establish for example that:

Theorem 5.3. *Let Δ be given by Equation 87. Then the stopping feedforward ILC problem is solvable with a linear convergence rate.*

Proof. This result is just a special case of Theorem 5.2. □

6 Concluding Remarks

The goal of this paper was to demonstrate how adaptive control designs can be fruitfully employed in iterative learning scenarios. The principle drawback of such designs is the requirement that the controller is updated on-line, rather than all computations occuring at the end of each pass. This means the implementation of such designs is more complex than many other ILC designs.

However, contrasting to most ILC designs, the control schemes are universal to parameter variations, involve no growth constraints on nonlinearities, are valid for any known relative degree and achieve quantifiable convergence rates. Convergence rate analysis is shown to be particularly simple. In particular a linear rate of convergence of the mean square error can be achieved for non- minimum phase linear systems. The new ILC problem of learning of unseen trajectories (generalisation) is easily achieved within this framework. Although the designs require an uncertain parameterisation, the existence of a parameterisation is assured in the case of finite dimensional linear systems, and can with appropriate handling of modelling errors also be assured in the nonlinear case by utilizing appropriate function approximators in the manner of eg. [12].[3] The ideas developed can easily be extended to other adaptive designs based on global Lyapunov functions such as the designs which have recently been developed for stochastic and distributed nonlinear systems.

Although this paper has concentrated primarily on theoretical issues, we believe that the designs considered are worth consideration in certain application domains. It can be expected that with appropriate robust modifications that the designs will work well in any iterative setting within domains where adaptive control has traditional success. Such areas would certainly include robotics, which is of course a traditional application area for ILC – particularly in manufacturing where the control tasks are naturally repetitive.

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