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Dual field modelling using tubes and slices

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TAS (Tubes And Slices) is the first computer inter-active graphics program, for calculation of electric circuit parameters (R, L, C) of distributed systems, based on the dual energy bounds approach known as the method of tubes and slices. Pre- and post-processing are simple and computing times extremely short as the method does not require a solution of a set of simultaneous equations. A useful combination of tubes and slices with finite elements is also reported. The method has been shown to be particularly suitable for teaching purposes.

Key words: computational electromagnetics, CAD in magnetics, electromagnetism.

THE METHOD OF TUBES AND SLICES

The method of tubes and slices is based on a dual energy formulation. Foundations of this approach may be traced back to Maxwell, who in his famous treatise on electricity and magnetism,[1] describes a variational method applied to the calculation of the resistance of conductors of varying cross-section. The method relies on subdivision of the conductor into slices of equipotential surfaces and tubes separated by very thin insulating sheets. The two calculations yield lower and upper bounds of the resistance, respectively. The approach is applicable to other types of vector fields as demonstrated by Hammond.[2–4] In electrostatics Maxwell's method of 'slices' is identical to the application of the variational principle

$$\langle(\rho - \operatorname{div}\mathbf{D}), \delta\phi\rangle = \langle(\rho + \operatorname{div}\epsilon\nabla\phi), \delta\phi\rangle = 0 \tag{1}$$

where the brackets $\langle\,\rangle$ indicate integration through the region of interest, and his method of 'tubes' is identical to the application of

$$\langle(\mathbf{E} + \nabla\phi), \delta\mathbf{D}\rangle = \left\langle\left(\frac{1}{\epsilon}\mathbf{D} + \nabla\phi\right), \delta\mathbf{D}\right\rangle = 0 \tag{2}$$

The two functionals are equivalent to introducing additional fictitious sources, divergence or curl respectively. In capacitance problems where the energy can be written as $\frac{1}{2}\phi^2 C$ or $\frac{1}{2}Q^2/C$, upper bounds of the capacitance are produced by the variational statement of eqn (1) (slices) and lower bounds result from the use of eqn (2) (tubes). Thus any potential map produces an upper bound of capacitance and any flux map produces a lower bound.

In magnetostatics the equilibrium conditions can be described by two variational principles

$$\langle(\nabla \times \mathbf{H} - \mathbf{J}'), \delta\mathbf{A}\rangle = 0 \tag{3}$$

and

$$\langle(\nabla \times \mathbf{A} - \mathbf{B}), \delta\mathbf{H}\rangle = 0 \tag{4}$$

where $\mathbf{J}'$ is the assigned current density. The first variational principle assumes that $\mathbf{B} = \operatorname{curl}\mathbf{A}$, so that $\operatorname{div}\mathbf{B} = 0$ and thus there are no divergence sources for the magnetic field. However, the expression $\nabla \times \mathbf{H} - \mathbf{J}'$ allows a small variation in $\nabla \times \mathbf{H}$ from its correct value, so that the variation allows a small additional distribution of curl sources. The product of this small fictitious current multiplied by the small variation of $\mathbf{A}$ gives an energy variation of the second order of small quantities which can be put to zero.

The second variational principle assumes that $\operatorname{curl}\mathbf{H} = \mathbf{J}'$, so that the curl sources of the magnetic field are correct. However, the expression $\nabla \times \mathbf{A} - \mathbf{B}$ allows a small variation in the divergence sources. The product of this small polarity distribution multiplied by the small variation of $\mathbf{H}$ gives an energy variation of the second order of small quantities which can be put to zero. The field energy can be expressed either in terms of the field vectors $\mathbf{H}$ and $\mathbf{B}$ by

$$U = \tfrac{1}{2}\langle\mathbf{B}, \mathbf{H}\rangle \tag{5}$$

or in terms of the interaction of the current sources with the vector potential $\mathbf{A}$ by

$$U = \tfrac{1}{2}\langle\mathbf{J}', \mathbf{A}\rangle + \tfrac{1}{2}[\mathbf{I}', \mathbf{A}] \tag{6}$$

where $\mathbf{I}'$ is the assigned line density of current on the surface, and the brackets [] represent integration over the closed boundary surface; $\mathbf{I}'$ is related to $\mathbf{J}'$ so as to make the total current in the system zero. This isolates the system and gives it a unique energy.

The first variational principle is applied to the energy in terms of $\mathbf{A}$ and $\mathbf{B} = \operatorname{curl} \mathbf{A}$ by writing

$$\delta U(\mathbf{A}) = \delta\left(\langle \mathbf{J}', \mathbf{A} \rangle + [\mathbf{I}', \mathbf{A}] - \tfrac{1}{2}\left\langle \mathbf{B}, \frac{\mathbf{B}}{\mu} \right\rangle \right) = 0 \qquad (7)$$

The second variation is therefore negative

$$\delta^2 U(\mathbf{A}) \le 0 \qquad (8)$$

The second variational principle is applied to the energy in terms of $\mathbf{H}$ by writing

$$\delta U(\mathbf{H}) = \delta\left(\tfrac{1}{2}\langle \mathbf{H}, \mu \mathbf{H} \rangle\right) = 0 \qquad (9)$$

Hence

$$\delta^2 U(\mathbf{H}) \ge 0 \qquad (10)$$

For simplicity μ has been assumed constant and this gives the factor $\frac{1}{2}$. However, the method is applicable to permeabilities which are single-valued functions of the field-strength. The second variations show the possibility of obtaining both upper and lower bounds for the energy. The first variational principle treats the field as a collection of tubes and the second one as a collection of slices.

Similar arguments may be presented for a steady current flow and calculation of resistance, where the analogue of permittivity ϵ or magnetic permeability μ is electric conductivity σ. Extension of the method to time-varying problems is possible and some interesting suggestions may be found in Refs 5–7.

DUAL BOUND CALCULATIONS

In electrostatic problems, for a given field solution, the flux map and the potential map may be represented using connections of simple parallel-plate capacitors, as demonstrated in Fig. 1. These component capacitors are connected in parallel and in series, as shown in Fig. 2, to form an equivalent circuit. Neither of the two representations is exact due to the approximations introduced. First, a number of subdivisions will necessarily be finite. Secondly, the flux or potential lines may not be in the correct position so that the orthogonality of the two field maps is violated. This second consideration is very important and leads to the following equation for tubes

$$C^- = \sum_{i=1}^{n} \frac{1}{\displaystyle\sum_{j=1}^{m} \frac{1}{\left(\frac{\epsilon S_{ij}}{l_{ij}}\right)}} \qquad (11)$$

whereas for slices:

$$C^+ = \frac{1}{\displaystyle\sum_{i=1}^{m} \frac{1}{\displaystyle\sum_{j=1}^{m} \left(\frac{\epsilon S_{ij}}{l_{ij}}\right)}} \qquad (12)$$

Thus dual bounds for the capacitance are established.

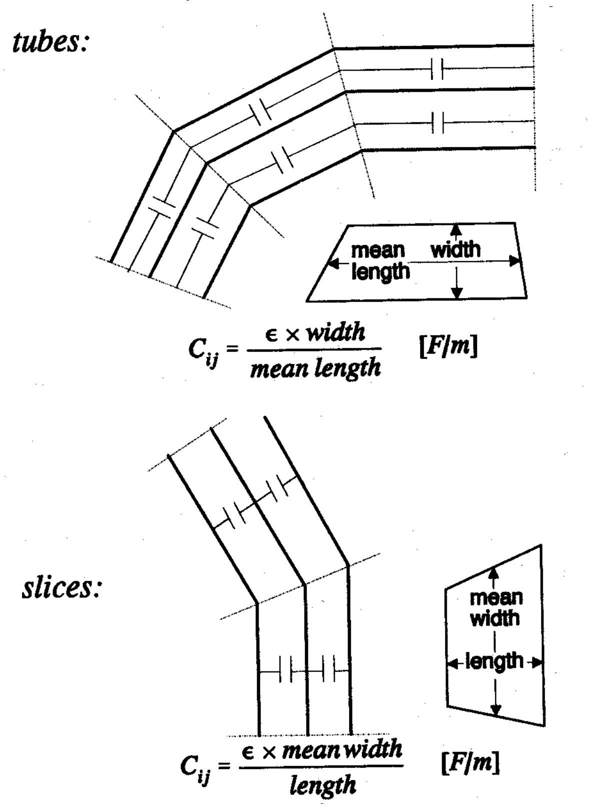

Fig. 1. Circuit representation of tubes and slices in electrostatics.

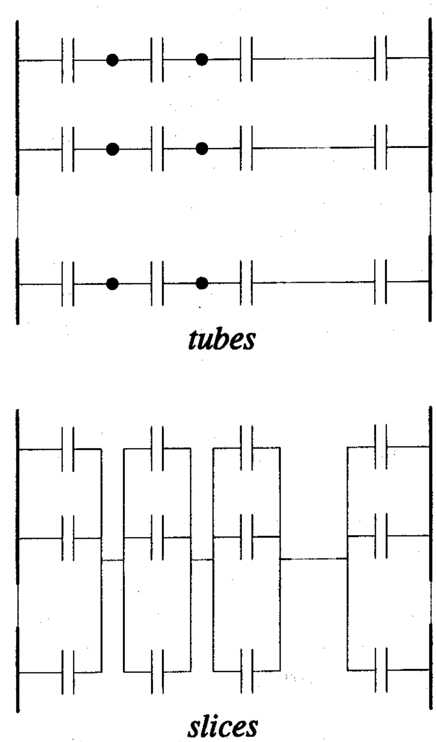

Fig. 2. Series/parallel connections of component capacitors.

For the steady current flow problems the resistance may be calculated in an analogous way. Physically, subdivision into tubes can be achieved by inserting thin insulating sleeves between the tubes, which must always increase the resistance unless the flow of current is undisturbed. Very thin sleeves in the correct direction everywhere will have negligible effect, but if such sleeves are not strictly in the direction of current flow they will increase the resistance. The undisturbed resistance is therefore a minimum. Equally, the insertion of infinitely conducting sheets for slices will reduce the resistance if they disturb the flow. The undisturbed resistance is therefore a maximum. Thus a division into tubes and slices enables one to calculate the upper and lower bounds for the unknown resistance. The appropriate formulae are

$$R^+ = \frac{1}{\sum_{i=1}^{m} \frac{1}{\sum_{j=1}^{n} \frac{l_j}{\sigma S_i}}} \qquad (13)$$

for the tubes, and

$$R^+ = \sum_{j=1}^{n} \frac{1}{\sum_{i=1}^{m} \frac{\sigma S_i}{l_j}} \qquad (14)$$

for the slices. For two-dimensional flow the area S becomes a line.

Finally, in many electrical devices the magnetic circuit is designed in such a way that very little mmf is absorbed in the iron core and attention is focused on the shape and dimensions of the air-gap. An unsaturated iron surface may be assumed to have a constant magnetic potential and thus becomes a slice. At the same time the flux distribution may be described in terms of tubes. Those tubes terminate on iron surfaces. One can work in terms of permeance which is the analogue of conductance in the electric circuit. However, for problems outside a current region the calculation of inductance is reduced to a calculation of permeance. Thus for calculating inductance, equations analogous to (11) and (12) may be used with μ substituted for ϵ. There is now a system of equations for calculating circuit parameters R, C or L for many practical problems under static conditions.

Calculation of an internal inductance, on the other hand, involves working in terms of energy instead of the circuit parameters. A possible approximation technique is suggested in Ref. 8.

The simplicity of the final expressions (eqns (11)–(14)) is very striking and should be contrasted with the complexity of other numerical formulations, such as finite elements. Although the TAS method, indeed like many other methods, usually calls for more than one calculation to achieve the desired accuracy, this should hardly matter in view of the simplicity of this computation.

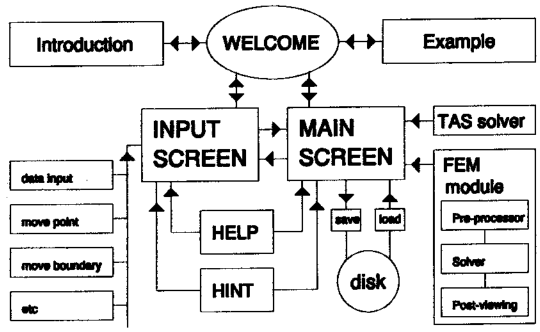

Fig. 3. TAS program flowchart.

THE TAS PROGRAM

The tubes and slices method is essentially a geometrical approach based not on solving equations but using sketching of fields as a means of finding the solution. Full advantage can be taken of inter-active graphics capabilities of modern computers. An appropriate computer package is described in Ref. 8.

In the TAS program the user is in constant interaction with the field solution provided in the form of approximate distributions of tubes and slices. These distributions may be easily modified by moving appropriate construction points and lines on the screen and the system responds almost instantaneously with a pair of bounded solutions given by simple calculations of eqns (11)–(14). The bounded values give confidence limits to the solution, whereas the orthogonality of the two field patterns, or lack of it, sets a criterion for further modifications.

The program works in a menu-driven mode and most operations are performed using a mouse, although the keyboard input is also available. The structure of the TAS program is illustrated in Fig. 3. Two menu screens handle most of the important operations. The input screen provides inter-active data input and mesh reshaping; the main screen is used for obtaining the solution and for post-viewing. Extensive help facilities are provided throughout the program. These include an introduction to the method and animated examples, as well as a built-in manual in the form of help and hint screens. Due to the simplicity of the computational scheme, the solution times are remarkably short: of the order of seconds on a personal computer.

NUMERICAL EXAMPLE

A tubular capacitor is shown in Fig. 4. Due to symmetry only a quarter of the system needs to be investigated (see Fig. 5). A possible set of 'construction lines' and

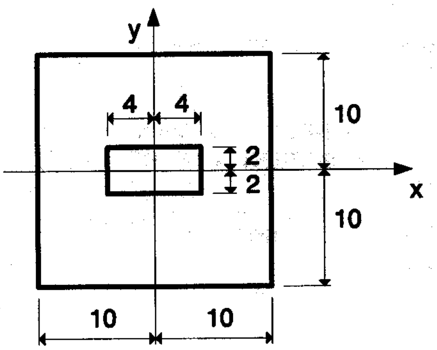

Fig. 4. A tubular capacitor.

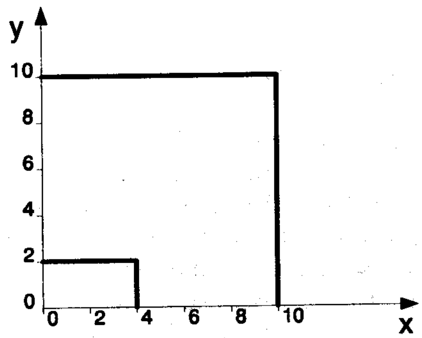

Fig. 5. Computational model of the capacitor.

'quadrilaterals' is demonstrated in Fig. 6. Such quadrilaterals (each with a diagonal line) are used to match a particular shape of system boundaries. These construction lines and quadrilaterals are at the same time used to generate tubes and slices of Fig. 7. The two distributions are also shown in Figs 8 and 9. Each tube and slice may be further subdivided into subtubes and subslices to improve the accuracy of computation. The whole process is fully automated, although the user may wish to change manually some distributions.

An even simpler solution may be found by 'forcing' the flux to go straight across the space between electrodes, and thus making two particularly simple tubes. For the potential map 'rectangular' slices are assumed. These are demonstrated in Fig. 10.

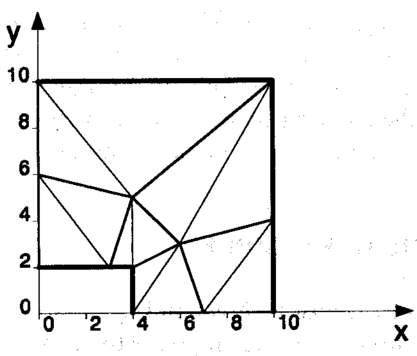

Fig. 6. Construction lines.

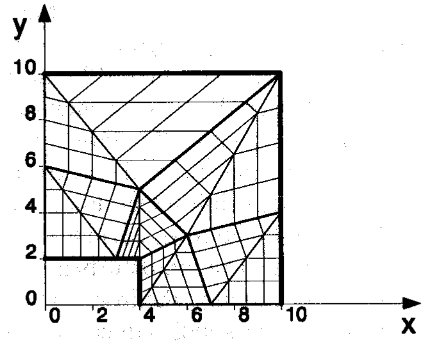

Fig. 7. Distribution of tubes and slices.

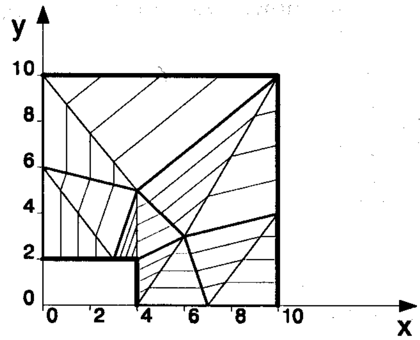

Fig. 8. Distribution of tubes.

This system is suitable for hand calculations. Thus for the two tubes one can write

$$C^- = \frac{4}{8} + \frac{2}{6} = 0{\cdot}833 \times \epsilon \quad \text{(F/m)} \tag{15}$$

and for the slices we have

$$\frac{1}{C^+} = \frac{1}{\dfrac{3}{1{\cdot}5} + \dfrac{4{\cdot}75}{2}} + \frac{1}{\dfrac{5}{1{\cdot}5} + \dfrac{6{\cdot}25}{2}} + \frac{1}{\dfrac{7}{1{\cdot}5} + \dfrac{7{\cdot}75}{2}} + \frac{1}{\dfrac{9}{1{\cdot}5} + \dfrac{9{\cdot}25}{2}} \tag{16}$$

so that

$$C^+ = 1{\cdot}682 \times \epsilon \quad \text{(F/m)} \tag{17}$$

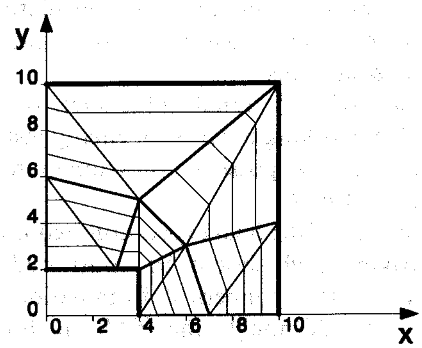

Fig. 9. Distribution of slices.

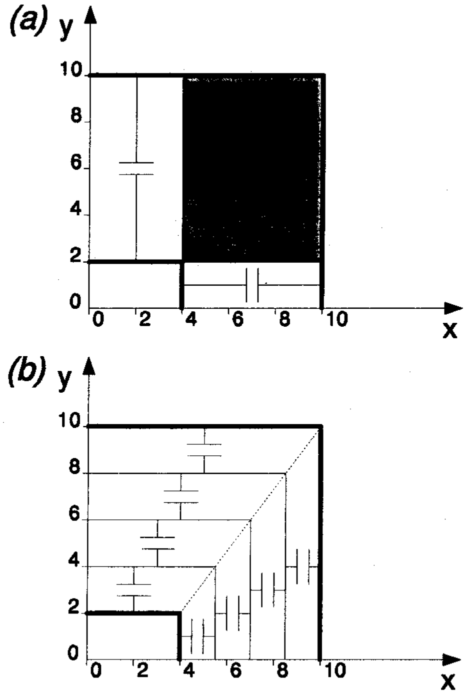

Fig. 10. Simple tubes (a) and slices (b) for hand calculations.

Taking the average yields

$$C_{\text{ave}} = \frac{C^{+} + C^{-}}{2} = 1{\cdot}258 \times \epsilon \pm 33{\cdot}7\% \quad (\text{F/m}) \quad (18)$$

Finally, an improved subdivision may be achieved by adding more construction lines to the basic distribution of Fig. 6 and repositioning them so that a more orthogonal system is obtained. An example of such an improved distribution is shown in Fig. 11.

The results are summarized in Table 1 and compared with a finite-element solution. It is interesting to notice that although the improved solution exhibits improved orthogonality, the average value of the capacitance is hardly changed and even for the very crude hand calculation shows remarkable accuracy. Nevertheless, the smaller error band of the improved solution gives additional confidence to the user.

Table 1. Summary of results

	Hand calculation	Coarse distribution	Improved distribution	Finite elements
C (F/m) (per unit length)	$1{\cdot}258 \times \epsilon$	$1{\cdot}3845 \times \epsilon$	$1{\cdot}3954 \times \epsilon$	$1{\cdot}3979 \times \epsilon$
Confidence limits	±33·7%	±11·1%	±5·0%	±0·32%
Error (against FE)	−10%	−0·96%	−0·18%	—

TUBES/SLICES AND FINITE ELEMENTS

The well-known and popular finite-element method divides regions into small elements (typically triangles in 2D problems) and minimizes the energy functional to obtain field approximation. The approach is substantially different from the tubes and slices method because the elements must always be considered as a complete set. This leads to the solution of a set of simultaneous equations, which is generally a cumbersome process. However, the calculations can be greatly accelerated by using approximate solutions provided by tubes and slices.

The TAS program has a built-in finite-element module. This combination of the two techniques has been found very useful. First, an approximate field distribution is calculated by the tube/slice process and then the accuracy is improved by means of a finite-element procedure. A much smoother distribution is obtained as shown in Fig. 12. It should also be noted that the finite-element method has been adapted to produce upper and lower bounds to the unknown exact values.

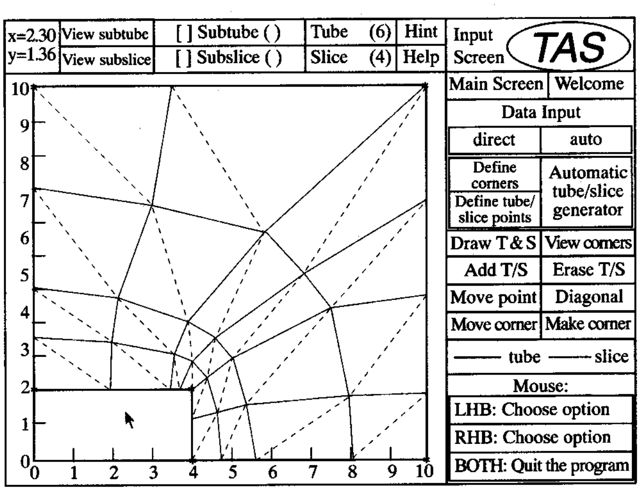

Fig. 11. Improved distribution of tubes and slices.

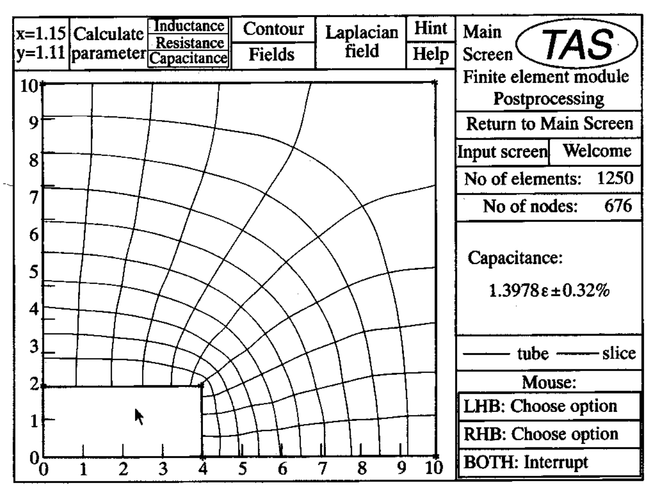

Fig. 12. Finite-element solution for the case of Fig. 4.

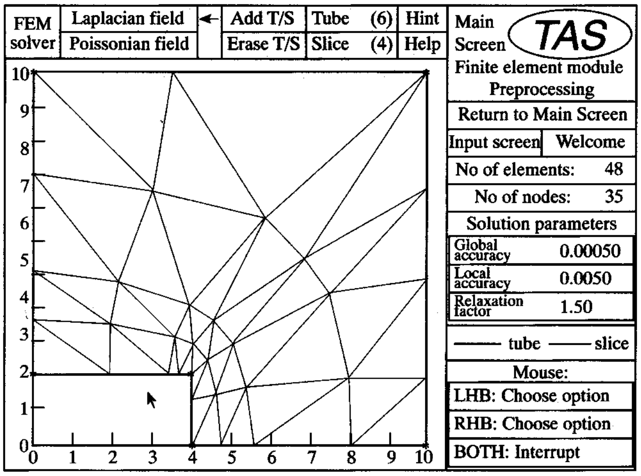

Fig. 13. A basic FE mesh.

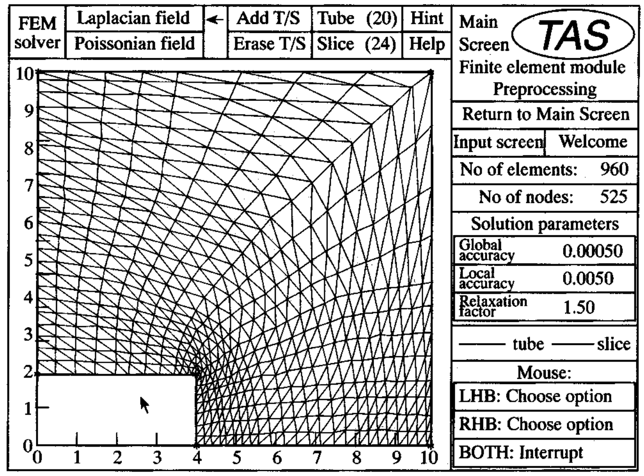

Fig. 14. A refined FE mesh.

The two methods complement each other. The construction lines and quadrilaterals from the TAS calculations are now used as a basis of a FE mesh as shown in Fig. 13. This mesh will typically be refined automatically to improve accuracy, as demonstrated in Fig. 14. The values at all nodes will be first calculated from the TAS solution so that a relatively small number of iterations will be required for the solution to converge. It is also worth noting that the mesh is very regular and that most mesh lines follow the direction of the field or the direction of equipotential surfaces.

TAS AS A TEACHING AID

The method stimulates the student's visual imagination and has been found very effective in introductory courses on fields as well as in advanced courses for design engineers. The method is based on an approach which describes the field in terms of an energy distribution having a geometrical structure. This enables the solution to be found without a direct reference to the partial differential equations which describe the field. Thus the subsequent discretization and solution of a set of simultaneous algebraic equations is not required. The structure allows the region to be subdivided into orthogonal systems of tubes and slices and the system energy parameters, such as resistance, inductance or capacitance, may be calculated as upper and lower bounds giving confidence limits to the solution. If the two bounds are close the local field will be correct. This can be checked visually by looking at the orthogonality locally.

Various teaching aspects of using the TAS approach have been discussed in Ref. 9.

CONCLUSION

The Tubes and Slices method in general, and the TAS program in particular, have been found helpful in field computation as an alternative or complement to other well established methods. TAS is based on a geometrical approach and uses the sketching of fields as a means of finding the solution. The range of applications is still rather restricted but the program has proved an efficient and low-cost CAD tool. It has also been found extremely useful in teaching.

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