# Comparative Study on Pole Sensitivity and Stability Radius Measures for Finite-Precision Digital Controller Realizations

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## Abstract

This paper compares the two approaches based on pole sensitivity and the complex stability radius measures for optimizing the closed-loop stability robustness of digital controllers implemented with finite word length (FWL). Design details and related optimization procedures are derived for the two methods. An example is used to verify the theoretical analysis and to illustrate the two designs for determining optimal FWL controller realizations.

*Keywords*—finite word length, closed-loop stability, complex stability radius, pole sensitivity.

## 1 Introduction

The current controller design methodology often assumes that the controller is implemented exactly, even though in reality a control law can only be realized in finite precision. It is now well-known that a designed stable control system may achieve a lower than predicted performance or even become unstable when the control law is implemented with a finite-precision device. The FWL effect on the closed-loop stability depends on the controller realization structure, and this property can be utilized to select controller realization in order to improve the FWL stability robustness. Currently, two approaches exist for determining the optimal controller realizations under the criteria of the pole-sensitivity measure [1]-[6] and the complex stability radius measure [7],[8].

In the first approach, a suitable pole sensitivity measure is used to quantify the FWL effect, leading to a nonlinear optimization problem to find an optimal FWL controller realization. Efficient global optimization techniques to solve for this optimization problem are readily available [3],[4],[9]. Fialho and Georgiou [8] used the complex stability radius measure to formulate an optimal FWL controller realization problem that can be represented as a special $H_\infty$ norm minimization problem and solved for with the method of linear matrix inequality (LMI) [10],[11].

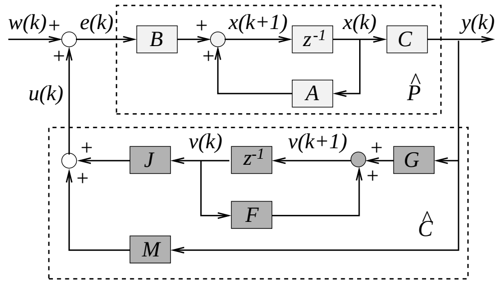


Figure 1: Discrete-time closed-loop control system.

This paper provides a comparative study on these two approaches for determining optimal FWL controller realizations[1].

## 2 Problem formulation

Consider the discrete-time closed-loop control system shown in Fig. 1, where the linear time-invariant plant $\hat{P}$ is described by

$$\begin{cases} \mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{e}(k) \\ \qquad \mathbf{y}(k) = \mathbf{C}\mathbf{x}(k) \end{cases} \tag{1}$$

which is completely state controllable and observable with $\mathbf{A} \in \mathcal{R}^{n\times n}$, $\mathbf{B} \in \mathcal{R}^{n\times p}$ and $\mathbf{C} \in \mathcal{R}^{q\times n}$; and the digital output-feedback controller $\hat{C}$ is described by

$$\begin{cases} \mathbf{v}(k+1) = \mathbf{F}\mathbf{v}(k) + \mathbf{G}\mathbf{y}(k) \\ \quad \mathbf{u}(k) = \mathbf{J}\mathbf{v}(k) + \mathbf{M}\mathbf{y}(k) \end{cases} \tag{2}$$

with $\mathbf{F} \in \mathcal{R}^{m\times m}$, $\mathbf{G} \in \mathcal{R}^{m\times q}$, $\mathbf{J} \in \mathcal{R}^{p\times m}$ and $\mathbf{M} \in \mathcal{R}^{p\times q}$. Assume that a realization $(\mathbf{F}_0, \mathbf{G}_0, \mathbf{J}_0, \mathbf{M}_0)$ of $\hat{C}$ has been designed. It is well-known that the realizations of $\hat{C}$ are not unique. All the realizations of $\hat{C}$ form the realization set:

$$\mathcal{S} = \{(\mathbf{F}, \mathbf{G}, \mathbf{J}, \mathbf{M}) : \mathbf{F} = \mathbf{T}^{-1}\mathbf{F}_0\mathbf{T}, \mathbf{G} = \mathbf{T}^{-1}\mathbf{G}_0, \mathbf{J} = \mathbf{J}_0\mathbf{T}, \mathbf{M} = \mathbf{M}_0\} \tag{3}$$

[1] Fialho and Georgiou's ACC99 paper [8] only contained the two-page summary. The material for the complex stability radius approach presented at this paper are our interpretation.

where $\mathbf{T} \in \mathcal{R}^{m \times m}$ is any real-valued non-singular matrix. Let $\mathbf{w_F} = \mathrm{Vec}(\mathbf{F})$, where $\mathrm{Vec}(\cdot)$ denotes the column stacking operator. Denote

$$\mathbf{w} = \begin{bmatrix} w_1 \\ \vdots \\ w_N \end{bmatrix} \triangleq \begin{bmatrix} \mathbf{w_F} \\ \mathbf{w_G} \\ \mathbf{w_J} \\ \mathbf{w_M} \end{bmatrix}, \quad \mathbf{w}_0 \triangleq \begin{bmatrix} \mathbf{w}_{\mathbf{F}_0} \\ \mathbf{w}_{\mathbf{G}_0} \\ \mathbf{w}_{\mathbf{J}_0} \\ \mathbf{w}_{\mathbf{M}_0} \end{bmatrix} \tag{4}$$

where $N = (m+p)(m+q)$. We also refer to $\mathbf{w}$ as a realization of $\hat{C}$. The stability of the closed-loop system in Fig. 1 depends on the poles of the matrix

$$\begin{aligned} \bar{\mathbf{A}}(\mathbf{w}) &= \begin{bmatrix} \mathbf{A}+\mathbf{BMC} & \mathbf{BJ} \\ \mathbf{GC} & \mathbf{F} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{T}^{-1} \end{bmatrix} \bar{\mathbf{A}}(\mathbf{w}_0) \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{T} \end{bmatrix} \end{aligned} \tag{5}$$

All the different realizations $\mathbf{w}$ achieve exactly the same set of closed-loop poles if they are implemented with infinite precision. Since the closed-loop system is designed to be stable, the eigenvalues

$$|\lambda_i(\bar{\mathbf{A}}(\mathbf{w}))| = |\lambda_i(\bar{\mathbf{A}}(\mathbf{w}_0))| < 1, \ \forall i \in \{1, \ldots, m+n\} \tag{6}$$

When a $\mathbf{w}$ is implemented with a fixed-point processor, it is perturbed into $\mathbf{w} + \Delta\mathbf{w}$ due to the FWL effect. Each element of $\Delta\mathbf{w}$ is bounded by $\pm\epsilon/2$,

$$\|\Delta\mathbf{w}\|_\infty \triangleq \max_{i \in \{1,\cdots,N\}} |\Delta w_i| \le \epsilon/2 \tag{7}$$

For a fixed point processor of $B_s$ bits, let $B_s = B_i + B_f$, where $2^{B_i}$ is a "normalization" factor to make the absolute value of each element of $2^{-B_i}\mathbf{w}$ no larger than 1. Thus, $B_i$ are bits required for the integer part of a number and $B_f$ are bits used to implement the fractional part of a number. It can be seen that

$$\epsilon = 2^{-B_f}, \tag{8}$$

With the perturbation $\Delta\mathbf{w}$, $\lambda_i(\bar{\mathbf{A}}(\mathbf{w}))$ is moved to $\lambda_i(\bar{\mathbf{A}}(\mathbf{w}+\Delta\mathbf{w}))$. It is critical to know when the FWL error will cause closed-loop instability. That is, we would like to know the largest open "sphere" in $\Delta\mathbf{w}$ space, within which closed-loop remains stable. The size or radius of this sphere is defined by:

$$\mu_0(\mathbf{w}) \triangleq \inf\{\|\Delta\mathbf{w}\|_\infty : \bar{\mathbf{A}}(\mathbf{w}+\Delta\mathbf{w}) \text{ is unstable}\} \tag{9}$$

The larger $\mu_0(\mathbf{w})$ is, the larger FWL error the closed-loop stability can tolerate. Let $B_s^{\min}$ be the smallest word length, when used to implement $\mathbf{w}$, can guarantee the closed-loop stability. $B_s^{\min}$ is generally unknown. An estimate of $B_s^{\min}$ can be obtained by

$$\hat{B}_{s0}^{\min} = B_i + \mathrm{Int}[-\log_2(\mu_0(\mathbf{w}))] - 1 \tag{10}$$

where the integer $\mathrm{Int}[x] \ge x$. It can easily be seen that the closed-loop system remains stable if $\mathbf{w}$ is implemented with a fixed-point processor of at least $\hat{B}_{s0}^{\min}$. Moreover, $\mu_0(\mathbf{w})$ is a function of the controller realization $\mathbf{w}$, we could search for an optimal realization that maximizes $\mu_0(\mathbf{w})$.

However, it is not known how to compute $\mu_0(\mathbf{w})$. A solution is to derive a lower bound of the stability measure $\mu_0(\mathbf{w})$, which is computationally tractable. This in effect defines a smaller but known stable "sphere" in the controller perturbation space. The closer a lower bound is to $\mu_0(\mathbf{w})$, the better. The pole sensitivity and the complex stability radius measures can both be regarded as such lower bounds.

# 3 Pole sensitivity approach

Roughly speaking, how easily the FWL error $\Delta\mathbf{w}$ can cause a stable control system to become unstable is determined by how close $|\lambda_i(\bar{\mathbf{A}}(\mathbf{w}))|$ are to 1 and how sensitive they are to the controller parameter perturbations. This leads to the following FWL stability measure [6]

$$\mu_p(\mathbf{w}) \triangleq \min_{i \in \{1,\cdots,m+n\}} \frac{1 - |\lambda_i(\bar{\mathbf{A}}(\mathbf{w}))|}{\alpha_i(\mathbf{w})} \tag{11}$$

with

$$\alpha_i(\mathbf{w}) \triangleq \sum_{\mathbf{X}=\mathbf{F},\mathbf{G},\mathbf{J},\mathbf{M}} \left\| \frac{\partial |\lambda_i(\bar{\mathbf{A}}(\mathbf{w}))|}{\partial \mathbf{w_X}} \right\|_1 \tag{12}$$

For a vector $\mathbf{x} \in \mathcal{C}^s$, the 1-norm $\|\mathbf{x}\|_1$ is defined as

$$\|\mathbf{x}\|_1 \triangleq \sum_{i=1}^{s} |x_i| \tag{13}$$

It can be proved that under certain conditions $\mu_p(\mathbf{w})$ is a lower bound of $\mu_0(\mathbf{w})$, that is, $\mu_p(\mathbf{w}) \le \mu_0(\mathbf{w})$.

The stability measure $\mu_p(\mathbf{w})$ is computationally tractable, as it can be shown that:

$$\frac{\partial |\lambda_i(\bar{\mathbf{A}}(\mathbf{w}))|}{\partial \mathbf{F}} = [\,\mathbf{0} \quad \mathbf{I}\,]\, L_i(\mathbf{w}) \begin{bmatrix} \mathbf{0} \\ \mathbf{I} \end{bmatrix} \tag{14}$$

$$\frac{\partial |\lambda_i(\bar{\mathbf{A}}(\mathbf{w}))|}{\partial \mathbf{G}} = [\,\mathbf{0} \quad \mathbf{I}\,]\, L_i(\mathbf{w}) \begin{bmatrix} \mathbf{C}^T \\ \mathbf{0} \end{bmatrix} \tag{15}$$

$$\frac{\partial |\lambda_i(\bar{\mathbf{A}}(\mathbf{w}))|}{\partial \mathbf{J}} = [\,\mathbf{B}^T \quad \mathbf{0}^T\,]\, L_i(\mathbf{w}) \begin{bmatrix} \mathbf{0} \\ \mathbf{I} \end{bmatrix} \tag{16}$$

$$\frac{\partial |\lambda_i(\bar{\mathbf{A}}(\mathbf{w}))|}{\partial \mathbf{M}} = [\,\mathbf{B}^T \quad \mathbf{0}^T\,]\, L_i(\mathbf{w}) \begin{bmatrix} \mathbf{C}^T \\ \mathbf{0} \end{bmatrix} \tag{17}$$

with

$$L_i(\mathbf{w}) = \frac{\mathrm{Re}\left[\lambda_i^*(\bar{\mathbf{A}}(\mathbf{w}))\mathbf{y}_i^*(\bar{\mathbf{A}}(\mathbf{w}))\mathbf{x}_i^T(\bar{\mathbf{A}}(\mathbf{w}))\right]}{|\lambda_i(\bar{\mathbf{A}}(\mathbf{w}))|} \tag{18}$$

where $\mathbf{x}_i(\bar{\mathbf{A}}(\mathbf{w}))$ and $\mathbf{y}_i(\bar{\mathbf{A}}(\mathbf{w}))$ are the right and reciprocal left eigenvectors related to the $\lambda_i(\bar{\mathbf{A}}(\mathbf{w}))$, $^*$ denotes the conjugate operation, $^T$ the transpose operator, and Re$[\cdot]$ the real part. Similar to (10), an estimate of $B_s^{\min}$ can be provided with $\mu_p(\mathbf{w})$ by

$$\hat{B}_{sp}^{\min} = B_i + \mathrm{Int}[-\log_2(\mu_p(\mathbf{w}))] - 1 \tag{19}$$

Given an initial design $\mathbf{w}_0$, the optimal FWL controller realization that maximizes the stability measure (11) is defined as

$$\mathbf{w}_{\mathrm{opt},p} = \arg\max_{\mathbf{w}\in\mathcal{S}} \mu_p(\mathbf{w}) \tag{20}$$

and the optimization procedure to find a $\mathbf{w}_{\mathrm{opt},p}$ can readily be derived. $\forall i \in \{1,\cdots,m+n\}$, partition

$$\mathbf{x}_i(\bar{\mathbf{A}}(\mathbf{w}_0)) = \begin{bmatrix} \mathbf{x}_{i,1}(\bar{\mathbf{A}}(\mathbf{w}_0)) \\ \mathbf{x}_{i,2}(\bar{\mathbf{A}}(\mathbf{w}_0)) \end{bmatrix} \tag{21}$$

$$\mathbf{y}_i(\bar{\mathbf{A}}(\mathbf{w}_0)) = \begin{bmatrix} \mathbf{y}_{i,1}(\bar{\mathbf{A}}(\mathbf{w}_0)) \\ \mathbf{y}_{i,2}(\bar{\mathbf{A}}(\mathbf{w}_0)) \end{bmatrix} \tag{22}$$

where $\mathbf{x}_{i,1}(\bar{\mathbf{A}}(\mathbf{w}_0)), \mathbf{y}_{i,1}(\bar{\mathbf{A}}(\mathbf{w}_0)) \in \mathcal{C}^n$, $\mathbf{x}_{i,2}(\bar{\mathbf{A}}(\mathbf{w}_0))$, $\mathbf{y}_{i,2}(\bar{\mathbf{A}}(\mathbf{w}_0)) \in \mathcal{C}^m$. It is easily seen from (5) that

$$\mathbf{x}_i(\bar{\mathbf{A}}(\mathbf{w})) = \begin{bmatrix} \mathbf{x}_{i,1}(\bar{\mathbf{A}}(\mathbf{w}_0)) \\ \mathbf{T}^{-1}\mathbf{x}_{i,2}(\bar{\mathbf{A}}(\mathbf{w}_0)) \end{bmatrix} \tag{23}$$

$$\mathbf{y}_i(\bar{\mathbf{A}}(\mathbf{w})) = \begin{bmatrix} \mathbf{y}_{i,1}(\bar{\mathbf{A}}(\mathbf{w}_0)) \\ \mathbf{T}^{T}\mathbf{y}_{i,2}(\bar{\mathbf{A}}(\mathbf{w}_0)) \end{bmatrix} \tag{24}$$

From (14)–(17), we have

$$\frac{\partial\left|\lambda_i(\bar{\mathbf{A}}(\mathbf{w}))\right|}{\partial \mathbf{F}} = \mathbf{T}^T L_{i,2,2}(\mathbf{w}_0)\mathbf{T}^{-T} \tag{25}$$

$$\frac{\partial\left|\lambda_i(\bar{\mathbf{A}}(\mathbf{w}))\right|}{\partial \mathbf{G}} = \mathbf{T}^T L_{i,2,1}(\mathbf{w}_0)\mathbf{C}^{T} \tag{26}$$

$$\frac{\partial\left|\lambda_i(\bar{\mathbf{A}}(\mathbf{w}))\right|}{\partial \mathbf{J}} = \mathbf{B}^T L_{i,1,2}(\mathbf{w}_0)\mathbf{T}^{T} \tag{27}$$

$$\frac{\partial\left|\lambda_i(\bar{\mathbf{A}}(\mathbf{w}))\right|}{\partial \mathbf{M}} = \mathbf{B}^T L_{i,1,1}(\mathbf{w}_0)\mathbf{C}^{T} \tag{28}$$

where

$$L_{i,j,l}(\mathbf{w}_0) = \frac{\mathrm{Re}\left[\lambda_i^*(\bar{\mathbf{A}}(\mathbf{w}_0))\mathbf{y}_{i,j}^*(\bar{\mathbf{A}}(\mathbf{w}_0))\mathbf{x}_{i,l}^T(\bar{\mathbf{A}}(\mathbf{w}_0))\right]}{\left|\lambda_i(\bar{\mathbf{A}}(\mathbf{w}_0))\right|}, \quad j,l = 1,2 \tag{29}$$

Define the following cost function:

$$f(\mathbf{T}) \triangleq \min_{i\in\{1,\cdots,m+n\}} \frac{1-|\lambda_i(\bar{\mathbf{A}}(\mathbf{w}_0))|}{\alpha_i(\mathbf{w})} = \mu_p(\mathbf{w}) \tag{30}$$

The optimal realization problem (20) can then be posed as the following optimisation problem:

$$\mathbf{T}_{\mathrm{opt},p} = \arg \max_{\substack{\mathbf{T}\in\mathcal{R}^{m\times m} \\ \det(\mathbf{T})\neq 0}} f(\mathbf{T}) \tag{31}$$

Although $f(\mathbf{T})$ is non-smooth and non-convex, efficient global optimisation methods exist for solving for this kind of optimisation problem. With $\mathbf{T}_{\mathrm{opt},p}$, we can obtain the optimal realization $\mathbf{w}_{\mathrm{opt},p}$.

# 4 Stability radius approach

Let $\partial E$ denote the unit circle in the complex plane, and $\bar{\sigma}(\mathbf{U})$ the maximal singular value of the complex-valued matrix $\mathbf{U}$. For a stable matrix $\tilde{\mathbf{A}} \in \mathcal{C}^{(n+m)\times(n+m)}$, i.e. $|\lambda_i(\tilde{\mathbf{A}})| < 1$ for $i = 1,\cdots,n+m$, the complex stability radius of a matrix triple $(\tilde{\mathbf{A}},\tilde{\mathbf{B}},\tilde{\mathbf{C}}) \in \mathcal{C}^{(n+m)\times(n+m)} \times \mathcal{C}^{(n+m)\times(p+m)} \times \mathcal{C}^{(q+m)\times(n+m)}$ is defined as

$$\begin{aligned} r_{\mathcal{C}}(\tilde{\mathbf{A}},\tilde{\mathbf{B}},\tilde{\mathbf{C}}) = \inf\{&\bar{\sigma}(\mathbf{\Delta}) : \mathbf{\Delta} \in \mathcal{C}^{(p+m)\times(q+m)} \text{ and} \\ &\tilde{\mathbf{A}} + \tilde{\mathbf{B}}\mathbf{\Delta}\tilde{\mathbf{C}} \text{ is unstable}\} \end{aligned} \tag{32}$$

From [12],[13], we have

$$r_{\mathcal{C}}(\tilde{\mathbf{A}},\tilde{\mathbf{B}},\tilde{\mathbf{C}}) = \frac{1}{\sup_{z\in\partial E} \bar{\sigma}\left(\tilde{\mathbf{C}}(z\mathbf{I}-\tilde{\mathbf{A}})^{-1}\tilde{\mathbf{B}}\right)} \tag{33}$$

Define the transfer function matrix $\hat{\mathbf{G}} = \tilde{\mathbf{C}}(z\mathbf{I} - \tilde{\mathbf{A}})^{-1}\tilde{\mathbf{B}}$ and the $H_\infty$-norm of $\hat{\mathbf{G}}$ [11]:

$$\|\hat{\mathbf{G}}\|_\infty = \sup_{z\in\partial E} \bar{\sigma}\left(\tilde{\mathbf{C}}(z\mathbf{I}-\tilde{\mathbf{A}})^{-1}\tilde{\mathbf{B}}\right) \tag{34}$$

Then,

$$r_{\mathcal{C}}(\tilde{\mathbf{A}},\tilde{\mathbf{B}},\tilde{\mathbf{C}}) = \frac{1}{\|\hat{\mathbf{G}}\|_\infty} \tag{35}$$

Let $\gamma > 0$ be a given scalar. According to [11] (page 158), the linear time-invariant discrete-time closed-loop transfer function $\hat{\mathbf{G}}$ satisfies $\|\hat{\mathbf{G}}\|_\infty < \gamma$ if and only if there exists a matrix $\mathbf{X} > 0$ such that

$$\begin{bmatrix} \mathbf{X} & \mathbf{0} \\ \mathbf{0} & \gamma^2\mathbf{I} \end{bmatrix} > \begin{bmatrix} \tilde{\mathbf{A}} & \tilde{\mathbf{B}} \\ \tilde{\mathbf{C}} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{X} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{A}} & \tilde{\mathbf{B}} \\ \tilde{\mathbf{C}} & \mathbf{0} \end{bmatrix}^T \tag{36}$$

Let $\bar{\mathbf{A}}_0$ be the closed-loop system matrix for an initial controller realization $(\mathbf{F}_0,\mathbf{G}_0,\mathbf{J}_0,\mathbf{M}_0)$. For $(\mathbf{F} = \mathbf{T}^{-1}\mathbf{F}_0\mathbf{T}, \mathbf{G} = \mathbf{T}^{-1}\mathbf{G}_0, \mathbf{J} = \mathbf{J}_0\mathbf{T}, \mathbf{M} = \mathbf{M}_0)$, consider the controller perturbation

$$\begin{bmatrix} \mathbf{M} & \mathbf{J} \\ \mathbf{G} & \mathbf{F} \end{bmatrix} + \mathbf{\Delta} \tag{37}$$

where $\mathbf{\Delta}$ is complex-valued. With (37), the closed-loop system matrix (5) becomes

$$\begin{aligned} \bar{\mathbf{A}} &= \begin{bmatrix} \mathbf{I}_n & \mathbf{0} \\ \mathbf{0} & \mathbf{T}^{-1} \end{bmatrix} \bar{\mathbf{A}}_0 \begin{bmatrix} \mathbf{I}_n & \mathbf{0} \\ \mathbf{0} & \mathbf{T} \end{bmatrix} \\ &+ \begin{bmatrix} \mathbf{B} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_m \end{bmatrix} \mathbf{\Delta} \begin{bmatrix} \mathbf{C} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_m \end{bmatrix} \end{aligned} \tag{38}$$

where $\mathbf{I}_s$ denotes the $s\times s$ identity matrix. Denote

$$\tilde{\mathbf{A}}(\mathbf{T}) = \begin{bmatrix} \mathbf{I}_n & \mathbf{0} \\ \mathbf{0} & \mathbf{T}^{-1} \end{bmatrix} \bar{\mathbf{A}}_0 \begin{bmatrix} \mathbf{I}_n & \mathbf{0} \\ \mathbf{0} & \mathbf{T} \end{bmatrix} \in \mathcal{R}^{(n+m)\times(n+m)} \tag{39}$$

$$\tilde{\mathbf{B}} = \begin{bmatrix} \mathbf{B} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_m \end{bmatrix} \in \mathcal{R}^{(n+m)\times(p+m)} \tag{40}$$

$$\tilde{\mathbf{C}} = \begin{bmatrix} \mathbf{C} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_m \end{bmatrix} \in \mathcal{R}^{(q+m)\times(n+m)} \tag{41}$$

$$\hat{\mathbf{G}}(\mathbf{T}) = \tilde{\mathbf{C}}\left(z\mathbf{I} - \tilde{\mathbf{A}}(\mathbf{T})\right)^{-1}\tilde{\mathbf{B}} \tag{42}$$

Then an alternative optimal FWL realization problem is defined as

$$\max_{\mathbf{T}} r_{\mathcal{C}}(\tilde{\mathbf{A}}(\mathbf{T}), \tilde{\mathbf{B}}, \tilde{\mathbf{C}}) = \frac{1}{\min\limits_{\mathbf{T}} \|\hat{\mathbf{G}}(\mathbf{T})\|_\infty} = \frac{1}{\mu} \tag{43}$$

Consider how to solve for the optimal realization problem (43). From (36), $\|\hat{\mathbf{G}}(\mathbf{T})\|_\infty < \gamma$ if and only if there exists a positive definite matrix $\mathbf{X} \in \mathcal{R}^{(n+m)\times(n+m)}$ such that:

$$\begin{bmatrix} \mathbf{P}_1 & & \\ & \mathbf{I}_q & \\ & & \mathbf{P}_2 \end{bmatrix} > \mathbf{M}_\gamma \begin{bmatrix} \mathbf{P}_1 & & \\ & \mathbf{I}_p & \\ & & \mathbf{P}_2 \end{bmatrix} \mathbf{M}_\gamma^T \tag{44}$$

subject to

$$\mathbf{P}_1 = \begin{bmatrix} \mathbf{I}_n & \mathbf{0} \\ \mathbf{0} & \mathbf{T} \end{bmatrix} \mathbf{X} \begin{bmatrix} \mathbf{I}_n & \mathbf{0} \\ \mathbf{0} & \mathbf{T}^T \end{bmatrix} > 0 \tag{45}$$

and

$$\mathbf{P}_2 = \mathbf{T}\mathbf{T}^T > 0 \tag{46}$$

where

$$\mathbf{M}_\gamma = \begin{bmatrix} \bar{\mathbf{A}}_0 & \tilde{\mathbf{B}} \\ \frac{1}{\gamma}\tilde{\mathbf{C}} & \mathbf{0} \end{bmatrix} \tag{47}$$

The inequality (44) with the constraints $\mathbf{P}_1 > 0$ and $\mathbf{P}_2 > 0$ is an LMI problem [10],[11], and numerical algorithms for solving for this kind of problems are readily available. Therefore, the optimal value of $\mu$ can be obtained together with the corresponding $\mathbf{P}_{1\,\mathrm{opt}}$ and $\mathbf{P}_{2\,\mathrm{opt}}$. This leads to

$$\mathbf{T}_{\mathrm{opt},r} = \mathbf{P}_{2\,\mathrm{opt}}^{1/2} \tag{48}$$

and

$$\mathbf{X}_{\mathrm{opt}} = \begin{bmatrix} \mathbf{I}_n & \\ & \mathbf{T}_{\mathrm{opt},r}^{-1} \end{bmatrix} \mathbf{P}_{1\,\mathrm{opt}} \begin{bmatrix} \mathbf{I}_n & \\ & \mathbf{T}_{\mathrm{opt},r}^{-T} \end{bmatrix} \tag{49}$$

With $\mathbf{T}_{\mathrm{opt},r}$, the corresponding optimal controller realization $\mathbf{w}_{\mathrm{opt},r}$ can be obtained.

Unlike the pole-sensitivity measure (11), the complex stability radius measure does not have a direct relationship with the word length, and a statistical word length was adopted to circumvent this difficulty [7]. Under certain assumptions, it can be shown that the closed-loop system is stable with probability no less than 0.9777, provided that the elements of $\mathbf{\Delta}$ are bounded absolutely by

$$\mu_r(\mathbf{w}) = \frac{r_{\mathcal{C}}}{\sqrt{\frac{N}{3} + 4\sqrt{\frac{N}{45}}}} \tag{50}$$

where $N$ is the nonzero elements in $\mathbf{\Delta}$. The measure (50) can be regarded as a lower bound of $\mu_0(\mathbf{w})$, and the statistical word length formula using the stability measure (50) leads to the following minimum bit length estimate

$$\hat{B}_{sr}^{\min} = B_i + \mathrm{Int}[-\log_2(\mu_r(\mathbf{w}))] - 1 \tag{51}$$

# 5 A numerical example

Both the pole sensitivity and complex stability radius approaches involve some approximations in estimating the true stability measure $\mu_0(\mathbf{w})$. Therefore, they are conservative measures. As conditions are different for them to be lower bounds of $\mu_0(\mathbf{w})$, it is difficult to say which measure is less conservative in estimating the true minimum bit length. It will be case dependent. In particular, the corresponding optimal controller realizations $\mathbf{w}_{\mathrm{opt},p}$ and $\mathbf{w}_{\mathrm{opt},r}$ will be different. An advantage of the complex stability radius measure is that the corresponding optimization problem can be posed as the LMI problem (44), and this LMI problem is easier to solve for than the nonlinear optimization problem (31).

A numerical example was used to compare the two approaches. The example was a torsional vibration control system given in [14]. Discretizing the continuous-time plant with the sampling period 0.001 yielded the discrete-time plant model:

$$\mathbf{A} = \begin{bmatrix} 0.0 & 0.0 & 1.0 \\ 1.0 & 0.0 & -2.97686 \\ 0.0 & 1.0 & 2.97686 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1.0 \\ 0.0 \\ 0.0 \end{bmatrix},$$

$$\mathbf{C} = \begin{bmatrix} 0.24863 & 0.24621 & 0.24143 \end{bmatrix}$$

and the initially designed controller:

$$\mathbf{F}_0 = \begin{bmatrix} 0.0 & -0.33333 \\ 1.0 & 1.33333 \end{bmatrix}, \quad \mathbf{G}_0 = \begin{bmatrix} 1.0 \\ 0.0 \end{bmatrix},$$

$$\mathbf{J}_0 = \begin{bmatrix} -1.20982 & -0.41278 \end{bmatrix}, \quad \mathbf{M}_0 = [1.35120]$$

With this initial controller realization $\mathbf{w}_0$, the two optimal controller realizations $\mathbf{w}_{\mathrm{opt},p}$ and $\mathbf{w}_{\mathrm{opt},r}$ obtained by solving for the two optimizations problem (31) and(44), respectively, are:

$$\mathbf{F}_{\mathrm{opt},p} = \begin{bmatrix} 0.71295 & -0.88451 \\ -0.12320 & 0.62038 \end{bmatrix},$$

$$\mathbf{G}_{\mathrm{opt},p} = \begin{bmatrix} 0.62934 \\ 0.33823 \end{bmatrix},$$

$$\mathbf{J}_{\mathrm{opt},p} = \begin{bmatrix} -0.62540 & -2.41321 \end{bmatrix}, \quad \mathbf{M}_{\mathrm{opt},p} = [1.35120]$$

and

$$\mathbf{F}_{\mathrm{opt},r} = \begin{bmatrix} 1.07316 & 0.16668 \\ -0.32475 & 0.26017 \end{bmatrix},$$

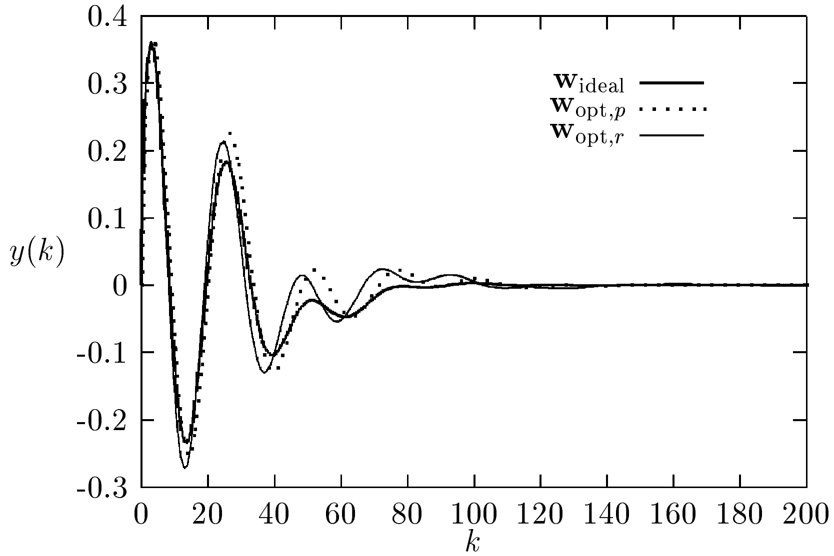


Figure 2: Comparison of unit impulse response for the infinite-precision controller implementation $\mathbf{w}_{\text{ideal}}$ with those for the two 6-bit implemented controller realizations $\mathbf{w}_{\text{opt},p}$ and $\mathbf{w}_{\text{opt},r}$.

$$\mathbf{G}_{\text{opt},r} = \begin{bmatrix} 0.01716 \\ -0.48973 \end{bmatrix},$$

$$\mathbf{J}_{\text{opt},r} = \begin{bmatrix} 1.24139 & 2.51388 \end{bmatrix}, \quad \mathbf{M}_{\text{opt},r} = [1.35120]$$

For the initial and two optimal controller realizations, the true minimal bit lengths $B_s^{\min}$ that can guarantee the closed-loop stability were also determined using a computer simulation method. Table 1 compares the values of the two stability measures $\mu_p$ and $\mu_r$, corresponding estimated minimum bit lengths and true minimum bit lengths for the initial and two optimal controller realizations.

We also computed the unit impulse response of the closed-loop control system when the controllers were the infinite-precision implemented $\mathbf{w}_0$ and various FWL implemented realizations. Notice that any realization $\mathbf{w} \in \mathcal{S}$, implemented in infinite precision, will achieve the exact performance of the infinite-precision implemented $\mathbf{w}_0$, which is the *designed* controller performance. For this reason, the infinite-precision implemented $\mathbf{w}_0$ is referred to as the *ideal* controller realization $\mathbf{w}_{\text{ideal}}$. Fig. 2 compares the unit impulse response of the plant output for the ideal controller $\mathbf{w}_{\text{ideal}}$ with those of two 6-bit implemented $\mathbf{w}_{\text{opt},p}$ and $\mathbf{w}_{\text{opt},r}$. For this example, although $\mathbf{w}_{\text{opt},p}$ and $\mathbf{w}_{\text{opt},r}$ are different, they both have similar good FWL characteristics in fixed-point implementation.

# 6 Conclusions

In this paper, we have compared the two approaches for obtaining optimal FWL controller realizations based on the pole sensitivity and complex stability radius measures, respectively. Design procedures for the both methods are provided. Although the motivations for these two approaches are different, they can be regarded as two methods of approximating a true but computationally intractable FWL closed-loop stability measure. An example is used to compare the two design procedures, and the results show that for the example tested the two approaches produce two different optimal controller realizations which have similar good FWL characteristics.

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| realization | $\mu_p$ | $\hat{B}_{sp}^{\min}$ | $r_C$ | $\mu_r$ | $\hat{B}_{sr}^{\min}$ | $B_s^{\min}$ |
|---|---|---|---|---|---|---|
| $\mathbf{w}_0$ | 9.8513e-4 | 10 | 5.3470e-3 | 2.4434e-3 | 9 | **7** |
| $\mathbf{w}_{\text{opt},p}$ | **8.9321e-3** | **8** | 2.0181e-2 | 9.2219e-3 | 8 | **6** |
| $\mathbf{w}_{\text{opt},r}$ | 5.02743e-3 | 9 | 2.63050e-2 | **1.20205e-2** | **8** | **6** |

Table 1: Comparison of the two stability measures, corresponding estimated minimum bit lengths and true minimum bit lengths for the initial and two optimal controller realizations.

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