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Stability conditions for a class of 2D continuous-discrete linear systems with dynamic boundary conditions

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Repetitive processes are a distinct class of 2D systems of both practical and theoretical interest. Their essential characteristic is repeated sweeps, termed passes, through a set of dynamics defined over a finite duration with explicit interaction between the outputs, or pass profiles, produced as the process evolves. Experience has shown that these processes cannot be studied/controlled by direct application of existing theory (in all but a few very restrictive special cases). This fact, and the growing list of applications areas, has prompted an on-going research programme into the development of a 'mature' systems theory for these processes for onward translation into reliable generally applicable controller design algorithms. This paper develops stability tests for a sub-class of so-called differential linear repetitive processes in the presence of a general set of initial conditions, where it is known that the structure of these conditions is critical to their stability properties.

1. Introduction

The essential unique characteristic of a repetitive, or multipass, process can be illustrated by considering machining operations where the material or workpiece involved is processed by a sequence of passes of the processing tool. Assuming that the pass length α (i.e. the duration of a pass of the processing tool) is finite and constant, the output, or pass profile, $y_k(t)$, $0 \le t \le \alpha$ (t being the independent spatial or temporal variable) produced on pass k acts as a forcing function on the next pass and hence contributes to the dynamics of the new pass profile $y_{k+1}(t)$, $0 \le t \le \alpha$, $k \ge 0$.

Industrial examples of repetitive processes include long-wall coal cutting and metal rolling operations (Edwards 1974, Smyth 1992, Benton 2000). Also cases exist where adopting a repetitive process setting for analysis has major advantages over alternatives—so-called algorithmic examples. This is especially true for classes of iterative learning control schemes (Amann *et al*. 1998) and iterative solution algorithms for nonlinear dynamic optimal control problems based on the maximum principle (Roberts 2000).

Repetitive processes clearly have a two-dimensional, or 2D, structure, i.e. information propagation occurs along a given pass (t direction) and from pass to pass (k direction). They are distinct from, in particular, the extensively studied 2D linear systems described by the Roesser (1975) and Fornasini and Marchesini (1978) state space models and other classes of so-called 2D continuous-discrete linear systems (see, for example, Kaczorek 1996) by the fact that information propagation along the pass only occurs over a finite and fixed interval—the pass length α.

The basic unique control problem for repetitive processes is that the output sequence of pass profiles can contain oscillations that increase in amplitude in the pass to pass direction (i.e. in the k-direction in the notation for variables used here). Early approaches to stability analysis and controller design for (linear single-input single-output (SISO)) repetitive processes and, in particular, long-wall coal cutting (Edwards 1974) was based on first converting the underlying dynamics into those of a so-called infinite-length single-pass process. This resulted, for example, in a scalar algebraic/delay system to which standard scalar inverse-Nyquist stability criteria then applied.

In general, however, it was soon established that this approach to stability analysis and controller design would, except in a few very restrictive special cases, lead to incorrect conclusions (Owens 1977). The basic reason for this is that such an approach effectively neglects their finite pass length repeatable nature and the effects of resetting the initial conditions before the start of each pass. To remove these difficulties, a rigorous stability theory has been developed (Rogers and Owens 1992, Rogers *et al*. 2002) based on an abstract model in a Banach space setting which includes all linear dynamics constant pass length processes as special cases.

Use of this theory confirms that the initial conditions at the start of each pass, termed the boundary conditions here, have a critical role in determining the stability properties of a given example. In Owens and Rogers (1999), a stability analysis for so-called differential linear repetitive processes, a sub-class of major interest in terms of both physical and algorithmic applications, was given in the presence of a general set of boundary

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conditions. This paper develops this work further in one particular case to produce conditions which can (in principle at least) be tested. These conditions are developed using a so-called 1D Lyapunov equation and, as an alternative, the so-called strictly bounded real lemma (Anderson and Vongpanitlerd 1973). We begin in the next section by giving the necessary background results.

2. Background

The state space model of the sub-class of differential linear repetitive processes considered here has the following form over $0 \leq t \leq \alpha$, $k \geq 0$

$$\dot{y}_{k+1}(t) = Ay_{k+1}(t) + Bu_{k+1}(t) + B_0 y_k(t) \qquad (1)$$

Here on pass k, $y_k(t)$ is the $n \times 1$ pass profile vector, and $u_k(t)$ is the $l \times 1$ vector of control inputs. To complete the process description it is necessary to specify the initial conditions, termed the boundary conditions here, i.e. the state initial vector on each pass and the initial pass profile. The simplest possible form for these is

$$\left.\begin{aligned} y_{k+1}(0) &= d_{k+1}, \qquad k \geq 0 \\ y_0(t) &= y(t), \qquad 0 \leq t \leq \alpha \end{aligned}\right\} \qquad (2)$$

where the $n \times 1$ vector d_{k+1} has constant entries and the entries in the $n \times 1$ vector $y(t)$ are known functions of t.

In certain cases, for examples see Smyth (1992), the boundary conditions of (2) and, in particular, the form of $y_{k+1}(0), k \geq 0$, is simply not strong enough to 'adequately model' (even for initial control related studies) the process dynamics. The most general form of boundary conditions for (1) are obtained by replacing (2) with

$$\left.\begin{aligned} y_{k+1}(0) &= d_{k+1} + \sum_{j=1}^{M} K_j y_k(t_j), \qquad k \geq 0 \\ y_0(t) &= y(t), \qquad 0 \leq t \leq \alpha \end{aligned}\right\} \qquad (3)$$

where $0 \leq t_1 < t_2 < \cdots < t_M \leq \alpha$ are M sample points along the previous pass profile, and $K_j, 1 \leq j \leq M$, are $n \times n$ matrices with constant entries. In this paper we will focus on the case when $M = 1$ and $t_1 = \alpha$ and $K_1 = I_n$ which is of particular interest in terms of links with delay differential systems and also repetitive control schemes.

The stability theory for linear constant pass length repetitive processes is based on the following abstract model of the underlying dynamics where E_α is a suitably chosen Banach space with norm $||\cdot||$ and W_α is a linear subspace of E_α

$$y_{k+1} = L_\alpha y_k + b_{k+1}, \qquad k \geq 0 \qquad (4)$$

In this model y_k is the pass profile on pass k and L_α is a bounded linear operator mapping E_α into itself. The term $L_\alpha y_k$ represents the contribution from pass k to pass $k+1$ and b_{k+1} represents known initial conditions, disturbances and control input effects. We denote this model by S.

In the case of (1) and (3) we choose $E_\alpha = L_2^n[0, \alpha] \cap L_\infty[0, \alpha]$ and it is routine to show that

$$(L_\alpha y)(t) = e^{At}\hat{y} + \int_0^t e^{A(t-\tau)} B_0 y(\tau)\, d\tau, \qquad 0 \leq t \leq \alpha \qquad (5)$$

where

$$\hat{y} = \sum_{j=1}^{M} K_j y(t_j) \qquad (6)$$

and

$$b_{k+1} = e^{At} d_{k+1} + \int_0^t e^{A(t-\tau)} Bu_{k+1}(\tau)\, d\tau, \quad 0 \leq t \leq \alpha, k \geq 0 \qquad (7)$$

The linear repetitive process S is said to be asymptotically stable (Rogers and Owens 1992, Rogers *et al.* 2002) if $\exists$ a real scalar $\delta > 0$ such that, given any initial profile y_0 and any disturbance sequence $\{b_k\}_{k \geq 1} \subset W_\alpha$ bounded in norm (i.e. $||b_k|| \leq c_1$ for some real constant $c_1 \geq 0$ and $\forall k \geq 1$), the output sequence generated by the perturbed process

$$y_{k+1} = (L_\alpha + \gamma) y_k + b_{k+1}, \qquad k \geq 0 \qquad (8)$$

is bounded in norm whenever $||\gamma|| \leq \delta$.

This definition is easily shown to be equivalent to the requirement that $\exists$ finite real scalars $M_\alpha > 0$ and $\lambda_\alpha \in (0, 1)$ such that

$$||L_\alpha^k|| \leq M_\alpha \lambda_\alpha^k, \qquad k \geq 0 \qquad (9)$$

(where $||\cdot||$ is also used to denote the induced operator norm).

A necessary and sufficient condition (Rogers and Owens 1992, Rogers *et al.* 2002) for (9) to hold is that the spectral radius, $r(L_\alpha)$, of L_α satisfies

$$r(L_\alpha) < 1 \qquad (10)$$

Introduce

$$M(z) := \sum_{j=1}^{M} K_j\, e^{\hat{A}(z)t_j} \qquad (11)$$

where

$$\hat{A}(z) = A + z^{-1} B_0, \qquad z \neq 0 \qquad (12)$$

Then the following result, proved in Owens and Rogers (1999), characterizes the spectral radius of L_α in the case considered here.

Theorem 1: *Suppose that the pair $\{A, B_0\}$ is controllable. Then the linear repetitive process S generated by (1) and (3) has operator L_α with spectral radius*

$$r(L_\alpha) = \max\{0, \sup\{|z| : z \neq 0 \,\&\, |zI_n - M(z)| = 0\}\} < 1 \tag{13}$$

Corollary 1: *The linear repetitive process S generated by (1) and (3) is asymptotically stable if, and only if, all solutions of*

$$|zI_n - M(z)| = 0 \tag{14}$$

have modulus strictly less than unity.

Consider now the case when the boundary conditions are of the simple form (2). Then we have the 'counter-intuitive' result that asymptotic stability is essentially independent of the process dynamics and, in particular, the eigenvalues of the matrix A. This is due entirely to the fact that the pass length α is finite and of constant value for all passes. This situation will change drastically if (as below) we let $\alpha \to +\infty$.

In general, Theorem 1 shows that the property of asymptotic stability for differential linear repetitive processes is critically dependent on the structure of $y_{k+1}(0), k \geq 0$. Suppose also that this sequence is incorrectly modelled as in (2) instead of a special case of the form given in (3). Then the process could well be interpreted as asymptotically stable when in actual fact it is asymptotically unstable!

The above analysis provides necessary and sufficient conditions for asymptotic stability but no really 'useful' information concerning transient behaviour and, in particular, about the behaviour of the output sequence of pass profiles as the process evolves from pass to pass (i.e. in the k direction). The limit profile provides a characterization of process behaviour after a 'large number' of passes have elapsed.

Suppose that the abstract model S is asymptotically stable and let $\{b_k\}_{k\geq 1}$ be a disturbance sequence that converges strongly to a disturbance b_∞. Then the strong limit

$$y_\infty := \lim_{k\to+\infty} y_k \tag{15}$$

is termed the limit profile corresponding to this disturbance sequence. Also, it can be shown (Rogers and Owens 1992, Rogers *et al.* 2002) that y_∞ is uniquely given by

$$y_\infty = (I - L_\alpha)^{-1} b_\infty \tag{16}$$

Note also that (16) can be obtained from (4) (which describes the dynamics of S) by replacing all variables by their strong limits.

In the case considered here, the limit profile is described by the following result.

Proposition 1: *In the case when* S *described by* (1) *and* (3) *is asymptotically stable, the resulting limit profile is*

$$\left.\begin{aligned} \dot{y}_\infty(t) &= (A + B_0) y_\infty(t) + B u_\infty(t) \\ y_\infty(0) &= (I_n - M(1))^{-1} d_\infty \end{aligned}\right\} \tag{17}$$

where d_∞ is the strong limit of $\{d_k\}_{k\geq 1}$ and the matrix inverse exists by asymptotic stability.

Asymptotic stability of processes described by (1) and (3) guarantees the existence of a limit profile which is described by a standard, or 1D, linear systems state space model. Hence, in effect, if the process under consideration is asymptotically stable, then its repetitive dynamics can, after a 'sufficiently large' number of passes, be replaced by those of a 1D linear time-invariant system. This result has obvious implications in terms of the design of control schemes for these processes.

Owing to the finite pass length (over which duration even an unstable 1D linear system can only produce a bounded output), asymptotic stability cannot guarantee that the resulting limit profile has 'acceptable' along the pass dynamics, where in this case the basic requirement is stability as a 1D linear system. As a simple example to demonstrate this fact, consider the case of $A = -1$, $B = 1, B_0 = 1 + \beta, y_{k+1}(0) = 0, k \geq 0$, where $\beta > 0$ is a real scalar. Then the resulting limit profile dynamics are described by the unstable 1D linear system

$$\dot{y}_\infty(t) = \beta y_\infty(t) + u_\infty(t), \qquad 0 \leq t \leq \alpha \tag{18}$$

The natural definition of stability along the pass for the above example is to ask that the limit profile is stable in the sense that $\beta < 0$ if we let the pass length α become infinite. This intuitively appealing idea is, however, not applicable to cases where the limit profile resulting from asymptotic stability is not described by a 1D linear systems state-space model. Consequently stability along the pass for the general model S has been defined in terms of the rate of approach to the limit profile as the pass length α becomes infinitely large. One of several equivalent formulations of this property is that S is said to be stable along the pass if, and only if, $\exists$ real numbers $M_\infty > 0$ and $\lambda_\infty \in (0, 1)$ which are independent of α and satisfy

$$||L_\alpha^k|| \leq M_\infty \lambda_\infty^k, \qquad \forall \alpha > 0, \qquad \forall k \geq 0 \tag{19}$$

Necessary and sufficient conditions (Rogers and Owens 1992, Rogers *et al.* 2002) for (19) are that

$$r_\infty := \sup_{\alpha\geq 0} r(L_\alpha) < 1 \tag{20}$$

and

$$M_0 := \sup_{\alpha\geq 0} \sup_{|z|\geq\lambda} ||(zI - L_\alpha)^{-1}|| < +\infty \tag{21}$$

for some real number $\lambda \in (r_\infty, 1)$.

In the case of S generated by (1) and (3), there are two possible cases which could be considered. The first of these is that as $\alpha \to +\infty$, we allow $M \to +\infty$ and $t_j \to +\infty$, and the second is that as $\alpha \to +\infty$ we keep M and t_j fixed. Given that $\alpha \to +\infty$ is a mathematical requirement, only the second of these cases is physically relevant. Hence as $\alpha \to +\infty$, we assume that M and $\{t_j\}_{1\le j\le M}$ remain unchanged.

The following result, established in Owens and Rogers (1999), gives the necessary and sufficient conditions for stability along the pass.

Theorem 2: *Suppose that $\{A, B_0\}$ is controllable and that all eigenvalues of the matrix A have strictly negative real parts. Then the linear repetitive process* S *generated by* (1) *and* (3) *is stable along the pass if, and only if,*

(a) *Corollary 1 holds $\forall \alpha \ge 0$, and*

(b)

$$\sup_{\omega\ge 0} r(G(i\omega)) < 1 \tag{22}$$

where

$$G(s) := (sI_n - A)^{-1}B_0 \tag{23}$$

3. 1D Lyapunov equation stability tests

In terms of applying Theorem 2 to a given example, first note that it is only condition (a) which cannot be tested by direct application of 1D linear systems tests. In this section, we develop a so-called 1D Lyapunov equation based interpretation of this condition in one case of practical interest. The designation of the Lyapunov equation used as 1D is to highlight the fact that the structure of this equation is identical to its well known counterpart for 1D differential linear systems but here the defining matrices are functions of a complex variable.

The case of (3) treated here is when $M = 1, t_1 = \alpha$, and $K_1 = I_n$, i.e.

$$y_{k+1}(0) = d_{k+1} + y_k(\alpha), \qquad k \ge 0 \tag{24}$$

In this case, condition (a) of Theorem 2 requires that all solutions of

$$|zI_n - e^{(A+z^{-1}B_0)\alpha}| = 0 \tag{25}$$

have modulus strictly less than unity $\forall \alpha \ge 0$. Now write $z = e^{s\alpha}$, and hence (25) reduces to the requirement that all solutions of

$$|sI_n - F(s)| = 0 \tag{26}$$

have strictly negative real parts where

$$F(s) = A + B_0\, e^{-s\alpha} \tag{27}$$

Also it can be shown, using results in Kamen (1980), that (26) reduces to the requirement that

$$|sI_n - F(e^{-i\omega\alpha})| \ne 0, \qquad \forall\ \mathrm{Re}\,(s) \ge 0, \qquad \forall \omega \in [0, 2\pi] \tag{28}$$

The following result now expresses the condition of (28) in terms of a so-called 1D Lyapunov equation (see Brierley *et al*. (1982) for a similar approach for a class of differential linear systems with commensurate time delays).

Theorem 3: *The condition of* (28) *holds if, and only if, for a given positive definite Hermitian, denoted PDH from this point onwards, matrix $Q(e^{i\omega})$, $\forall \omega \in [0, 2\pi]$, the solution, $P(e^{i\omega})$, of the 1D matrix Lyapunov equation*

$$F^*(e^{i\omega})P(e^{i\omega}) + P(e^{i\omega})F(e^{i\omega}) = -Q(e^{i\omega}) \tag{29}$$

is PDH $\forall \omega \in [0, 2\pi]$, where $*$ denotes the complex conjugate transpose operation.

Proof: To show sufficiency, first note that for any fixed $\omega_o \in [0, 2\pi]$, the matrix $F(e^{i\omega_o})$ is an $n \times n$ matrix with complex elements. Also let λ_o be an eigenvalue of this matrix and x_o the corresponding eigenvector. Then

$$F(e^{i\omega_o})x_o = \lambda_o x_o \tag{30}$$

$$x_o^* F^*(e^{i\omega_o}) = \overline{\lambda}_o x_o^* \tag{31}$$

where the bar denotes the complex conjugate operation. Now pre-multiply (29) by x_o^* and then post-multiply this same equation by x_o to yield

$$\left.\begin{aligned} x_o^*Q(e^{i\omega_o})x_o &= -x_o^*(F^*(e^{i\omega_o})P(e^{i\omega_o}) + P(e^{i\omega_o})F(e^{i\omega_o}))x_o \\ &= -(\overline{\lambda}_o + \lambda_o)x_o^*P(e^{i\omega_o})x_o \end{aligned}\right\} \tag{32}$$

Now if $Q(e^{i\omega})$ and $P(e^{i\omega})$ are PDH for $\forall \omega \in [0, 2\pi]$, it follows that

$$\mathrm{Re}(\lambda) = \frac{1}{2}(\overline{\lambda} + \lambda) = -\frac{1}{2}\left(\frac{x^*Q(e^{i\omega})x}{x^*P(e^{i\omega})x}\right) < 0 \tag{33}$$

where now λ is any eigenvalue of $F(e^{i\omega})$ and x is the corresponding eigenvector. Hence condition (a) of Theorem 2 holds.

To show necessity, consider (29) with an arbitrary PDH matrix $Q(e^{i\omega})$ on $[0, 2\pi]$. Then if (28) holds, it can be shown (Kamen 1980) that all eigenvalues of the matrix $F(e^{i\omega})$ have strictly negative real parts $\forall \omega \in [0, 2\pi]$. Now define

$$P(e^{i\omega}) = \int_0^\infty e^{F^*(e^{i\omega})t}Q(e^{i\omega})e^{F(e^{i\omega})t}\,dt \tag{34}$$

which is well defined since the eigenvalues of $F(e^{i\omega})$ (and $F^*(e^{i\omega})$) are in the left-half of the complex plane. Also $P^*(e^{i\omega}) = P(e^{i\omega}), \forall\omega \in [0, 2\pi]$ and

$$\begin{aligned} F^*(e^{i\omega})P(e^{i\omega}) + P(e^{i\omega})F(e^{i\omega}) &= \int_0^\infty (F^*(e^{i\omega})e^{F^*(e^{i\omega})t}Q(e^{i\omega})e^{F(e^{i\omega})t} \\ &\quad + e^{F^*(e^{i\omega})t}Q(e^{i\omega})e^{F(e^{i\omega})t}F(e^{i\omega}))\,dt \\ &= \int_0^\infty \left(\left(\frac{d}{dt}e^{F^*(e^{i\omega})t}\right)Q(e^{i\omega})e^{F(e^{i\omega})t} \right. \\ &\quad \left. + e^{F^*(e^{i\omega})t}Q(e^{i\omega})\left(\frac{d}{dt}e^{F(e^{i\omega})t}\right)\right)dt \\ &= \int_0^\infty \frac{d}{dt}(e^{F^*(e^{i\omega})t}Q(e^{i\omega})e^{F(e^{i\omega})t})\,dt \\ &= -Q(e^{i\omega}) \end{aligned} \tag{35}$$

where this last equality follows from the fact that $e^{F(i\omega)t} \to 0$, and $e^{F^*(i\omega)t} \to 0$, as $t \to +\infty$. To complete the proof, we now have that

$$F^*(e^{i\omega})P(e^{i\omega}) + P(e^{i\omega})F(e^{i\omega}) = -Q(e^{i\omega}), \qquad \forall\omega \in [0, 2\pi] \tag{36}$$

and $P^*(e^{i\omega}) = P(e^{i\omega}), \forall\omega \in [0, 2\pi]$.

Now define

$$F(z)|_{z=e^{i\omega}} = F_1(\omega) + iF_2(\omega) \tag{37}$$

where $F_1(\omega)$ and $F_2(\omega)$ are real $n \times n$ matrices. Also for a fixed $\omega_o \in [0, 2\pi]$, $F(e^{i\omega_o})$ is an $n \times n$ matrix with complex entries which can be written as

$$F(e^{i\omega_o}) = F_1(\omega_o) + iF_2(\omega_o) \tag{38}$$

The system $\dot{y} = F(e^{i\omega_o})y$ can be rewritten as

$$\dot{y}_r + i\dot{y}_i = (F_1(\omega_o) + iF_2(\omega_o))(y_r + iy_i) \tag{39}$$

where y_r and y_i denote the real and imaginary parts of y respectively, and separating (39) into real and imaginary parts now yields the equivalent expression

$$\begin{bmatrix} \dot{y}_r \\ \dot{y}_i \end{bmatrix} = \begin{bmatrix} F_1(\omega_o) & -F_2(\omega_o) \\ F_2(\omega_o) & F_1(\omega_o) \end{bmatrix} \begin{bmatrix} y_r \\ y_i \end{bmatrix} = \hat{F}(\omega_o) \begin{bmatrix} y_r \\ y_i \end{bmatrix} \tag{40}$$

Now consider the SISO case and write

$$\hat{F}(\omega) = \begin{bmatrix} f_1(\omega) & -f_2(\omega) \\ f_2(\omega) & f_1(\omega) \end{bmatrix} \tag{41}$$

Then in this case a necessary and sufficient condition for condition (a) of Theorem 2 to hold (i.e. corollary 1) is that $f_1(i\omega) < 0\ \forall\omega \in [0, 2\pi]$, i.e. 1D stability of the real part of $F(e^{i\omega})$. This follows immediately from

$$\det(sI - \hat{F}(\omega)) = s^2 - 2f_1(\omega)s + f_1^2(\omega) + f_2^2(\omega) \tag{42}$$

As an example to illustrate this last result, consider the following unforced process

$$\dot{y}_{k+1}(t) = \begin{bmatrix} 0 & 1 \\ -a & -b \end{bmatrix} y_{k+1}(t) + \begin{bmatrix} 0 & 0 \\ 0 & -c \end{bmatrix} y_k(t) \tag{43}$$

where a, b, and c are positive real numbers. Then in this case

$$F(z) = \begin{bmatrix} 0 & 1 \\ -a & -b - cz \end{bmatrix}, \qquad z = e^{i\omega} \tag{44}$$

The solution of the Lyapunov equation (29) with $Q = I_2$ is

$$\left.\begin{aligned} P(z) &= \frac{1}{ay}\begin{bmatrix} |b + cz|^2 + a(a+1) & b + c\bar{z} \\ b + cz & a + 1 \end{bmatrix} \\ y &= 2(b + c\cos\omega) \end{aligned}\right\} \tag{45}$$

and

$$\det(P(z)) = \frac{|b + cz|^2 + (a+1)^2}{ay^2} \tag{46}$$

Hence $P(e^{i\omega})$ is PDH $\forall\omega \in [0, 2\pi]$ if, and only if, $y > 0$, i.e.

$$b + c\cos\omega > 0, \qquad \forall\omega \in [0, 2\pi] \tag{47}$$

and therefore (43) satisfies (a) of Theorem 2 if, and only if, $b > c$.

Finally, note that it is easy to generate examples which demonstrate that a generalization of these last results for the SISO case is not possible. One such process is that with

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad B_0 = \begin{bmatrix} -1 & -1 \\ 4 & -1 \end{bmatrix}$$

where both A and B_0 are 1D stable but $A + iB_0$ does not satisfy (a) of Theorem 2.

4. Strict positive realness based tests

In this section, we use results from the theory of strict bounded realness to develop equivalent formulations of the stability along the pass conditions considered in this paper. This will lead to Riccati equation based conditions which can be checked numerically. Prior to this, however, it is useful to consider the development of conditions based on the Hermite matrix approach from 'classical' root clustering theory.

First note that (28) is equivalent, on setting $z = e^{i\omega}$, to

$$\Delta(s, z) \equiv \det(sI_n - F(z)) \neq 0, \qquad \mathrm{Re}(s) \geq 0, \qquad |z| = 1 \tag{48}$$

or

$$\Delta(s, e^{i\omega}) \neq 0, \qquad \mathrm{Re}(s) \geq 0, \qquad \omega \in [0, 2\pi] \tag{49}$$

This is an equation with complex coefficients which are polynomials in $e^{i\omega}$ and it is required that all of its roots should lie in the open-left half of the s-plane. Using 'classical' root clustering theory, the condition for this (see, e.g. Jury 1973 and the relevant cited references) is that the Hermite matrix obtained from the coefficients in $\Delta(s, e^{i\omega})$ is positive definite or, alternatively, the inner-wise matrix obtained from the coefficients must be positive inner-wise.

Consider the complex polynomial

$$B(s) = \sum_{i=0}^{n} b_i s^i \tag{50}$$

Then the associated Hermite matrix, H, is obtained as

$$H = \{h_{p,q}\}, \qquad h_{p,q} = h_{q,p} \tag{51}$$

where

$$\left.\begin{aligned} h_{p,q} &= 2(-1)^{(p+q)/2}\sum_{j=1}^{p}(-1)^j \mathrm{Re}(b_{n-j-1}\overline{b}_{n-p-q+j}) \\ p+q &= \text{even}, \qquad p \leq q \\ h_{p,q} &= 2(-1)^{(p+q-1)/2}\sum_{j=1}^{p}(-1)^j \mathrm{Im}(b_{n-j-1}\overline{b}_{n-p-q+j}) \\ p+q &= \text{odd}, \qquad p \leq q \end{aligned}\right\} \tag{52}$$

(where Re and Im denote the real and imaginary parts of a complex number respectively). Also it can be shown that the requirement that H is positive definite $\forall \omega \in [0, 2\pi]$ (or $|e^{i\omega}| \in [-1, 1]$) is equivalent to the conditions

$$H(e^{i0}) = H(1) > 0 \tag{53}$$

$$\det(H(e^{i\omega})) > 0, \qquad \forall \omega \in [0, 2\pi] \tag{54}$$

To establish (53) and (54) (which are, in fact, well known results), first note that if $H(e^{i\omega})$ is positive definite $\forall \omega \in [0, 2\pi]$ then these two conditions obviously hold. Conversely, if (53) and (54) hold then by (54) all eigenvalues of $H(e^{i\omega})$ are non-zero $\forall \omega \in [0, 2\pi]$. These are real continuous functions of ω which are positive at $\omega = 0$ by (53) and hence positive $\forall \omega \in [0, 2\pi]$.)

The checking of (53) is straightforward and to check the more difficult condition of (54) it is possible to use a positivity test. This is based on the fact that $\det(H(e^{i\omega}))$ is a function of $\cos\omega, \cos 2\omega, \ldots$ and on setting $x = \cos\omega$, $\det(H(e^{i\omega}))$ becomes a function of x and its powers. Hence (54) becomes

$$\det(H(e^{i\omega})) = E(x) > 0, \qquad x \in [-1, 1] \tag{55}$$

This last condition holds provided $E(x)$ has no real roots in the interval $[-1, 1]$. Also introduce the change of variable (a bilinear transform)

$$x = \frac{u-1}{u+1} \tag{56}$$

into (55) to yield the equivalent condition that

$$E_1(u) > 0, \qquad u \in [0, \infty] \tag{57}$$

This condition can be checked computationally using any of the computational positivity tests (see, for example, Jury 1973 and the relevant cited references).

In the remainder of this section we develop a computationally more feasible alternative to the approach just presented. The starting point is to note that the condition to be tested here can be expressed as the requirement that a two variable polynomial of the general form

$$a(s, z) = s^p + \sum_{j=0}^{p-1}\sum_{i=0}^{q} a_{ij} s^j z^i \tag{58}$$

should satisfy

$$a(s, z) \neq 0, \qquad \mathrm{Re}(s) \geq 0, \qquad |z| \leq 1 \tag{59}$$

Next we describe how to reduce (59) to a one-dimensional problem by showing that this condition is equivalent to positive realness of a certain 1D rational transfer function matrix. This leads to a numerically efficient testing algorithm and requires, as background, the results summarized next relating to the so-called strictly bounded real lemma (Anderson and Vongpanitlerd 1973).

Definition 1: A real rational transfer function matrix $G(s) = C_1(sI - A_1)^{-1}B_1$ is termed strictly bounded real if, and only if, the matrix A_1 is Hurwitz (i.e. all its eigenvalues have strictly negative real parts) and

$$I - G^{\mathrm{T}}(-i\omega)G(i\omega) > 0, \qquad \forall \omega \in \mathbb{R} \tag{60}$$

The well known strictly bounded real lemma takes the following form here.

Definition 2: Suppose that $G(s)$ is a proper rational transfer function matrix. Suppose also that $\{A_1, B_1, C_1, D_1\}$ is an associated minimal realization. Then this transfer function matrix is strictly bounded real if, and only if, $\exists$ a real symmetric positive definite matrix P such that

$$M = \begin{bmatrix} A_1^{\mathrm{T}}P + PA_1 + C_1^{\mathrm{T}}C_1 & PB_1 + C_1^{\mathrm{T}}D_1 \\ (PB_1 + C_1^{\mathrm{T}}D_1)^{\mathrm{T}} & D_1^{\mathrm{T}}D_1 - I \end{bmatrix} < 0 \tag{61}$$

One characterization of this strictly bounded real property (for the proof see, e.g. Gu and Lee 1989) is that $G(s)$

has this property if, and only if, for any given real symmetric matrix $Q > 0$, $\exists \epsilon > 0$ such that:

(a) $I - D_1^T D_1 > 0$

(b) the algebraic Riccati equation

$$A_1^T P + PA_1 + (PB_1 + C_1^T D_1)(I - D_1^T D_1)^{-1} \times (B_1^T P + D_1^T C_1) + C_1^T C_1 + \epsilon Q = 0 \quad (62)$$

has a positive definite solution P.

Also the requirement for a minimal realization can be relaxed by the following result (also proved in Gu and Lee 1989).

Lemma 1: *Suppose that $G(s)$ is strictly proper and let $\{A_1, B_1, C_1\}$ be a state space realization with the pair $\{A_1, B_1\}$ controllable. Then $G(s)$ is strictly bounded real if, and only if, for any given real symmetric matrix $Q > 0$, $\exists$ a scalar $\epsilon > 0$ such that the algebraic Riccati equation*

$$A_1^T P + PA_1 + PB_1 B_1^T P + C_1^T C_1 + \epsilon Q = 0 \quad (63)$$

has a positive definite solution P.

Note: If (63) has a solution $P > 0$ for $\epsilon^* > 0$ then for any $\epsilon \in [0, \epsilon^*]$, this equation admits at least one positive definite solution.

If $G(s)$ is not strictly proper, the following result (again from Gu and Lee 1989) can be used.

Lemma 2: *Suppose that $\{A_1, B_1, C_1, D_1\}$ is a minimal realization of $G(s)$. Then $G(s)$ is strictly bounded real if, and only if, $G_m(s)$ is strictly bounded real where $G_m(s)$ is realized by $\{A_m, B_m, C_m\}$ where*

$$\left.\begin{aligned} A_m &= A_1 + B_1(I - D_1^T D_1)^{-1} D_1^T C_1 \\ B_m &= B_1(I - D_1^T D_1)^{-1/2} \\ C_m &= (I - D_1 D_1^T)^{-1/2} D_1^T C_1 C_1 \end{aligned}\right\} \quad (64)$$

The key point here is that if A_1 is Hurwitz then this implies that A_m is Hurwitz and also the controllability of $\{A_1, B_1\}$ implies the controllability of $\{A_m, B_m\}$.

To apply these results, first note the following result (proved in Gu and Lee 1986).

Lemma 3: *Consider the two variable polynomial $a(s, z)$ and suppose that $a(0, z) \neq 0$, $\forall |z| = 1$. Then (59) holds if, and only if:*

(a) $a(s, 0)$ is Hurwitz, and

(b)

$$a(s, z) \neq 0, \qquad \mathrm{Re}(s) = 0, \qquad |z| \leq 1 \quad (65)$$

Clearly it is the second of these last conditions which is the most difficult to test. In what follows we develop a numerically efficient test based on treating $a(s, z)$ as a polynomial, denoted $a_s(z)$, in z with coefficients which are polynomials in s with s taking values on the extended imaginary axis of the complex plane.

The key point to note now is that (65) is true if, and only if, $a_s(z)$ has all its roots outside the unit circle for all s on the imaginary axis. Hence we can apply a 1D stability test to this condition using a point-wise approach, and here we use the Schur–Cohn test expressed in the following form (from Ptak and Young 1980).

Lemma 4: *Let $a(z) = a_0 + a_1 z + \cdots + a_n z^n$, $a_0 \neq 0$, $a_n \neq 0$, be a polynomial with complex coefficients $a_k, k = 0, 1, \ldots, n$. Define also the triangular Toeplitz matrices*

$$D = \begin{bmatrix} \overline{a}_0 & \overline{a}_1 & \cdots & \overline{a}_{n-2} & \overline{a}_{n-1} \\ 0 & \overline{a}_0 & \overline{a}_1 & \cdots & \overline{a}_{n-2} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \overline{a}_0 & \overline{a}_1 \\ 0 & 0 & \cdots & 0 & \overline{a}_0 \end{bmatrix} \quad (66)$$

and

$$N = \begin{bmatrix} a_n & a_{n-1} & \cdots & a_2 & a_1 \\ 0 & a_n & a_{n-1} & \cdots & a_2 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & a_n & a_{n-1} \\ 0 & 0 & \cdots & \cdots & a_n \end{bmatrix} \quad (67)$$

Then $a(z) \neq 0, \forall |z| \leq 1$, if, and only if, the Hermitian matrix

$$\Phi = D^* D - N^* N \quad (68)$$

is strictly positive (where again $*$ denotes the complex conjugate transpose operation).

Note also that if $a_0 \neq 0$ then if Φ is PDH $\Leftrightarrow$ the matrix $G = ND^{-1}$ is a strict contraction.

In the case under consideration here, the coefficient a_k is a polynomial in s, $s = i\omega$. Hence $\overline{a}_k = a_k(-s)$, $k = 0, 1, \ldots, n$. Also the triangular Toeplitz matrices D and N of (66) and (67) respectively can be constructed for this case. Similarly, define

$$\Phi(s) = D^T(-s) D(s) - N^T(-s) N(s) \quad (69)$$

and

$$G(s) = N(s) D^{-1}(s) \quad (70)$$

Then a simple controllable realization for $G(-s)$ is defined as

$$\left.\begin{aligned} \hat{A} &= \begin{bmatrix} -\hat{A}_1 & -\hat{A}_2 & -\hat{A}_3 & \cdots & -\hat{A}_p \\ I_q & 0 & 0 & \cdots & 0 \\ 0 & I_q & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \cdots & \cdots & 0 & I_q & 0 \end{bmatrix} \\ \hat{B} &= \begin{bmatrix} I_q \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \qquad \hat{C}^{\mathrm{T}} = \begin{bmatrix} \hat{C}_1^{\mathrm{T}} \\ \vdots \\ \hat{C}_p^{\mathrm{T}} \end{bmatrix} \end{aligned}\right\} \tag{71}$$

where

$$\hat{A}_{p-j} = \begin{bmatrix} a_{0j} & a_{ij} & a_{2j} & \cdots & a_{q-1j} \\ 0 & a_{0j} & a_{1j} & \cdots & a_{q-2j} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & a_{0j} & a_{1j} \\ 0 & 0 & \cdots & 0 & a_{0j} \end{bmatrix} \tag{72}$$

$$\hat{C}_{p-j} = (-1)^j \begin{bmatrix} a_{qj} & a_{q-1j} & a_{q-2j} & \cdots & a_{1j} \\ 0 & a_{qj} & a_{q-1j} & \cdots & a_{2j} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & a_{qj} & a_{q-1j} \\ 0 & 0 & \cdots & 0 & a_{qj} \end{bmatrix} \tag{73}$$

are upper triangular Toeplitz matrices with a_{ij} real as defined in (58).

The next stage is to show that the conditions of Lemma 3 are equivalent to $G(-s)$ being bounded real. To do this, first take $G(s) = \hat{C}(sI - \hat{A})^{-1}\hat{B}$ as defined by (72). Then (a) of Lemma 3 implies that $\det(sI - \hat{A})) = \det(D(-s)) = (a(s,0))^q$ is Hurwitz and hence $G(-s)$ is stable. Using (b) of Lemma 3 we now have that $\Phi(i\omega)$ is PDH $\forall \omega \in \mathbb{R}$ and this, in turn, is equivalent to $G(-i\omega)$ being a strict contraction for each $\omega \in \mathbb{R}$. Hence $G(-s)$ is strictly bounded real.

Suppose now that $G(-s)$ is strictly bounded real. Then $\det(sI - \hat{A})$ is Hurwitz and hence (a) of Lemma 3 holds. Also since $G(-i\omega)$ is a strict contraction for each ω implies, by the Schur–Cohn test, that (b) of Lemma 3 holds.

The arguments just given establish the following result.

Theorem 4: *Consider the two-variable polynomial $a(s,z)$ defined by* (58) *and $G(-s)$ defined by the state space matrices of* (71). *Suppose also that $a(0,z) \neq 0$, $\forall |z| = 1$. Then this polynomial satisfies* (59) *if, and only if, $G(-s)$ is strictly bounded real.*

This leads immediately to the following algorithm for testing (59).

(1) Input p, q and a_{ij} as defined in (58).

(2) Test if $a(s,0)$ is Hurwitz and if not then stop since (59) does not hold (and hence the example under consideration is not stable along the pass).

(3) Construct the matrices $\hat{A}, \hat{B}, \hat{C}$ and choose a positive-definite matrix $\hat{Q}$ and a positive real scalar ϵ to solve the algebraic Riccati equation (63) applied to this data set, i.e.

$$\hat{A}^{\mathrm{T}}P + P\hat{A} + P\hat{B}\hat{B}^{\mathrm{T}}P + \hat{C}^{\mathrm{T}}\hat{C} + \epsilon\hat{Q} = 0 \tag{74}$$

If this equation has a solution then (59) holds. In which case proceed to test the other conditions for stability along the pass.

Note that the realization defined by (71) may not be minimal and hence there could be numerical problems in solving the algebraic Riccati equation if the product pq is large. Hence an input normal realization (see Moore 1981) should be used to obtain a minimal realization prior to testing $G(-s)$ for the strictly bounded realness property.

It is possible to avoid computing the solution of the algebraic Riccati equation here. This is based on the fact that since $G(-s)$ is strictly proper, it is guaranteed to be strictly bounded real if $\det(I - G^{\mathrm{T}}(-s)G(s)) \neq 0$, $\forall \mathrm{Re}(s) = 0$ or, equivalently, $\det(\Phi(s)) \neq 0, \forall \mathrm{Re}(s) = 0$. Hence this transfer function matrix is strictly bounded real if, and only if, the Hamiltonian matrix

$$H_a := \begin{bmatrix} \hat{A} & \hat{B}\hat{B}^{\mathrm{T}} \\ -\hat{C}^{\mathrm{T}}\hat{C} & -\hat{A}^{\mathrm{T}} \end{bmatrix} \tag{75}$$

has no purely imaginary eigenvalues. Note that the dimensions of this matrix are $2pq \times 2pq$ and hence if pq is 'large' then the eigenvalue computation cannot be expected to produce 'high accuracy' results.

As an example, suppose that

$$a(s,z) = s + \gamma + (\beta + \lambda s)z \tag{76}$$

where $|\gamma| \neq |\beta|$ and $\gamma > 0$. In this case, (59) clearly only holds if, and only if

$$G(-s) = \frac{\lambda s + \beta}{s + \gamma} \tag{77}$$

is strictly bounded real. Now set

$$A_m = \frac{\beta\lambda - \gamma}{1 - \lambda^2}, \qquad B_m = 1, \qquad C_m = \frac{\beta - \gamma\lambda}{1 - \lambda^2} \tag{78}$$

(as per (64)) and hence strict bounded realness of $G(-s)$ implies $A_m < 0$. Also let $\hat{P}$ be the solution of (74) with $\hat{Q} = 1$ and then

$$\hat{P}^2 + 2A_m\hat{P} + C_m^2 + \epsilon = 0 \qquad (79)$$

Now we have that $\hat{P} > 0$ requires that $A_m^2 > C_m^2 + \epsilon$ which holds if, and only if, $\gamma > |\beta|$ (since $A_m < 0$, $\gamma > 0$). Hence we have stability when

$$|\lambda| < 1, \gamma > |\beta| \geq 0 \qquad (80)$$

5. Conclusions

Previous work had shown that the stability of differential linear repetitive processes is critically dependent on the structure of the boundary conditions and, in particular, on the structure of the pass state initial vector sequence. This was established by defining a general set of such boundary conditions and then applying the stability theory based on the abstract model in a Banach space setting. In this paper the problem of testing these conditions has been addressed in one case of particular interest.

The results given here have resulted in tests based on a so-called 1D Lyapunov equation and on the well known concept of a real rational strictly bounded real transfer function matrix. Of these, the tests based on the latter approach are more suited to numerical computation whereas the 1D Lyapunov equation based tests offer the potential of developing performance bounds. Note that such performance bounds have already been reported for the simplest possible set of pass state initial vectors. Work is proceeding in this and related areas and will be reported on in due course.

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