A Recursive Algorithm for the Error Probability Evaluation of M-QAM

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***Abstract*—A general recursive algorithm for the efficient and accurate computation of the bit error rate (BER) of square-shaped M-QAM constellations over additive white Gaussian noise (AWGN) channels is derived. We take advantage of the relationship amongst different square-shaped M-QAM constellations using Gray coded bit mapping.**

***Index Terms*—AWGN, BER computation, M-QAM, square constellation.**

I. Introduction

SINCE its discovery in the early 1960s, quadrature amplitude modulation (QAM) has continued to gain interest and practical applications. M-ary QAM (M-QAM) is a bandwidth efficient scheme, which has been employed for digital video broadcasting (DVB) [1], while adaptive M-QAM has been applied for high-speed data transmission over fading channels [2], [3]. Conventionally, the bit error rate (BER) computation of M-QAM has been performed by either calculating the symbol error probability first [4], [5] or simply estimating it using lower/upper bounds [4], [6]. In [1] the BER has been directly estimated for 16-QAM and 64-QAM constellations using Gray coded bit mapping. However, no general algorithm of satisfactory accuracy exists for directly estimating the BER of arbitrary M-QAM constellations.

The objective of this letter is to present a general algorithm for the efficient and accurate computation of the BER of coherent square-shaped Gray coded M-QAM constellations over additive white Gaussian channels (AWGN). In contrast to conventional approaches, which treat M-QAM having different values of M separately, in our proposed approach, we take advantage of the relationship between different square M-QAM constellations. Explicitly, a simple recursive algorithm is derived and the BER performance of an arbitrary square M-QAM constellation is estimated.

This letter is organized as follows. Section II describes the M-QAM system model. As a special case, the BER of a 16-QAM constellation is analyzed in Section III. The approach used for 16-QAM is then generalized for arbitrary square M-QAM constellations in Section IV. In Section V we present numerical examples, which is followed by our conclusions in Section VI.

II. System Model

The square M-QAM signal constellation is exemplified in Fig. 1 for $M = 16$ [1]. The M-QAM signal can be mathematically represented by:

$$s(t) = A_c \cos \omega_c t - A_s \sin \omega_c t, \quad 0 \le t \le T_s \qquad (1)$$

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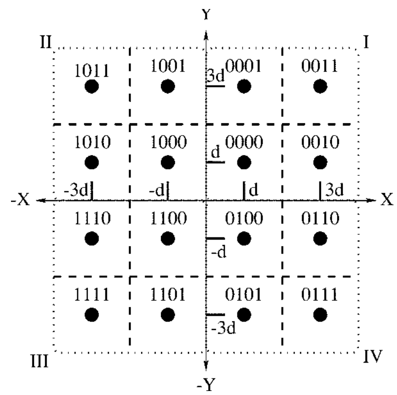

Fig. 1. 16-QAM square constellation.

where T_s is the symbol interval, which is related to the bit duration by $T_s = mT_b$ where $m = \log_2 M$ represents the number of bits per symbol, the quadrature amplitudes A_c and A_s range over the set of $\{\pm d, \pm 3d, \cdots, \pm(\sqrt{M}-1)\, d\}$, in which $2d$ represents the minimum Euclidean distance of constellation points, while d can be computed according to [5]:

$$d = \sqrt{\frac{3m \cdot E_b}{2(M-1)}} \qquad (2)$$

where E_b represents the average energy per bit.

Throughout, we will restrict our considerations to Gray coded bit mapping [1]. With reference to Fig. 1 as an example, it can be seen that according to the Gray coded bit mapping two adjacent m-bit symbols differ in a single bit. As a result, an erroneous decision resulting in an adjacent symbol is accompanied by one and only one bit error. Furthermore, we will assume optimum coherent detection with perfect carrier tracking, perfect frequency tracking, and symbol synchronization.

III. BER of 16-QAM Constellation

We commence by considering the simple case of a square 16-QAM signal constellation, as shown in Fig. 1, where each phasor can be represented by a 4-bit symbol, $i_1 q_1 i_2 q_2$, where i_1 and i_2 indicate the inphase (I) bits, while q_1 and q_2 the quadrature-phase (Q) bits. The M-QAM signal can be decomposed into two independent $\sqrt{M}$-AM signals [5]. These two $\sqrt{M}$-AM signals have the same error probability and can be treated independently. Consequently, considering the in-phase (I) 4-AM signal the average bit error probability can be computed by considering the i_1th bit and the i_2th bit, respectively [1].

Referring to Fig. 1 and [1, Ch. 5], for the i_1 bit the error probability can be expressed as:

$$P_{16}^1(\gamma) = \tfrac{1}{2}\left[Q\left(\sqrt{\gamma}\right) + Q\left(3\sqrt{\gamma}\right)\right] \qquad (3)$$

where $Q(\cdot)$ is the Q-function, which is defined as $Q(x) = (1/\sqrt{2\pi})\int_x^{\infty} e^{-t^2/2}\,dt$, and

$$\gamma = \frac{d^2}{N_0/2}. \tag{4}$$

Substituting (2) into the above equation, we arrive at $\gamma = (4/5)\cdot(E_b/N_0)$ for 16-QAM, where E_b/N_0 represents the average signal-to-noise ratio (SNR) per bit. Similarly, according to Fig. 1 and following the approach of [1] the error probability of the i_2 bit can be expressed as

$$P_{16}^2(\gamma) = \tfrac{1}{2}\left[2Q(\sqrt{\gamma}) + Q(3\sqrt{\gamma}) - Q(5\sqrt{\gamma})\right]. \tag{5}$$

Finally, the exact average BER for the square 16-QAM signal is computed by averaging the error probabilities given by (3) and (5), yielding

$$P_{16}(\gamma) = \tfrac{1}{2}\left[P_{16}^1(\gamma) + P_{16}^2(\gamma)\right]. \tag{6}$$

In conjunction with Gray coded bit mapping [1], in addition to two adjacent m-bit symbols differing in a single bit, another important property is that the $M/4$ constellation points of the four quadrants in Fig. 1 in the mirror-symmetric positions have identical bits assigned to them with respect to the X and Y axes, if we ignore the i_1 and q_1 bits. Hence, without considering bits i_1 and q_1, the points in each quadrant actually constitute an $(M/4)$-QAM constellation. Consequently, the average bit error probability of an M-QAM constellation can be expressed with the aid of the BER expression of an $(M/4)$-QAM constellation, if we ignore the bits i_1 and q_1. Let us now invoke this property in order to investigate the approximate BER expression of 16-QAM. Throughout, let $P_4(\gamma)$ represent the BER expression of a 4-QAM constellation but with γ computed according to the 16-QAM constellation, i.e. using $\gamma = (4/5)\cdot(E_b/N_0)$, as shown previously.

A. Approximation 1

In practical terms it is reasonable to assume that a bit error is most frequently caused by a noise sample exceeding d, while the probability of exceeding $3d$ is insignificant, when the signal-to-noise ratio is sufficiently high. Consequently, the BER expression of (3) for the bit i_1 can be approximated as

$$P_{16}^1(\gamma) \approx \tfrac{1}{2}Q(\sqrt{\gamma}). \tag{7}$$

Upon invoking the BER expression of 4-QAM—which is expressed as $P_4(\gamma) = Q(\sqrt{\gamma})$—the BER of (5) for the i_2 bit can be expressed as

$$P_{16}^2(\gamma) \approx P_4(\gamma). \tag{8}$$

Finally, the average BER of square 16-QAM can be approximated with the aid of (6), where $P_{16}^1(\gamma)$ and $P_{16}^2(\gamma)$ were given by (7) and (8).

B. Approximation 2

The approximate formulae of (7) and (8) are suitable for 16-QAM upon assuming sufficiently high SNR's. However, if the SNR is low, a more close approximation is required. This can be achieved by considering also the case, when the noise exceeds $3d$. In this case the BER of the i_1 bit is given by (3), while that of the i_2 bit can be expressed as

$$P_{16}^2(\gamma) = Q(\sqrt{\gamma}) + \frac{1}{2}Q(3\sqrt{\gamma}) = P_4(\gamma) + \frac{1}{2}Q(3\sqrt{\gamma}) \tag{9}$$

upon neglecting the probability of the noise exceeding $5d$ in (5). And finally, the average BER for the square 16-QAM signal can approximated by (6) with $P_{16}^1(\gamma)$ and $P_{16}^2(\gamma)$ given by (3) and (9), respectively.

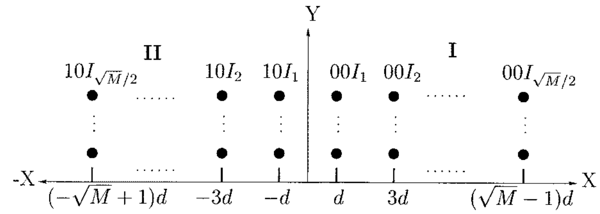

Fig. 2. Simplified representation of an arbitrary square M-QAM constellation using Gray coded bit mapping.

Above, we have used 16-QAM as an example and investigated its average exact and approximate BER. The above approach of combining the BER expressions of M-QAM having different values of M can be extended to arbitrary square M-QAM constellations and consequently can be used to simplify the associated BER computations. Let us now consider the general algorithm suitable for computing the BER of an arbitrary square M-QAM constellation.

IV. BER of Arbitrary Square M-QAM Constellations

The points of an M-QAM constellation in quadrants I and II can be simply portrayed, as seen in Fig. 2, where the first two bits indicate the bits i_1 and q_1 encountered in the quadrants I and II, while the remaining $(m-2)$ number of bits of the M-QAM symbol are represented by $I_1, I_2, \cdots$, or $I_{\sqrt{M}/2}$. Referring to Fig. 1 and [1], it is explicit that $I_1, I_2, \cdots, I_{\sqrt{M}/2}$ actually represent the corresponding points of a row in the $(M/4)$-QAM constellation. Let $i_1q_1i_2q_2\cdots i_{m/2}q_{m/2}$ represent the m-bit M-QAM symbol. Then, the average BER can be derived as follows.

A. Approximation 1

Based on Approximation 1, where the probability of noise exceeding $3d$ was neglected, the BER of the i_1 bit of an M-QAM constellation can be expressed as:

$$P_M^1(\gamma) = \frac{2}{\sqrt{M}}Q(\sqrt{\gamma}) \tag{10}$$

where γ is given by (4), while d is given by (2) for M-QAM. The average BER of bits $i_2, i_3, \cdots, i_{m/2}$ of an M-QAM constellation can be expressed as:

$$P_M^2(\gamma) = P_{M/4}(\gamma). \tag{11}$$

Consequently, the average BER of M-QAM can be derived by averaging the BER given by (10) and (11), yielding

$$P_M(\gamma) = \frac{1}{m/2}\cdot\frac{2}{\sqrt{M}}Q(\sqrt{\gamma}) + \frac{m/2-1}{m/2}\cdot P_{M/4}(\gamma). \tag{12}$$

According to (12) $P_{M/4}(\gamma)$ has to be determined first, in order to derive $P_M(\gamma)$. Hence the general BER expression of (12) can be implemented with the aid of a recursive algorithm, which is described as follows. Let $P_4(\gamma) = \lambda = Q(\sqrt{\gamma})$ and evaluate:

$$P_K(\gamma) = \frac{1}{k/2}\cdot\frac{2}{\sqrt{K}}\lambda + \frac{k/2-1}{k/2}\cdot P_{K/4}(\gamma) \tag{13}$$

for $K = 16, 4\times 16, \cdots$, until $K = M$, which is the average BER of the considered square M-QAM scheme, where $k = \log_2 K$ represents the number of bits per symbol of a K-QAM constellation. It can be shown on the basis of (13) that we have to evaluate only one integral, in order to determine the resulting BER of an M-QAM constellation.

Approximation 2

When using Approximation 2, where the probability of the noise exceeding $5d$ was neglected $P_M^1(\gamma)$ and $P_M^2(\gamma)$ can be expressed as:

$$P_M^1(\gamma) = \frac{2}{\sqrt{M}}\left[Q(\sqrt{\gamma}) + Q(3\sqrt{\gamma})\right], \tag{14}$$

$$P_M^2(\gamma) = P_{M/4}(\gamma) + \frac{2}{\sqrt{M}} \cdot \frac{1}{m/2-1} Q(3\sqrt{\gamma}) \tag{15}$$

where the former is the probability of an i_1 bit error, while the latter is the average BER of bits $i_2, i_3, \cdots, i_{m/2}$. Note that in the computation of $P_M^2(\gamma)$ there exists only one term including $Q(3\sqrt{\gamma})$, except those included in $P_{M/4}(\gamma)$. This term is the second part at the right-hand side of (15), which is derived from $00I_1$ and $10I_2$ by considering that I_1 and I_2 have a one-bit difference according to Fig. 2, due to the associated Gray coding.

When invoking Approximation 2, let $P_4(\gamma) = \lambda = Q(\sqrt{\gamma})$, $\beta = Q(3\sqrt{\gamma})$. Then the average BER of the square M-QAM signal can be recursively determined according to:

$$P_K(\gamma) = \frac{1}{k/2} \cdot \frac{2}{\sqrt{K}}[\lambda + \beta] + \frac{k/2-1}{k/2} \cdot \left[P_{K/4}(\gamma) + \frac{2}{\sqrt{K}} \cdot \frac{1}{k/2-1}\beta\right] \tag{16}$$

for $K = 16, 4 \times 16, \cdots$, until $K = M$, which is the average BER of the considered square-shaped M-QAM constellation. Equation (16) implies that only two integrals have to be evaluated at the first step, in order to obtain the resulting BER of the square M-QAM constellation.

V. Numerical Examples

In this section we use the above algorithms, in order to evaluate the BER performance of various square M-QAM schemes using different values of M. Fig. 3 is used to show the accuracy of the BER estimated by the proposed approximation approaches, while Fig. 4 implies that we can readily estimate the BER of an arbitrarily high-order square M-QAM constellation. In Fig. 3, the BER of 16-QAM and 64-QAM computed according to the recursive expression of (13) and (16) were compared with the exact BER of 16-QAM—which was obtained from (6)—and the exact BER of 64-QAM—which was computed according to the related equations in [1]. These results were also compared with the "standard" approximation derived from the division of the symbol error probability by the number of bits per symbol, $\log_2(M)$ [5, p. 630]. It can be shown that both Approximation 1 and 2 are closer to the exact BER than the "standard" approximation within the range of SNR per bit considered. An accurate BER can be achieved by Approximation 1, when the SNR per bit is sufficiently high. However, when the SNR per bit is too low (for example lower than 4dB for 64-QAM), the more accurate Approximation 2 has to be invoked, in order to achieve a satisfactory BER approximation. As shown in Fig. 3, we cannot distinguish the exact BER from that computed by Approximation 2 for 16-QAM and 64-QAM over the SNR range of interest.

Fig. 4 shows the BER performance of various square M-QAM constellations for $M = 4, 16, 64, 256, 1024$ and 4096. Note that for $M = 4$ the BER formula is reduced to that of quadrature phase shift keying (QPSK), yielding $P_4 = Q(\sqrt{(2E_b/N_0)})$, while the BER of the remaining M-QAM constellations was estimated according to (13) and (16), upon invoking Approximations 1 and 2, respectively. It can be shown that at the average BER of 10^{-3}, 4–5 dB of SNR per bit has to be invested, in order to transmit an extra 2 bits/symbol by employing a higher-order square M-QAM constellation.

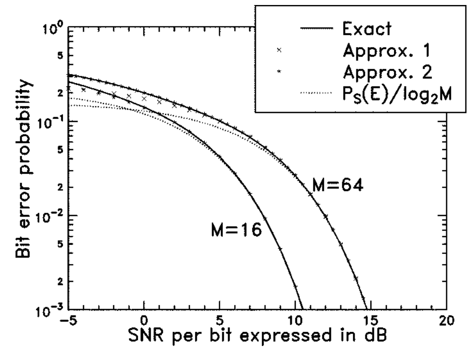

Fig. 3. BER performance of square M-QAM using Gray coded bit mapping over AWGN channels, where $P_s(E)$ represents the average symbol error probability.

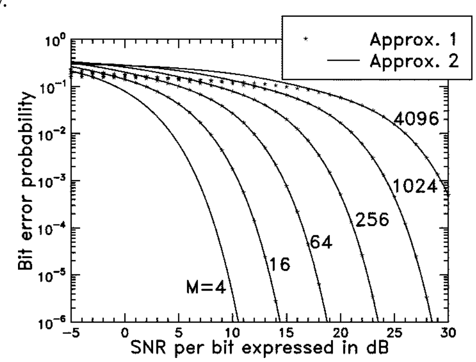

Fig. 4. BER performance of square M-QAM using Gray coded bit mapping over AWGN channels.

VI. Conclusions

A simple recursive algorithm was proposed, which is based on the symmetry of the different square M-QAM constellations using Gray coded bit mapping for evaluating the BER of arbitrarily high-order square M-QAM constellations. The numerical examples show that the BER of any square M-QAM constellation can be accurately estimated by the proposed algorithm.

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