An Analytical Comparison Between the Nonsingular Quadratic Performance of Robust and Adaptive Backstepping Designs

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Abstract—**Robust and adaptive backstepping designs for an uncertain strict feedback system are compared with respect to a cost functional which is based on an instantaneous quadratic penalty measuring both the output transient and the control effort. It is shown that the adaptive design out-performs the robust design when the actual uncertainty level is sufficiently high and the a-priori known uncertainty level is sufficiently conservative.**

Index Terms—**Backstepping, nonsingular performance, robust control.**

I. Introduction

A major open field in control theory concerns the definition of and relation between the two main branches of the subject: namely adaptive control and robust control [10]. There are many reasons as to why this field remains so open, including the following.

- The lack of a clear focus in adaptive control as to the very definition of an adaptive controller [9].
- The fact that the domain of adaptive control is largely restricted to that of parametric uncertainties, whilst robust control theory encompasses much wider classes of uncertainties: perhaps primarily it is focused on the case of un-modeled dynamics.
- Whilst the performance theory in robust control is highly developed, the corresponding adaptive performance and robustness theory is less developed. Adaptive theory is largely limited to the basic performance requirement of closed loop stability and the analysis of the transient state signal, see, eg. [9], [8], [5], (with some notable exceptions, see, for example, [4], [1][1], [6])[2].

So there are two main problems in developing any comparative results: firstly we must find a problem domain in which both robust and adaptive control designs can both be meaningfully considered; secondly we must measure performance in a manner which is both meaningful and for which analytical results can be derived. The recent framework of constructive nonlinear control [5] is an ideal setting for the development of such results. In this note, we will develop a set of results which allow analytical comparisons to be made between adaptive and robust designs: in particular we will establish results which indicate when adaptive designs can be expected to out-perform their robust counterparts. The dual theory, namely conditions for when robust controllers out-perform adaptive controllers will be considered in a forthcoming paper.

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[1]Optimal adaptive results based on the solution of Isaacs equations such as [1] utilize worst case costs which contain terms directly measuring the size of the uncertainty, differing from the results in this paper and generally in robust control where the uncertainty level solely enters the cost functional via a worst case supremum over all possible systems.

[2]Although note that the inverse optimal results of [6] are not concerned with integral performance *per-se*, as the cost functional is not determined *a priori*, rather they are concerned with guaranteeing desirable gain and phase margins. See also [6] for a discussion of the limitations of earlier work on optimal LQ adaptive control.

Necessarily for such a comparison to be made on a level playing field, we are hampered largely by the state of art in adaptive control. As noted above, adaptive control theory is weak in the presence of unmodeled dynamics; thus in our comparative scenario we will only consider static uncertainties: those that arise from bounded external disturbances, or internal static uncertainties of the plant. In all other manners, we will weight the situation in favor of the robust control theory: namely, we will consider arbitrarily fast time variations and nonparametric uncertainties. Performance will be measured by a integral performance cost functional which penalizes both the state and the control effort.

The main result of this note establishes that an adaptive backstepping design out-performs its robust counterpart when the actual uncertainty level is sufficiently high and the *a priori* known uncertainty level is sufficiently conservative. This is undoubtedly a "folklore" result which is known to control practioners: *adaptive control should be used when the uncertainty is high*, but this note establishes the first such mathematical result.

II. Systems, Uncertainties and Performance Criteria

Let $\mathcal{U}, \mathcal{Y}$ be function spaces representing the input and output signal spaces. A system is denoted by Σ and is a causal operator $\Sigma : \mathcal{U} \to \mathcal{Y}$. The set of all admissable causal systems Σ is denoted by $\mathcal{S} = \mathcal{S}(\mathcal{U}, \mathcal{Y})$. The basic problem considered in this paper is the control of a parameterized set of systems $\Sigma_p(\{f\})$ where $p \in P$ generally represents eg. an initial condition and $f \in \mathcal{F}$ represents, e.g. a system function. In particular, we will consider systems in a (time varying) strict feedback form

$$\begin{aligned}\Sigma_{x_0}(\{f\}) : \dot{x}_i &= x_{i+1} + f_i(x_1, \ldots, x_i, t) \quad 1 \le i \le n-1 \\ \dot{x}_n &= u + f_n(x_1, \ldots, x_n, t) \quad x(0) = x_0 \\ y &= x_1. \end{aligned} \tag{1}$$

The parameterised set of systems is denoted by Σ_P, ie. $\Sigma_P = \{\Sigma_p(\{f\}) \mid p \in P\}$. For concreteness, we define P to be the initial condition set

$$P = \{x_0 \in \mathbb{R}^n \mid x_0^T x_0 \le 4\eta^2\}. \tag{2}$$

To model uncertainties, we define $\{\Delta(\delta)\}_{\delta \ge 0}$ to be a set of subsets of $\mathcal{F}$ such that

$$\begin{aligned} &1.\ \Delta(0) = \left\{f^0\right\} \text{ for some } f^0 \in \mathcal{F} \\ &2.\ \Delta(\delta_1) \subset \Delta(\delta_2) \text{ if } \delta_1 \le \delta_2. \end{aligned} \tag{3}$$

We say $\Sigma_p(\Delta(\delta))$ has uncertainty $\Delta(\delta)$ with an uncertainty level δ. For the systems given by (1), we consider pointwise uncertainty models of the form

$$\begin{aligned} f &= (f_1, \ldots, f_n)^T \in \Delta(\delta) = \Delta\left(L^\infty\left(\mathbb{R}^n \times \mathbb{R}_+; \frac{1}{w}\right), f^0, \delta\right) \\ &= \{f \in C(\mathbb{R}^n \times \mathbb{R}_+, \mathbb{R}^n) \mid \forall t \ge 0 \\ &\qquad \|f_i(\cdot, t) - f_i^0(\cdot)\|_{L^\infty(\mathbb{R}^i; 1/w_i)} \le \delta,\ 1 \le i \le n\} \\ &= \left\{ f \in C(\mathbb{R}^n \times \mathbb{R}_+, \mathbb{R}^n) \;\middle|\; \left|\frac{(f_i(x,t) - f_i^0(x))}{w_i(x)}\right| \right. \\ &\qquad \left. \le \delta, \forall x \in \mathbb{R}^i, \forall t \in \mathbb{R}_+,\ 1 \le i \le n \right\} \end{aligned} \tag{4}$$

where $f^0 = (f_1^0, \ldots, f_n^0)^T$ is the nominal model and the continuous functions $(f_i^0 : \mathbb{R}^i \to \mathbb{R}\ 1 \le i \le n)$ satisfy $f_i^0(0) = 0$ for $1 \le i \le n$.

Furthermore we assume that the weight w is continuous and $w_i(x) > 0\ \forall x \in \mathbb{R}^i$, $1 \le i \le n$. We have thus defined the system $\Sigma_P(\Delta(\delta))$. $\{\Sigma_p(\Delta(\delta))\}_{\delta \ge 0}$ is a sequence of subsets of $\mathcal{S}$ and is defined in the natural manner, likewise $\{\Sigma_P(\Delta(\delta))\}_{\delta \ge 0}$.

A controller is denoted by Ξ and is a causal operator $\Xi : \mathcal{Y} \to \mathcal{U}$. The controllers we will be considering are defined in Section IV. The set of all admissable controllers is denoted by $\mathcal{C} = \mathcal{C}(\mathcal{Y}, \mathcal{U})$. Finally we define an interconnection $[\Sigma, \Xi]$ of a system Σ and controller Ξ as $[\Sigma, \Xi] = (y, u)$ where y, u are the closed-loop signals (ie. the solutions of $\Sigma u = y,\ u = \Xi y$).

Performance of a closed loop is measured by a functional of the output and input signals

$$\mathcal{J} : \mathcal{Y} \times \mathcal{U} \to \mathbb{R}_+. \tag{5}$$

Throughout, we consider a quadratic cost functional which penalizes the nonsingular transient performance of the system and is given by

$$\mathcal{J}[y(\cdot), u(\cdot)] = \int_{\mathcal{T}_\eta} y^2(t) + u^2(t)\, dt \tag{6}$$

where the time set $\mathcal{T}_\eta$ is defined as $\mathcal{T}_\eta = \{t \ge 0 \mid |y(t)| > \eta\}$. Such a cost penalizes the response of the system whilst $y(t) \notin [-\eta, \eta]$, hence for a closed loop whose goal is to stabilise y to any closed subset of $(-\eta, \eta)$, whilst keeping y, u bounded, this cost is a reasonable penalty on the transient behavior. Note that also a finite cost implies the required stabilization.

Performance of a controller Ξ will be measured in this paper with respect to a worst case cost, ie. $\mathcal{P} : P(\mathcal{S}) \times \mathcal{C} \to \mathbb{R}_+$,[3] where $P(\mathcal{S})$ denotes the power set of $\mathcal{S}$ and where (7), shown at the bottom of the page, holds true.[4]

We now make a crucial definition.

Definition2.1: A $\mathcal{P}$ stable control design is a mapping $\Gamma : \mathbb{R}_+ \to \mathcal{C}$ such that

$$\mathcal{P}(\Sigma_P(\Delta(\delta)), \Gamma(\delta)) < \infty, \quad \forall \delta \ge 0. \tag{8}$$

Thus, we are concerned with the behavior of a class of controllers $\{\Gamma(\delta)\}_{\delta \ge 0}$ as specified by the design function Γ, which defines a (different) controller for each uncertainty level δ. Examples of design operators Γ will be given later by, e.g., (10), (11), (30), (19) , (20), and (40).

III. A Partial Classification of Control Designs

First, we make two definitions. The first is a stronger version of the concept of universality in adaptive control, namely we demand that for all uncertainty levels, a single controller gives a finite cost.

Definition 3.1: Γ *is said to be a type A control design if*

1) Γ *is* $\mathcal{P}$ *stable*;
2) *there exists* $\Xi_a \in \mathcal{C}$ *such that for all* $\delta \ge 0, \Gamma(\delta) = \Xi_a$.

[3]We also admit the possibility that either $\mathcal{J}$ or $\mathcal{P}$ may not be defined for all their respective domains.

[4]The final supremum is taken over all solutions to the closed loop, this is required since our differential equations have discontinuous right-hand sides, hence although existence of solutions will be guaranteed, uniqueness will not.

The second definition captures the essential feature of robust designs, namely that the performance degrades as the uncertainty description becomes more conservative. Γ

Definition 3.2: Γ *is said to be a type R control design if*

1) Γ *is* $\mathcal{P}$ *stable;*
2) *there exists* $\delta \ge 0$ *such that for all* $\delta_1 > \delta$, $\mathcal{P}(\Sigma_P(\Delta(\delta_1)), \Gamma(\delta_2)) \to \infty \quad \text{as } \delta_2 \to \infty$.

It is straightforward to observe that control designs of type A and R are mutually exclusive as: R implies (not A). However, this does not provide a complete classification for the reverse implication does not hold and so there are controllers which are of neither type. The key performance relation between the two types of design is given by the following lemma:

Lemma 3.3: Let Γ_a, Γ_r *be type A and R designs respectively. Then* $\exists \delta \ge 0$ *such that* $\forall \delta_1 \ge \delta\ \exists \delta' \ge \delta_1$ *such that* $\forall \delta_2 \ge \delta'$ *we have*

$$\mathcal{P}\left(\Sigma_P(\Delta(\delta_1)), \Gamma_r(\delta_2)\right) > \mathcal{P}\left(\Sigma_P(\Delta(\delta_1)), \Gamma_a(\delta_2)\right). \tag{9}$$

Proof: This is a simple consequence of the definition of type A and type R control designs. □

The interpretation of this lemma is as follows. Think of δ_1 as the 'actual' uncertainty level in the system and δ_2 as the a-priori known uncertainty level in the system. Typically δ_2 is a conservative estimate of δ_1. The lemma states that providing the actual uncertainty level is sufficiently high, then as the *a priori* estimated uncertainty level becomes more conservative, the type A design neccessarily beats the type R design. This is because the type A design is independant of δ_2, whereas the performance of the type R controller degrades as δ_2 increases.

IV. A Comparison Between Adaptive and Robust Backstepping

In this section, we will demonstrate how the framework described previously can be used to explicitly compare the performance of two backstepping control designs.

A. Robust Controller (ISS Controller)

The robust controller is a variant on robust backstepping [5], [7] and is defined recursively as follows. Let $c > 0$ and $z_0 = \alpha_0 = 0$. Then, for $1 \le i \le n$, define

$$\begin{aligned} z_i &= x_i - \alpha_{i-1} \\ \alpha_i(x_1, \ldots, x_i) &= -cz_i - z_{i-1} + \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_j}(x_{j+1}) \\ &\quad - \kappa z_i \left(w_i^2 + \sum_{j=1}^{i-1} \left(\frac{\partial \alpha_{i-1}}{\partial x_j} w_j \right)^2 \right) \\ &\quad - \left(f_i^0 - \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_j} f_j^0 \right). \end{aligned} \tag{10}$$

We let $T_1 : \mathbb{R}^n \to \mathbb{R}^n$ denote the mapping $T_1(x) = z$. The controller $\Xi_r(\kappa)$ is then taken to be

$$\Xi_r(\kappa) : u = \alpha_n(x). \tag{11}$$

Relevant properties of this well-known control design are summarized as follows.

$$\mathcal{P}([\Sigma_P(\Delta(\delta)), \Xi]) = \sup_{f \in \Delta(\delta)} \sup_{x_0 \in P} \sup_{\text{solns } [\Sigma_{x_0}(\{f\}), \Xi]} \mathcal{J}[y(\cdot), u(\cdot)]. \tag{7}$$

Proposition 4.1: For the closed-loop system $(\Sigma_{x_0}(\Delta(\delta)), \Xi_r(\kappa))$, *where* $x_0 \in \mathbb{R}^n$ *we have*

1) y, x, u *are uniformly bounded for all* $t \geq 0$ *as a function of* x_0, δ *and* κ;
2) $\lim\sup_{t\to\infty} |y(t)| \leq \sqrt{n(n+1)}\delta/\sqrt{8c\kappa}$;
3) *if* $T_1(x_0)^T T_1(x_0) \geq \eta^2 > (n(n+1)\delta^2)/8c\kappa$, *then*[5]

$$m(\mathcal{T}_\eta) \leq \frac{4\kappa\left(T_1(x_0)^T T_1(x_0) - \eta^2\right)}{8c\kappa\eta^2 - n(n+1)\delta^2}; \tag{12}$$

4) *if* $T_1(x_0)^T T_1(x_0) \geq \eta^2 > n(n+1)\delta^2/8c\kappa$, *then*

$$\int_{\mathcal{T}_\eta} z^2(t)dt \leq \left(c + \frac{\kappa}{2}\epsilon^2 + \frac{n(n+1)\delta^2}{4\kappa\eta^2}\right)^{-1}\left(\frac{z_0^T z_0 - \eta^2}{2}\right) \tag{13}$$

where

$$\epsilon = \min_{1\leq i\leq n} \inf\Big\{w_i(x_1,\ldots,x_i)\,|x \in \mathbb{R}^n \quad |T_1(x)| \leq |T_1(x_0)|\Big\}. \tag{14}$$

Proof: (Sketch). Let $V = (1/2)z^T z$. Then, it is routine to compute

$$\begin{aligned}\dot V =& -cz^T z + \sum_{i=1}^n \Bigg[z_i\left(f_i - f_i^0\right) - \kappa z_i^2 w_i^2 \\ &+ \sum_{j=1}^{i-1}\left(z_i\frac{\partial\alpha_{i-1}}{\partial x_j}\left(f_j - f_j^0\right) - \kappa z_i^2\left(\frac{\partial\alpha_{i-1}}{\partial x_j}w_j\right)^2\right)\Bigg] \\ \leq& -cz^T z + \frac{n(n+1)\delta^2}{8\kappa}.\end{aligned} \tag{15}$$

1) and 2) now follow from standard arguments. 3) is established as follows. First, note that the $\eta^2/2$ level set of V is invariant, so defining $\mathcal{T}_\eta^* = \{t \geq 0 | z^T z \geq \eta^2\}$, we must have $\mathcal{T}_\eta^* = [0, t^*)$ for some $t^* \in [0,\infty]$. Now, the inequalities

$$\begin{aligned}(\mathcal{T}_\eta) \leq& \frac{\int_{\mathcal{T}_\eta^*} -\dot V\,dt}{\inf_{t\in\mathcal{T}_\eta^*}|\dot V(t)|} \\ \leq& \frac{V(0) - V(t^*)}{c\eta^2 - \frac{n(n+1)\delta^2}{8\kappa}} = \frac{4\kappa\left(T_1(x_0)^T T_1(x_0) - \eta^2\right)}{8c\kappa\eta^2 - n(n+1)\delta^2}\end{aligned} \tag{16}$$

establish 3) as required. We establish 4) as follows. Alternatively to (15), we can bound $\dot V$ as

$$\dot V \leq -\left(c + \frac{\kappa}{2}\epsilon^2\right)z^T z + \frac{n(n+1)\delta^2}{4\kappa}. \tag{17}$$

In particular, on $[0, t^*)$ we have $z^T z \geq \eta^2$, so

$$\begin{aligned}\dot V \leq& -\left(c + \frac{\kappa}{2}\epsilon^2\right)z^T z + \frac{n(n+1)\delta^2}{4\kappa}\frac{z^T z}{\eta^2} \\ =& -\left(c + \frac{\kappa}{2}\epsilon^2 + \frac{n(n+1)\delta^2}{4\kappa\eta^2}\right)z^T z.\end{aligned} \tag{18}$$

[5]Here $m(T)$ denotesthe Lebesgue measure of the set $T \subset \mathbb{R}_+$.

Then, 4) follows as required:

$$\begin{aligned}\int_{\mathcal{T}_\eta} y^2(t)\,dt \leq& \int_{\mathcal{T}_\eta^*} z^T z\,dt \\ \leq& \left(c + \frac{\kappa}{2}\epsilon^2 + \frac{n(n+1)\delta^2}{4\kappa\eta^2}\right)^{-1}\int_0^{t^*} -\dot V\,dt \\ =& \left(c + \frac{\kappa}{2}\epsilon^2 + \frac{n(n+1)\delta^2}{4\kappa\eta^2}\right)^{-1}(V(0) - V(t^*)) \\ =& \left(c + \frac{\kappa}{2}\epsilon^2 + \frac{n(n+1)\delta^2}{4\kappa\eta^2}\right)^{-1}\left(\frac{z_0^T z_0 - \eta^2}{2}\right).\end{aligned}$$

B. Adaptive Controller

The adaptive controller is also based on a backstepping idea. However, the adaptive estimates are of the uncertainty level δ, rather than any physical parameter of the system as in more standard designs, e.g., [5]. This controller operates by increasing it's gains until the state of the system is sufficiently small and can be thought of as an adaptive counterpart to the previous robust controller. Later, in Section IV-E we will consider conventional parametric adaptive controllers under stronger assumptions on the system uncertainty.

We define the adaptive controller as follows. Let $c > 0$, $z_0 = \alpha_0 = 0$. For $1 \leq i \leq n$, define

$$\begin{aligned}z_i =& x_i - \alpha_{i-1} \\ \alpha_i&\left(x_1,\ldots,x_i,\hat\theta_1,\ldots,\hat\theta_i\right) \\ =& -cz_i - z_{i-1} + \sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial\hat\theta_j}\beta_j - \left(f_i^0 - \sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_j}f_j^0\right) \\ &+ \sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_j}x_{j+1} - \hat\theta_i z_i\left(w_i^2 + \sum_{j=1}^{i-1}\left(\frac{\partial\alpha_{i-1}}{\partial x_j}w_j\right)^2\right) \\ \beta_i =& z_i^2\left(w_i^2 + \sum_{j=1}^{i-1}\left(\frac{\partial\alpha_{i-1}}{\partial x_j}w_j\right)^2\right).\end{aligned} \tag{19}$$

We let $T_2 : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ denote the mapping $T_2(x,\hat\theta) = (z,\hat\theta)$. The controller Ξ_a is then taken to be

$$\begin{aligned}\Xi_a : u =& \alpha_n\left(x,\hat\theta\right), \\ \dot{\hat\theta}_i =& D\left(B\left(0,\frac{\eta}{2}\right), z\right)\right)\beta_i \\ \hat\theta_i(0) =& 0 \quad 1 \leq i \leq n\end{aligned} \tag{20}$$

where D is a dead-zone function defined to be such that: $D(\Omega, z) = 0$ if $z \in \Omega$ and $D(\Omega, z) = 1$ if $z \notin \Omega$ and $B(x,r)$ denotes the Euclidean ball centred at x, of radius r. As the closed loop will be governed by an equation with a discontinuous RHS, we adopt the Fillipov notion of a solution, [2]. Relevant properties of this controller are summarized in the following.

Proposition 4.2: For the closed-loop system $(\Sigma_{x_0}(\Delta(\delta)), \Xi_a)$, *where* $x_0 \in \mathbb{R}^n$ *we have*

1) $y, x, u, \hat\theta$ *are uniformly bounded for all* $t \geq 0$ *as a function of* x_0 *and* δ;
2) $\lim\sup_{t\to\infty} |y(t)| \leq \eta/2 < \eta$;
3) *if* $T_1(x_0)^T T_1(x_0) \geq \eta^2$, *then*

$$m(\mathcal{T}_\eta) \leq \frac{2\left(T_1(x_0)^T T_1(x_0) - \eta^2\right)}{3c\eta^2}. \tag{21}$$

Proof: Whenever $z \notin B(0, \eta/2)$ we can write the system in the z-coordinates in the form $(1 \le i \le n)$

$$\begin{aligned}\dot z_i =& z_{i+1} - cz_i - z_{i-1} + \left(f_i - f_i^0\right) \\ &+ \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_j}\left(f_j - f_j^0\right) \\ &- \hat\theta_i z_i \left(w_i^2 + \sum_{j=1}^{i-1}\left(\frac{\partial \alpha_{i-1}}{\partial x_j} w_j\right)^2\right) \\ =& z_{i+1} - cz_i - z_{i-1} \\ &+ \left(\theta - \hat\theta_i\right) z_i \left(w_i^2 + \sum_{j=1}^{i-1}\left(\frac{\partial \alpha_{i-1}}{\partial x_j} w_j\right)^2\right) \\ &+ \left(\left(f_i - f_i^0\right) - \theta z_i w_i^2\right) \\ &+ \sum_{j=1}^{i-1}\left(\frac{\partial \alpha_{i-1}}{\partial x_j}\left(f_j - f_j^0\right) - \theta z_i \left(\frac{\partial \alpha_{i-1}}{\partial x_j} w_j\right)^2\right).\end{aligned} \tag{22}$$

Now, we define

$$V\left(z, \hat\theta\right) = \frac{1}{2} z^T z + \frac{1}{2}\sum_{i=1}^{n}\left(\theta - \hat\theta_i\right)^2, \quad \theta = \frac{n(n+1)\delta^2}{2c\eta^2}. \tag{23}$$

Since

$$\begin{aligned}\frac{d}{dt}\frac{1}{2} z^T z =& -cz^T z \\ &+ \sum_{i=1}^{n}\left(\theta - \hat\theta_i\right) z_i^2 \left(w_i^2 + \sum_{j=1}^{i-1}\left(\frac{\partial \alpha_{i-1}}{\partial x_j} w_j\right)^2\right) \\ &+ z_i \left(\left(\left(f_i - f_i^0\right) - \theta z_i w_i^2\right)\right. \\ &\left. + \sum_{j=1}^{i-1}\left(\frac{\partial \alpha_{i-1}}{\partial x_j}\left(f_j - f_j^0\right) - \theta z_i \left(\frac{\partial \alpha_{i-1}}{\partial x_j} w_j\right)^2\right)\right)\end{aligned} \tag{24}$$

it follows that:

$$\begin{aligned}\dot V \le& -cz^T z \\ &+ \sum_{i=1}^{n} z_i \left(\left(\left(f_i - f_i^0\right) - \theta z_i w_i^2\right)\right. \\ &\left. + \sum_{j=1}^{i-1}\left(\frac{\partial \alpha_{i-1}}{\partial x_j}\left(f_j - f_j^0\right) - \theta z_i \left(\frac{\partial \alpha_{i-1}}{\partial x_j} w_j\right)^2\right)\right).\end{aligned} \tag{25}$$

By repeated application of the inequality $ab - b^2 \le a^2/4$, we can establish $\forall z \notin B(0, \eta/2)$

$$\begin{aligned}\dot V \le& \sum_{i=1}^{n}\left(-cz_i^2 + \frac{\left(f_i - f_i^0\right)^2}{4\theta w_i^2} + \sum_{j=1}^{i-1}\frac{\left(f_j - f_j^0\right)^2}{4\theta w_j^2}\right) \\ \le& \sum_{i=1}^{n}\left(-cz_i^2 + \frac{\delta^2}{4\theta} + \sum_{j=1}^{i-1}\frac{\delta^2}{4\theta}\right) \le \frac{c\eta^2}{4} - \sum_{i=1}^{n} cz_i^2.\end{aligned} \tag{26}$$

As the closed-loop system has a discontinuous RHS, it is also neccessary to check for the absence of destabilising sliding solutions on the boundary of the dead-zone region. Thus, it suffices to check that $D_-V \le 0$ for $z \in \partial B(0, \eta/2)$ and $D_- z^T z = 0$[6]. Now, by definition of a Fillipov solution, $\dot{\hat\theta}_i = \lambda \beta_i$ for some $\lambda \in [0, 1]$. So since $D_- z^T z = 0$, it follows from (24) that:

$$\dot{\hat\theta}_i = \lambda\left(cz^T z - \sum_{i=1}^{n}\left(\theta - \hat\theta_i\right) z_i^2 \cdot\left(w_i^2 + \sum_{j=1}^{i-1}\left(\frac{\partial \alpha_{i-1}}{\partial x_j} w_j\right)^2\right)\right). \tag{27}$$

Consequently

$$\begin{aligned}D_-V =& D_-\frac{1}{2} z^T z + D_-\frac{1}{2}\left(\theta - \hat\theta\right)^T\left(\theta - \hat\theta\right) \\ =& -\frac{1}{2}\left(\theta - \hat\theta\right)^T \dot{\hat\theta} \le \lambda\left(\frac{c\eta^2}{4} - cz^T z\right) \le 0\end{aligned} \tag{28}$$

where the last two inequalities follow from (25) and (26), as required. Consequently, it can be established that $z \to B(0, \eta/2)$, (hence, $\limsup_{t\to\infty}|y(t)| \le \eta/2 < \eta$), z is bounded and that $\hat\theta$ is bounded. By standard arguments we have the uniform boundedness of y, x, u, $\hat\theta$. 3) is established by the inequalities

$$m\left(\mathcal{T}_\eta\right) \le \frac{\int_{\mathcal{T}_\eta} -\dot V\, dt}{\inf_{t\in\mathcal{T}_\eta}\left|\dot V(t)\right|} \le \frac{2\left(T_1\left(x_0\right)^T T_1\left(x_0\right) - \eta^2\right)}{3c\eta^2}. \tag{29}$$

■

C. Comparative Result

1) Robust Backstepping is a Type R Design: From the previous properties of the robust controller, we define a robust control design by

$$\Gamma_r\left(\delta_2\right) = \Xi_r\left(\frac{n(n+1)\delta_2^2}{4c\eta^2}\right) \quad \forall \delta_2 \ge 0. \tag{30}$$

The crux of this note is the following result which allows us to show that the robust design Γ_r is type R with respect to the transient cost.

Proposition 4.3: Consider the integrator chain $\Sigma_{x_0}(\{0\})$

$$\begin{aligned}\Sigma_\kappa : \dot x_i =& x_{i+1} \quad 1 \le i \le n-1 \\ \dot x_n =& u_\kappa \quad y_\kappa = x_1\end{aligned} \tag{31}$$

with initial condition of the form

$$x(0) = (0, \ldots, 0, 2\eta) \tag{32}$$

where $\eta > 0$ *and* $u_\kappa = \Xi_r(\kappa)$. *Then*

1)
$$\int_{\mathcal{T}_\eta}\left(y_\kappa^2 + u_\kappa^2\right) dt < \infty \quad \forall \kappa > 0; \tag{33}$$

2)
$$\int_{\mathcal{T}_\eta}\left(y_\kappa^2 + u_\kappa^2\right) dt \to \infty \quad \text{as } \kappa \to \infty. \tag{34}$$

Proof: 1) follows from 1) and 3) of Proposition 4.1. To establish 2), define $t_\kappa = \inf\{t \ge 0 | x_n(t_\kappa) = \eta\}$, $t'_\kappa = \sup\{t \le t_k | x_n(t_\kappa) = 2\eta\}$, $e_\kappa = t_\kappa - t'_\kappa$ and let $H(\kappa, i)$ denote the interval $H(\kappa, i) = [t'_\kappa + (1 - 1/2^i)e_\kappa, t_\kappa]$. For a contradiction suppose $e_\kappa \not\to 0$ as $\kappa \to \infty$, ie. there exists an $e_* > 0$ and a subsequence $\{e_{\kappa_m}\}_{m\ge 1}$ such that $e_{\kappa_m} \ge e_* \, \forall m \ge 1$.

For an induction, claim for $0 \le i \le n$ that

$$x_{n-i}(t) \ge 2^{-(i)(i+1)/2} e_*^i \eta \quad \forall t \in H(\kappa, i). \tag{35}$$

[6]Here D_- denotes the left-hand derivative.

By construction, $\forall m \geq 1$, $e_{\kappa_m} \geq e_*$, so $x_n(t) \geq \eta \, \forall t \in H(\kappa, 0)$, so claim is valid for $i = 0$. Assume the claim is valid for $i = j$. Then

$$\dot{x}_{n-j-1}(t) = x_{n-j}(t) \geq 2^{-(j)(j+1)/2} e_*^j \eta \quad \forall t \in H(\kappa, j). \tag{36}$$

Since all components of x are increasing on $[0, t_\kappa]$ by definition of the initial condition, it follows that for all $t \in H(\kappa, j+1)$:

$$\begin{aligned} x_{n-j-1}(t) \geq & m\left(H(\kappa, j) \setminus H(\kappa, j+1)\right) \\ & \times \left(\inf_{H(\kappa, j+1) \setminus H(\kappa, j)} \dot{x}_{n-j-1} \right) \\ = & 2^{-(j)(j+1)/2} e_*^j \eta \frac{e_*}{2^{j+1}} \\ = & 2^{-(j+1)(j+2)/2} e_*^{j+1} \eta \end{aligned} \tag{37}$$

thus completing the proof of the claim.

Now, since $H(\kappa, n-1) \subset \mathcal{T}_\eta$, it follows that:

$$\begin{aligned} \int_{\mathcal{T}_\eta} z^2 dt \geq & \int_{H(\kappa, n-1)} x_1^2 dt \\ \geq & \left(2^{-(n)(n-1)/2} e_*^{n-1} \eta\right)^2 \frac{e_*}{2^{n-1}} \\ = & 2^{-(n+1)(n-1)/2} e_*^{2n-1} \eta^2. \end{aligned} \tag{38}$$

This is a contradiction since by 4 of Proposition 4.1, $\int_{\mathcal{T}_\eta} z^2 dt \to 0$ as $\kappa \to \infty$ (as $T_1(x_0) = x_0$ by choice of the initial condition and the equilibrium assumption on the nominal model). Therefore, $e_\kappa \to 0$ as $\kappa \to \infty$.

Now, the following estimate holds by Cauchy–Schwartz:

$$\begin{aligned} \int_{\mathcal{T}_\eta} u_\kappa^2 dt \geq & \int_{H(\kappa, n-1)} u_\kappa^2 dt \\ \geq & \frac{\left(\int_{H(\kappa, n-1)} x_n \dot{x}_n dt\right)^2}{\int_{H(\kappa, n-1)} x_n^2 dt} \\ = & \frac{\left(x_n^2(t'_\kappa) - x_n^2(t_\kappa)\right)^2}{4 \int_{H(\kappa, n-1)} x_n^2 dt} \\ \geq & \frac{9\eta^4}{4 \int_{H(\kappa, n-1)} x_n^2 dt}. \end{aligned} \tag{39}$$

Finally, the result follows since $\int_{H(\kappa, n-1)} x_n^2 dt \leq 4 e_\kappa \eta^2 \to 0$, as $\kappa \to \infty$. ■

Proposition 4.4: *Γ_R [(30)] is a type-R control design with respect to the performance cost $\mathcal{P}$ defined by (6).*

Proof: Take $\delta > \max_{1 \leq i \leq n} \|f_i^0\|_{L^\infty(\mathbb{R}^i, 1/w_i)}$, so $0 \in \Delta$. The fact that Γ_R is $\mathcal{P}$ stable follows from 1) and 3) of Proposition 4.1. The divergent performace property now follows from Proposition 4.3. ■

Adaptive Backstepping is a Type A Design: The adaptive control design denoted by Γ_a is defined by

$$\Gamma_a(\delta) = \Xi_a, \quad \forall \delta \geq 0. \tag{40}$$

It is simple to show that the adaptive design is a type A design.

Proposition 4.5: *Γ_a [(40)] is a type A control design with respect to the performance cost $\mathcal{P}$ defined by (6).*

Proof: This is a simple consequence of Proposition 4.2. ■

D. Main Result

The main result of this note now follows by an application of Lemma 3.3.

Theorem 4.6: *Suppose the initial condition set P is given by 2, and the uncertainty model $\Delta(\delta) = \Delta(L^\infty(\mathbb{R}^n \times \mathbb{R}_+; 1/w), f^0, \delta)$ is given by (4). Suppose the performance $\mathcal{P}(\Sigma_P(\Delta(\delta), \Xi)$ is defined by (6). Let Γ_r, Γ_a be defined by (30), (40) respectively. Then $\exists \delta \geq 0$ such that $\forall \delta_1 \geq \delta \, \exists \delta' \geq \delta_1$ such that $\forall \delta_2 \geq \delta'$ we have*

$$\mathcal{P}\left(\Sigma_P\left(\Delta\left(\delta_1\right)\right), \Gamma_r\left(\delta_2\right)\right) > \mathcal{P}\left(\Sigma_P\left(\Delta\left(\delta_1\right)\right), \Gamma_a\left(\delta_2\right)\right). \tag{41}$$

Proof: The result follows from a direct application of Lemma 3.3 and Propositions 4.4 and 4.5.

E. Further Generalizations and Applications

- Under greater structural assumptions on the uncertainty Δ, similar results comparing the robust controller (10), (11), (30) with conventional parametric adaptive controllers can be obtained. For example, consider an uncertainty Δ of the form

$$\begin{aligned} f \in & \Delta(\text{parametric}, \delta) \\ & = \{f \in C\left(\mathbb{R}^n \times \mathbb{R}_+, \mathbb{R}^n\right) | \forall t \geq 0, f_i\left(x_1, \ldots, x_i, t\right) \\ & \quad = \varphi_i\left(x_1, \ldots, x_i\right)^T \theta, v\theta \in \mathbb{R}^m, \theta^T \theta = \delta^2 \Big\}. \end{aligned} \tag{42}$$

  This describes an parametrically uncertain time invariant strict feedback system. The same robust control design (10), (11), (30) can be utilized, since for a suitable choice of w, (eg. $w_i = 1 + |\varphi_i|$), we have

$$\Delta(\text{parametric}, \delta) \subset \Delta\left(L^\infty\left(\mathbb{R}^n \times \mathbb{R}_+; \frac{1}{w}\right), 0, \delta\right). \tag{43}$$

  Furthermore, it is straightforward to prove that standard adaptive controllers such as the adaptive backstepping design [5] or tuning function design [5] have the type A property. Hence the analogue of Theorem 4.6 is also valid for these controller comparisons.
- Theorem 4.6 requires the actual uncertainty to be sufficiently large so that we can ensure $0 \in \Delta$ and hence apply Proposition 4.3. To remove the assumption on the high uncertainty level, we require further restrictions on the nominal nonlinearity f^0. For example, this assumption can be removed if $f^0 = 0$, or if we have

$$f_i^0\left(x_1, \ldots x_i\right) \geq 0, \quad \forall x_i \geq 0, \quad 1 \leq i \leq n-1, \quad f_n^0 = 0. \tag{44}$$

  This is summarized in the following theorem.

Theorem 4.7: *Suppose the initial condition set P is given by 2 and the uncertainty model $\Delta(\delta) = \Delta(L^\infty(\mathbb{R}^n \times \mathbb{R}_+; 1/w), f^0, \delta)$ is given by (4), where the nominal nonlinearity f^0 satisfies (44). Suppose the performance $\mathcal{P}(\Sigma_P(\Delta(\delta), \Xi)$ is defined by (6). Let Γ_r, Γ_a be defined by (30), (40) respectively. Then $\forall \delta_1 \geq 0 \, \exists \delta' \geq \delta_1$ such that $\forall \delta_2 \geq \delta'$ we have*

$$\mathcal{P}\left(\Sigma_P\left(\Delta\left(\delta_1\right)\right), \Gamma_r\left(\delta_2\right)\right) > \mathcal{P}\left(\Sigma_P\left(\Delta\left(\delta_1\right)\right), \Gamma_a\left(\delta_2\right)\right). \tag{45}$$

Proof: The proof is similar to that of Theorem 4.6, by noting that Proposition 4.3 can be extended to the case of Σ_{f^0} by noting that the equivalent inequality (36) holds by the sign assumption on f_i^0. ■

- The results can also be extended to a number of alternative cost functionals, here we remark that the integrand can easily be changed: Let $q(t) \geq 0$ be the instantaneous cost occured at time $t \geq 0$. If q is of the form $q(t) = Q(y(t), u(t))$, where Q is radially unbounded (so that q penalizes both the output and the control effort), then by definition of Q, we have $q(t) \geq \gamma(y^2(t) + u^2(t))$ for some class $\mathcal{K}_\infty$ function γ. The enables us to give the same results for the cost functional with integrand q.

V. Concluding Remarks

We have demonstrated that an adaptive backstepping design outperforms its robust counterpart provided that the uncertainty in the system is sufficiently high and that the a-priori estimate of the uncertainty is sufficiently conservative. The performance was measured in a worst case sense penalising both the output and the control effort. The techniques developed in this paper for the comparison of these two specific schemes can be extended to compare many other control designs. It was illustrated how to obtain similar results for systems with parametric uncertainties when comparing the same robust design to a parametric adaptive design: as the type A nature of an adaptive design is simple to verify.

Although we have only stated qualitative results here, using recent quantitative upper bounding techniques developed for adaptive control performance [3], [4], bounds for the regions in which the adaptive design outperforms the robust design bounds can be constructed. However, as we have made no effort to optimize the lower bound developed in this note, we have not exhibited these regions, but leave this for future work.

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An LMI Condition for the Robust Stability of Uncertain Continuous-Time Linear Systems

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***Abstract*—A new sufficient condition for the robust stability of continuous-time uncertain linear systems with convex bounded uncertainties is proposed in this note. The results are based on linear matrix inequalities (LMIs) formulated at the vertices of the uncertainty polytope, which provide a parameter dependent Lyapunov function that assures the stability of any matrix inside the uncertainty domain. With the aid of numerical procedures based on unidimensional search and the LMIs feasibility tests, a simple and constructive way to compute robust stability domains can be established.**

***Index Terms*—Linear matrix inequalities (LMIs), linear uncertain systems, parameter dependent Lyapunov functions, robust stability, time-invariant uncertain parameters.**

I. INTRODUCTION

The investigation of stability domains for state space models has been addressed in many papers during the last years. The use of Lyapunov functions is certainly the main approach for this kind of analysis, since bounds for the stability domains can be obtained in terms of the associated Lyapunov matrix and the allowed perturbation directions [1]. As is well known, the use of a parameter independent Lyapunov function to investigate the stability domain of a linear system is only a sufficient condition for robust stability. Denominated *quadratic stability* in the literature (see, for instance, [2]), this kind of Lyapunov stability analysis can be used to design robust state feedback control gains [3], [4], being specially adequate when time-varying uncertain parameters are considered (providing not too conservative results when the parameters vary fast).

Less conservative results based on parameter dependent Lyapunov functions have been obtained [5]–[7]. In most of these cases, the uncertainty must verify some matching condition or must have a particular structure and intensive computation is needed to test the robust stability, sometimes involving scaling parameters. It is worth of mentioning that scaling parameters can be useful in some situations, mainly when stability domains can be computed without evaluating all the vertices of the polytope. Another recent and interesting approach for analysis and control design of uncertain systems is based on the use of piecewise Lyapunov functions [8] but the numerical solution of the problems also requires a high level of computational complexity.

Among the more recent papers on this subject, it is worth to mention the Lyapunov dependent parameter function approaches presented in [9]–[11] and also the LMI formulations given in [12]. In [9], sufficient conditions for diagonal and simultaneous stability of a class of system matrices have been proposed in terms of LMIs which are related to passivity and real positiveness conditions. In [10], the sufficient conditions for the existence of a parameter dependent Lyapunov

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