ELSEVIER Linear Algebra and its Applications 353 (2002) 33–52

LINEAR ALGEBRA AND ITS APPLICATIONS

www.elsevier.com/locate/laa

Kronecker product based stability tests and performance bounds for a class of 2D continuous–discrete linear systems

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Received 2 April 2001; accepted 9 January 2002

Submitted by V. Mehrmann

Abstract

This paper reports further development of the so-called 1D Lyapunov equation based approach to the stability analysis of differential linear repetitive processes which are a distinct class of 2D continuous–discrete linear systems of both practical and theoretical interest. In particular, it is shown that this approach leads to stability tests which can be implemented by computations with matrices which have constant entries. Also if the example under consideration is stable then physically meaningful information concerning one key aspect of transient performance is available for no extra cost.

Keywords: 2D dynamics; Stability; Kronecker product

1. Introduction

Repetitive processes (also termed multipass processes in the early literature) can be characterized by considering machining operations where the material, or workpiece, involved is processed by a series of sweeps, termed passes, of the processing tool over a finite duration known as the pass length. Assuming the pass length,

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PII: S0024-3795(02)00287-2

$\alpha < \infty$, to be constant, the output vector, or pass profile, $y_k(t)$, $0 \leqslant t \leqslant \alpha$ (t being the independent spatial or temporal variable) generated on the kth pass acts as a forcing function on the next pass and hence contributes to the dynamics of the new pass profile $y_{k+1}(t)$, $0 \leqslant t \leqslant \alpha$, $k \geqslant 0$.

Industrial examples include long-wall coal cutting and metal rolling operations [1,2]. In recent years so-called algorithmic examples have arisen where adopting a repetitive process setting for analysis/controller design has clear advantages over alternatives. This is particularly true for linear model based iterative learning control schemes [3] and iterative solution algorithms for nonlinear dynamic optimal control problems based on the maximum principle [4].

The interaction between successive pass profiles is the source of the unique control problem for repetitive processes in that the sequence of pass profiles generated can contain oscillations that increase in amplitude in the pass to pass direction (i.e. in the k direction in the notation for variables used here). Such behaviour is easily generated in simulation studies and in experiments on scaled models of industrial examples [1,2]. For example, in long-wall coal cutting this behaviour is caused by the effect of the cutting machine's weight as it is 'pushed over' to rest on the newly cut pass profile ready for the start of the next pass along the coal face. This results in a build up of severe undulations in succeeding passes with the result that cutting, i.e. productive work, must be halted to enable their manual removal, thereby causing the 'stop/start' cutting pattern of a typical working cycle in a coal mine.

Early attempts [1] to control these processes in the case of linear single-input single-output (SISO) dynamics focused on first 'converting' them into an equivalent standard, termed 1D here, process to enable the application of classical analysis/design techniques such as the inverse Nyquist stability criterion. In general, however, this approach will produce erroneous results since it completely ignores their inherent 2D systems structure, i.e. information propagation along a given pass (t) and from pass to pass (k) coupled with a resetting of the initial conditions at the start of each pass. In particular, so-called differential linear repetitive processes, which are of both theoretical and practical interest, are a distinct class of 2D continuous–discrete linear systems where, in contrast to other classes of such systems (see, for example, [5]) information is propagated in one direction as a function of a continuous variable over the finite and constant pass length and as a function of a discrete variable in the other direction, i.e. from pass to pass. These facts mean that these processes cannot be studied/controlled by direct application of systems theory/algorithms developed for other areas but, of course, elements of such theory, e.g. stability tests, can be used either directly or after appropriate modifications where justified.

To remove this fundamental difficulty, the starting point must be to apply the stability theory of [6]. This theory is based on an abstract model in a Banach space setting which includes all examples with linear dynamics and a constant pass length as special cases. The resulting necessary and sufficient stability conditions are expressed in terms of the bounded linear operator which describes the contribution of the previous pass dynamics to the current one in the abstract model.

Differential linear repetitive processes are a sub-class of the general model which are of both systems theoretic and applications interest. The stability theory in this case results in conditions that can be tested by direct application of standard (or 1D) linear systems tests compatible with computer aided analysis. This, however, necessitates the computation of the eigenvalues of a transfer function matrix in the Laplace variable s over the imaginary axis of the complex plane and can be computationally intensive even for simple (low order in a well defined sense) examples. Also, unlike the 1D linear systems case, these Nyquist like tests do not provide any information as to expected process performance.

This paper uses the matrix Kronecker product to develop stability tests involving only computations with constant coefficient matrices and which lead naturally to computable information on a key aspect of expected system performance. The route to these is to first express stability in terms of a so-called 1D Lyapunov equation where the entries in the defining matrices are functions of a complex variable. In the following section we give the required background on differential linear repetitive processes with full details, including proofs, again in [6].

2. Background

The differential linear repetitive processes considered here are described by a state space model of the form

$$\begin{aligned} \dot{x}_{k+1}(t) &= Ax_{k+1}(t) + Bu_{k+1}(t) + \sum_{j=1}^{M} B_{j-1} y_{k+1-j}(t), \\ y_{k+1}(t) &= Cx_{k+1}(t) + \sum_{j=1}^{M} D_j y_{k+1-j}(t). \end{aligned} \tag{1}$$

Here on pass k, $x_k(t)$ is the $n \times 1$ state vector, $y_k(t)$ is the $m \times 1$ vector pass profile, $u_k(t)$ is the $l \times 1$ vector of control inputs, and the integer $M \geqslant 1$ is termed the memory length, i.e. the number of previous passes which (explicitly) contribute to the current one.

To complete the process description, it is necessary to specify the 'boundary conditions', i.e. the state initial vector on each pass and the initial pass profiles. The simplest possible form of these is assumed here and are of the form

$$\begin{aligned} x_{k+1}(0) &= d_{k+1}, \quad k \geqslant 0, \\ y_{1-j}(t) &= f_{1-j}(t), \quad 0 \leqslant t \leqslant \alpha, \ \ 1 \leqslant j \leqslant M, \end{aligned} \tag{2}$$

where d_{k+1} is an $n \times 1$ column vector with known constant entries and $f_{1-j}(t)$ is an $m \times 1$ column vector whose entries are known functions of t. These boundary conditions cover a wide range of cases where (1) is an 'adequate model' of the underlying dynamics on which to base further analysis. There are, however, cases where this is not true and instead the state initial vector on each pass must be an explicit function of the previous pass profiles. This is a critical issue since it is known [7] that the structure of the pass initial state vector sequence alone can result in instability.

Processes described by (1) and (2) are termed non-unit memory and are the natural generalization of the unit memory case when $M = 1$. A practical example is the bench mining system [2] used to extract coal from so-called 'relatively rich seam' mines (basically, much larger, and hence more efficient, cutting heads can be used but the supporting infrastructure must lie over more than just the previous pass floor profile). For the purposes of the results developed here, however, it is convenient to write the non-unit memory process in unit memory form as follows.

Define the so-called pass profile supervector as

$$z_k(t) = \left[y_{k+1-M}^{\mathrm{T}}(t), \ldots, y_k^{\mathrm{T}}(t)\right]^{\mathrm{T}}$$

and introduce the matrices

$$E = [B_{M-1}, \ldots, B_0],$$

$$F = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ C \end{bmatrix}$$

and

$$G = \begin{bmatrix} 0 & I_m & \cdots & 0 \\ \vdots & \vdots & \ddots & I_m \\ D_M & D_{M-1} & \cdots & D_1 \end{bmatrix}.$$

Then it follows immediately that the dynamics of the non-unit memory process (1) and (2) can be expressed in the unit memory form

$$\begin{aligned} \dot{x}_{k+1}(t) &= Ax_{k+1}(t) + Bu_{k+1}(t) + Ez_k(t), \\ z_{k+1}(t) &= Fx_{k+1}(t) + Gz_k(t), \\ x_{k+1}(0) &= 0, \quad 0 \leqslant t \leqslant \alpha, \quad k \geqslant 0, \end{aligned}$$

where no loss of generality arises here from setting $d_{k+1} = 0$ from this point onwards.

Consider now the application of the abstract model based stability theory of [6] to this case. Then the problem here can be considered in the context of the product space $E_\alpha^M = E_\alpha \times E_\alpha \times \cdots \times E_\alpha$ (M times), where $E_\alpha = C_m(0, \alpha)$—the space of bounded continuous mappings of the interval $0 \leqslant t \leqslant \alpha$ into the space of complex m vectors $\mathbb{C}^m$ with norm

$$\|y\| = \sup_{0 \leqslant t \leqslant \alpha} \|y(t)\|_m,$$

and $\|\cdot\|_m$ is any convenient norm on $\mathbb{C}^m$. To define the norm on E_α^M consider $Y := [y_1^{\mathrm{T}}, \ldots, y_M^{\mathrm{T}}]^{\mathrm{T}}$ and, without loss of generality, define this quantity as

$$\|Y\|_M := \max_{1 \leqslant j \leqslant M} \|y_j\|.$$

Given this setting, it is routine to show that the dynamics of (1) and (2) can be written in the form

$$z_{k+1} = L_\alpha z_k + b_{k+1}, \quad k \geqslant 0, \tag{3}$$

where L_α is the bounded linear map in E_α^M defined by

$$L_\alpha = \begin{bmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & I \\ L_\alpha^M & L_\alpha^{M-1} & L_\alpha^{M-2} & \cdots & L_\alpha^1 \end{bmatrix},$$

with for, $1 \leqslant j \leqslant M$,

$$(L_\alpha^j y)(t) = C \int_0^t \mathrm{e}^{A(t-\tau)} B_{j-1} y(\tau)\, \mathrm{d}\tau + D_j y(t), \quad 0 \leqslant t \leqslant \alpha.$$

Also

$$b_{k+1}(t) = \begin{bmatrix} 0 & \cdots & 0 & \hat{b}_{k+1}^{\mathrm{T}}(t) \end{bmatrix}^{\mathrm{T}},$$

where

$$\hat{b}_{k+1}(t) = C \int_0^t \mathrm{e}^{A(t-\tau)} B u_{k+1}(\tau)\, \mathrm{d}\tau, \quad 0 \leqslant t \leqslant \alpha, \ \ k \geqslant 0,$$

and it is assumed that $\hat{b}_{k+1} \in W_\alpha$—a linear subspace of E_α. If the control inputs are assumed to be piecewise continuous, then $W_\alpha \subset E_\alpha$ is obtained by evaluating the expression given above for this quantity for all such $u_{k+1}(t)$. We denote this abstract model by S.

Given the unique characteristic of repetitive processes, the natural intuitive approach to a definition of asymptotic stability is to ask (in terms of (3)) that, for any initial profile z_0 and any disturbance sequence $\{b_k\}_{k \geqslant 1}$ that ‘settles down’ to a steady disturbance b_∞ as $k \to +\infty$, the sequence of pass profiles $\{z_k\}_{k \geqslant 1}$ settles down to a steady, or so-called limit, profile as $k \to +\infty$. The major problem with such an approach is that it does not explicitly include the intuitive idea that asymptotic stability is retained if the model is perturbed slightly due to modelling errors or simulation approximations. For this reason, asymptotic stability has been defined as follows which ensures that the ‘set of stable processes’ is (in a well-defined sense) open in the class of all linear repetitive processes. Here, however, we do not deal with the case when modelling errors or simulation approximations are significant enough to require explicit treatment.

In more formal terms, asymptotic stability for these processes requires, in effect, that bounded sequences $\{b_k\}_{k \geqslant 1}$ applied to S produces bounded sequences of pass profiles $\{z_k\}_{k \geqslant 1}$, where the bounded property is defined in terms of the norm on the underlying function space. This can be shown to be equivalent to the existence of finite real scalars $M_\alpha > 0$ and $\lambda_\alpha \in (0, 1)$ (which depend on α) such that

$$\|L_\alpha^k\| \leqslant M_\alpha \lambda_\alpha^k, \quad k \geqslant 0.$$

The process S has this property [6] if, and only if, the spectral radius, $r(L_\alpha)$, of L_α satisfies

$$r(L_\alpha) < 1.$$

Also if this condition holds, z_∞—the resulting limit profile—is the unique solution of the linear equation

$$z_\infty = L_\alpha z_\infty + b_\infty.$$

In the case of processes described by (1) and (2), it can be shown [6] that asymptotic stability holds if, and only if,

$$r(G) < 1.$$

If this condition holds, and the control input sequence applied $\{u_k\}_{k\geqslant 1}$ converges strongly to u_∞, then it is straightforward to show that the corresponding limit profile is described by the state space model

$$\begin{aligned} \dot{x}_\infty(t) &= (A + \hat{B}(I_m - \hat{D})^{-1}C)x_\infty(t) + Bu_\infty(t), \\ y_\infty(t) &= (I_m - \hat{D})^{-1}Cx_\infty(t), \\ x_\infty(0) &= 0, \quad 0 \leqslant t \leqslant \alpha, \end{aligned} \tag{4}$$

where

$$\hat{B} = \sum_{j=1}^{M} B_{j-1}, \quad \hat{D} = \sum_{j=1}^{M} D_j.$$

In effect, we now have that if a process described by (1) and (2) is asymptotically stable then its repetitive dynamics can, after a 'sufficiently large' number of passes, be replaced by those of a standard, or 1D, linear system. Note also its counter-intuitive nature, i.e. asymptotic stability is largely independent of the underlying dynamics and, in particular, of the eigenvalues of the state matrix A which clearly govern the dynamics along any pass.

This last fact is due entirely to the finite pass length and is possible for the resulting limit profile to be 'unstable along the pass' in the standard 1D sense, i.e. contain exponentially growing dynamics as t evolves over $[0, \alpha]$. A simple example here is the following SISO process:

$$\begin{aligned} \dot{x}_{k+1} &= -x_{k+1}(t) + u_{k+1}(t) + (1+\beta)y_k(t), \\ y_{k+1}(t) &= x_{k+1}(t), x_{k+1}(0) = 0, \quad 0 \leqslant t \leqslant \alpha, \quad k \geqslant 0, \end{aligned} \tag{5}$$

where $\beta > 0$ is a real scalar. This process is clearly asymptotically stable with limit profile state space model

$$\dot{y}_\infty(t) = \beta y_\infty(t) + u_\infty(t), \quad y_\infty(0) = 0, \quad 0 \leqslant t \leqslant \alpha,$$

which is unstable as a 1D linear system.

The reason for this situation is the finite duration of the pass length—over which even an unstable 1D linear system can only produce a bounded output in response to a bounded input. Applications do, however, exist where asymptotic stability is all that is required or can be achieved [3,4]. In general, however, it is the stronger property of stability along the pass discussed next that will be required.

Stability along the pass demands that the BIBO property which defines asymptotic stability holds uniformly with respect to the pass length and hence the existence of finite real numbers $M_\infty > 0$ and $\lambda_\infty \in (0, 1)$ which are independent of α and satisfy

$$\|L_\alpha^k\| \leqslant M_\infty \lambda_\infty^k,$$

$k \geqslant 0$, and for all $\alpha > 0$.

Necessary and sufficient conditions for stability along the pass of S can again be found in [6] but here we move immediately to giving their interpretation for the processes considered. We simply note that asymptotic stability for all pass lengths (and hence the existence of a limit profile) is a necessary condition for stability along the pass.

In actual fact, several equivalent sets of necessary and sufficient conditions for stability along the pass of processes described by (1) and (2) can be derived [6] but here we use the following which involve the transfer function matrix:

$$H(s) = F(sI_n - A)^{-1}E + G. \tag{6}$$

Theorem 1. *The linear repetitive process S generated by* (1) *and* (2) *is stable along the pass if, and only if,*
(a) $r(G) < 1$,
(b) *all eigenvalues of the matrix A have strictly negative real parts*,
(c) *all eigenvalues of* $H(\imath\omega)$, $\omega \geqslant 0$, *of* (6) *have modulus strictly less than one.*

It is easy to show that stability along the pass guarantees that the corresponding limit profile of (4) is stable as a 1D linear system, i.e. all eigenvalues of the state matrix $A + \hat{B}(I_m - \hat{D})^{-1}C$ have strictly negative real parts. Also, by (b) of Theorem 1, stability of the matrix A (i.e. a uniformly bounded first pass profile) is, in general, only a necessary condition for stability along the pass.

All three conditions of Theorem 1 have well-defined physical interpretations and, unlike equivalents [2], can be tested by direct application of 1D linear time invariant systems tests which are compatible with computer aided analysis. The only difficulty which can arise is the computational cost associated with, in particular, condition (c) where even in the SISO case it is necessary to work with an $M \times M$ frequency response matrix (in some applications M could in the range of 40–50). Also, unlike the 1D case, the Nyquist like tests which can be applied to the conditions of Theorem 1 do not provide any useful indicators as to expected system performance in terms of, for example, the rate of approach of the output sequence of pass profiles to the resulting limit profile.

3. 1D Lyapunov equation based stability characterization and performance bounds

The following result is the starting point for the new stability tests developed in this paper. Its central feature—the so-called 1D Lyapunov equation for differential non-unit memory linear repetitive processes was first introduced in [8] for the unit memory case. Hence only the main steps in the proof in the non-unit memory case is given here since this is essential to the establishment of the subsequent performance bounds.

Theorem 2. *The linear repetitive process S generated by* (1) *and* (2) *is stable along the pass if, and only if,*
(a) $r(G) < 1$ *and all eigenvalues of the matrix A have strictly negative real parts*;
(b) *there exists a rational polynomial matrix solution* $P(s)$ *of the Lyapunov equation*

$$H^{\mathrm{T}}(-s)P(s)H(s) - P(s) = -I, \tag{7}$$

bounded in an open neighbourhood of the imaginary axis with the properties that $P(s) \equiv P^{\mathrm{T}}(-s)$ *and for all* $\omega \geqslant 0$

$$\beta_2^2 I \geqslant P(\imath\omega) = P^{\mathrm{T}}(-\imath\omega) \geqslant \beta_1^2 I, \tag{8}$$

for some choices of real scalars $\beta_i \geqslant 1$, $i = 1, 2$.

Proof. Suppose first that stability along the pass holds. Then the necessary conditions of (a) here ensure that $H(s)$ is bounded in an open neighbourhood of the imaginary axis. Also by (c) of Theorem 1, it follows that for each finite $\omega \geqslant 0$ there exists a unique positive definite Hermitian (denoted PDH) solution $P(\imath\omega)$ of the equation

$$H^{\mathrm{T}}(-\imath\omega)P(\imath\omega)H(\imath\omega) - P(\imath\omega) = -I. \tag{9}$$

This argument shows that a solution of (7) exists on the imaginary axis where also $\lim_{|s|\to+\infty} H(s) = G$. Also a simple argument based on the uniqueness and continuity of the solutions shows that $\lim_{|\omega|\to+\infty} P(\imath\omega) = P_\infty$, where $P_\infty = P_\infty^{\mathrm{T}} \geqslant 0$ is real and solves

$$G^{\mathrm{T}} P_\infty G - P_\infty = -I. \tag{10}$$

The fact that $r(G) < 1$ now yields $P_\infty > 0$ and hence $P(\imath\omega)$ is bounded for $\omega \geqslant 0$. Further, a routine continuity and compactness argument establishes the necessity of (8), where the proof for β_2 is obvious and that for β_1 follows from noting that $r(H(\imath\omega)) < 1$, for all ω and hence $H^{\mathrm{T}}(-\imath\omega)P(\imath\omega)G(\imath\omega) + I = P(\imath\omega) \geqslant I$. Also the fact that this solution also extends to an open neighbourhood of the imaginary axis is easily established by, in effect, standard linear algebraic arguments.

It now remains to show that (a) and (b) here imply that $r(H(\imath\omega)) < 1$, for all $\omega \geqslant 0$. This is trivially verified point-wise from (7). □

Note. The structure of the Lyapunov equation arising in Theorem 2 is identical to that for 1D discrete linear time invariant systems except for the fact that the entries in the coefficient matrices are functions of a complex variable. Hence it is termed 1D to distinguish it from an alternative Lyapunov equation interpretation for 2D linear systems/repetitive processes. This is the so-called 2D Lyapunov equation characterization of stability where the entries in the coefficient matrices are again constants. (See, for example [9], who established that, in general, only sufficient, but not necessary, conditions for stability of 2D discrete linear systems described by the extensively studied Roesser state space model [10] (or equivalents) can be achieved with such a Lyapunov equation.)

The numbers β_i, $i = 1, 2$, of Theorem 2 play no role in the stability analysis but, together with the solution matrix $P(s)$ of the 1D Lyapunov equation they are the key to obtaining bounds on expected performance of a stable along the pass example. This is developed further below.

Suppose that a process described by (1) and (2) is stable along the pass, i.e. there exists a PDH solution matrix $P(s)$ to the associated 1D Lyapunov equation of Theorem 2. Then standard factorization techniques enable this matrix to be written as

$$P(s) = F^{\mathrm{T}}(-s)F(s),$$

where without loss of generality,

$$\lim_{|s|\to+\infty} F(s) = P_{\infty}^{1/2},$$

and the matrix on the right-hand side here is the unique symmetric positive definite square root of P_∞ and, as an immediate consequence of (9), $P_\infty = P_\infty^{\mathrm{T}}$ is the unique positive definite solution matrix of the (1D) Lyapunov equation (10). Also $F(s)$ is stable and minimum phase and hence has a stable and minimum phase inverse.

Return now to (1) and delete the current pass input term from the state equation. Then (see [2] for the details of how the technical difficulties arising from the finite pass length can be avoided) the process dynamics can be written as

$$z_{k+1}(s) = H(s)z_k(s),$$

where $z_k(s)$ is the Laplace transform of $z_k(t)$. Also let

$$\hat{z}_k(s) = F(s)z_k(s), \quad k \geqslant 0,$$

denote 'filtered' (by properties of $F(s)$) outputs. Then the following results give bounds on expected system performance. Again, this result was established in [8] for the unit memory case and the proof in the non-unit memory case follows identical steps. Hence it is omitted here.

Theorem 3. *Suppose that S generated by* (1) *and* (2) *is stable along the pass and set $N = mM$, and $E_\alpha^N = L_2^N(0,+\infty) = L_2(0,+\infty) \times \cdots \times L_2(0,+\infty)$ (N times). Then for all $k \geqslant 0$,*

$$\|\hat{z}_{k+1}\|^2_{L_2^N(0,+\infty)} = \|\hat{z}_k\|^2_{L_2^N(0,+\infty)} - \|z_k\|^2_{L_2^N(0,+\infty)},$$

and hence the 'filtered' output sequence $\{\|\hat{z}_k\|_{L_2^N(0,+\infty)}\}_{k\geqslant 0}$ *is strictly monotonically decreasing to zero and satisfies, for* $k \geqslant 0$, *the inequality*

$$\|\hat{z}_k\|_{L_2^N(0,+\infty)} \leqslant \lambda^k \|\hat{z}_0\|_{L_2^N(0,+\infty)},$$

where

$$\lambda := (1-\beta_2^{-2})^{1/2} < 1.$$

Also the actual output sequence $\left\{\|z_k\|_{L_2^N(0,+\infty)}\right\}_{k\geqslant 0}$ *is bounded by*

$$\|z_k\|_{L_2^N(0,+\infty)} \leqslant \tilde{M}\lambda^k \|z_0\|_{L_2^N(0,+\infty)},$$

where

$$\tilde{M} := \beta_2\beta_1^{-1} \geqslant 1.$$

Theorem 3 gives the following computable (see the next section) information concerning the rate of approach of the output sequence of pass profiles of a differential non-unit memory linear repetitive process to the resulting limit profile under stability along the pass (stated here in terms of $\{y_k\}_{k\geqslant 0}$).

- The sequence of 'filtered' pass profiles $\{\hat{y}_k\}_{k\geqslant 0}$ consists of monotone signals converging to zero at a computable rate in $L_2^N(0,+\infty)$.
- The actual output sequence $\{y_k\}_{k\geqslant 0}$ converges at the same geometric rate but this is no longer necessarily monotonic. Also the deviation from monotonicity is described by the parameter $\tilde{M}$ computed from the solution of the 1D Lyapunov equation.

4. Solving the 1D Lyapunov equation

In this section procedures for solving the 1D Lyapunov equation are developed which are, of course, also stability tests when combined with those for the other conditions of Theorem 2. Also if they hold then they release the performance information of Theorem 3. These solution procedures are either explicit, i.e. solve for $P(\imath\omega)$ and then test it for the PDH property, or implicit, i.e. avoid the need to solve for $P(\imath\omega)$.

Consider first the explicit route where in computational, or testing, terms only the imaginary axis of the complex plane needs to be considered in (7). Hence if the conditions listed under (a) of Theorem 2 hold, the essential task is to solve for $P(\imath\omega)$ and if it has the PDH property, then the example under consideration is stable along the pass.

Suppose now that $P(\imath\omega) = P^{\mathrm{T}}(-\imath\omega)$ has been obtained as the solution of (9). Then it follows immediately that the PDH requirement on $P(\imath\omega)$ is equivalent to it having the so-called axis positivity property—see, for example, the work of Siljak [11]. In particular, the following result is an immediate consequence of Siljak's criteria for axis positivity of $P(\imath\omega)$.

Lemma 1. *The linear repetitive process S generated by* (1) *and* (2) *is stable along the pass if*, *and only if*,

(a) $r(G) < 1$ *and all eigenvalues of the matrix A have strictly negative real parts*;

(b) *the solution matrix* $P(\imath\omega)$ *of the* 1D *Lyapunov equation* (9) *satisfies*

$$P(0) > 0,$$

and

$$\det(P(\imath\omega)) > 0, \tag{11}$$

for all $\omega \geqslant 0$.

The condition on $P(0)$ in this lemma is easily tested, and in the case of that of (11), the fact that $P(\imath\omega)$ is a Hermitian matrix means that its determinant must be real for all values of ω. Hence this determinant must be a real even order polynomial of the form

$$g(\omega^2) = \sum_{i=0}^{r} g_{2i}\omega^{2i},$$

and therefore (11) is equivalent to the requirement that this polynomial has the so-called positive realness property

$$g(\omega^2) > 0$$

for all $\omega \geqslant 0$.

This last condition is easily seen to be equivalent to the requirement that $g(\omega^2)$ has no positive real roots and $g(\omega^2) > 0$ for some $\omega \geqslant 0$. Hence to proceed further with this approach requires a means of determining the location of the real roots (if any) of the real polynomial $g(\omega^2)$. This is a well-researched problem and numerous algorithms exist which avoid the need to compute the roots. For example, algorithms have been developed from the concept of a matrix inner (see [12]) and the so-called modified Routh array [11]. Such tests are not considered further here since clearly their major use is in low order 'synthesis type' problems where some, or all, of the matrices defining the example under consideration contain design parameters.

Later in this section, it will be shown that the 1D Lyapunov equation condition for stability along the pass is equivalent to one expressed in terms of an eigenvalue problem for matrices with constant entries. This uses the well-known Kronecker product for matrices and, in effect, the basic starting point is the matrices which define the transfer function matrix $H(s)$. Prior to this analysis, however, further special cases are analysed.

Consider the case when (1) is unit memory ($M = 1$) and SISO. Suppose also that $|D_1| < 1$ and that the matrix A has eigenvalue–eigenvector decomposition

$$T^{-1}AT = \Lambda = \mathrm{diag}\{\lambda_i\}_{1\leqslant i\leqslant n}.$$

Then the conditions listed under (a) of Theorem 2 for stability along the pass hold if, and only if, all eigenvalues of A have strictly negative real parts, i.e. $\mathrm{Re}(\lambda_i) < 0$, $1 \leqslant i \leqslant n$.

Now focus on the case when λ_i, $1 \leqslant i \leqslant n$, is real and introduce the so-called augmented plant matrix for (1) as

$$Q = \begin{bmatrix} A & B_0 \\ C & D_1 \end{bmatrix}.$$

Also define $\hat{T}$ as $\hat{T} = \operatorname{diag}\{T, 1\}$ and transform Q to

$$\hat{Q} = \begin{bmatrix} A & \hat{B}_0 \\ \hat{C} & D_1 \end{bmatrix},$$

where

$$\hat{B}_0 = T^{-1}B_0 = [b_1, \dots, b_n]^{\mathrm{T}}$$

and

$$\hat{C} = CT = [c_1, \dots, c_n].$$

Consider also the case when all of the numbers

$$\delta_i := b_i c_i, \quad 1 \leqslant i \leqslant n, \tag{12}$$

have the same sign. Then the following result holds.

Lemma 2. *Suppose that S generated by* (1) *and* (2) *is SISO, unit memory,* $|D_1| < 1$, *and the matrix A is diagonalizable with real eigenvalues* λ_i, $1 \leqslant i \leqslant n$, *which all lie on the negative real axis. Suppose also that all the numbers of* (12) *have the same sign. Then this process is stable along the pass if, and only if,*

$$|D_1 - CA^{-1}B_0| < 1. \tag{13}$$

Proof. First note that the 1D Lyapunov equation for this case is

$$P(\imath\omega)(1 - |D_1 + C(\imath\omega I_n - A)^{-1}B_0|^2) = 1.$$

Hence $P(\imath\omega) > 0$ for all ω if, and only if,

$$|D_1 + C(\imath\omega I_n - A)^{-1}B_0| < 1,$$

for all ω and necessity is immediate.

To prove sufficiency, first note that

$$\begin{aligned} |\gamma(\imath\omega)| &:= |D_1 + C(\imath\omega I_n - A)^{-1}B_0| \\ &= \left|D_1 + \hat{C} \operatorname{diag} \{\imath\omega - \lambda_i\}^{-1}_{1\leqslant i\leqslant n} \hat{B}_0\right| \\ &= \left|D_1 + \operatorname{sgn}(\delta_1) \sum_{i=1}^{n} \gamma_i(\imath\omega)\right|, \end{aligned}$$

where

$$\gamma_i(\imath\omega) = \frac{|\delta_i|}{\imath\omega - \lambda_i}, \quad 1 \leqslant i \leqslant n,$$

and each of these functions maps the imaginary axis in the s-plane onto a circle whose centre is on the real axis. This means that the maximum value can only occur when $\omega = 0$ and (11) follows immediately. □

In the case when the numbers δ_i of (12) have different signs, the following result gives a sufficient condition for stability along the pass.

Lemma 3. *Suppose that S generated by* (1) *and* (2) *is SISO, unit memory,* $|D_1| < 1$, *and the matrix A is diagonalizable with real eigenvalues λ_i, $1 \leqslant i \leqslant n$, which all lie on the negative real axis. Suppose also that in the numbers of* (12) δ_i, $1 \leqslant i \leqslant h$, *have the same sign and so do the remainder δ_i, $h+1 \leqslant i \leqslant n$. Then this process is stable along the pass if*

$$\max\{h_1 + h_2,\ h_2 + h_3\} < 1,$$

where

$$h_1 = \left| D_1 + \sum_{i=1}^{h} \frac{|\delta_i|}{\lambda_i} \right|,$$

$$h_2 = \left| \sum_{i=h+1}^{n} \frac{|\delta_i|}{-\lambda_i} \right|,$$

$$h_3 = \left| D_1 - \sum_{i=1}^{h} \frac{|\delta_i|}{\lambda_i} \right|.$$

Proof. In this case

$$\begin{aligned} &|D_1 + C(sI_n - A)^{-1}B_0| \\ &\quad = \left| D_1 + \sum_{i=1}^{h} \frac{\delta_i}{s-\lambda_i} + \sum_{i=h+1}^{n} \frac{\delta_i}{s-\lambda_i} \right| \\ &\quad \leqslant \max_{s=\imath\omega} \left| D_1 + \operatorname{sgn}(\delta_1) \sum_{i=1}^{h} \frac{|\delta_i|}{s-\lambda_i} \right| + \max_{s=\imath\omega} \left| \sum_{i=h+1}^{n} \frac{|\delta_i|}{s-\lambda_i} \right| \end{aligned}$$

and the result follows immediately. □

Consider now the general case when the conditions listed under (a) of Theorem 2 hold and the remaining task is to determine if condition (b) of this result—the 1D Lyapunov equation—holds and hence stability along the pass. The analysis which follows uses the matrix Kronecker product, denoted $\otimes$ to develop a solution to this problem expressed in terms of the locations of the eigenvalues of constant matrices.

Using $\otimes$, the 1D Lyapunov equation (9) can be written as

$$(I - H^{\mathrm{T}}(-\imath\omega) \otimes H^{\mathrm{T}}(\imath\omega))S_s\,[P(\imath\omega)] = S_s\,[I],$$

where $S_s[\ ]$ denotes the stacking operator. This now leads to the following set of necessary and sufficient conditions for stability along the pass of S generated by (1) and (2), which are then further developed to give stability tests which can be implemented using only operations on matrices with constant entries.

Theorem 4. *The linear repetitive process S generated by* (1) *and* (2) *is stable along the pass if, and only if,*

(a) $r(G) < 1$ *and all eigenvalues of the matrix A have strictly negative real parts*;

(b) $P \equiv P(\imath\omega_0)$, *the solution of*

$$H^{\mathrm{T}}(-\imath\omega_0)PH(\imath\omega_0) - P = -I,$$

is positive definite for some $\omega_0 \geqslant 0$, *or*, *equivalently*, $r(H(\imath\omega_0)) < 1$;

(c) $$\det\left(I - H^{\mathrm{T}}(-\imath\omega) \otimes H^{\mathrm{T}}(\imath\omega)\right) \neq 0 \tag{14}$$

for all ω.

Proof. It is clearly required to prove that (b) and (c) here are, together, to (b) of Theorem 2. First note, therefore, that (14) here guarantees the existence of a unique solution $P(\imath\omega)$ of (9). Also $P(\imath\omega)$ is PDH if, and only if, its eigenvalues are positive for all ω. These are continuous functions of ω and will always be positive if $P(\imath\omega_0)$ is positive definite for some $\omega_0 \geqslant 0$ and (14) holds. Hence (b) and (c) here are equivalent to (b) of Theorem 2 and the proof is complete. □

To apply Theorem 4 to a given example, it is necessary to test three constant matrices for stability, i.e. G and $H(\imath\omega_0)$ in the discrete 1D sense and A in the differential 1D sense, and (14) for all ω. Clearly therefore increased computational efficiency (relative to the necessary and sufficient conditions of Theorem 1 (or equivalents [2]) will only be achieved by (if possible) further development (i.e. reducing the computational complexity) of the condition of (14). The following result now expresses stability along the pass of processes described by (1) and (2) in terms of the eigenvalues of constant matrices.

Theorem 5. *The linear repetitive process S generated by* (1) *and* (2) *is stable along the pass if, and only if,*

(a) $r(G) < 1$, *all eigenvalues of the matrix A have strictly negative real parts*;

(b) *condition* (b) *of Theorem* 4 *holds*;

(c) $$\det(\lambda^2 X_1 + \lambda X_2 + X_3) \neq 0, \quad \lambda = \imath\omega, \tag{15}$$

for all ω, *where*

$$X_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -I \end{bmatrix},$$

$$X_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & -I & 0 \\ 0 & 0 & 0 & I \otimes A^{\mathrm{T}} - A^{\mathrm{T}} \otimes I \end{bmatrix}$$

and

$$X_3 = \begin{bmatrix} I - G^{\mathrm{T}} \otimes G^{\mathrm{T}} & I \otimes E^{\mathrm{T}} & E^{\mathrm{T}} \otimes G^{\mathrm{T}} & E^{\mathrm{T}} \otimes E^{\mathrm{T}} \\ G^{\mathrm{T}} \otimes F^{\mathrm{T}} & -I \otimes A^{\mathrm{T}} & 0 & 0 \\ F^{\mathrm{T}} \otimes I & 0 & -A^{\mathrm{T}} \otimes I & 0 \\ F^{\mathrm{T}} \otimes F^{\mathrm{T}} & 0 & 0 & A^{\mathrm{T}} \otimes A^{\mathrm{T}} \end{bmatrix}.$$

Proof. First set $\lambda = \imath\omega$ and introduce

$$f(\lambda) := \det\left(I - H^{\mathrm{T}}(-\lambda) \otimes H^{\mathrm{T}}(\lambda)\right).$$

Then

$$\begin{aligned} f(\lambda) = {} & \det\left(I - G^{\mathrm{T}} \otimes G^{\mathrm{T}} - E^{\mathrm{T}} \otimes G^{\mathrm{T}}\right)((-\lambda I - A^{\mathrm{T}}) \otimes I)^{-1}(F^{\mathrm{T}} \otimes I) \\ & -(I \otimes E^{\mathrm{T}})(I \otimes (\lambda I - A^{\mathrm{T}})^{-1})(G^{\mathrm{T}} \otimes F^{\mathrm{T}}) \\ & - \left((E^{\mathrm{T}} \otimes E^{\mathrm{T}})h(\lambda)^{-1}(F^{\mathrm{T}} \otimes F^{\mathrm{T}})\right), \end{aligned} \tag{16}$$

where the matrix $h(\lambda)$ whose inverse appears in the last term of (16) is given by

$$h(\lambda) = -\lambda^2 I + \lambda(I \otimes A^{\mathrm{T}} - A^{\mathrm{T}} \otimes I) + A^{\mathrm{T}} \otimes A^{\mathrm{T}}.$$

Hence

$$f(\lambda) = \det\left(I - G^{\mathrm{T}} \otimes G^{\mathrm{T}} - UV^{-1}W\right),$$

where

$$U = \left[(I \otimes E^{\mathrm{T}}), (E^{\mathrm{T}} \otimes G^{\mathrm{T}}), (E^{\mathrm{T}} \otimes E^{\mathrm{T}})\right],$$

$$V = \begin{bmatrix} \lambda I - I \otimes A^{\mathrm{T}} & 0 & 0 \\ 0 & -\lambda I - A^{\mathrm{T}} \otimes I & 0 \\ 0 & 0 & h(\lambda) \end{bmatrix}$$

and

$$W = \begin{bmatrix} G^{\mathrm{T}} \otimes F^{\mathrm{T}} \\ F^{\mathrm{T}} \otimes I \\ F^{\mathrm{T}} \otimes F^{\mathrm{T}} \end{bmatrix}.$$

Also it is easy to show (all eigenvalues of A have strictly negative real parts) that

$$\det\left(\lambda I - I \otimes A^{\mathrm{T}}\right) \times \det\left(-\lambda I - A^{\mathrm{T}} \otimes I\right) \times \det\left(h(\lambda)\right) \neq 0, \quad \lambda = \imath\omega, \tag{17}$$

for all ω.

At this stage, suppose that $f(\lambda)$ is pre-multiplied by the left-hand side of (17). In which case, it follows immediately that $f(\lambda) \neq 0, \lambda = \imath\omega$, for all ω is equivalent to

$$\det\left(\begin{bmatrix} I - G^{\mathrm{T}} \otimes G^{\mathrm{T}} & I \otimes E^{\mathrm{T}} & E^{\mathrm{T}} \otimes G^{\mathrm{T}} & E^{\mathrm{T}} \otimes E^{\mathrm{T}} \\ G^{\mathrm{T}} \otimes F^{\mathrm{T}} & \lambda I - I \otimes A^{\mathrm{T}} & 0 & 0 \\ F^{\mathrm{T}} \otimes I & 0 & -\lambda I - A^{\mathrm{T}} \otimes I & 0 \\ F^{\mathrm{T}} \otimes F^{\mathrm{T}} & 0 & 0 & h(\lambda) \end{bmatrix}\right) \neq 0. \tag{18}$$

Finally, it is easy to see that (18) and (15) are equivalent. □

To examine Theorem 5 for a given example, it is necessary to test three matrices with constant entries for stability in the 1D linear systems sense (G and $H(\imath\omega_0)$ in the discrete 1D sense and A in the differential 1D sense) and the second order matrix polynomial of (15). The only remaining difficulty is the fact that the matrix X_1 here is singular and therefore the solutions of (15) cannot be obtained directly using existing software packages. In the remainder of this paper it will be shown that this stability condition can be further modified to result in a condition which is easily tested.

Theorem 6. *The linear repetitive process S generated by* (1) *and* (2) *is stable along the pass if, and only if,*

(a) *the conditions listed under* (a) *and* (b) *of Theorem* 5 *hold*;

(b) $\det(\lambda I + L_{12} - L_{13}L_7^{-1}L_8) \neq 0$ (19)

for $\lambda = \imath\omega$, *for all* ω, *where*

$$L_7 = I - G^{\mathrm{T}} \otimes G^{\mathrm{T}},$$

$$L_8 = \begin{bmatrix} I \otimes E^{\mathrm{T}} & E^{\mathrm{T}} \otimes G^{\mathrm{T}} & E^{\mathrm{T}} \otimes E^{\mathrm{T}} & 0 \end{bmatrix},$$

$$L_{12} = \begin{bmatrix} -I \otimes A^{\mathrm{T}} & 0 & 0 & 0 \\ 0 & A^{\mathrm{T}} \otimes I & 0 & 0 \\ 0 & 0 & 0 & -I \\ 0 & 0 & -A^{\mathrm{T}} \otimes A^{\mathrm{T}} & A^{\mathrm{T}} \otimes I - I \otimes A^{\mathrm{T}} \end{bmatrix}$$

and

$$L_{13} = \begin{bmatrix} G^{\mathrm{T}} \otimes F^{\mathrm{T}} \\ -F^{\mathrm{T}} \otimes I \\ 0 \\ -F^{\mathrm{T}} \otimes F^{\mathrm{T}} \end{bmatrix}.$$

Proof. The determinant of the matrix polynomial defining the condition of (15) can be rewritten as

$$\det(\lambda^2 X_1 + \lambda X_2 + X_3) = \det\begin{pmatrix} \lambda L_1 + L_2 & L_3 \\ L_4 & -\lambda^2 I + \lambda L_5 + L_6 \end{pmatrix},$$

where

$$L_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & -I \end{bmatrix},$$

$$L_2 = \begin{bmatrix} I - G^{\mathrm{T}} \otimes G^{\mathrm{T}} & I \otimes E^{\mathrm{T}} & E^{\mathrm{T}} \otimes G^{\mathrm{T}} \\ G^{\mathrm{T}} \otimes F^{\mathrm{T}} & -I \otimes A^{\mathrm{T}} & 0 \\ F^{\mathrm{T}} \otimes I & 0 & -A^{\mathrm{T}} \otimes I \end{bmatrix},$$

$$L_3 = \begin{bmatrix} E^{\mathrm{T}} \otimes E^{\mathrm{T}} \\ 0 \\ 0 \end{bmatrix},$$

$$L_4 = \begin{bmatrix} F^{\mathrm{T}} \otimes F^{\mathrm{T}} & 0 & 0 \end{bmatrix},$$

$$L_5 = \begin{bmatrix} I \otimes A^{\mathrm{T}} - A^{\mathrm{T}} \otimes I \end{bmatrix},$$

$$L_6 = \begin{bmatrix} A^{\mathrm{T}} \otimes A^{\mathrm{T}} \end{bmatrix}.$$

This implies that (15) is equivalent to

$$\det(-\lambda^2 I + \lambda L_5 + L_6) \times \det(\lambda L_1 + L_2 - L_3(-\lambda^2 I + \lambda L_5 + L_6)^{-1} L_4) \neq 0$$

for $\lambda = \imath\omega$, for all ω.

Routine algebraic manipulations applied to this last equation now shows that the condition of (15) is equivalent to

$$\det\left(\begin{bmatrix} \lambda L_1 + L_2 & L_3 & 0 \\ 0 & \lambda I & -I \\ L_4 & L_6 & -\lambda I + L_5 \end{bmatrix}\right) \neq 0$$

for $\lambda = \imath\omega$, for all ω. At this stage, the second order polynomial in (15) has been transformed to a first order one, which can now be written in the form

$$\det\left(\begin{bmatrix} L_7 & L_8 \\ L_9 & \lambda L_{10} + L_{11} \end{bmatrix}\right) \neq 0$$

for $\lambda = \imath\omega$, for all ω, where

$$L_9 = \begin{bmatrix} G^{\mathrm{T}} \otimes F^{\mathrm{T}} \\ F^{\mathrm{T}} \otimes I \\ 0 \\ F^{\mathrm{T}} \otimes F^{\mathrm{T}} \end{bmatrix},$$

$$L_{10} = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & -I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & -I \end{bmatrix}$$

and

$$L_{11} = \begin{bmatrix} -I \otimes A^{\mathrm{T}} & 0 & 0 & 0 \\ 0 & -A^{\mathrm{T}} \otimes I & 0 & 0 \\ 0 & 0 & 0 & -I \\ 0 & 0 & A^{\mathrm{T}} \otimes A^{\mathrm{T}} & I \otimes A^{\mathrm{T}} - A^{\mathrm{T}} \otimes I \end{bmatrix}.$$

The result now follows immediately on applying the block determinant formula. □

Using Theorem 6 the stability along the pass properties of an example can be checked by direct application of basic 1D linear systems tests. These are checking that: (i) $r(G) < 1$, (ii) checking that all eigenvalues of the matrix A have strictly negative real parts, (iii) checking that $r(H(0)) < 1$, and (iv) checking that the matrix $-L_{12} + L_{13}L_7^{-1}L_8$ has no purely imaginary eigenvalues. Also in terms of the computational load incurred it is clearly the test of (iv) which is the most demanding but, crucially, in comparison to other stability tests, such as those for the conditions of Theorem 1 in this paper, only computations on matrices with constant entries are required. In order to illustrate these tests, and provide a benchmark example for further work (see also the conclusions section), consider the following unit memory process (where the matrices B and D are not detailed since they play no role in stability analysis)

$$\begin{aligned} \dot{x}_{k+1}(t) &= \begin{bmatrix} -0.1831 & 0.0649 & -0.0243 \\ -0.1464 & -0.0648 & -0.2281 \\ 0.0536 & 0.0376 & -0.2364 \end{bmatrix} x_{k+1}(t) + Bu_{k+1}(t) \\ &\quad + \begin{bmatrix} -0.0937 & 0.0916 & 0.0562 \\ -0.2436 & -0.2036 & 0.0543 \\ -0.0580 & -0.2323 & -0.2421 \end{bmatrix} y_k(t) \\ y_{k+1}(t) &= \begin{bmatrix} -0.2418 & -0.2212 & 0.1088 \\ -0.1550 & -0.0662 & 0.0963 \\ 0.0435 & 0.0657 & -0.2080 \end{bmatrix} x_{k+1}(t) + Du_{k+1}(t) \\ &\quad + \begin{bmatrix} -0.0228 & -0.1732 & 0.1138 \\ -0.0291 & 0.0878 & -0.0108 \\ -0.0734 & 0.0996 & 0.0274 \end{bmatrix} y_k(t). \end{aligned} \tag{20}$$

Then (the numerical results are in [13]) performing the tests developed here it is concluded that this example is stable along the pass. Moreover, this requires much less computational effort than the testing of the conditions of Theorem 1 for this case (again see [13] for the details). The tests developed here also provide the computational information re-performance.

5. Conclusions

This paper has used the matrix Kronecker product to develop tests for stability along the pass of differential linear repetitive processes which can be implemented by computations on matrices with constant entries. The starting point is a 1D Lyapunov equation interpretation of stability along the pass for these processes which if it holds also provides information concerning one key aspect of the performance of these processes, i.e. the rate of convergence of the pass profile output sequence of a stable along the pass example to the resulting limit profile which is described by a 1D time invariant linear systems state space model. The extraction of performance information from the 1D (and 2D) Lyapunov equation characterization of stability along the pass for linear repetitive processes is a unique feature in relation to similar approaches for other classes of 2D linear (discrete or continuous–discrete) systems. Work is in progress as to how this information can be exploited in terms of the specification and design of control schemes for these processes and some preliminary ideas/results in this direction can again be found in [13]. Another major area which clearly requires investigation is the numerical aspects of, in particular, the requirement to test a (potentially) very large dimension matrix for the absence of purely imaginary eigenvalues. Note, however, that until a 'mature' control systems theory for these processes is available then the numerical examples encountered are highly unlikely to generate a large matrix in (b) of Theorem 6. In such cases, it is apparent that this result will be more efficient to use than Theorem 1. Note also that even in the SISO case the Nyquist like frequency domain tests which can be applied to (b) and (c) of Theorem 1 here do not provide useful indicators of expected process performance (e.g. the repetitive process equivalents of gain and phase margins).

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