On Homomorphisms of n-D Behaviors

Harish Pillai, Jeffrey Wood, and Eric Rogers

***Abstract*—Different modeling procedures applied to a physical system may result in behaviors which are distinct but nevertheless share many structural properties. Such behaviors are isomorphic in a sense which we formalize and characterize in this paper. More generally, we introduce a natural notion of homomorphisms between behaviors of multidimensional systems, generalizing recent work of Fuhrmann. A generalization of strict system equivalence (in the sense of Fuhrmann) is shown to describe the relationship between generalized state-space descriptions in the nD case.**

***Index Terms*—Abstract behavior, behaviors, isomorphism of behaviors, module theory, multidimensional systems, system equivalence.**

I. Introduction

THE process of describing a physical system by an element of a prescribed family of mathematical models is generally nondeterministic. For example, in selecting the physical variables to be modeled one chooses a basis, often arbitrarily; one may also include for convenience, additional variables which in fact can be determined from the others. In the behavioral setting, one therefore obtains many highly distinct behaviors as models of a single physical system. We expect these behaviors nevertheless to a very large extent to share the same structure.

In this paper, we formalize in a natural way, the concept of two distinct behaviors having 'essentially the same structure'. Further, we generalize recent work on 1-D behaviors by Fuhrmann in [5] introducing notions of homomorphisms and isomorphisms between behaviors. The concept of isomorphism essentially is that the 'modules of observables' of the two systems are isomorphic, a standard concept recently deeply explored in a system-theoretic context by Pommaret and Quadrat [11]; see also [10, II.2.3, V p. 656]. We also introduce here a distinction between a concrete behavior (a behavior in the usual sense, i.e., contained in some specified trajectory space) and an abstract behavior (effectively, an isomorphism class of behaviors).

Two special cases of behavior isomorphism are unimodular transformation, and elimination or addition of observable variables. In fact, we show by dualizing a result in algebra that two isomorphic behaviors can always be related by a sequence of such operations. Nevertheless, the form of an arbitrary isomorphism of behaviors is very general.

The paper is arranged as follows. In Section II we recall some basic notions from behavioral theory, and also motivate the distinction between abstract and concrete behaviors. In Section III we define homomorphisms and isomorphisms of nD behaviors, and characterize them by various means. In the case where the behaviors are specified by given kernel representations which have full row rank, we derive a direct generalization of the zero left prime and zero right prime conditions which are familiar from the work of Fuhrmann. Section IV looks at the special cases of unimodular equivalence and elimination of observable variables, and shows that any isomorphism can be expressed by combining these two operations (essentially a generalization of Rosenbrock's strict system equivalence). Then, in Section V we study isomorphisms of a special type acting on latent variable descriptions of a behavior. This leads to a generalization of strict system equivalence in the sense of Fuhrmann, which can be applied to any Rosenbrock system matrix and admits a behavioral interpretation. The final Section VI contains various additional results.

II. Abstract and Concrete Behaviours

Recall [14] that a behavior $\mathcal{B}$ of a system is the set of its trajectories, which we view a subspace of $\mathcal{W}^q$, where $\mathcal{W}$ is the signal space of the system (e.g., $\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$), and q is the number of (dependent) system variables. We consider in this paper only behaviors defined by linear partial differential equations with constant coefficients, or the discrete equivalent (difference equations). For a signal space we consider one of the following: in the continuous case, $\mathcal{W} = \mathcal{C}^\infty(\mathbb{R}^n, k)$ or $\mathcal{W} = \mathcal{D}'(\mathbb{R}^n, k)$, or in the discrete case $\mathcal{W} = k^{\mathbb{Z}^n}$ or $\mathcal{W} = k^{\mathbb{N}^n}$, where k is either $\mathbb{R}$ or $\mathbb{C}$. Oberst has shown [8, 2.54] that these signal spaces have the important algebraic property of being injective cogenerators over the ring of partial differential operators $k[\partial/\partial t_1, \ldots, \partial/\partial t_n]$ (or partial difference operators, analogously). For convenience, we identify this ring with the polynomial ring $\mathcal{D} = k[z_1, \ldots, z_n]$ in n indeterminates (though in the discrete case $\mathcal{W} = k^{\mathbb{Z}^n}$ it is necessary instead to use the Laurent polynomial ring $k[z_1, \ldots, z_n, z_1^{-1}, \ldots, z_n^{-1}]$, since then, the shift operators are invertible). Our behaviors are therefore of the type

$$\mathcal{B} = \ker_{\mathcal{W}} R := \{w \in \mathcal{W}^q \mid Rw = 0\} \tag{1}$$

where $R \in \mathcal{D}^{g,q}$ is a polynomial "**kernel representation**" matrix and Rw is interpreted according to the usual partial differential (partial difference) action of R on $w = (w_i)$. For a differential operator (polynomial matrix) $A \in \mathcal{D}^{q',q}$ and a behavior

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H. Pillai is with the Department of Electrical Engineering, IIT-Bombay, Powai, Mumbai 400 076, India (e-mail: hp@ee.iitb.ac.in).

J. Wood is with the Department of Electronics and Computer Science, University of Southampton, Southampton SO17 1BJ, U.K. and also with the Royal Society University, U.K. (e-mail: jjw@ecs.soton.ac.uk).

E. Rogers is with the Department of Electronics and Computer Science, University of Southampton, Southampton SO17 1BJ, U.K. (e-mail: etar@ecs.soton.ac.uk).

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$\mathcal{B} \subseteq \mathcal{W}^q$, we also use the notation $A\mathcal{B}$ for the set $\{Aw \mid w \in \mathcal{B}\}$. For brevity we will henceforth refer only to the continuous case.

It is easily observed that kernel representations are highly nonunique. Indeed, any two matrices R_1, R_2 which have the same row-span (over $\mathcal{D}$) generate the same set of partial differential equations, so define the same behavior. For the signal spaces we have identified, the converse is also true: two matrices which define the same behavior must have the same row span [8, 2.23,2.48–9,2.61], which is equal to the **module of system equations**

$$\mathcal{B}^\perp = \{v \in \mathcal{D}^{1,q} \mid vw = 0 \text{ for all } w \in \mathcal{W}^q\}. \tag{2}$$

It is also valuable to consider the factor module $\mathcal{D}^{1,q}/\mathcal{D}^{1,g}R$, which we denote by $\mathcal{M}$ and refer to as the **module of observables** of $\mathcal{B}$. This module is generated by $e_i + \mathcal{B}^\perp, i = 1, \ldots, q$, where $e_1, \ldots, e_q$ are the natural basis vectors in $\mathcal{D}^{1,q}$. The module element $e_i + \mathcal{B}^\perp$ can be identified with the formal quantity w_i (note $(e_i + v)w = w_i$ for any $v \in \mathcal{B}^\perp$). $\mathcal{M}$ can now be considered as the module of distinct formal quantities ("observables") associated to the system; each element is a polynomial combination (conceptually, a differential linear combination) of the given generating formal quantities. For example, $z_1(e_2 + \mathcal{B}^\perp) - 2(e_1 + \mathcal{B}^\perp)$ corresponds to the formal quantity $\partial w_2/\partial t_1 - 2w_1$, where t_1 is the first independent variable. This is the approach taken in [10], [11]. In fact, any finitely generated $\mathcal{D}$-module can be written in the form $\mathcal{D}^{1,q}/\mathcal{B}^\perp$ for some behavior $\mathcal{B}$. In what follows, we make no distinction between isomorphic modules.

The module $\mathcal{M}$ is directly related to the behavior $\mathcal{B}$, according to

$$\mathcal{B} \cong \mathrm{Hom}_{\mathcal{D}}(\mathcal{M}, \mathcal{W}). \tag{3}$$

Here, $w = (w_1, \ldots, w_q) \in \mathcal{B}$ is identified with the homomorphism determined by $e_i + \mathcal{B}^\perp \mapsto w_i, i = 1, \ldots, q$ (an assignment of signals to the formal quantities in a manner consistent with the system laws). Note however that this correspondence is not canonical, as the choice of generators $\{e_i + \mathcal{B}^\perp\}$ of $\mathcal{M}$ is arbitrary.

The central result of Oberst in [8, 2.54], is that the signal spaces identified are injective cogenerators. One restatement of this result is that $\mathrm{Hom}_{\mathcal{D}}(-, \mathcal{W})$ is a faithfully exact contravariant functor, and therefore we have a *categorical duality* between finitely generated $\mathcal{D}$-modules and behaviors of the type discussed above. We are not concerned in the present paper with the precise meaning of these terms. However, this categorical duality is very powerful, since it allows us to translate any structural statement about modules to a corresponding statement about behaviors, and frequently vice versa. From this duality, it is possible to prove that a behavior is controllable if and only if the corresponding module of observables is torsionfree, that it is autonomous if and only if the corresponding module is torsion, etc. See [15] for a survey of results of this type. Furthermore, the module theory comes equipped with a comprehensive set of algorithms for system-theoretic constructions, based on Gröbner bases or similar techniques.

However, as discussed in the introduction, a behavior is a highly noncanonical model of a given physical system. For example, one may add an additional physical variable to the model which in fact is a differential linear combination of the existing variables, thus changing not only the behavior but the trajectory space $\mathcal{W}^q$ in which it lies. Alternatively, one may change basis in the space of physical (dependent) variables Such actions often appear to completely change the behavior; however clearly much of the structure must remain the same. In particular, the set of all formal quantities associated with the system, i.e., the module $\mathcal{M}$, is unchanged. (However, the module of system equations $\mathcal{B}^\perp$ generally is changed, though by the classical Schanuel's Lemma it is in fact determined up to 'projective equivalence'). Furthermore, the object $\mathrm{Hom}_{\mathcal{D}}(\mathcal{M}, \mathcal{W})$ is both canonical and meaningful: it is the set of all assignments of signals to the physical variables (formal quantities) of the system in a manner which respects the algebraic relationships between those quantities. These considerations motivate the following distinction:

Definition 1: A **concrete behavior** is a behavior of the type $\mathcal{B} = \ker_{\mathcal{W}} R$, that is a subset of $\mathcal{W}^q$ for some q, defined by a linear partial differential operator $R \in \mathcal{D}^{g,q}$.

An **abstract behavior** is a set of the type $D(\mathcal{M}) := \mathrm{Hom}_{\mathcal{D}}(\mathcal{M}, \mathcal{W})$ for some finitely generated $\mathcal{D}$-module $\mathcal{M}$ (recall that we identify together isomorphic modules, and therefore the corresponding abstract behaviors).

In the case where R is a presentation of $\mathcal{M}$, or equivalently $\mathcal{M}$ is the module of observables of $\mathcal{B}$, we will say that the concrete behavior $\mathcal{B} = \ker_{\mathcal{W}} R$ is a **manifestation** of the abstract behavior $D(\mathcal{M})$.

We now summarize the relationships between these concepts (note that this discussion applies only to signal spaces $\mathcal{W}$ which are injective cogenerators; the picture is complicated by the consideration of other $\mathcal{W}$). A (concrete) behavior $\mathcal{B}$ implicitly specifies an inclusion map $\iota : \mathcal{B} \hookrightarrow \mathcal{W}^q$. The dual object is the natural projection map $\rho : \mathcal{D}^{1,q} \to \mathcal{M}$; that is, concrete behaviors are in one-to-one correspondence with such pairs $(\mathcal{M}, \rho)$. Abstract behaviors are in one-to-one correspondence with modules $\mathcal{M}$ themselves (identified of course only up to isomorphism). Thus an abstract behavior is essentially an equivalence class of concrete behaviors, consisting of all concrete behaviors obtained by choosing a generating set of the module of observables and embedding the abstract behavior accordingly in a trajectory space. A natural problem, to which we devote much of this paper, is to characterize the concrete behaviors which correspond to the same abstract behavior (we will see that this corresponds to a natural notion of isomorphism).

For most purposes, it is not necessary to make this distinction between abstract and concrete behaviors and to introduce the mapping ρ etc. However our formalization of these ideas, which (like the map ι) are normally implicit, will be useful in what follows. We will retain the term "behavior" in the usual sense, i.e., to mean "concrete behavior".

A. Background

We now recall some important concepts and results from the literature which will be useful in what follows. We give references only for less well-known results.

Recall that a polynomial matrix R is said to be **zero right prime** (**minor right prime**) if it has full column rank and the ideals of highest order minors have no common root in $\mathbb{C}^n$ (have no common factor in $\mathcal{D}$). A zero right prime matrix is precisely one which admits a polynomial left inverse. Zero/ minor left primeness is defined by transposition. A **unimodular matrix** is one which is square with a constant determinant.

A polynomial matrix $R \in \mathcal{D}^{g,q}$ is said to be a **minimal left annihilator (MLA)** of another polynomial matrix $M \in \mathcal{D}^{q,h}$ if the rows of R generate all the syzygies (polynomial relations) on the rows of the matrix M. Equivalently, we have the condition

$$\ker_{\mathcal{W}} R = \mathrm{im}_{\mathcal{W}} M := \{w \in \mathcal{W}^q \mid \exists v \in \mathcal{W}^h \text{ s.t. } w = Mv\}.$$

We say that R is a **minimal right annihilator (MRA)** of M if R^T is an MLA of M^T.

A behavior $\mathcal{B} = \ker_{\mathcal{W}} R$ is controllable, as defined in [9] for the continuous case, if and only if R is an MLA of some matrix M (and in this case, clearly $\mathcal{B} = \mathrm{im}_{\mathcal{W}} M$).

Lemma 2: (e.g., [17]): A polynomial matrix R is an MLA of M if and only if:

1) R is an MLA (of some matrix);
2) $RM = 0$;
3) $\mathrm{Rank}\ R + \mathrm{Rank}\ M = q$.

Also, a full row-rank matrix is an MLA (of some matrix) if and only if it is minor left prime.

The **number of free variables** of a behavior $\mathcal{B}$ is the maximum number l such that l dependent variables of $\mathcal{B}$ can be independently freely chosen within $\mathcal{B}$. This number is denoted by $m(\mathcal{B})$, and, for a given kernel representation $\mathcal{B} = \ker_{\mathcal{W}} R$ with q columns, equals $q - \mathrm{rank} R$. The number $m(\cdot)$ is also **additive**, which signifies that, given an exact sequence of behaviors

$$0 \to \mathcal{B}_1 \to \mathcal{B}_2 \to \mathcal{B}_3 \to 0$$

(for example where $\mathcal{B}_1$ is a subbehavior of $\mathcal{B}_2$ with factor $\mathcal{B}_3$), we have $m(\mathcal{B}_2) = m(\mathcal{B}_1) + m(\mathcal{B}_3)$. In particular, for any two behaviors $\mathcal{B}_1, \mathcal{B}_2 \in \mathcal{W}^q$, we have [13, 2.7)]

$$m(\mathcal{B}_1 + \mathcal{B}_2) + m(\mathcal{B}_1 \cap \mathcal{B}_2) = m(\mathcal{B}_1) + m(\mathcal{B}_2). \quad (4)$$

A **latent variable description** of a behavior $\mathcal{B}$ is a representation of the form

$$\mathcal{B} = \mathcal{B}_w := \{w \mid \exists\, \ell \text{ with } Rw = M\,\ell\} \quad (5)$$

where ℓ are the **latent variables**. The latent variable description is called **observable** if M is zero right prime, since this implies that ℓ is uniquely determined (indeed, by a differential operator) from w. The **full behavior** is the set of solutions (w, ℓ) of $Rw = M\ell$ with respect to the given signal space. We also denote by $\mathcal{B}_{0,\ell}$ the subbehavior of all solutions $(0, \ell)$, which is the kernel of the projection map $\mathcal{B}_{w,\ell} \mapsto \mathcal{B}_w$. To eliminate the latent variables, we can apply the algorithm given e.g., in [8, Cor. 2.38)], : compute an MLA C of M; then $\mathcal{B}_w = \ker_{\mathcal{W}} CR$.

A special case of elimination is the computation of $A\mathcal{B}$ for some differential operator A and behavior $\mathcal{B} = \ker_{\mathcal{W}} R$. We have [13 , Lem. 2.13)].

Lemma 3: A kernel representation of $A\mathcal{B}, \mathcal{B} = \ker_{\mathcal{W}} R$, is given by C, where (CD) is any MLA of $\begin{pmatrix} A \\ R \end{pmatrix}$.

III. Homomorphisms and Isomorphisms of Behaviours

Having distinguished between concrete and abstract behaviors, an obvious question is "What is the relationship between different manifestations of the same abstract behavior?". The unsurprising answer is that they are isomorphic, in a natural sense now to be introduced.

Definition 4: A **homomorphism** from a (concrete) behavior $\mathcal{B} \subseteq \mathcal{W}^q$ to a concrete behavior $\mathcal{B}' \subseteq \mathcal{W}^{q'}$ is a mapping $\phi : \mathcal{B} \mapsto \mathcal{B}'$ which is represented by a (linear, constant coefficient) partial differential operator $A : \mathcal{W}^q \mapsto \mathcal{W}^{q'}$. The homomorphism is called a **monomorphism** (**epimorphism**, **isomorphism**) when ϕ is injective (surjective, bijective respectively).

Thus a homomorphism of behaviors $\phi : \mathcal{B} \mapsto \mathcal{B}'$ is simply a differential operator (rather, an equivalence class of them) which maps $\mathcal{B}$ to $\mathcal{B}'$. Two behaviors $\mathcal{B}, \mathcal{B}'$ are isomorphic precisely when there are homomorphisms $\phi : \mathcal{B} \mapsto \mathcal{B}'$ and $\phi' : \mathcal{B}' \mapsto \mathcal{B}$ which are mutual inverses. Note that this does not require that the representing matrices A and A' are mutual inverses.

It is immediate from the Oberst duality that homomorphisms $\mathcal{B} \mapsto \mathcal{B}'$ are in one-to-one correspondence with homomorphisms $\mathcal{M}' \mapsto \mathcal{M}$ (note the reversal of direction), where $\mathcal{M}$ and $\mathcal{M}'$ are the modules of observables of $\mathcal{B}$ and $\mathcal{B}'$ respectively. For if $\psi : \mathcal{M}' \mapsto \mathcal{M}$ is a module homomorphism then it can be extended (nonuniquely) to a map $\hat{\psi} : \mathcal{D}^{1,q'} \mapsto \mathcal{D}^{1,q}$, where q' and q are the numbers of dependent variables in $\mathcal{B}'$ and $\mathcal{B}$, respectively. The map $\hat{\psi}$ can be represented by a polynomial matrix A, which defines a differential operator from $\mathcal{W}^q$ to $\mathcal{W}^{q'}$, which restricts to a behavior homomorphism $\phi : \mathcal{B} \mapsto \mathcal{B}'$. Alternatively, ϕ can be obtained directly as $\mathrm{Hom}_{\mathcal{D}}(\psi, \mathcal{W})$. Thus, the homomorphisms of behaviors (in the sense of Definition 4) are precisely the "duals" of the homomorphisms of the corresponding finitely generated modules. Furthermore, by exactness of the functor $\mathrm{Hom}_{\mathcal{D}}(-, \mathcal{W}), \phi$ is injective if and only if ψ is surjective and vice versa. The following lemma is immediate.

Lemma 5: Two concrete behaviors are manifestations of the same abstract behavior if and only if they are isomorphic.

Proof: Both statements are equivalent to the condition that the modules of observables of the two behaviors are isomorphic. ■

Since an abstract behavior is a natural abstract model of a 'real system' which requires no noncanonical selection of physical variables $w_1, \ldots, w_q$ from the infinite number of those available, the concept of isomorphism as just introduced is also very natural, and exactly captures the concept of two behaviors 'describing the same system'. Isomorphic behaviors therefore have in all important ways the same structure (see Section III-B).

We now finally have the language to speak in precise terms about factor behaviors. Given a behavior $\mathcal{B}$ and sub-behavior $\mathcal{B}'$ (i.e., a subset which is also a behavior), the factor space $\mathcal{B}/\mathcal{B}'$ admits the structure of an abstract behavior; it is equal to $D(\mathcal{M}')$, where $\mathcal{M}'$ is the submodule of the module of observables of $\mathcal{B}$ consisting of all observables which are identically zero on $\mathcal{B}'$ [15, Cor. 2]. Thus $\mathcal{B}/\mathcal{B}'$ is called the **factor behavior**. The manifestations of $\mathcal{B}/\mathcal{B}'$ include the behaviors of the form $R'\mathcal{B}$, where R' is any kernel representation of $\mathcal{B}'$ (note that the kernel of the surjection $\mathcal{B} \mapsto R'\mathcal{B}$ is simply $\mathcal{B}'$, so that $R'\mathcal{B} \cong \mathcal{B}/\mathcal{B}'$ as k-spaces or $\mathcal{D}$-modules). These concrete behaviors live in different trajectory spaces, but are manifestations of the same abstract behavior. They are therefore isomorphic, and so share many properties such as those listed in Section III-B. These properties are effectively properties of the abstract behavior $\mathcal{B}/\mathcal{B}'$, and for many purposes it is sufficient to discuss this factor without specifying a manifestation of it. Factor behaviors can be used to describe succinctly many properties, such as set-controllability [13], regular interconnection [13] and input decoupling [16], amongst others.

A. Representations of Homomorphisms

We now characterize homomorphisms of behaviors in terms of polynomial matrix properties.

Theorem 6: Let $\mathcal{B} = \ker_{\mathcal{W}} R, \mathcal{B}' = \ker_{\mathcal{W}} R'$ be two (concrete) behaviors with given kernel representations $R \in \mathcal{D}^{g,q}, R' \in \mathcal{D}^{g',q'}$, and let $A \in \mathcal{D}^{q',q}$ be a given differential operator. Then, we have

1) $A : w \mapsto Aw$ defines a homomorphism from $\mathcal{B}$ to $\mathcal{B}'$ if and only if there exists a matrix $Y \in \mathcal{D}^{g',g}$ with

$$R'A = YR. \tag{6}$$

2) A defines a monomorphism from $\mathcal{B}$ to $\mathcal{B}'$ if and only if condition (6) holds for some Y, and $\begin{pmatrix} A \\ R \end{pmatrix}$ is zero right prime.

3) The following are equivalent:

   a) A defines an epimorphism from $\mathcal{B}$ to $\mathcal{B}'$.

   b) Condition (6) holds for some Y, and for any Y satisfying (6), we have

$$\ker_{\mathcal{W}} \begin{pmatrix} R' & -Y \\ 0 & R_1 \end{pmatrix} = \operatorname{im}_{\mathcal{W}} \begin{pmatrix} A \\ R \end{pmatrix} \tag{7}$$

   where R_1 is a given MLA of R.

   c) There exists a $Y \in \mathcal{D}^{g',g}$ such that (7) is satisfied.

4) A defines the zero map on $\mathcal{B}$ if and only if there exists $L \in \mathcal{D}^{q',g}$ with $A = LR$.

Proof:

1) A defines a homomorphism from $\mathcal{B}$ to $\mathcal{B}'$ if and only if $A\mathcal{B}$ is contained in $\mathcal{B}'$. Equivalently, $\ker_{\mathcal{W}} R \subseteq \ker_{\mathcal{W}} R'A$, which by [8, Th. 2.61] is equivalent to the given condition.

2) Monomorphisms are precisely the homomorphisms A for which $\mathcal{B} \cap \ker_{\mathcal{W}} A = 0$. By for example [15, Th. 9)], this is equivalent to the zero right primeness of $\begin{pmatrix} A \\ R \end{pmatrix}$.

3) We begin by showing

$$\ker_{\mathcal{W}} \begin{pmatrix} R' & -Y \\ 0 & R_1 \end{pmatrix} = \operatorname{im}_{\mathcal{W}} \begin{pmatrix} A \\ R \end{pmatrix} + \ker_{\mathcal{W}} \begin{pmatrix} R' & 0 \\ 0 & I \end{pmatrix} \tag{8}$$

for any R, R', A, Y, R_1, where R_1 is an MLA of R. It is clear that each behavior on the right-hand side, and therefore their sum, is contained in the left-hand side. Conversely, if $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \ker_{\mathcal{W}} \begin{pmatrix} R' & -Y \\ 0 & R_1 \end{pmatrix}$, then $y_2 \in \ker_{\mathcal{W}} R_1 = \operatorname{im}_{\mathcal{W}} R$, say $y_2 = Rw$, and now $0 = R'y_1 - YRw = R'y_1 - R'Aw$, proving $y_1 - Aw \in \ker_{\mathcal{W}} R'$. Thus

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} A \\ R \end{pmatrix} w + \begin{pmatrix} y_1 - Aw \\ 0 \end{pmatrix}$$

establishes the decomposition (8).

Now suppose that A is a homomorphism which moreover is epic, and let Y be a matrix satisfying (6). Then, it is easy to see that $\ker_{\mathcal{W}} \begin{pmatrix} R' & 0 \\ 0 & I \end{pmatrix} \subseteq \operatorname{im}_{\mathcal{W}} \begin{pmatrix} A \\ R \end{pmatrix}$, and (8) therefore yields condition 3b. It is trivial that 3b $\Rightarrow$ 3c.

Finally, suppose that 3c holds, and let $w' \in \mathcal{B}'$ be arbitrary. Then, $\begin{pmatrix} w' \\ 0 \end{pmatrix} \in \ker_{\mathcal{W}} \begin{pmatrix} R' & -Y \\ 0 & R_1 \end{pmatrix}$, so by supposition there exists w with $w' = Aw$ and $0 = Rw$, i.e., $w' \in A\mathcal{B}$ as required.

4) Both conditions are equivalent to: $\mathcal{B} \subseteq \ker_{\mathcal{W}} A$. ∎

Theorem 6 is a generalization of Fuhrmann's result [5, Theorem 3.6], which is given for 1D behaviors with full row rank representations. The only difference from Fuhrmann's results [5, 3.6.1–3.6.3] is in the epimorphism condition; we explore the full row rank case further in Corollary 9.

Any homomorphism $\phi : \mathcal{B} \mapsto \mathcal{B}'$ can be extended to a map of complexes (exact sequences); this can be seen as follows. Let R and R' be kernel representations of $\mathcal{B}$ and $\mathcal{B}'$ respectively, and let R_1 be an MLA of R, R'_1 and MLA of R', R_2 an MLA of R_1, R'_2 an MLA of R'_1, etc. Let the domains of R, R_1, R_2, etc. be $\mathcal{W}^q, \mathcal{W}^{q_1}, \mathcal{W}^{q_2}, \ldots$ and the domains of R', R'_1, R'_2, etc. be $\mathcal{W}^{q'}, \mathcal{W}^{q'_1}, \mathcal{W}^{q'_2}, \ldots$. Now if A is a polynomial matrix representing the homomorphism ϕ, and Y a corresponding matrix with $R'A = YR$, we find that

$$\begin{aligned} Y(\ker_{\mathcal{W}} R_1) &= Y(\operatorname{im}_{\mathcal{W}} R) = \operatorname{im}_{\mathcal{W}} YR \\ &= \operatorname{im}_{\mathcal{W}} R'A \subseteq \operatorname{im}_{\mathcal{W}} R' = \ker_{\mathcal{W}} R'_1. \end{aligned}$$

Hence, Y defines a homomorphism from the behavior $\ker_{\mathcal{W}} R_1 = \operatorname{im}_{\mathcal{W}} R$ to the behavior $\ker_{\mathcal{W}} R'_1 = \operatorname{im}_{\mathcal{W}} R'$. Thus there exists another matrix Z with $R'_1 Y = ZR_1$, and so on up the sequence of maps. It is convenient to write $Y = A_1, Z = A_2$, etc.; then these maps (A_i) define a map of complexes (rather exact sequences), as described by the exact commutative diagram

$$\begin{array}{ccccccccccc} 0 \to \mathcal{B} \to \mathcal{W}^q & \overset{R}{\to} & \mathcal{W}^{q_1} & \overset{R_1}{\to} & \mathcal{W}^{q_2} & \overset{R_2}{\to} & \cdots & \overset{R_{r-1}}{\to} & \mathcal{W}^{q_r} \to 0 \\ \phi \downarrow \quad A \downarrow & & A_1 \downarrow & & A_2 \downarrow & & & & A_r \downarrow \\ 0 \to \mathcal{B}' \to \mathcal{W}^{q'} & \overset{R'}{\to} & \mathcal{W}^{q'_1} & \overset{R'_1}{\to} & \mathcal{W}^{q'_2} & \overset{R'_2}{\to} & \cdots & \overset{R'_{r-1}}{\to} & \mathcal{W}^{q'_r} \to 0. \end{array} \tag{9}$$

There is, of course, a standard dual form in the language of finitely generated $\mathcal{D}$-modules.

Note that two operators A and $\hat{A}$ define the same map on a behavior $\mathcal{B}$, if and only if their difference is the zero map on $\mathcal{B}$, which from claim 4 of Theorem 6 is equivalent to the formula $\hat{A} = A + L_0 R$ for some L_0. This can easily be extended to give conditions on a map of complexes $A, A_1, A_2, \ldots$ to give the same map of behaviors as another collection $\hat{A}, \hat{A}_1, \hat{A}_2, \ldots$. We obtain the necessary and sufficient condition that there must exist matrices (L_i) with

$$\hat{A}_i = A_i + R'_{i-1} L_{i-1} + L_i R_i \tag{10}$$

for all $i \geq 2$. The relationship (10) between the map of complexes (A_i) and the map $(\hat{A}_i)$, is well-known in algebra and called a homotopy (see e.g., [4, A3.6]. The existence of a homotopy is proved in a noncommutative module-theoretic setting in [10], [11].

Note 7: By Theorem 6, ϕ is an epimorphism if and only if $\begin{pmatrix} R' & -A_1 \\ 0 & R_1 \end{pmatrix}$ is an MLA of $\begin{pmatrix} A \\ R \end{pmatrix}$. Several plausible-looking variations on this statement are false. Given R, R' and an epimorphism A from $\ker_{\mathcal{W}} R$ to $\ker_{\mathcal{W}} R'$, it is not necessarily the case that there exists an A_1 with $(R' - A_1)$ an MLA of $\begin{pmatrix} A \\ R \end{pmatrix}$, or an A_1 with $R'A = A_1 R$ and $(R' - A_1)$ zero left prime. For example, take

$$R = \begin{pmatrix} -z_1 \\ z_2^2 - 1 \end{pmatrix} = \begin{pmatrix} z_1 & z_2 \\ 1 & 0 \end{pmatrix} R'$$
$$R' = \begin{pmatrix} z_2^2 - 1 \\ -z_1 z_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ z_2 & 0 \end{pmatrix} R$$

which have the same kernel, and $A = (1)$ (trivially an isomorphism). Then, the matrices A_1 with $R'A = A_1 R$ are parameterized by $A_1 = \begin{pmatrix} 0 & 1 \\ z_2 & 0 \end{pmatrix} + ER_1$ where $R_1 = (z_2^2 - 1 z_1)$ and $E = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ is any polynomial matrix. The determinant of this parametric A_1 is $-\alpha z_1 z_2 - \beta(z_2^2 - 1) - z_2$. By consideration of the powers of z_2 appearing in this expression, it is easy to show that A_1 can never be unimodular. It follows that $(R' - A_1)$ can never be an MLA of $\begin{pmatrix} A \\ R \end{pmatrix}$, for if it were then standard results on the determinantal ideals of matrices satisfying such a relationship would yield that A be nonunimodular also. Also, $(R' - A_1)$ can never be zero left prime and satisfy $R'A = A_1 R$, for if it were then by Lemma 2 it would also have to be an MLA of $\begin{pmatrix} A \\ R \end{pmatrix}$.

By Lemma 2, a necessary condition for the epimorphism condition (7) in Theorem 6 to hold is that the ranks of the two matrices should add up to the number of rows of $\begin{pmatrix} A \\ R \end{pmatrix}$; we now investigate this condition.

Lemma 8: Consider the maps and behaviors as in diagram (9). The following are equivalent.

1) $\operatorname{rank}\begin{pmatrix} R' & -A_1 \\ 0 & R_1 \end{pmatrix} + \operatorname{rank}\begin{pmatrix} A \\ R \end{pmatrix} =$ number of rows of $\begin{pmatrix} A \\ R \end{pmatrix}$.
2) $m(\mathcal{B}') = m(A\mathcal{B})$ (or $(\mathcal{B}')^c \subseteq A\mathcal{B}$, or $\mathcal{B}'/A\mathcal{B}$ is autonomous).
3) $m(\mathcal{B}') = m(\mathcal{B}) - m(\ker_{\mathcal{W}}\begin{pmatrix} A \\ R \end{pmatrix})$.

Proof: Using the identity (4) and (8), we find

$$\begin{aligned} m&\left(\ker_{\mathcal{W}}\begin{pmatrix} R' & -A_1 \\ 0 & R_1 \end{pmatrix}\right) \\ &= m\left(\operatorname{im}_{\mathcal{W}}\begin{pmatrix} A \\ R \end{pmatrix}\right) + m\left(\ker_{\mathcal{W}}\begin{pmatrix} R' & 0 \\ 0 & I \end{pmatrix}\right) \\ &\quad - m\left(\operatorname{im}_{\mathcal{W}}\begin{pmatrix} A \\ R \end{pmatrix} \cap \ker_{\mathcal{W}}\begin{pmatrix} R' & 0 \\ 0 & I \end{pmatrix}\right) \\ &= \operatorname{rank}\begin{pmatrix} A \\ R \end{pmatrix} + m(\mathcal{B}') - m(A\mathcal{B}). \end{aligned}$$

Equivalence of 1. and 2. is now clear from the above equation. Equivalence of 2. and 3. follows from the fact that $m(\cdot)$ is additive, so that $m(A\mathcal{B}) = m(\mathcal{B}) - m(\ker\phi)$. ■

We make the following observations. If the map $A : \mathcal{B} \to \mathcal{B}'$ is injective, then condition 1. in the above lemma holds if and only if $m(\mathcal{B}) = m(\mathcal{B}')$. Moreover, in the case where $\mathcal{B}'$ is controllable, $\begin{pmatrix} R' & -A_1 \\ 0 & R_1 \end{pmatrix}$ must itself be an MLA, and we easily obtain that a given $A : \mathcal{B} \to \mathcal{B}'$ is an isomorphism if and only if it is injective and $m(\mathcal{B}) = m(\mathcal{B}')$.

Corollary 9: Suppose that $\mathcal{B} = \ker_{\mathcal{W}} R$ and $\mathcal{B}' = \ker_{\mathcal{W}} R'$, and let ϕ be a homomorphism from $\mathcal{B}$ to $\mathcal{B}'$ represented by a differential operator A, so that

$$R'A = YR$$

for some Y. Suppose further that R and R' have full row rank. Then, ϕ is an epimorphism if and only if $m(\mathcal{B}') = m(\mathcal{B}) - m(\ker\phi)$ and also $(R' - Y)$ is minor left prime.

Proof: Since R has full row rank, any MLA R_1 equals 0, and the condition in Theorem 6 for ϕ to be an epimorphism becomes that $(R' - Y)$ should be an MLA of $\begin{pmatrix} A \\ R \end{pmatrix}$. By Lemma 2, and the fact that $(R' - Y)$ has full row rank, this is equivalent to the conditions that $(R' - Y)$ is minor left prime, $R'A = YR$, and finally $\operatorname{rank}(R' - Y) = q' + g - \operatorname{rank}\begin{pmatrix} A \\ R \end{pmatrix}$. Applying Lemma 8 gives us the required condition on $m(\mathcal{B})$ and $m(\mathcal{B}')$. ■

Corollary 10: With notation and suppositions as in Corollary 9, suppose further that $q - g = q' - g'$, where $R \in \mathcal{D}^{g,q}$ and $R' \in \mathcal{D}^{g',q'}$. Then, ϕ is an isomorphism from $\mathcal{B}$ to $\mathcal{B}'$ if and only if $\begin{pmatrix} A \\ R \end{pmatrix}$ is zero right prime, and $(R' - Y)$ is zero left prime.

Proof: Suppose ϕ is an isomorphism; then $\ker\phi = 0$ and by Theorem 6, $\begin{pmatrix} A \\ R \end{pmatrix}$ is zero right prime, whereas by Theorem 6, $(R' - Y)$ must be a (full row rank) MLA of $\begin{pmatrix} A \\ R \end{pmatrix}$, which means by e.g., [18, Lem. 3] that it is in fact zero left prime. Conversely, if these conditions hold then by Theorem 6, ϕ is a monomorphism. Since $q - g = q' - g'$, and $g = \operatorname{rank} R, g' = \operatorname{rank} R'$, we have that $m(\mathcal{B}) = m(\mathcal{B}')$, so by Corollary 9, ϕ is also epic. ■

Note that Corollary 10 specializes in the 1D case to the condition familiar in Fuhrmann strict system equivalence (in which context the condition $q - g = q' - g'$, which effectively means that $\mathcal{B}$ and $\mathcal{B}'$ have the same number of free variables, is standard).

B. System-Theoretic Properties Shared by Isomorphic Behaviors

By Lemma 5, two (concrete) behaviors are isomorphic if and only if they are manifestations of the same abstract behavior, i.e., if and only if their modules of observables are isomorphic. Thus, any property of a concrete behavior $\mathcal{B}$ which can be shown to be equivalent to some intrinsic property of the module $\mathcal{M}$, is also shared by any isomorphic copy of $\mathcal{B}$. On the other hand, properties of $\mathcal{B}$ which depend upon the inclusion of $\mathcal{B}$ in the trajectory space $\mathcal{W}^q$, which may be describable in terms of $\mathcal{M}$ but only relative to the map $\mathcal{D}^{1,q} \mapsto \mathcal{M}$, are not generally preserved by isomorphism (e.g., the number of outputs).

If ϕ is an isomorphism from $\mathcal{B}$ to $\mathcal{B}'$, then the following properties are shared by $\mathcal{B}$ and $\mathcal{B}'$.

1) Controllability of $\mathcal{B}$. Also, ϕ restricts to an isomorphism from $\mathcal{B}^c$ to $(\mathcal{B}')^c$, as is easily seen from the module-theoretic analogue. Similarly, the so-called obstructions to controllability, $\mathcal{B}/\mathcal{B}^c$, and $\mathcal{B}'/(\mathcal{B}')^c$, are isomorphic. Furthermore, the equivalence class $w + \mathcal{B}^c$ of $w \in \mathcal{B}$ is precisely the set of trajectories which can be concatenated

with w [19], and this is mapped to the equivalence class of $\phi(w)$ in $\mathcal{B}'/\phi(\mathcal{B}')^c$. In other words, the diagram

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{\phi} & \mathcal{B}' \\ \downarrow & & \downarrow \\ \mathcal{B}/\mathcal{B}^c & \rightarrow & \mathcal{B}'/(\mathcal{B}')^c \end{array}$$

commutes. Therefore two trajectories can be concatenated (with respect to each pair of open sets) if and only if their images under the isomorphism can be concatenated (moreover any concatenating trajectory is mapped onto a concatenating trajectory).

2) Autonomy of $\mathcal{B}$, and more generally the characteristic variety. The ideals of minors of any kernel representation R of order $q-j$, where q is the number of columns of R, are also preserved by isomorphism, for any j[4, Cor. 20.4]
3) The number of free variables of $\mathcal{B}$.
4) The uncontrollable pole variety (which describes input decoupling properties [16]), since it is the characteristic variety of the obstruction to controllability.
5) The $\mathcal{Q}(\mathcal{D})$-space which is the factor of $\mathcal{Q}(\mathcal{D})^{1,q}$ by the row span of R over $\mathcal{Q}(\mathcal{D})$. This is of interest since the $\mathcal{Q}(\mathcal{D})$-span of the rows R is also the $\mathcal{Q}(\mathcal{D})$-span of the rows of $(G - I_p)$, where without loss of generality we take the first m variables to be a maximal free set (the inputs), the remaining variables to be outputs, and the corresponding transfer matrix $G \in \mathcal{Q}(\mathcal{D})^{q-m,m}$. We might call this subspace of $\mathcal{Q}(\mathcal{D})^{1,q}$ the **transfer matrix space**. The factor is $\mathcal{Q}(\mathcal{D}) \otimes \mathcal{M}$, which is entirely determined by $\mathcal{M}$ and so invariant under isomorphism (however the transfer matrix space itself is not).
6) For any subbehavior $\mathcal{B}_1$ of $\mathcal{B}, \mathcal{B}$ is set-controllable to $\mathcal{B}_1$ [13] if and only if $\mathcal{B}'$ is set-controllable to $\phi(\mathcal{B}_1)$. This is due to the fact that $\mathcal{B}/\mathcal{B}_1$ is isomorphic to $\mathcal{B}'/\phi(\mathcal{B}_1)$, and set-controllability is characterized by controllability of this factor [13].

IV. Isomorphisms and Unimodular Equivalence

We now look at two special cases of behavior isomorphism, namely unimodular equivalence and elimination of observable variables. We first state a general result, that we use subsequently.

Lemma 11: If $\phi : \mathcal{B}_1 \rightarrow \mathcal{B}_2$ and $\psi : \mathcal{B}_2 \rightarrow \mathcal{B}_1$ are both injective homomorphisms of behaviors, then they are in fact both isomorphisms.

Proof: Set $\tau = \psi \circ \phi : \mathcal{B}_1 \rightarrow \mathcal{B}_1$. Define $V_i \subseteq \mathcal{B}_1, i \in \mathbb{N}$ by $V_i := \tau(V_{i-1})$ with $V_0 = \mathcal{B}_1$. The V_i's form a descending sequence of behaviors, with a corresponding ascending sequence $V_i^\perp$ of submodules of $\mathcal{D}^{1,q}$, which must stabilize as the ring $\mathcal{D}$ is Noetherian. Hence, $V_{i-1} \neq V_i = V_{i+1}$ for some i. If $i > 0$, then there exists $w \in V_{i-1} \backslash V_i$. So $w = \tau^{i-1}v$ for some $v \in \mathcal{B}_1$. $\tau(w) = \tau^i(v) \in V_i$. Since $V_i = V_{i+1}$, so $\tau(w) = \tau^{i+1}(v')$ for some $v' \in \mathcal{B}_1$. Thus $w - \tau^i(v') \in \ker \tau$. But both ψ and ϕ are injective and so τ is injective. Hence, $w = \tau^i(v') \in V_i$, which is a contradiction. It must therefore be that $i = 0$, i.e., $\operatorname{im} \tau = V_1 = V_0 = \mathcal{B}_1$. Hence τ is surjective. Thus ψ must be surjective and so is an isomorphism. Repeating the above argument using $\tau' = \phi \circ \psi$, we obtain that ϕ is also an isomorphism. ∎

A. Unimodular Equivalence and Elimination of Observable Variables

It is not surprising that unimodularly equivalent behaviors are isomorphic, i.e., if $U\mathcal{B} = \mathcal{B}'$, where U is a unimodular matrix, then $\mathcal{B}$ and $\mathcal{B}'$ are isomorphic. For such a U is monic (on any behavior), and the equation says that the map is epic also. A special case of unimodular transformation is change of basis in the space of dependent variables.

We now give another trivial example of isomorphic behaviors. Let $\mathcal{B} = \ker_{\mathcal{W}} R$ and $\hat{\mathcal{B}} = \ker_{\mathcal{W}} \hat{R}, \hat{R} = \begin{pmatrix} I & 0 \\ 0 & R \end{pmatrix}$ with $R \in \mathcal{D}^{g,q}$. Then, by Lemma 11 $\mathcal{B}$ and $\hat{\mathcal{B}}$ are isomorphic, since $\begin{pmatrix} 0 \\ I_g \end{pmatrix} R = \hat{R} \begin{pmatrix} 0 \\ I_q \end{pmatrix}$ and $(0\ I_g)\hat{R} = R(0\ I_q)$, with the matrices $\begin{pmatrix} R \\ \begin{pmatrix} 0 \\ I_q \end{pmatrix} \end{pmatrix}$ and $\begin{pmatrix} \hat{R} \\ 0\ I_q \end{pmatrix}$ being zero right prime. Note that this particular case is essentially the addition/ subtraction to the behavior of a variable(s) which is identically zero. The dual process is that of adding (subtracting) a trivial complex to the projective resolution of the module $\mathcal{M}$.

Suppose now that we are given a behavior $\mathcal{B} = \ker_{\mathcal{W}} R$, where R is of the form $R = (R_1\ R_2)$, describing the equations $R_1 w_1 + R_2 w_2 = 0$. We can ask the question, "Under what conditions is the projection map $\mathcal{B} \mapsto \mathcal{B}_{w_1}, (w_1, w_2) \mapsto w_1$, an isomorphism?". This projection map is represented by the matrix $(I\ 0)$ and so is a homomorphism, and $\mathcal{B}_{w_1}$ is defined as the image of this projection, so it is epic. Clearly, the kernel of the map is the set $\mathcal{B}_{0,w_2} := \{(0, w_2) \in \mathcal{B}\}$. Thus the projection map is an isomorphism if and only if $\mathcal{B}_{0,w_2} = 0$, i.e., if and only if the variables w_2 are **observable** from the variables w_1 (i.e., w_1 determines w_2 uniquely). In matrix terms, this means that R_2 is zero right prime (e.g.,[15, Theorem 9]).

Recall from the elimination algorithm that a kernel representation of $\mathcal{B}_{w_1}$ is given by $\mathcal{B}_{w_1} = \ker_{\mathcal{W}} C_2 R_1$, where C_2 is an MLA of R_2. In the case where R_2 is zero right prime, we can choose C_2 to be the lower part of a unimodular matrix satisfying $\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} R_2 = \begin{pmatrix} I \\ 0 \end{pmatrix}$, and we obtain

$$\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} (R_1 \quad R_2) \begin{pmatrix} 0 & I \\ I & -C_1 R_1 \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & C_2 R_1 \end{pmatrix}$$

Thus, $\mathcal{B}$ equals $\ker_{\mathcal{W}} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} (R_1\ R_2)$, which is unimodularly equivalent to $\ker_{\mathcal{W}} \begin{pmatrix} I & 0 \\ 0 & C_2 R_1 \end{pmatrix}$, and therefore we again find that $\mathcal{B}$ is isomorphic to $\ker_{\mathcal{W}} C_2 R_1$. Moreover, this isomorphism can be decomposed into a unimodular transformation followed by deletion of a zero variable (these two operations are enough), or as removal of a trivial complex. Note that by deleting all observable system variables in this way we obtain a minimally embedded system in the sense of Kleon and Oberst [7].

Another application of these ideas is to latent variable descriptions of a behavior $\mathcal{B}_w$: the projection map from the full behavior $\mathcal{B}_{w,\ell}$ to $\mathcal{B}_w$ is an isomorphism if and only if the latent variables are observable. The natural question to ask next is whether every full behavior $\mathcal{B}_{w,\ell}$ that is isomorphic to $\mathcal{B}_w$ is of this form—that is, do all full behaviors $\mathcal{B}_{w,\ell}$ which are isomorphic to $\mathcal{B}_w$ have observable latent variables? It turns out that one can have full behaviors $\mathcal{B}_{w,\ell}$ with unobservable latent variables which are isomorphic to $\mathcal{B}_w$.

Example 12: Consider the full behavior $\mathcal{B}_{w,\ell}$ given by latent variable representation

$$(1 \quad s)\begin{pmatrix} w \\ \ell \end{pmatrix} = 0$$

of the manifest behavior $\mathcal{B}_w = \ker_{\mathcal{W}} 0$. Clearly, the latent variable ℓ is not observable from the manifest variable w, and therefore the projection map $(1\ 0) : \mathcal{B}_{w,\ell} \to \mathcal{B}_w$ is not an isomorphism. Now consider the map $(1s+1) : \mathcal{B}_{w,\ell} \to \mathcal{B}_w$. Since $Y = 0$ satisfies the equation $0(1s+1) = Y(1s)$, so by Theorem 6, we know that the map $(1s+1)$ is a homomorphism. Again, by Theorem 6 it is easy to check that this homomorphism is both epic and monic, therefore an isomorphism. In fact, the inverse map is given by $\begin{pmatrix} -s \\ 1 \end{pmatrix} : \mathcal{B}_w \to \mathcal{B}_{w,\ell}$. Thus, we have a latent variable representation of a behavior (with unobservable latent variables) that is isomorphic to the original behavior.

B. Isomorphisms and Extended Unimodular Equivalences

In fact, elimination of observable variables, together with unimodular equivalence, characterize all isomorphisms in the following sense.

Theorem 13: Two behaviors $\mathcal{B} \subseteq \mathcal{W}^q$ and $\mathcal{B} \subseteq \mathcal{W}^{q'}$ are isomorphic if and only if the extensions $\hat{\mathcal{B}} = \mathcal{B} \oplus 0^{q'}$ and $\hat{\mathcal{B}}' = 0^q \oplus \mathcal{B}'$ are unimodularly equivalent, where 0^q and $0^{q'}$ denote the zero behaviors in $\mathcal{W}^q$ and $\mathcal{W}^{q'}$, respectively.

Proof: Let $\mathcal{B} = \ker_{\mathcal{W}} R$ and $\mathcal{B}' = \ker_{\mathcal{W}} R'$, with the corresponding modules of observables $\mathcal{M}$ and $\mathcal{M}'$. It is shown in [3,Ex. 5.33] that $\mathcal{M}$ and $\mathcal{M}'$ are isomorphic if and only if there exist unimodular matrices U and V with

$$V\begin{pmatrix} R & 0 \\ 0 & I_{q'} \\ 0 & 0 \\ 0 & 0 \end{pmatrix} U = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ I_q & 0 \\ 0 & R' \end{pmatrix}.$$

Thus, the behaviors with representations

$$V\begin{pmatrix} R & 0 \\ 0 & I_{q'} \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ I_q & 0 \\ 0 & R' \end{pmatrix}$$

are unimodularly equivalent. The matrix V is irrelevant from a behavioral point of view, since it does not effect the module of equations. Hence, we obtain the desired result. ■

V. Latent Variable Descriptions and Strict System Equivalence

We now look at maps between latent variable descriptions of some fixed behavior $\mathcal{B} = \mathcal{B}_w$. This is a generalization of the study of equivalences of generalized or pseudostate representations as described by the Rosenbrock system matrix. In Section V.A we will apply these results to the special case of behaviors defined by Rosenbrock system matrices.

Lemma 14: Let $(R_1 \ \ -M_1)(w,\ell) = 0$ and $(R_2 \ \ -M_2)(w,\ell') = 0$ be two latent variable descriptions of a behavior $\mathcal{B}$. Denote the full behaviors by $\mathcal{B}_{w,\ell}$ and $\mathcal{B}'_{w,\ell'}$ respectively. Then, any homomorphism from $\mathcal{B}_{w,\ell}$ to $\mathcal{B}'_{w,\ell'}$ which moreover fixes w can be represented by a differential operator of the form

$$\begin{pmatrix} I & 0 \\ K & A \end{pmatrix} : \begin{pmatrix} w \\ \ell \end{pmatrix} \mapsto \begin{pmatrix} w \\ \ell' \end{pmatrix}$$

for some K, A.

Proof: Let the given homomorphism ϕ be represented by a matrix $\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}$. Then, we have $A_1 w + A_2 \ell = w$ for any $(w,\ell) \in \mathcal{B}_{w,\ell}$, so in particular $A_2 \mathcal{B}_{0,\ell} = 0$, or $\mathcal{B}_{0,\ell} \subseteq \ker_{\mathcal{W}} A_2$. Using [8, 2.61], this implies that there exists an L with $A_2 = LM_1$. Now for any $(w,\ell) \in \mathcal{B}_{w,\ell}$ we find

$$\begin{aligned} (I - A_1 - LR_1)w &= w - A_1 w - LM_1 \ell \\ &= w - A_1 w - A_2 \ell = 0 \qquad (11). \end{aligned}$$

Now, by the elimination algorithm, a kernel representation of $\mathcal{B}_w$ is given by CR_1, where C is any MLA of M_1. Equation (11) then gives us $I - A_1 - LR_1 = KCR_1$ for some K. We now have

$$\begin{pmatrix} I & 0 \\ A_3 & A_4 \end{pmatrix} - \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} = \begin{pmatrix} KC + L \\ 0 \end{pmatrix} (R_1 \ \ -M_1),$$

which by the discussion following (9) ensures that the two matrices on the left hand side represent the same map on $\mathcal{B}_{w,\ell}$. ■

Corollary 15: Given two latent variable descriptions

$$\begin{aligned} \mathcal{B}_{w,\ell}: &\quad R_1 w = M_1 \ell \\ \mathcal{B}'_{w,\ell'}: &\quad R_2 w = M_2 \ell' \end{aligned}$$

of the same behavior $\mathcal{B}_w$, there exists an isomorphism from $\mathcal{B}_{w,\ell}$ to $\mathcal{B}'_{w,\ell'}$ which moreover fixes w, if and only if there exist matrices A, K and Y, with $\begin{pmatrix} A \\ M_1 \end{pmatrix}$ zero right prime and such that $\begin{pmatrix} M_2 & -Y \\ 0 & C_1 \end{pmatrix}$ is an MLA of $\begin{pmatrix} A \\ M_1 \end{pmatrix}$ (where C_1 is any MLA of M_1), satisfying the law

$$(R_2 \ \ -M_2)\begin{pmatrix} I & 0 \\ K & A \end{pmatrix} = Y(R_1 \ \ -M_1). \qquad (12)$$

Proof: Suppose that an isomorphism exists; then by Lemma 14 it can be represented by a matrix of the form $\begin{pmatrix} I & 0 \\ K & A \end{pmatrix}$. Now consider the exact commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{B}_{0,\ell} & \overset{\begin{pmatrix} 0 \\ I \end{pmatrix}}{\longrightarrow} & \mathcal{B}_{w,\ell} & \overset{(I\ 0)}{\longrightarrow} & \mathcal{B}_w & \longrightarrow & 0 \\ & & \downarrow : A & & \downarrow \begin{pmatrix} I & 0 \\ K & A \end{pmatrix} & & \downarrow I & & . \\ 0 & \longrightarrow & \mathcal{B}'_{0,\ell'} & \overset{\begin{pmatrix} 0 \\ I \end{pmatrix}}{\longrightarrow} & \mathcal{B}'_{w,\ell'} & \overset{(I\ 0)}{\longrightarrow} & \mathcal{B}_w & \longrightarrow & 0 \end{array} \qquad (13)$$

Applying the Snake Lemma (e.g.,[4, Ex. A3.10]) to this diagram, we have that $\begin{pmatrix} I & 0 \\ KA & \end{pmatrix} : \mathcal{B}_{w,\ell} \mapsto \mathcal{B}'_{w,\ell'}$ is injective if and only if $A : \mathcal{B}_{0,\ell} \mapsto \mathcal{B}'_{0,\ell'}$ is, and similarly for surjectivity. The conclusion then follows from Theorem 6. Conversely, if the conditions of the corollary are satisfied, then $\begin{pmatrix} I & 0 \\ K & A \end{pmatrix}$ defines a homomorphism from $\mathcal{B}_{w,\ell}$ to $\mathcal{B}'_{w,\ell'}$ which fixes w, and is an isomorphism by the same arguments as used in the first half of the proof. ■

Note that the matrix A in (12) is actually an isomorphism from $\mathcal{B}_{0,\ell}$ to $\mathcal{B}'_{0,\ell'}$. That the conditions on the existence of an isomorphism are given entirely in terms of A comes from the fact that $\mathcal{B}_{0,\ell}$ and $\mathcal{B}'_{0,\ell'}$ are isomorphic if and only if $\mathcal{B}_{w,\ell}$ and $\mathcal{B}'_{w,\ell'}$ are, as explained in the proof of Corollary 15. Note also that the conditions given in the corollary can be broken down into monomorphism and epimorphism conditions.

A. Isomorphisms and Strict System Equivalence

Let us now apply our results to the special case of behaviors defined by Rosenbrock system matrices, i.e., behaviors of the form

$$\mathcal{B}_{x,u,y} = \ker_{\mathcal{W}} \begin{pmatrix} T & -U & 0 \\ V & W & -I \end{pmatrix}. \quad (14)$$

Rosenbrock system matrices for nD systems have been well-studied by Pugh and co-workers (e.g., [12]). No assumptions on the structure of (T, U, V, W) are needed; hence the result which follows is an extension of Theorem 4 of Zerz in [18], which also links Rosenbrock system matrices with behaviors. By identifying (u, y) with the manifest variable w, and x with the latent variable ℓ, we see that we have a natural special case of a latent variable description. Further, since y is an observable variable, $\mathcal{B}_{x,u,y}$ is actually isomorphic to its projection $\mathcal{B}_{x,u}$ onto the (x, u) variables. This projection is given by

$$\mathcal{B}_{x,u} = \ker_{\mathcal{W}}(T - U). \quad (15)$$

Definition 16: Two Rosenbrock systems $(T_1, U_1, V_1, W_1), (T_2, U_2, V_2, W_2)$ are **generalized strictly system equivalent in the sense of Fuhrmann (GFSSE)** if there exist polynomial matrices Q_l, R_l, Q_r, R_r such that

$$\begin{pmatrix} Q_l & 0 \\ R_l & I \end{pmatrix} \begin{pmatrix} T_1 & -U_1 \\ V_1 & W_1 \end{pmatrix} = \begin{pmatrix} T_2 & -U_2 \\ V_2 & W_2 \end{pmatrix} \begin{pmatrix} Q_r & R_r \\ 0 & I \end{pmatrix} \quad (16)$$

holds, where moreover $\begin{pmatrix} Q_r \\ T_1 \end{pmatrix}$ is zero right prime, and $\begin{pmatrix} T_2 & -Q_l \\ 0 & C_1 \end{pmatrix}$ is an MLA of $\begin{pmatrix} Q_r \\ T_1 \end{pmatrix}$, where C_1 is a given MLA of T_1.

The definition of generalized strict system equivalence in the sense of Fuhrmann is slightly ugly in that the symmetry between the conditions on Q_l and Q_r is broken; however we believe following Note 7 that this is necessary for the full generality of the results which follow.

Theorem 17: The Rosenbrock systems (T_1, U_1, V_1, W_1) and (T_2, U_2, V_2, W_2) are GFSSE if and only if there is an isomorphism from the behavior $\mathcal{B}^{(1)}_{x,u,y}$ to the behavior $\mathcal{B}^{(2)}_{x',u,y}$, which furthermore fixes (u, y). Hence GFSSE is an equivalence relation.

Proof: Suppose that there is an isomorphism ϕ from $\mathcal{B}^{(1)}_{x,u,y}$ to $\mathcal{B}^{(2)}_{x',u,y}$. By Lemma 14, this can be represented by a matrix of the form

$$\begin{pmatrix} Q_r & R_r & S_r \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}.$$

Furthermore, since y is a function of x and u, we see that without loss of generality we can take $S_r = 0$. The equation for a homomorphism now gives us the identity

$$\begin{pmatrix} T_2 & -U_2 & 0 \\ V_2 & W_2 & -I \end{pmatrix} \begin{pmatrix} Q_r & R_r & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} = \begin{pmatrix} Q_l & Q'_l \\ R_l & R'_l \end{pmatrix} \begin{pmatrix} T_1 & -U_1 & 0 \\ V_1 & W_1 & -I \end{pmatrix} \quad (17)$$

where we see further that Q'_l must vanish and R'_l must equal I. Now, as in the proof of Corollary 15, there is an induced isomorphism ϕ_0 from $\mathcal{B}^{(1)}_{x,u=0}$ to $\mathcal{B}^{(2)}_{x',u=0}$, which we can easily see is represented by the differential operator Q_r, and the equation

$$T_2 Q_r = Q_l T_1$$

holds. Theorem 6, applied to ϕ_0, now gives us the required conditions on these matrices. It remains to observe that (17), together with the fact that $Q'_l = 0$ and $R'_l = I$, gives us the formula

$$\begin{pmatrix} Q_l & 0 \\ R_l & I \end{pmatrix} \begin{pmatrix} T_1 & -U_1 \\ V_1 & W_1 \end{pmatrix} = \begin{pmatrix} T_2 & -U_2 \\ V_2 & W_2 \end{pmatrix} \begin{pmatrix} Q_r & R_r \\ 0 & I \end{pmatrix}$$

so that we indeed have that the Rosenbrock systems are GFSSE.

Conversely, suppose that the Rosenbrock systems are GFSSE. Then, from the formula for GFSSE, we can derive an equation of the form (17), where $Q'_l = 0$ and $R'_l = I$. This equation specifies a homomorphism ϕ from $\mathcal{B}^{(1)}_{x,u,y}$ to $\mathcal{B}^{(2)}_{x',u,y}$, and moreover this homomorphism clearly fixes (u, y). Now consider the induced maps ϕ_1 from $\mathcal{B}^{(1)}_{x,u}$ to $\mathcal{B}^{(2)}_{x',u}$, and ϕ_0 from $\mathcal{B}^{(1)}_{x,u=0}$ to $\mathcal{B}^{(2)}_{x',u=0}$, the latter of which is represented by the polynomial matrix Q_r. From the conditions of GFSSE, and by Theorem 6, this mapping ϕ_0 is an isomorphism. Since ϕ fixes $(u, y), \phi_1$ fixes u, so we have an exact commutative diagram of the form

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{B}^{(1)}_{x,u=0} & \overset{\begin{pmatrix} I \\ 0 \end{pmatrix}}{\longrightarrow} & \mathcal{B}^{(1)}_{x,u} & \overset{(0\ I)}{\longrightarrow} & \mathcal{B}_u & \longrightarrow & 0 \\ & & \downarrow \phi_0 & & \downarrow \phi_1 & & \downarrow \text{id} & & \\ 0 & \longrightarrow & \mathcal{B}^{(2)}_{x',u=0} & \overset{\begin{pmatrix} I \\ 0 \end{pmatrix}}{\longrightarrow} & \mathcal{B}^{(2)}_{x',u} & \overset{(0\ I)}{\longrightarrow} & \mathcal{B}_u & \longrightarrow & 0 \end{array}$$

where ϕ_0 and the identity mapping on $\mathcal{B}_u$ are isomorphisms. It then follows from the Snake Lemma that ϕ_1 is also an isomorphism, and now it is easy to see that ϕ itself is an isomorphism. This completes the proof. ■

The preceding result is a generalization of [6, Lem. 5.10] by Hinrichsen and Prätzel-Wolters, which gives coprimeness conditions for a homomorphism between 1-D Rosenbrock system matrices to be monic/epic.

VI. Further Results

Finally, we look at various other aspects of homomorphisms and isomorphisms of behaviors.

A. Isomorphic Copies of a Fixed Behavior

In the previous sections, we considered the properties of maps between two behaviors with given kernel representations.

We now consider the behaviors which can be obtained as isomorphic copies of a given behavior, i.e., we allow ourselves to choose the kernel representation of the image behavior $\phi(\mathcal{B})$. This leads to a very rich structure reminiscent of the classical doubly coprime factorization, and is called a **doubly unimodular extension** by Fuhrmann [5]. Our result here is a generalization of Fuhrmann's Theorem 3.6.4. We do not require the condition that the given kernel representation of $\mathcal{B}$ has full row rank.

Theorem 18: Given a behavior $\mathcal{B} \subseteq \mathcal{W}^q$ and a differential operator $A : \mathcal{W}^q \to \mathcal{W}^h$ such that $A : \mathcal{B} \to A\mathcal{B}$ is an isomorphism, one obtains a commutative diagram: shown in (18) at the bottom of the page where $c = h + g - q, \mathcal{B}' = A\mathcal{B}$ and $\phi : \mathcal{B} \to \mathcal{B}'$ is the isomorphism represented by A. The diagram is to be understood in the following way: picking either horizontal direction ($\leftarrow$ or $\rightarrow$) and either vertical direction ($\uparrow$ or $\downarrow$), and retaining in the above diagram only the maps pointing in the chosen directions, we obtain an exact commutative diagram. The maps $\alpha, \alpha^{-1}, \beta, \beta^{-1}, \psi$ and ψ^{-1} are induced, respectively, from the maps R', C, Y, X, R and D, and are mutual inverses as indicated, although none of the corresponding matrix pairs are necessarily inverses.

Proof: Since the map $A : \mathcal{B} \to \mathcal{B}'$ is an isomorphism, it is injective (it is surjective by definition of $\mathcal{B}'$) and so, with R a kernel representation of $\mathcal{B}$, we have a zero right prime matrix $\binom{A}{R}$, which we can embed in a unimodular product

$$\begin{pmatrix} B & D \\ R' & -Y \end{pmatrix} \begin{pmatrix} A & C \\ R & -X \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \tag{19}$$

$$\begin{pmatrix} A & C \\ R & -X \end{pmatrix} \begin{pmatrix} B & D \\ R' & -Y \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}. \tag{20}$$

Now, by Lemma 3 we obtain the identities

$$\left.\begin{array}{rr} A(\ker_{\mathcal{W}} R) = \ker_{\mathcal{W}} R' & Y(\ker_{\mathcal{W}} D) = \ker_{\mathcal{W}} C \\ B(\ker_{\mathcal{W}} R') = \ker_{\mathcal{W}} R & X(\ker_{\mathcal{W}} C) = \ker_{\mathcal{W}} D \\ R(\ker_{\mathcal{W}} A) = \ker_{\mathcal{W}} Y & D(\ker_{\mathcal{W}} Y) = \ker_{\mathcal{W}} A \\ R'(\ker_{\mathcal{W}} B) = \ker_{\mathcal{W}} X & C(\ker_{\mathcal{W}} X) = \ker_{\mathcal{W}} B \end{array}\right\} \tag{21}$$

and the commutativity laws

$$\left.\begin{array}{rr} BC = DX & AD = CY \\ R'A = YR & RB = XR' \end{array}\right\} \tag{22}$$

which give us diagram (18). Finally, given $w \in \ker_{\mathcal{W}} R$ we have $BAw = (I - DR)w = w$ so that $A : \ker_{\mathcal{W}} R \to \ker_{\mathcal{W}} R'$ is the right inverse of $B : \ker_{\mathcal{W}} R' \to \ker_{\mathcal{W}} R$. The other left/right inverse relationships all follow from similar Bézout identities.■

The existence of a diagram of the form (18) is clearly sufficient as well as necessary for $A : \mathcal{B} \mapsto A\mathcal{B}$ to be an isomorphism. Another characterization for A to be an isomorphism is that $a_1 + \mathcal{B}^\perp, \ldots, a_{g'} + \mathcal{B}^\perp$ generate $\mathcal{M}$, where $a_1, \ldots, a_{g'}$ are the rows of A [15, Cor. 4]. Thus, looking for isomorphic copies of a concrete behavior is equivalent to looking for generating sets of its module of observables.

Note that the diagram (18) can be considered as an interpretation of any matrix product of the doubly unimodular extension form (19)–(20). The map B is the representation as a polynomial matrix of the inverse homomorphism $\phi^{-1} : \mathcal{B}' \mapsto \mathcal{B}$, and the other matrices admit interpretations as explained in the diagram.

B. The Space of Homomorphisms

We now show how the space of homomorphisms from a given behavior $\mathcal{B}$ to another given behavior $\mathcal{B}'$ can be computed explicitly. Essentially, this involves construction of a presentation of the module $\mathrm{Hom}_{\mathcal{D}}(\mathcal{M}', \mathcal{M})$, which is a standard problem however not usually expressed in polynomial matrix terms.

We have seen that homomorphisms $\phi : \mathcal{B} = \ker_{\mathcal{W}} R \mapsto \mathcal{B}' = \ker_{\mathcal{W}} R'$, $R \in \mathcal{D}^{g,q}, R' \in \mathcal{D}^{g',q'}$, are characterized by matrices A such that a matrix Y exists satisfying $R'A = YR$, which we can rewrite in the following form [2, 2.3]using the Kronecker product of matrices

$$(R^T \otimes I_{g'} - I_q \otimes R') \begin{pmatrix} \mathrm{col}(Y) \\ \mathrm{col}(A) \end{pmatrix} = 0. \tag{23}$$

Here, $\mathrm{col}(Y)$ denotes the $g'g \times 1$ vector obtained by writing out the columns of Y, in order, in a long column vector, and

$$\begin{array}{ccccccccccccc}
 & & & & 0 & & 0 & & & & \\
 & & & & \downarrow & & \downarrow & & & & \\
 & & & & \ker_{\mathcal{W}} A & \underset{\psi^{-1}}{\overset{\psi}{\rightleftarrows}} & \ker_{\mathcal{W}} Y & & & & \\
 & & & & \downarrow & & \downarrow & & & & \\
0 & \rightarrow & \mathcal{B} & \rightarrow & \mathcal{W}^q & \underset{D}{\overset{R}{\rightleftarrows}} & \mathcal{W}^g & \leftarrow & \ker_{\mathcal{W}} D & \leftarrow & 0 \\
 & & \phi^{-1} \uparrow\downarrow \phi & & B \uparrow\downarrow A & & X \uparrow\downarrow Y & & \beta^{-1} \uparrow\downarrow \beta & & \\
0 & \rightarrow & \mathcal{B}' & \rightarrow & \mathcal{W}^h & \underset{C}{\overset{R'}{\rightleftarrows}} & \mathcal{W}^c & \leftarrow & \ker_{\mathcal{W}} C & \leftarrow & 0 \\
 & & & & \uparrow & & \uparrow & & & & \\
 & & & & \ker_{\mathcal{W}} B & \underset{\alpha^{-1}}{\overset{\alpha}{\rightleftarrows}} & \ker_{\mathcal{W}} X & & & & \\
 & & & & \uparrow & & \uparrow & & & & \\
 & & & & 0 & & 0 & & & &
\end{array} \tag{18}$$

similarly for $\mathrm{col}(A)$. Therefore, the possible homomorphisms are entirely determined by the right syzygies of the matrix

$$(R^T \otimes I_{g'} - I_q \otimes R') \tag{24}$$

which we can compute by Gröbner basis methods. However there is redundancy in this description of homomorphisms, since as explained in Section III, the same homomorphism of behaviors can be represented by multiple differential operator matrices A and corresponding matrices Y. As discussed earlier, the pair (A, Y) defines the zero map from $\mathcal{B}$ to $\mathcal{B}'$ if and only if there exist matrices L and L_1 with

$$A = LR, \quad Y = R'L + L_1 R_1$$

where R_1 is an MLA of R. Under the Kronecker product representation above, these correspond to syzygies of the form

$$\begin{pmatrix} I_g \otimes R' & R_1^T \otimes I_{g'} \\ R^T \otimes I_{q'} & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

for some polynomial vectors α, β. To obtain a precise description of the homomorphism space, we must then factor out such syzygies. Computing a factor of modules is a standard problem easily solvable using Gröbner bases.

In the case where $\mathcal{B}$ is controllable, say that $\mathcal{B} = \mathrm{im}_{\mathcal{W}} M$, it is much easier to construct all possible maps A, for (23) can be written

$$(-I_q \otimes R')\mathrm{col}(A) \in \mathrm{im}_{\mathcal{D}}(R^T \otimes I_g). \tag{25}$$

We also have that $(R^T \otimes I_g)$ is an MRA of $(M^T \otimes I_g)$, so an equivalent formula is

$$0 = (M^T \otimes I_g)(-I_q \otimes R')\mathrm{col}(A) = (M^T \otimes R')\mathrm{col}(A). \tag{26}$$

To solve this, we need only compute the right syzygies of $(M^T \otimes R')$.

A further natural question to ask is: given two behaviors $\mathcal{B} = \ker_{\mathcal{W}} R$ and $\mathcal{B}' = \ker_{\mathcal{W}} R'$, how can one test whether they are isomorphic? By Lemma 11, this reduces to the ability to test for the existence of a monomorphism in each direction. Fortunately, from the discussion above we can compute an explicit description of the space of possible matrices (A, Y) which describe homomorphisms from $\mathcal{B}$ to $\mathcal{B}'$, and by parameterizing this space we obtain a 'generic homomorphism'. We can then test whether the parameters can be chosen such that this matrix is zero right prime. Such an argument in terms of generic maps is used to test for graded module isomorphism in the Macaulay [1] script `module_iso` by D. Eisenbud.

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Harish Pillai obtained the Ph.D. degree from the Indian Institute of Technology, Bombay, India, in December 1997.

He was a Postdoctoral Fellow at the University of Groningen, Groningen, The Netherlands and at the University of Southampton, U.K. Currently, he is an Assistant Professor at the Indian Institue of Technology, Bombay, India, where he works in systems theory, especially the behavioral approach to systems theory and its applications. He is also interested in algorithms associated to the theory.

Jeffrey Wood obtained the B.Sc. and Ph.D. degrees from Royal Holloway University, London, U.K.

Since 1995, he is with the University of Southampton, Southampton, U.K., where he is currently the holder of a Royal Society University Research Fellowship. His research interests are in mathematical systems theory, particularly algebraic and behavioral approaches.

Eric Rogers was born in 1956 near Dungannon. Northern Ireland, U.K. He received the undergraduate degree in mechanical engineering from The Queen's University, Belfast, U.K., and the Ph.D. degree from The University of Sheffield, Sheffield, U.K., for a thesis in the area of multidimensional systems theory.

He was a Lecturer at The Queen's University from 1984 to 1987, and at the University of Strathclyde, Strathclyde, U.K., from 1988 to 1990. He has been with The University of Southampton, U.K., since 1990, where he is currently Professor of Control Systems Theory and Design in The Department of Electronics and Computer Science. His current major research interests include multidimensional systems theory and applications, with particular emphasis on behavioral systems theory approaches and systems with repetitive dynamics, iterative learning control, flow control, and active control of microvibrations. He is currently the Editor of *The International Journal of Control*, an Associate Editor of *Multidimensional Systems and Signal Processing*, and Joint Editor of *The Taylor and Francis Research Book Series on Systems and Control*. In addition, he has served extensively on IEEE, IFAC, and IEE technical committees and acted as a consultant to numerous companies and government agencies in the U.K. and abroad.