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A HYBRID MODEL FOR SHARING INFORMATION BETWEEN FUZZY, UNCERTAIN AND DEFAULT REASONING MODELS IN MULTI-AGENT SYSTEMS

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This paper develops a hybrid model which provides a unified framework for the following four kinds of reasoning: 1) Zadeh's fuzzy approximate reasoning; 2) truth-qualification uncertain reasoning with respect to fuzzy propositions; 3) fuzzy default reasoning (proposed, in this paper, as an extension of Reiter's default reasoning); and 4) truth-qualification uncertain default reasoning associated with fuzzy statements (developed in this paper to enrich fuzzy default reasoning with uncertain information). Our hybrid model has the following characteristics: 1) basic uncertainty is estimated in terms of words or phrases in natural language and basic propositions are fuzzy; 2) uncertainty, linguistically expressed, can be handled in default reasoning; and 3) the four kinds of reasoning models mentioned above and their combination models will be the special cases of our hybrid model. Moreover, our model allows the reasoning to be performed in the case in which the information is fuzzy, uncertain and partial. More importantly, the problems of sharing the information among heterogeneous fuzzy, uncertain and default reasoning models can be solved efficiently by using our model. Given this, our framework can be used as a basis for information sharing and exchange in knowledge-based multi-agent systems for practical applications such as automated group negotiations. Actually, to build such a foundation is the motivation of this paper.

Keywords: Fuzzy reasoning; Uncertain reasoning; Default reasoning; Linguistic truth; Expert system; Knowledge-based system; Agent; Automated negotiation.

1. Introduction

The issues of information sharing and exchange are fundamental to work in area of in multi-agent systems [68,26,27,101,18,19,77,82]. They lie at the very heart of the most work on cooperation, coordination and negotiation. However, such sharing is generally difficult to attain because the agents are highly heterogeneous; developed at different times, by different people, for different purposes. This has led to much work in the area of ontologies [70,24,25,23,41] as a way of securing meaningful semantic inter-operation. One particularly difficult aspect of this problem is dealing with the case in which the agents have heterogeneous reasoning models (*e.g.*, one agent uses a uncertain reasoning model, another agent uses a default reasoning model, while the third one uses a fuzzy approximate reasoning model). Against this background, the aim of this paper is to make it possible to share heterogeneous information among a number of uncertainty reasoning models, default reasoning systems and fuzzy approximate reasoning models in multi-agent systems. In the reminder of this section, we will further justify the motivation of this paper. More precisely, we justify that agents need reasoning models in general, uncertain reasoning models in particular, and, moreover there is a need to share information among uncertain, fuzzy and default reasoning models.

1.1. *Expert Systems Work for Intelligent Agents*

Though the term *intelligent agent* lacks a widely accepted and precise definition, it is generally reserved for software/hardware entities which have some degree of reactivity, autonomy, and adaptability [102]. In this sense, expert systems are typically less complex than intelligent agents. However, an expert system could be an ingredient of an intelligent agent.*According to Brenner, Zarnekow and Wittig (see pages 24-25 of [7]), in a multi-agent system each agent should have a certain minimum degree of intelligence, and this intelligence is derived from three main components: its internal knowledge base, its reasoning capabilities based on the contents of the knowledge base,†and its ability to learn or adapt to changes to the environment. The basic components of an expert system are its internal knowledge base and its reasoning capabilities based on the contents of the knowledge base. Accordingly, an expert system may be the easiest way to give an agent some intelligence (see page 39 of [35]).‡

Given this view, it can be seen that it is the reasoning component that provides the intelligence of an intelligent agent. Actually, this is not difficult to understand. Generally speaking, an agent must perform two functions: perceive changes in the environment in which it is situated, and perform actions that affect changes in the environment. Here the problem is how to choose a proper action given the changes in the environment. Hayes-Roth argues that agents reason during the process of action selection (see page 10 of [67]). McCauley-Bell [64] also holds the similar view that agents deduce their further moves from their knowledge about prior scenarios

*It is interesting that inversely an agent could be integrated into an expert system. For example, Brown, Santos and Banks [9] integrate their intelligent interface agents into an expert system shell called PESKI (Probabilities, Experts Systems, Knowledge, and Inference) [29,30,8].

†Some researcher [64] even argues that the capability for reasoning distinguishes intelligent agents from other more "robotic" softbots.

‡In [35], Knapik and Johnson further discuss how agents can make use of expert systems that are up and running in most domains in which agents are likely to be used.

and their resulting actions. Moreover, Wooldridge [109] believes that an agent can be built as a particular type of knowledge-based system that contains a symbolic model of the environment, and that decides what actions to perform in a given situation through symbolic reasoning. Further, he [110] illustrates an agent of this kind by theorem proving agents, which expert system for the action selection comprises of four components: 1) ρ: the knowledge base (typically a set of rules), 2) Δ: the logical database representing the current situation of the environment, 3) AC: the set of actions the agent can take, and 4) d: the decision algorithm, for example:

for each action a in AC do
  if $Do(a)$ can be proved from Δ using ρ then return a end-if
end-for

A typical example of reasonings employed for the action selection is the BDI-logic based negotiation system developed by Parson, Sierra and Jennings [75]. For example, the system can realise the human negotiation like that as follows:

A: Please give me a nail. I need it to hang a picture.
B: I can't give you a nail because I need it to hang a mirror.
A: You may use screw to hang the mirror.
B: Why not? OK, I give you the nail.

During the course of such a negotiation, the reasoning is used by a negotiation agent to decide what should be responded to its opponent agent.

In short, we could have:

$$\text{Intelligent Agent} = \text{Expert System}+\text{Sensor}+\text{Effector}+\text{Communicator}.$$

In other words, intelligent agents can be knowledge-based systems. In fact, as mentioned above, several such systems have been developed. Besides, the following are further examples:

- In the GRATE system developed by Jennings [33], three independent expert systems are employed for action selection during interactions between agents.

- Poggi [81] integrates object-oriented and rule-based programming for building agent systems. In his system, rule-based expert systems are used for action selection during the interactions between the agents.

- The vivid agent system developed by Schroeder and Wagner [94] is also such a system without communicator. The knowledge sub-system of a vivid agent has an update operator and an inference operator, and allows various forms of knowledge representation including 1) relational database, 2) relational fact-base, 3) factbase with deduction rules, 4) temporal, disjunctive, fuzzy fact-bases, 5) deduction rules with negation-as-failure, 6) default rules with two kinds of negation, and 7) the rule-based specification of inter-agent cooperation. Again, the action selection of a vivid agent is based on its knowledge.

1.2. *Uncertain Reasoning in Intelligent Agents*

Given the fact that expert systems can be viewed as the basis of intelligent agents, and expert systems usually embody with some of form of uncertain reasoning; this raises the question: is uncertain reasoning necessary for intelligent agents? We believe the answer is affirmative because of the following reasons and facts.

Knapik and Johnson make it clear why an intelligent agent must be able to deal gracefully with these uncertainties (see page 32 of [35]):

> *As Russell and Novig point out, agents' actions based on first-order logic alone "...almost never have access to the whole truth about their environment"* [91]. *Therefore, agents cannot always know for certain what is the correct, rational action to take in the real world. There are too many uncertainties: environmental factors such as location, where to go next in the case of a mobile agent, resource uncertainties, unclear or wrong objectives and goals, faulty communication links, and so forth.*

In other words, the state of the world is often uncertain, and the sensors of an agent may themselves be uncertain (*e.g.*, fuzzy sensors) [16], thus the agent's belief about the world is also like to be uncertain. Given this, uncertainty about the selection and consequences of actions is unavoidable [37]. On the other hand, as argued in the previous subsection, action selection can be based on reasoning. Therefore, uncertain reasoning is necessary for an intelligent agent. For example, an uncertain reasoning can be employed to predict the future state of the environment based on its current situation. Clearly, such predictions are often uncertain. Then, an agent selects an action, which can maximises its utility, to perform upon the environment. This procedure is clearly shown in Figure 1. Notice that if we replace environment, sensor and effector in this figure by another agent, receiver and sender respectively, then we obtain a reasoning pattern that can be used for multi-agent systems (*e.g.*, automated negotiation systems).

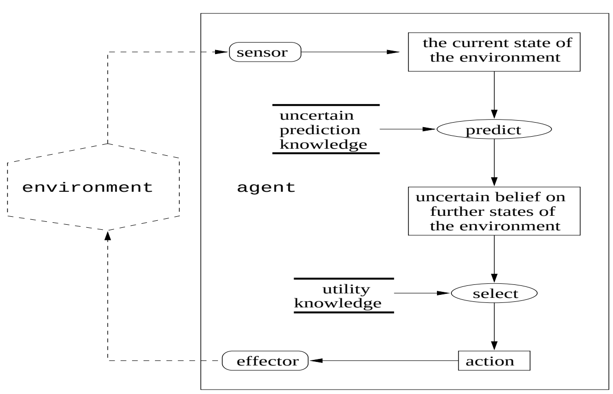

Fig. 1. A pattern of using uncertain reasonings in agent systems.

Moreover, a number of researchers have already employed various uncertain reasoning models to handle uncertainty in agents. The followings are some examples.

- *Evidence theory.* 1) Parsons and Giorgini [74,76] extend the work of Parsons, Sierra and Jennings [75] on the use of argumentation in BDI agents (namely agents capable of having Beliefs, Desires and Intentions) [87,88] to include degrees of belief. In their work, the degree of belief is expressed as a mass assignment in Dempster and Shafer's *evidence theory* [96]. If a belief is received from another agent, the recipient's degree of belief reflects the known reliability of that agent. If a belief is a new observation that an agent makes, the reliability of its sensors is used in place of the reliability of other agents. On updating its belief set when new information is received or sensed, the combination rule of evidence theory is used to calculate the degree of each piece of information in the belief set. 2) Similarly, Li and Zhang [39] employ *evidence theory* [96] to handle the issue of fusing an agent's belief with the information obtained from its sensors. Further, they [40] utilise *evidence theory* to fuse the information that is from an agent's own sensors, and the information that it is told by other agents. In their work, these two pieces of information are uncertain. The work of Li and Zhang [39,40] deals with the belief fusion under uncertainty, but disregards the belief revision when the new information is approximate, incomplete or erroneous; while the issue of belief revision is dealt with in the work of Parsons and Giorgini [74,76]. However, both do not include degrees of desire and intention (two of three mental attitudes in the BDI theory [87,88]).

- *Probability theory.* 1) Thiébaux, Hertzberg, Shoaff and Schneider [103] exploit Nilsson's *probabilistic logic* [71] to handle uncertainty about the environment in which an agent is situated and the uncertainty about the effect of an action that the agent performs. Later on, Nilsson uses a Bayesian network [78] to handle the similar problem (see Section 20.2 of [72]). 2) Zeng and Sycara [121] develop an agent system for bilateral multi-issue negotiation. In this system, a Bayesian network [78] is used to update the knowledge and beliefs each agent has about the environment and other agents, and offers and counter offers between agents are generated based on Bayesian probabilities. 3) Lee and McCartney [37] use *probabilistic models* to handle the issue of intelligent interface agents' acquiring plans of using resources from user's uncertain behaviours. It is also worth noting that Shoham [98] points out that intelligent agent could be based on *probabilistic reasoning.* 4) Mudgal and Vassileva [66] develop an agent system for bilateral negotiation with incomplete and uncertain information. In this system, decision making during the course of a negotiation is modelled as an influence diagram [95] which is an extension of a Bayesian network with both decision and utility nodes. In their influence diagram, the monetary importance for the traders, the urgency of the item being traded, the risk attitude of the traders and the opponent's actions are modelled as random nodes; its decision node can then take three values:

accept, reject and counter-offer; its utility node represents the trader's preference on these actions; and the monetary importance node, the urgency node and the risk attitude node are linked towards the decision node, and the decision node and opponent's action node are linked towards the utility node. By means of the influence diagram, the trader agent can find the action with the highest expected utility and then take this action. 5) Maes, Truyls and Manderick [59] also propose a generic framework using influence diagrams [95] to model agents. In their framework, random nodes of an influence diagram represent the agent's possible uncertain belief about the world; its decision nodes hold the choice of actions that an agent is able to take; its utility nodes represents the agent's preferences; and the links between the nodes represent their dependence relationships. The main character of this proposal is that it can solve the combinatorial explosion problem of actions of all agents in the environment.

- *Fuzzy theory.* 1) For a similar issue to the one that Thiébaux, Hertzberg, Shoaff and Schneider [103] use Nilsson's *probabilistic logic* [71] to handle, Pereira, Carcia, Lang and Martin-Clouaire [79] give another solution based on the *possibility theory* proposed by Dubois and Prade [12]. 2) Possibility-based approaches, proposed by Garcia-Calvés, Giménez-Funes, Godo, Rodríguez-Aguilar, Matos and Sierra [17,20,61], also provide a number of ways to perform multi-agent reasoning under uncertainty. In these approaches, uncertainty due to the lack of knowledge about other agents' behaviours is modelled by possibility distributions. Based on information which is induced from a case base composed of previous negotiation behaviours, the possibility distributions are generated by choosing the most similar situation and the most similar price from the case base with the current market environment. Sometimes, approximate reasoning techniques are used to fine tune the distributions. Finally, the possibilistic decision model [12] is used to choose the most preferred decision with the highest global utility. These approaches can handle negotiation based on a set of mutually influencing two parties and many issues in the world. 3) El-Nasr, Yen and Ioerger [15] use *fuzzy logic* to model emotions in agents. In their model, a fuzzy logic representation is used to map emotion states and events to behaviours. For example, for a pet, if its anger is $High\_Intensity$ and its fear is $Medium\_Intensity$ and the event is $dish\_was\_taken\_away$ then its behaviour is *growl*, where $High\_Intensity$ and $Medium\_Intensity$ are fuzzy linguistic terms. 4) He, Leung and Jennings [31,32] designed and implemented fuzzy logic based bidding strategies used by seller and buyer agents in a continuous double auction. In fact, the agents use fuzzy control rules to select their bidding actions (*i.e.*, how much they should bid) according to the market bidding history and the current market situation.

In addition to the above mentioned work, a number of researchers have developed uncertain reasoning models specifically for multi-agent reasoning. 1) Kraus and Subrahmanian [36] develop a family of logics that a reasoning agent may use to perform successively more sophisticated types of reasoning about uncertainty in the world, about the actions that may occur in the world (either due to the agent or those initiated by other agents), about the probabilistic beliefs of other agents, and how these probabilistic beliefs change over time. 2) Xiang [111] proposes a probabilistic framework for cooperative multi-agent distributed interpretation and optimisation of communication. 3) Wong and Butz [107] propose the multi-agent probabilistic reasoning model. Unlike Xiang's model, their model is able to process input in a *truly* asynchronous fashion. Further, they [108] propose an automated process for constructing a multi-agent probabilistic network from any known and conditionally independent information supplied by each domain expert. 4) Bloemeke [5] introduces agent encapsulated Bayesian networks. In such a network, each agent uses a single Bayesian network as its model of the world. Each agent's network is made up of three groups of variables: *input* variables (about which other agents have better knowledge); *local* variables (which are only used within the agent); and *output* variables (which the agent has the best knowledge about and other agent may want to know). The agents exchange probability distributions on variables shared between the individual agents' networks. Later on, Valtorta, Kim and Vomlel [105] further investigated the Bayesian network multi-agent system and designed the two algorithms to update a probability distribution in the light of soft evidence (a set of probability distributions on common variables).

In addition, various other forms of uncertainties in agents have also been dealt with. The following are some examples. 1) Xuan and Lesser [112] incorporate uncertainty in agent commitments. 2) The work of Luo, Leung and Lee [57], and the work of Tyan, Wang and Bahler [104] both bridge fuzzy constraint satisfaction problems [92,13] and multi-agent systems for different purposes. 3) Pinto, A. Sernadas, C. Sernadas and Mateus [80] extend the situation calculus with actions that have a non-deterministic or uncertain nature. The situation calculus is originally proposed by McCarthy [62,63] as a logical framework for the representing knowledge about actions and change they provoke on the world. 4) Based on fuzzy set theory, Yager [118] reveals the possibility that individual agents can strategically manipulate the information they provide such that their most preferred alternative is selected in multi-agent decision making. He also suggests ways to discourage such a strategic manipulation by the participating agents.

1.3. *Heterogeneous Uncertain Information Sharing*

Given the fact that agents need to represent and reason with uncertain information, and the fact that agents interact with one another (either to achieve their individual goals or the goals of collective), it follows that agents will need to exchange and share uncertain information with one another. Actually, this is not difficult to understand. For example, suppose two agents that cooperate to achieve a common goal have different sensors or have the same sensors but they are geographically separated. That is, some necessary information cannot be sensed by one agent, but can be done by another agent. Thus, one agent must get some information through another agent. Now an agent gets some information via its own sensors, then it uses its own uncertain reasoning model to analyse the information. Further, it sends another agent the result of the analysis.

We can give two examples of such systems. One is a vehicle tracking system implemented as a scenario of agents that are each responsible for a particular physical region [38]. In the system, the agents need to share hypotheses about the various vehicles as they move between regions. By definition, these hypotheses contain uncertainty (about the vehicle type, its direction, its velocity, etc.) and this uncertainty must be understood by the recipient agent and incorporated into its own reasoning model. Another example is a team of agents that possess different categories of knowledge cooperate to negotiate with its negotiation party. Actually, bilateral business negotiations are often this kind. In fact, since this kind of negotiations is sometimes very complex, one side needs to be a team consisting of a group of agents representing a professional negotiator, a financial expert, a business expert, a technique expert and a legal expert[§] If this side needs to cooperate to analyse the opponent's offers and thus make its counter-offers, these agents must exchange and share information drawn from their individual knowledge bases (see Figure 2). Further, if they use different information models, for example, heterogeneous uncertain reasoning models, there must be a kind of mechanism for them to understand each other. This is something like that there must be a sort of mechanism for different language speakers to understand each other.

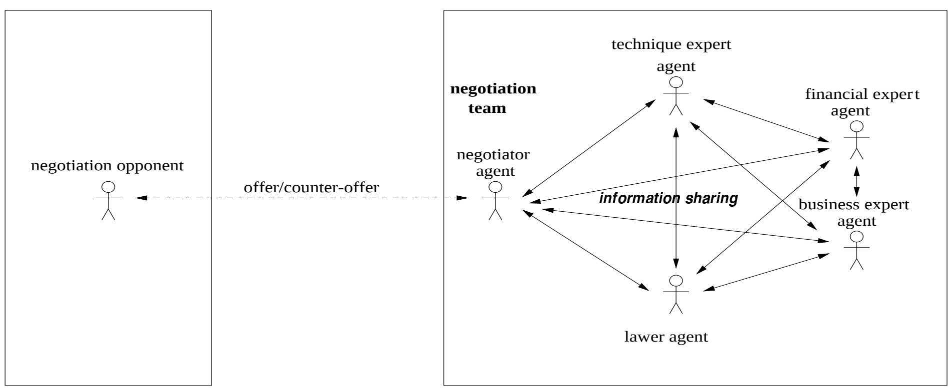

Fig. 2. A example of information sharing between multi-agent system with uncertain reasonings.

In a multi-agent system, sometimes it is also necessary for an agent using default reasoning to share information with another agent using uncertain reasoning model. A default reasoning is performed in the absence of negative evidence. In Reiter's default reasoning model [89], a default rule is in the form of $\alpha : \beta \rightarrow \gamma$, where α, β and γ are well-formed formulae; α is the prerequisite of the default rule, β is its

[§]Such an example of bilateral business negotiation can further be studied in the area of automated business negotiation [34,42], which is a typical and active area of agent technique's application.

justification, and γ is its consequence. The intuitive meaning of the rule is: unless there is information contrary to β, if α is true, then γ is inferred. Notice that in this case, the prerequisite α is required to be absolutely true and the consequence γ is also absolutely true though it is tentative. In other words, uncertainty cannot be handled in this system of default reasoning. Nevertheless, in a cooperative multi-agent environment, we sometimes have to face the problem of uncertainty in default reasoning. In fact, an agent using a default reasoning model may have to make use of some not completely certain information from another agent using an uncertain reasoning model. For instance, suppose in a multi-agent system we have two heterogeneous reasoning agents: a default reasoning agent and an uncertain reasoning agent. Now an inference process in the uncertain reasoning agent outputs a piece of information "α is very true", while in the default reasoning agent there is a default rule $\alpha : \beta \to \gamma$. If the output of the uncertain reasoning agent is used as an input for a further inference by using $\alpha : \beta \to \gamma$ in the default reasoning agent, the corresponding default reasoning has to be performed in the case where the prerequisite of the default rule is not absolutely true. Also, perhaps the reasoning has to be performed in the situation where the available information from another agent using an uncertain reasoning model is neither completely in contradiction to, nor absolutely consistent with, the justification of the default rule. Hence, the conclusion drawn is not guaranteed to be absolutely true, either.

In a multi-agent system, there may also be the requirement for sharing information among fuzzy approximate reasoning models, uncertain reasoning models and default reasoning models. In fuzzy approximate reasoning, although propositions are fuzzy, the belief in a proposition is certain (it is either true or false). In a cooperative multi-agent environment, in order to cooperate with an agent using an uncertain reasoning model, the agent using a fuzzy approximate reasoning model has to deal with these uncertain beliefs in its reasoning model. If further uncertain beliefs in fuzzy propositions are sent to an agent with a default reasoning system, the default reasoning system has to process such fuzzy and uncertain information. Furthermore in such systems, an agent using a default reasoning model may need to receive information from an agent using a fuzzy approximate reasoning model, and so fuzzy information needs to be handled in the default reasoning model.

Against this background, this paper presents a framework for solving the problem of information sharing between agents that use the uncertainty reasoning models, default reasoning model, and fuzzy approximate reasoning models.

1.4. *The Structure of the Paper*

The remainder of the paper is structured as follows. Section 2 recaptures the basic definitions, notions and methods from fuzzy mathematics that underpin the work described in this paper. Sections 3, 4 and 5 develop a hybrid model for reasoning with fuzzy, uncertain and incomplete information. In particular, we discuss some of its properties, which reveal the propagation mechanism in the absence of negative information, and show how to use fuzzy default rules without uncertainties to perform reasoning under uncertainty. Section 6 illustrates how to use our model to deal with the issue of information sharing in heterogeneous reasoning multi-agent systems. In fact, we show how our hybrid model enables agents employing heterogeneous reasoning models to take results from one another and to incorporate them into their own models. Section 7 compares our model with related work. Finally, Section 8 presents our conclusion and outlines the directions for further research.

2. Preliminaries

In this section, we recapture the basic concepts, notions and methods in fuzzy mathematics, which will be used throughout the reminder of this paper.

2.1. *Linguistic Truth*

A linguistic variable is one whose value is a word or a phrase in natural language [120]. The term set of a linguistic variable is a set of its linguistic values. The truth of a proposition can also be a linguistic variable taking values on the Linguistic Truth Term Set (LTTS):

$$\begin{aligned}LTTS = \{&\textit{absolutely-true}, \textit{very-true}, \textit{true}, \textit{fairly-true},\\ &\textit{undecided}, \textit{fairly-false}, \textit{false}, \textit{very-false}, \textit{absolutely-false}\}.\end{aligned} \tag{1}$$

In [1], the semantics of the terms in this term set are defined as shown in Table 1 (and as drawn graphically in Figure 3).

Table 1. Linguistic truth.

$\mu_{absolutely\text{-}true}(x) = \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{otherwise} \end{cases}$	$\mu_{absolutely-false}(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}$
$\mu_{very-true}(x) = \mu_{true}^2(x)$	$\mu_{very-false}(x) = \mu_{false}^2(x)$
$\mu_{true}(x) = x, \forall x \in [0,1]$	$\mu_{false}(x) = 1 - x, \forall x \in [0,1]$
$\mu_{fairly-true}(x) = \mu_{true}^{1/2}(x)$	$\mu_{fairly-false}(x) = \mu_{false}^{1/2}(x)$
$\mu_{undecided}(x) = \begin{cases} 1 & \text{if } x \in (0,1) \\ 0 & \text{otherwise} \end{cases}$	

2.2. *The Extension Principle*

The operations on linguistic variables are defined according to the corresponding operations on numerical variables by using the extension principle [119]. Suppose f is a function with n arguments $x_1, \cdots, x_n$, denoted by $\vec{x}$. Let $\mu_i(x_i)$ be the membership function of argument x_i ($1 \leq i \leq n$). Then:

$$\mu(y) = \sup\{\mu_1(x_1) \wedge \cdots \wedge \mu_n(x_n) \mid f(\vec{x}) = y\}. \tag{2}$$

Here *sup* denotes the supremum operation on a set. Let the fuzzy set corresponding to μ be B, and let the fuzzy set corresponding to μ_i be A_i. For convenience, we denote the operation of the extension principle as $\bigotimes$. That is,

$$B = \bigotimes(A_1, \cdots, A_n, f). \tag{3}$$

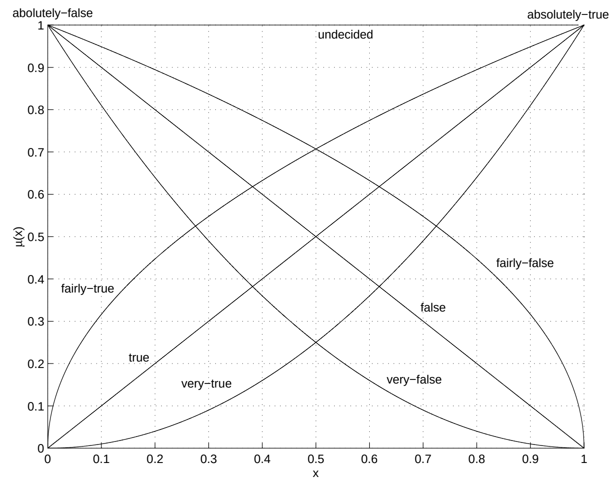

Fig. 3. Linguistic truth.

2.3. *Linguistic Approximation*

If an operation on some linguistic terms is not closed within the predefined linguistic term set, a linguistic approximation technique is necessary in order to find a term in the term set, whose *meaning* (membership function) is the *closest* to the *meaning* (membership function) of the result of the operation. The most straight forward approach, the *Best FIT* [93], uses the Euclidean Distance (ED):

$$ED(A, B) = \sqrt{\sum\{(\mu_A(x) - \mu_B(x))^2 \mid x \in [0, 1]\}} \tag{4}$$

between fuzzy sets A and B defined on $[0, 1]$, to evaluate which one in the term set is the *closest* to the set being approximated. Namely, $\tau \in LTTS$, being the closest to τ'', should satisfy

$$\forall \tau' \in LTTS, ED(\tau, \tau'') \leq ED(\tau', \tau''). \tag{5}$$

For convenience, we denote the above operation of linguistic approximation as $\bigodot$, namely,

$$\tau = \bigodot(\tau''). \tag{6}$$

2.4. *Truth-Qualification*

If τ is a linguistic truth term of $LTTS$, then the statement

"it is τ that X is A"

or

$$\text{“}(X \text{ is } A) \text{ is } \tau\text{”}$$

is interpreted as

$$\text{“}X \text{ is } \tau \circ A\text{”} \tag{7}$$

where

$$\mu_{\tau \circ A}(x) = \mu_\tau(\mu_A(x)), \forall x \in [0,1]. \tag{8}$$

Here τ is a truth-qualification of the proposition “X is A” [11]. For example, let $\tau =$“*true*”, then “(X is A) is *true*” is defined by

$$\text{“}X \text{ is } \tau \circ A\text{”}=\text{“}X \text{ is } A\text{”},$$

because

$$\mu_{\tau \circ A}(x) = \mu_{true}(\mu_A(x)) = \mu_A(x), \tag{9}$$

for each x $\in [0,1]$. Namely, the fuzzy proposition

$$\text{“}(X \text{ is } A) \text{ is } true\text{”}$$

can be reduced into the fuzzy proposition

$$\text{“}X \text{ is } A\text{”}.$$

We will see that (9) plays a critical role in interactions among different reasoning systems.

If we know two fuzzy propositions “X is A^*” and “(X is A) is τ”, how can we obtain the truth-qualification, τ, of the proposition “X is A” such that these two propositions are equivalent? That is, to find $\tau \in LTTS$ such that

$$\text{“}X \text{ is } A^*\text{”} = \text{“}(X \text{ is } A) \text{ is } \tau\text{”}.$$

For this problem, Godo et al. [22] gave the following formula:

$$\mu_\tau(z) = \begin{cases} \sup\{\mu_{A^*}(u) \mid u \in U, \mu_A(u) = z\} & \text{if } \mu^{-1}(z) \neq \emptyset, \\ 0 & \text{otherwise,} \end{cases} \tag{10}$$

where U is the universe of discourse of fuzzy sets A and A^*. Notice that if $\mu_A(u)$ is a one-to-one mapping, then the above expression turns into $\mu_\tau(z) = \mu_{A^*}(\mu_A^{-1}(z))$, *i.e.*

$$\mu_\tau = \mu_{A^*} \circ \mu_A^{-1}. \tag{11}$$

For convenience, we still use the above formula to stand for (10) (except in cases where this would cause confusion). Clearly, this operation can be regarded as the inverse operation of (8).

2.5. *Reasoning with Fuzzy Truth-Qualification*

The Generalised Modus Ponens inference rule with respect to truth-qualification is given by Mantaras and Godo [60] as follows:

$$\frac{\begin{array}{l}(\text{If } X \text{ is } A \text{ then } Y \text{ is } B) \text{ is } \eta \\ (X \text{ is } A) \text{ is } \tau\end{array}}{(Y\ is\ B)\ is\ \upsilon} \tag{12}$$

where, under some very general continuity conditions, the truth-qualification υ for the consequence of the rule is given by the following formula [21]:

$$\mu_\upsilon(z) = \sup\{m_I(\mu_\tau(x), \mu_\eta(I(x,z))) \mid x \in [0,1]\}. \tag{13}$$

Here the function m_I is defined, by Valverde and Trillas [106], as:

$$m_I(x,y) = \inf\{z \in [0,1] \mid I(x,z) \geq y\}, \forall x, y \in [0,1]\}. \tag{14}$$

Here inf denotes the infimum operation on a set. Examples of pairs (I, m_I) are as shown in Table 2. For convenience, the operator given by the above Generalised

Table 2. Examples of pairs (I, m_I)

$I(x,y) = \max\{1-x, y\}$	$m_I(x,y) = \begin{cases} 0 & \text{if } y \leq 1-x \\ y & \text{otherwise} \end{cases}$
$I(x,y) = 1 - x + xy$	$m_I(x,y) = \begin{cases} 0 & \text{if } x = 0 \\ \frac{\max\{x+y-1,0\}}{x} & \text{otherwise} \end{cases}$
$I(x,y) = \min\{1, 1-x+xy\}$	$m_I(x,y) = \max\{0, x+y-1\}$
$I(x,y) = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{otherwise} \end{cases}$	$m_I(x,y) = \min\{x,y\}$
$I(x,y) = \begin{cases} 1 & \text{if } x \leq y \\ \frac{y}{x} & \text{otherwise} \end{cases}$	$m_I(x,y) = xy$

Modus Ponens inference rule (12) is denoted as $**$. That is,

$$\upsilon = **(\tau, \eta). \tag{15}$$

And we call $**$ the sequential propagation operation.

3. Representing of Knowledge and Its Uncertainty

Our framework consists of various rules and the corresponding mechanisms for propagating uncertainties along the inference network. In this section, we give the representations of the various rules and their uncertainties. The propagation mechanisms will be discussed in the next two sections. Notice that throughout this paper all propositions are fuzzy, and sometimes, for the sake of simplicity, we also denote a fuzzy proposition "X is A" as "A".

The uncertainty measure, um, of a proposition is of the following form:

$$um(E, S) = \tau \tag{16}$$

where $\tau \in LTTS$. This means that under the observation S, (X is E) is τ.

Within our framework, various rules (along with the corresponding rule uncertainty measure RUM) are classified into two types: the ordinary type (discussed in Subsection 3.1) and the default type (discussed in Subsection 3.2).

3.1. *The Ordinary Type*

This type of rule is of the form

$$E \to H.$$

Two kinds of measures for uncertainty of this type of rule can be identified:

3.1.1. *Classical Subtype*

$$RUM(E \to H) = \tau_{E \to H} \tag{17}$$

where $\tau_{E \to H} \in LTTS$. That is,

$$(\textsf{If } X \textsf{ is } E \textsf{ then } Y \textsf{ is } H) \textsf{ is } \tau_{E \to H}.$$

For example:

It is very-true that if *price is high* then *profit is good.*

If $\tau_{E \to H} =$"*true*", by (9) the form of rule degenerates into the following form:

$$\textsf{If } X \textsf{ is } E \textsf{ then } Y \textsf{ is } H.$$

This is a typical fuzzy inference rule. So, a knowledge base merely containing such fuzzy rules can be a subsystem of our model.

3.1.2. *Fuzzy Subtype*

$$RUM(E \to H) = \tau_E \xrightarrow{\tau_{RS}} \tau_H \tag{18}$$

where $\tau_E, \tau_H, \tau_{RS} \in LTTS$. That is,

$$(\textsf{If } (X \textsf{ is } E) \textsf{ is } \tau_E \textsf{ then } (Y \textsf{ is } H) \textsf{ is } \tau_H) \textsf{ is } \tau_{RS}.$$

For example:

It is very-true that if *it is very-true that price is high*
then *it is fairly-true that profit is good.*

This is a truth-qualification fuzzy rule mentioned in [60]. So, a knowledge base simply containing such fuzzy rules can also be a subsystem of our model. In other words, the approach developed in [60] is actually a subsystem of ours. In addition, if $\tau_E = \tau_H =$"*true*", by (9) the type of rule is reduced to the classic subtype of the rule together with its uncertainty measure.

3.2. *The Default Type*

This subsection couples various default rules with linguistically expressed uncertainty (we call these uncertain default rules). Within our framework, the intuitive interpretation of an uncertain default rule is: when to a degree the uncertain justification of this rule is consistent with the available information, if the uncertain prerequisite of this rule is matchable with the available information, then to a degree of belief the conclusion is inferred. As a result, our system can include Reiter's default system, in a fuzzy situation, as its special case. Naturally, if a knowledge base contains only such uncertain default rules, it forms an uncertain default reasoning model with respect to a fuzzy proposition, in which reasoning can be performed with ambiguous, uncertain and incomplete information.

In this paper, we only discuss three basic types of uncertain default rules: no-prerequisite subtype, normal subtype and semi-normal subtype.

3.2.1. *No-Prerequisite Subtype*

This type of rule is of the form

$$:H \to H$$

and its uncertainty measure is in the form of

$$RUM(:H \to H) = (\tau_{:H\to H}, \tau_{RS}, \varepsilon), \tag{19}$$

where $\tau_{:H\to H}, \tau_{RS} \in LTTS$ and $\varepsilon \in (0,1]$. That is,

$$:((Y \text{ is } H) \text{ is } \tau_{:H\to H})_\varepsilon \xrightarrow{\tau_{RS}} ((Y \text{ is } H) \text{ is } \tau_{:H\to H}).$$

Its intuitive meaning is that if to the extent ε the available information is consistent with

$$\text{“}(Y \text{ is } H) \text{ is } \tau_{:H\to H}\text{”},$$

then the following statement is τ_{RS}

$$\text{“}(Y \text{ is } H) \text{ is } \tau_{:H\to H}\text{”}.$$

For example:

> If *to the extent* 0.2 *the available information is consistent with*
> *that temperature is high is fairly-true*
> then *it is very-true that temperature is high is fairly-true*.

Clearly, when $\varepsilon = 1$, by (9), if $\tau_{RS} = \tau_{:H\to H} =$ "*true*", we have the following special case of this type of default rule:

$$:(Y \text{ is } H) \to (Y \text{ is } H).$$

The above rule is the no-prerequisite type of default rule in Reiter's system in fuzzy situation. In addition, notice that because $\varepsilon = 1$, for the virtue of simplicity, we omit ε in the above rule (hereafter, we always do this).

In Section 5, we discuss the measure of the consistency between the available information and the justification. We will see that when there is an absence of negative information and the available information is the same as the justification, the measure takes 1, which means there is complete consistency. Obviously, Reiter's default reasoning system considers these two cases only. In contrast, in our model, in addition to these two cases, we also considered the case where to an extent the available information is consistent with the justification. This is one of the important ways in which our model can be seen as an extension of Reiter's system.

3.2.2. *Normal Subtype*

This type of rule is of the form

$$E : E_\beta \to H$$

and its uncertainty measure is in the form of

$$RUM(E : E_\beta \to H) = (\tau_E, \tau_{E_\beta}, \tau_{RS}, \tau_H, \varepsilon), \tag{20}$$

where $\tau_E, \tau_{E_\beta}, \tau_H, \tau_{RS} \in LTTS$, $\varepsilon \in (0, 1]$. That is,

$$((X \text{ is } E) \text{ is } \tau_E) : ((Y \text{ is } E_\beta) \text{ is } \tau_{E_\beta})_\varepsilon \xrightarrow{\tau_{RS}} ((Z \text{ is } H) \text{ is } \tau_H).$$

Its intuitive meaning is that if to the extent ε the available information is consistent with

$$\text{"}(Y \text{ is } E_\beta) \text{ is } \tau_\beta\text{"},$$

then the following statement is τ_{RS}:

$$\text{"if } (X \text{ is } E) \text{ is } \tau_E\text{" then "}(Z \text{ is } H) \text{ is } \tau_H\text{"}.$$

For example:

If *to the extent 0.3 the available information is consistent with that the old man has experience is very-true*
then *it is very-true that* if *John is old* then *it is very-false that he is fool.*

Table 3. Some special cases of uncertain normal default

Condition	Special Case
$\tau_{RS} =$"*true*"	$((X \text{ is } E) \text{ is } \tau_E) : ((Y \text{ is } E_\beta) \text{ is } \tau_{E_\beta}) \to ((Z \text{ is } H) \text{ is } \tau_H)$
$\tau_E =$"*true*"	$((X \text{ is } E) : ((Y \text{ is } E_\beta) \text{ is } \tau_{E_\beta}) \to ((Z \text{ is } H) \text{ is } \tau_H)) \text{ is } \tau_{RS}$
$\tau_E = \tau_{E_\beta} =$"*true*"	$((X \text{ is } E) : (Y \text{ is } E_\beta) \to ((Z \text{ is } H) \text{ is } \tau_H)) \text{ is } \tau_{RS}$
$\tau_E = \tau_H =$"*true*"	$((X \text{ is } E) : ((Y \text{ is } E_\beta) \text{ is } \tau_{E_\beta}) \to (Z \text{ is } H)) \text{ is } \tau_{RS}$
$\tau_E = \tau_{E_\beta} = \tau_H =$"*true*"	$((X \text{ is } E) : (Y \text{ is } E_\beta) \to (Z \text{ is } H)) \text{ is } \tau_{RS}$
$\tau_E = \tau_{E_\beta} = \tau_H = \tau_{RS} =$"*true*"	$(X \text{ is } E) : (Y \text{ is } E_\beta) \to (Z \text{ is } H)$

Clearly, if $\varepsilon = 1$, by (9) we have the special cases of this subtype of rule, as shown in Table 3. The first case in this table is a truth-qualification subtype of a fuzzy default rule without correlation strength. We have focused on the above form of the rule in [97]. The last case in this table is the normal type of default rule in Reiter's system in a fuzzy situation.

3.2.3. *Semi-Normal Subtype.*

This type of rule is of the form

$$E : E_\beta \wedge H \rightarrow H$$

and its uncertainty measure is in the form of

$$RUM(E : E_\beta \wedge H \rightarrow H) = (\tau_E, \tau_{E_\beta}, \tau_H, \tau_{RS}, \varepsilon), \tag{21}$$

where $\tau_E, \tau_{E_\beta}, \tau_H, \tau_{RS} \in LTTS$, $\varepsilon \in (0, 1]$. That is,

$$((X \text{ is } E) \text{ is } \tau_E) : (((Y \text{ is } E_\beta) \text{ is } \tau_{E_\beta}) \wedge ((Z \text{ is } H) \text{ is } \tau_H))_\varepsilon \xrightarrow{\tau_{RS}} ((Z \text{ is } H) \text{ is } \tau_H).$$

The intuitive meaning is that if to the extent ε the available information is consistent with

$$\text{“}(Y \text{ is } E_\beta) \text{ is } \tau_\beta\text{” and “}(Z \text{ is } H) \text{ is } \tau_H\text{”},$$

then the following statement is τ_{RS}:

$$\text{“if } (X \text{ is } E) \text{ is } \tau_E \text{ then } (Z \text{ is } H) \text{ is } \tau_H\text{”}.$$

For example:

If *to the extent* 0.5 *the available information is consistent with*
that an old man has experience is fairly-true
and it can be possible that he is a fool is very-false
then *it is fairly-true that* if *it is very-true that John is old*
then *it is very-false that he is a fool.*

Clearly, if $\varepsilon = 1$, by (9) we have the special cases of this type of default rule, as shown in Table 4. The first case in this table is a truth-qualification subtype of a fuzzy default rule without correlation strength. We have focused on the above form of rule in [97]. The last case in this table is the semi-normal type of default rule in Reiter's system in a fuzzy situation.

Table 4. Some special cases of uncertain semi-normal default

When $\tau_{RS} =$“*true*”, $((X \text{ is } E) \text{ is } \tau_E) : ((Y \text{ is } E_\beta) \text{ is } \tau_{E_\beta}) \wedge ((Z \text{ is } H) \text{ is } \tau_H) \rightarrow ((Z \text{ is } H) \text{ is } \tau_H)$.
When $\tau_E =$“*true*”, $((X \text{ is } E) : ((Y \text{ is } E_\beta) \text{ is } \tau_{E_\beta}) \wedge ((Z \text{ is } H) \text{ is } \tau_H) \rightarrow ((Z \text{ is } H) \text{ is } \tau_H))$ is τ_{RS}.
When $\tau_E = \tau_{E_\beta} =$“*true*”, $((X \text{ is } E) : (Y \text{ is } E_\beta) \wedge ((Z \text{ is } H) \text{ is } \tau_H) \rightarrow ((Z \text{ is } H) \text{ is } \tau_H))$ is τ_{RS}.
When $\tau_E = \tau_H =$“*true*”, $((X \text{ is } E) : ((Y \text{ is } E_\beta) \text{ is } \tau_{E_\beta}) \wedge (Z \text{ is } H) \rightarrow (Z \text{ is } H))$ is τ_{RS}.
When $\tau_E = \tau_{E_\beta} = \tau_H =$“*true*”, $((X \text{ is } E) : (Y \text{ is } E_\beta) \wedge (Z \text{ is } H) \rightarrow (Z \text{ is } H))$ is τ_{RS}.
When $\tau_E = \tau_{E_\beta} = \tau_H = \tau_{RS} =$“*true*”, $(X \text{ is } E) : (Y \text{ is } E_\beta) \wedge (Z \text{ is } H) \rightarrow (Z \text{ is } H)$.

We refer to the above three types of default rules, quantified with a degree τ_{RS} of correlation, as *uncertain* default rules because some parts of their prerequisites and justifications may be omitted by human experts. Here the degree, τ_{RS}, of correlation actually represents the extent to which the prerequisite and justification of a default rule approximately support its conclusion. Also, τ_{RS} can be regarded as the representation of confidence from human experts when they are specifying an uncertain default rule.

In summary, the following items for default rules are considered in our framework:

- the uncertainty of its prerequisite, justification and its conclusion, *i.e.* τ_E, τ_{E_β} and τ_H;
- the uncertainty of the correlations among its prerequisite, justification and its conclusion, *i.e.* τ_{RS}; and
- the basic uncertainty, estimated by the means of linguistic terms, *i.e.* all τ_E, τ_{E_β}, τ_H and τ_{RS} are linguistic truth values.

The common characteristic of the various type rules presented in this paper is that their general representation can be specialised into the corresponding rules without uncertainty. Concretely, fuzzy rules, truth-qualification rules and fuzzy default rules can be viewed as the special cases of the corresponding rules in our framework. Furthermore, if a knowledge base does not contain all types of rules, it can form some special models, such as the fuzzy approximate reasoning model [120], the truth-qualification uncertain reasoning model [60], the fuzzy default reasoning model, and the truth-qualification uncertain default reasoning model.

4. Propagation Using Ordinary Uncertain Rules

Based on the fuzzy mathematical theory reviewed in Section 2, this section will develop the formulae for the combination and aggregation of uncertainties when using ordinary uncertain rules to perform reasonings.

4.1. *Boolean Combination*

For the ordinary type of rule $E \to H$, if E is a Boolean expression, we can obtain the estimate for the uncertainty of this proposition from estimates for the uncertainties of its sub-propositions. Let $um(E_1, S) = \tau_{E_1} \in LTTS$, $um(E_2, S) = \tau_{E_2} \in LTTS$ and $um(E, S) = \tau_E \in LTTS$, where $LTTS$ is the linguistic truth term set given by (1). Then by the extension principle $\bigotimes$ and the linguistic approximation technique $\bigodot$, we have the formulae for Boolean combination, as shown in Table 5. Here,

Table 5. Boolean Combination

	formula	meaning
AND	$um(E_1 \wedge E_2, S) = \bigodot(\bigotimes(\tau_{E_1}, \tau_{E_2}, \triangle))$	(X is E_1 and Y is E_2) is $\bigodot(\bigotimes(\tau_{E_1}, \tau_{E_2}, \triangle))$
OR	$um(E_1 \vee E_2, S) = \bigodot(\bigotimes(\tau_{E_1}, \tau_{E_2}, \triangledown))$	(X is E_1 or Y is E_2) is $\bigodot(\bigotimes(\tau_{E_1}, \tau_{E_2}, \triangledown))$
Not	$um(\neg E, S) = \tau_{\neg E}$, $\mu_{\tau_{\neg E}}(x) = \mu_{\tau_E}(1-x), \forall x \in [0,1]$	(X is not E) is $\tau_{\neg E}$

according to [6], $\triangle$ denotes the operator given by one of $\triangle_1$, $\triangle_2$ and $\triangle_3$ defined as follows

$$\triangle_1(x,y) = \max\{0, x+y-1\},\ \triangle_2(x,y) = x \times y,\ \triangle_3(x,y) = \min\{x,y\}.$$

and $\bigtriangledown$ denotes the operator given by one of $\bigtriangledown_1$, $\bigtriangledown_2$ and $\bigtriangledown_3$ defined as follows

$$\bigtriangledown_1(x,y) = \min\{1, x+y\},\ \bigtriangledown_2(x,y) = x+y-x \times y,\ \bigtriangledown_3(x,y) = \max\{x,y\}.$$

For convenience, we still denote the operation of *AND* as $\triangle$, that is,

$$um(E_1 \wedge E_2, S) = \tau_{E_1} \triangle \tau_{E_2}; \tag{22}$$

denote the operation of *OR* as $\bigtriangledown$, that is,

$$um(E_1 \vee E_2, S) = \tau_{E_1} \bigtriangledown \tau_{E_2}; \tag{23}$$

and denote the operation of negation, as $\neg$, that is,

$$um(\neg E, S) = \neg \tau_E. \tag{24}$$

4.2. *Parallel Combination*

For a hypothesis H, if we have two assessments for uncertainty $\tau_1 \in LTTS$ and $\tau_2 \in LTTS$ from different rules, then by the extension principle $\bigotimes$, and the linguistic approximation technique $\bigodot$, its combined um is:

$$um(H,S) = \bigodot(\bigotimes(\tau_1, \tau_2, ++)) \tag{25}$$

where ++ denotes the operator given by one of $\bigtriangledown_1$, $\bigtriangledown_2$ and $\bigtriangledown_3$ according to [6].

Intuitively, if from different rules we get two beliefs which negate each other completely, no decision can be made. That is, if $\tau_1 = \neg\tau_2$, the result of their parallel combination should be "*undecided*". Furthermore, if from a source we get no new idea on the belief of a hypothesis, the current belief on this hypothesis should not be changed. Namely, if one of τ_1 and τ_2 is "*undecided*", the result of their parallel combination should be the other one. In order to remedy this, (25) should be modified into

$$um(H,S) = \begin{cases} \text{“}undecided\text{”} & \text{if } \tau_1 = \neg\tau_2, \\ \tau_1 & \text{if } \tau_2 = \text{“}undecided\text{”}, \\ \tau_2 & \text{if } \tau_1 = \text{“}undecided\text{”}, \\ \bigodot(\bigotimes(\tau_1, \tau_2, ++)) & \text{otherwise.} \end{cases} \tag{26}$$

For convenience, we still denote the operator given by (26) with ++, that is,

$$um(H,S) = ++(\tau_1, \tau_2). \tag{27}$$

4.3. *Sequential Propagation*

The basic procedure along a fuzzy rule is as follows:

(i) When its fuzzy premise is not the same as the fuzzy evidence known, we use the operation, related to truth-qualification, to make them the same. We thereby get the uncertainty estimate of the fuzzy premise of the rule.

(ii) Then, we use the sequential propagation operator $**$ to propagate the corresponding uncertainties.

(iii) Finally, if the result of the operation is not closed in the linguistic truth term set, the linguistic approximation technique $\bigodot$ needs to be used.

In the reminder of this subsection, we will discuss this process in more detail.

4.3.1. *Classical Subtype*

For the rule $E \to H$, suppose $um(E', S) = \tau_{E'|S} \in LTTS$ and $RUM(E \to H) = \tau_{E\to H} \in LTTS$. We will find $um(H, S)$. Namely, we will perform the following reasoning:

$$\frac{\begin{array}{l}(\text{If } X \text{ is } E \text{ then } Y \text{ is } H) \text{ is } \tau_{E\to H}\\ (X \text{ is } E') \text{ is } \tau_{E'|S}\end{array}}{(Y \text{ is } H) \text{ is } \tau_{H|S}}$$

Notice that in the above inference pattern, the evidence "X is E'" is different from the premise "X is E" of the rule, but in (12) they are the same. Hence, if we want to use the sequential propagation operator $**$ given by (12), **firstly**, we have to make them the same. That is, we need to transform the statement "(X is E') is $\tau_{E'|S}$" into the equivalent statement "(X is E) is $\tau_{E|S}$". So, we need to find the value $\tau_{E|S}$ of $um(E, S)$ from the value $\tau_{E'|S}$ of $um(E', S)$ such that

$$\text{“}(X \text{ is } E) \text{ is } \tau_{E|S}\text{”} = \text{“}(X \text{ is } E') \text{ is } \tau_{E'|S}\text{”}.$$

Thus,

$$\begin{aligned} & \text{“}(X \text{ is } E) \text{ is } \tau_{E|S}\text{”} = \text{“}(X \text{ is } E') \text{ is } \tau_{E'|S}\text{”} \\ \Rightarrow\ & \tau_{E|S} \circ E = \tau_{E'|S} \circ E' \\ \Rightarrow\ & \tau_{E|S} = \tau_{E'|S} \circ E' \circ E^{-1}. \end{aligned}$$

Notice that here we also use a fuzzy set to represent its membership function. In other words, if A is a fuzzy set, then A denotes this fuzzy set itself and also denotes its membership function. Hereafter, we always proceed in this way.

Then we perform a sequential propagation. By using the sequential propagation operator $**$ given by (12), and the linguistic approximation technique $\bigodot$, we have

$$um(H, S) = \bigodot(**\,(\tau_{E|S}, \tau_{E\to H})) = \bigodot(**\,(\tau_{E'|S} \circ E' \circ E^{-1}, \tau_{E\to H})). \tag{28}$$

Clearly, if we have no idea about the premise of the rule, that is,

$$\tau_{E|S} = \bigodot(\tau_{E'|S} \circ E' \circ E^{-1}) = \text{“}\textit{undecided}\text{”},$$

the rule should not go into effect, and so the result drawn from this rule should be "*undecided*". In order to remedy this, the formula (28) should be modified into

$$um(H,S)=\begin{cases} \text{“}\mathit{undecided}\text{”} & \text{if } \bigodot(\tau_{E'|S}\circ E'\circ E^{-1})=\text{“}\mathit{undecided}\text{”},\\ \bigodot(**\,(\tau_{E'|S}\circ E'\circ E^{-1},\tau_{E\to H})) & \text{otherwise.}\end{cases} \tag{29}$$

For convenience, we still denote the operator given by (29) with $**$, that is,

$$um(H,S) = **\,(\tau_{E'|S},\tau_{E\to H}). \tag{30}$$

4.3.2. *Fuzzy Subtype*

For the rule $E \to H$, suppose $um(E,S) = \tau_{E|S} \in LTTS$ and (18). We will find the value of $um(H,S)$. Namely, we will perform the following reasoning:

$$\begin{array}{l} (\text{If } (X \text{ is } E) \text{ is } \tau_E \text{ then } (Y \text{ is } H) \text{ is } \tau_H) \text{ is } \tau_{RS}\\ (X \text{ is } E') \text{ is } \tau_{E'|S}\\ \hline (Y \text{ is } H) \text{ is } \tau_{H|S}\end{array}$$

or

$$\begin{array}{l} (\text{If } X \text{ is } \tau_E\circ E \text{ then } Y \text{ is } \tau_H\circ H) \text{ is } \tau_{RS}\\ (X \text{ is } E') \text{ is } \tau_{E'|S}\\ \hline (Y \text{ is } H) \text{ is } \tau_{H|S}\end{array}$$

Comparing the above pattern with (12), we can see that in the above the evidence "X is E'" is different from the premise "X is $\tau_E \circ E$" of the rule, but in (12) they are the same. Hence, if we want to use the sequential propagation operator $**$ given by (12), **firstly** we have to make them the same. That is, we have to transform the statement "$(X$ is $E')$ is $\tau_{E'|S}$" into the equivalent statement "$(X$ is $\tau_E \circ E)$ is $\tau_{E|S}$". So, we need to find the value $\tau_{E|S}$ of $um(\tau \circ E, S)$ such that

$$\text{“}(X \text{ is } \tau_E\circ E) \text{ is } \tau_{E|S}\text{”} = \text{“}(X \text{ is } E') \text{ is } \tau_{E'|S}\text{”}.$$

Thus,

$$\begin{array}{rl} & \text{“}(X \text{ is } \tau_E\circ E) \text{ is } \tau_{E|S}\text{”} = \text{“}(X \text{ is } E') \text{ is } \tau_{E'|S}\text{”}\\ \Rightarrow & \tau_{E|S}\circ\tau_E\circ E = \tau_{E'|S}\circ E'\\ \Rightarrow & \tau_{E|S} = \tau_{E'|S}\circ E'\circ E^{-1}\circ\tau_E^{-1}.\end{array}$$

Then, noticing that τ_{RS} is the strength of the following rule:

$$\text{“if } X \text{ is } \tau_E\circ E \text{ then } Y \text{ is } \tau_H\circ H\text{”},$$

by using the sequential propagation operator $**$ given by (12), we have

$$um(\tau\circ H,S) = **\,(\tau_{E|S},\tau_{RS}) = **\,(\tau_{E'|S}\circ E'\circ E^{-1}\circ\tau_E^{-1},\tau_{RS}),$$

that is,

$$((Y \text{ is } H) \text{ is } \tau_H) \text{ is } **(\tau_{E'|S} \circ E' \circ E^{-1} \circ \tau_E^{-1}, \tau_{RS}),$$

thus,

$$(Y \text{ is } H) \text{ is } **(\tau_{E'|S} \circ E' \circ E^{-1} \circ \tau_E^{-1}, \tau_{RS}) \circ \tau_H.$$

Finally, by using the linguistic approximation technique $\bigodot$, we have

$$um(H,S) = \bigodot(**\,(\tau_{E'|S} \circ E' \circ E^{-1} \circ \tau_E^{-1}, \tau_{RS}) \circ \tau_H). \tag{31}$$

For convenience, we denote the operator given by (31) with $\star\star$, that is,

$$um(H,S) = \star\star\,(\tau_{E|S}, \tau_E \xrightarrow{\tau_{RS}} \tau_H). \tag{32}$$

Theorem 1 *If $\tau_E = \tau_H =$ "true", then*

$$\star\star\,(\tau_{E|S}, \tau_E \xrightarrow{\tau_{RS}} \tau_H) = **\,(\tau_{E'|S}, \tau_{RS}). \tag{33}$$

Proof. If $\tau_E = \tau_H =$ "*true*", then,

$$\mu_{\tau_E}^{-1}(x) = \mu_{\tau_H}(x) = x,$$

thus, for any map $f : [0,1] \to [0,1]$, we have

$$f \circ \mu_{\tau_E}^{-1}(x) = f \circ \mu_{\tau_H}(x) = f(x).$$

Hence, from (31) we can get

$$\begin{aligned} um(H,S) &= \bigodot(**\,(\tau_{E'|S} \circ E' \circ E^{-1} \circ \tau_E^{-1}, \tau_{RS}) \circ \tau_H) \\ &= \bigodot(**\,(\tau_{E'|S} \circ E' \circ E^{-1}, \tau_{RS})) \\ &= **\,(\tau_{E'|S} \circ E' \circ E^{-1}, \tau_{RS})). \end{aligned}$$

By (30) and (32), we can express the above result as (33). □

This theorem reveals that, when the form of the fuzzy-type rule with uncertainty degenerates into the classic-type rule with uncertainty, its propagation mechanism also degenerates into that of the classic-type rule.

5. Propagation Using Uncertain Default Rules

Based on the fuzzy mathematical theory introduced in Section 2, this section develops the formulae for the combination and aggregation of uncertainties when uncertain default rules are used to perform reasoning. The development of propagation formulae is based on the following basic procedure:

- Firstly, if the fuzzy proposition known is not the same as the fuzzy justification of an uncertain default rule, the corresponding operation related to truth-qualification is used to make them the same, and thus the uncertainty estimate for the fuzzy justification is obtained from the available information.

- Secondly, determine the belief in the conclusion of rules according to the consistency between the available information and the rule's justification:
  - If the justification is completely consistent with the available information, then:
    - for a no-prerequisite subtype rule, the current belief in the hypothesis does not change,
    - for a normal rule or a semi-normal rule, it is reduced into an ordinary uncertain rule.
  - If the consistency degree between the justification and the available information is less than a threshold, since in real life this is usually regarded as a contrary case, this rule should not go into effect.
  - Otherwise, this consistency degree is used to reduce the belief in the drawn conclusion.
- Finally, if the results of the above operations are not closed in the linguistic truth term set $LTTS$, the linguistic approximation technique $\odot$ is employed to select a linguistic truth from term set $LTTS$.

In what follows, we discuss some essential properties. In particular, for various uncertain default rules, also we give properties associated with the following three cases:

- there is no information contradictory to the justification,
- the available information is the same as the justification, and
- some uncertainty estimates take the special linguistic truth "*true*".

From the discussion in the third case above, we actually obtain a fuzzy default reasoning system.

5.1. *Consistency Degree*

Above, we mentioned the consistency degree between the available information and the justification of a default rule. Here let us discuss this conception. Let

$$TS = \{absolutely\text{-}true, very\text{-}true, true, fairly\text{-}true\}, \tag{34}$$
$$FS = \{fairly\text{-}false, false, very\text{-}false, absolutely\text{-}false\}, \tag{35}$$
$$US = \{undecided\}. \tag{36}$$

Formally, we define the conception of a consistency degree between two linguistic truth-values as follows:

Definition 1 *The function $\rho : LTTS \times LTTS \to [0,1]$ is called a consistency degree on $LTTS$, if*

(i) $\forall \tau \in LTTS, \rho(\tau, \tau) = 1$;

(*ii*) $\forall\tau_1, \tau_2 \in LTTS, \rho(\tau_1, \tau_2) = \rho(\tau_2, \tau_1)$;
(*iii*) $\forall\tau_1, \tau_2, \tau_3 \in LTTS$, *if* $\tau_1 \subseteq \tau_2 \subseteq \tau_3$, *then* $\rho(\tau_1, \tau_3) \leq \rho(\tau_2, \tau_3)$;
(*iv*) $\rho(\tau, \text{“}\textit{undecided}\text{”}) = 1, \forall\tau \in LTTS$;
(*v*) $\rho(\tau_1, \tau_2) = 0, \forall\tau_1 \in TS, \tau_2 \in FS$.

In the above definition, items (i), (ii) and (iii) are easily understood. We will therefore explain only items (iv) and (v). The idea behind item (iv) is that the linguistic truth value "*undecided*" is completely consistent with any linguistic truth value. Actually, it ensures that in the absence of negative information, default reasoning can be carried out. Item (v) means that any two linguistic truth values which are contrary are absolutely inconsistent. This agrees with our intuition.

Concretely, we have:

Lemma 1 *The map* $\rho : LTTS \times LTTS \to [0, 1]$, *given by*

$$\rho(\tau_1, \tau_2) = 1 - \sup\{|\mu_{\tau_1}(x) - \mu_{\tau_2}(x)| \mid x \in [0, 1]\} \times (\mu_{\tau_1}(0) + \mu_{\tau_1}(1))(\mu_{\tau_2}(0) + \mu_{\tau_2}(1)), \tag{37}$$

is a consistency degree on $LTTS$.

Proof. We will check that function (37) satisfies the definition of a consistency degree on $LTTS$.

1) Clearly, items (i) and (ii) of Definition 1 are satisfied.

2) Clearly, we have

$$\mu_{undecided}(0) + \mu_{undecided}(1) = 0 + 0 = 0,$$

thus, $\forall\tau \in LTTS$

$$\begin{aligned} & \rho(\tau, \text{“}\textit{undecided}\text{”}) \\ = \;& 1 - \sup\{|\mu_\tau(x) - \mu_{undecided}(x)| \mid x \in [0, 1]\} \times (\mu_\tau(0) + \mu_\tau(1)) \\ & \quad \times(\mu_{undecided}(0) + \mu_{undecided}(1)) \\ = \;& 1 - \sup\{|\mu_\tau(x) - \mu_{undecided}(x)| \mid x \in [0, 1]\} \times (\mu_\tau(0) + \mu_\tau(1)) \times 0 \\ = \;& 1 - 0 \\ = \;& 1. \end{aligned}$$

So, item (iv) of Definition 1 is satisfied.

3) If $\tau_1 \in TS$ and $\tau_2 \in FS$, then from Figure 3 we can see

$$\begin{aligned} & \sup\{|\mu_{\tau_1}(x) - \mu_{\tau_2}(x)| \mid x \in [0, 1]\} = 1, \\ & \mu_{\tau_1}(0) + \mu_{\tau_2}(1)) = 0 + 1 = 1, \\ & \mu_{\tau_1}(0) + \mu_{\tau_2}(1)) = 0 + 1 = 1. \end{aligned}$$

Thus,

$$\begin{aligned} & \rho(\tau_1, \tau_2) \\ = \;& 1 - \sup\{|\mu_{\tau_1}(x) - \mu_{\tau_2}(x)| \mid x \in [0, 1]\} \times (\mu_{\tau_1}(0) + \mu_{\tau_1}(1)) \times (\mu_{\tau_2}(0) + \mu_{\tau_2}(1)) \\ = \;& 1 - 1 \times 1 \times 1 \\ = \;& 0. \end{aligned}$$

So, item (v) of Definition 1 is satisfied.

4) The relationship

$$\tau_1 \subseteq \tau_2 \subseteq \tau_3 \tag{38}$$

can hold only under one of the following situations:

(i) $\tau_1, \tau_2, \tau_3 \in TS$,
(ii) $\tau_1, \tau_2, \tau_3 \in FS$,
(iii) $\tau_1, \tau_2 \in TS, \tau_3 =$ "*undecided*", and
(iv) $\tau_1, \tau_2 \in FS, \tau_3 =$ "*undecided*".

In the first and second cases, from (38) we can derive

$$\sup\{|\mu_{\tau_1}(x) - \mu_{\tau_3}(x)| \mid x \in [0,1]\} \geq \sup\{|\mu_{\tau_2}(x) - \mu_{\tau_3}(x)| \mid x \in [0,1]\},$$

thus,

$$\rho(\tau_1, \tau_3) \leq \rho(\tau_2, \tau_3).$$

So, item (iii) of Definition 1 is satisfied in the first and second cases. In the third and fourth cases, by 2) we have

$$\rho(\tau_1, \tau_3) = \rho(\tau_2, \tau_3) = 1.$$

So, item (iii) of Definition 1 is also satisfied in the third and fourth cases. □

After understanding the conception of the consistency degree, we are ready to discuss the propagation mechanism for various kinds of uncertain default rules.

5.2. *No-Prerequisite Subtype*

For the rule

$$: H \to H,$$

suppose $um(H', S) = \tau_{H'|S} \in LTTS$, and (19). We will find the value of $um(H, S)$.

1) We will transform the fuzzy proposition "(Z is H') is $\tau_{H'|S}$" into the equivalent fuzzy proposition "(Z is H) is $\tau_{H|S}$". That is, to find the value $\tau_{H|S}$ of $um(H, S)$ from the value $\tau_{H'|S}$ of $um(H', S)$ such that "(Z is H) is $\tau_{H|S}$" = "(Z is H) is $\tau_{H'|S}$". Thus,

$$\begin{aligned} & \text{“}(Z \text{ is } H) \text{ is } \tau_{H|S}\text{”} = \text{“}(Z \text{ is } H) \text{ is } \tau_{H'|S}\text{”} \\ \Rightarrow \quad & \tau_{H|S} \circ H = \tau_{H'|S} \circ H' \\ \Rightarrow \quad & \tau_{H|S} = \tau_{H'|S} \circ H' \circ H^{-1}. \end{aligned}$$

And by the linguistic approximation technique $\bigodot$, we get the current belief in H as follows:

$$um(H, S) = \bigodot(\tau_{H'|S} \circ H' \circ H^{-1}). \tag{39}$$

In other words, the available information is

$$\text{“}Z \text{ is } \bigodot(\tau_{H'|S} \circ H' \circ H^{-1})\text{”}.$$

2) By (8), the statement

> "if to an extent ε the available information is consistent with
> that (Z is H) is $\tau_{:H\to H}$
> then it is τ_{RS} that (Z is H) is $\tau_{:H\to H}$"

is equivalent to the statement

> "if to an extent ε, the available information is consistent with
> that (Z is H) is $\tau_{:H\to H}$
> then (Z is H) is $\tau_{RS} \circ \tau_{:H\to H}$".

3) Generally speaking, when the consistency degree between the justification

$$\text{"}Z \text{ is } \tau_{:H\to H} \circ H\text{"}$$

and the available information

$$\text{"}Z \text{ is } \bigodot(\tau_{H'|S} \circ H' \circ H^{-1})\text{"}$$

is less than a predefined threshold ε, this is usually regarded as a contrary case. That is, if

$$\rho(\tau_{:H\to H} \circ H, \tau_{H'|S} \circ H' \circ H^{-1}) < \varepsilon,$$

then this rule should not go into effect. In this case, the current belief in H should not be changed, namely the assessment of uncertainty of H continues to be $\bigodot(\tau_{H'|S} \circ H' \circ H^{-1})$.

4) Otherwise, this rule should go into effect. Clearly, the less the consistency degree is, the less the belief in H should become. In other words, the current belief $\tau_{RS} \circ \tau_{:H\to H}$ should be updated by the consistency degree.

5) Here we will consider the way to use the consistency degree to decrease the belief.

- By Subsection 2.1 and (34), we know that for any $\tau \in TS$,

  $$\mu_\tau(x) = x^\alpha.$$

  Here when α is getting bigger, the belief represented by τ is getting stronger. Especially, when $\alpha = \infty$, τ represents "*absolutely-true*". So, we can reduce the belief represented by τ, as long as we reduce the value of α. Notice that the consistency degree takes its values on $[0, 1]$. Hence, clearly, if we use the consistency degree to multiply α, the value of α is decreased.

- By Subsection 2.1 and (35), we know for any $\tau \in FS$,

  $$\mu_\tau(x) = (1 - x)^\alpha.$$

Here when α is getting bigger, the belief represented by τ is getting weaker. Specially, when $\alpha = \infty$, τ represents "***absolutely-false***". So, we can reduce the belief represented by τ, as long as we amplify the value of α. Notice that the consistency degree takes its values on $[0,1]$. Hence, clearly, if we use the consistency degree to divide α, the value of α is increased.

Therefore, from 1), 2), 3), 4) and 5), we have

$$um(H,S) = \begin{cases} \bigodot(\tau_{H'|S} \circ H' \circ H^{-1}) & \text{if } \rho(\tau_{:H\to H} \circ H, \tau_{H'|S} \circ H' \circ H^{-1}) < \varepsilon, \\ \bigodot(\tau_\rho) & \text{otherwise;} \end{cases} \tag{40}$$

$$\mu_{\tau_\rho}(x) = \begin{cases} (\mu_{RS}(\mu_{\tau_{:H\to H}}(x)))^{\rho(\tau_{:H\to H}\circ H, \tau_{H'|S}\circ H'\circ H^{-1})} & \text{if } \bigodot(\tau_{RS} \circ \tau_{:H\to H}) \in TS, \\ (\mu_{RS}(\mu_{\tau_{:H\to H}}(x)))^{\frac{1}{\rho(\tau_{:H\to H}\circ H, \tau_{H'|S}\circ H'\circ H^{-1})}} & \text{if } \bigodot(\tau_{RS} \circ \tau_{:H\to H}) \in FS. \end{cases} \tag{41}$$

For convenience, we denote the operator, given by (40) and (41), with ∞. That is,

$$um(H,S) = \infty\,(\tau_{H'|S}, \tau_{:H\to H}, \tau_{RS}, \varepsilon). \tag{42}$$

In the case where there is an absence of negative evidence against the justification, or in the situation where the available information is the same as the justification, the following theorem demonstrates how this type of uncertain default rule works. From this theorem, we can easily see that the way it works agrees with our intuition.

Theorem 2 *If* $\tau_{H'|S} =$ "*undecided*" *or* $\tau_{:H\to H} \circ H = \tau_{H'|S} \circ H' \circ H^{-1}$, *then*

$$\infty\,(\tau_{H'|S}, \tau_{H:\to H}, \tau_{RS}, \varepsilon) = \bigodot(\tau_{RS} \circ \tau_{:H\to H}). \tag{43}$$

Proof. In the case $\tau_{H'|S} =$ "*undecided*", we have

$$\mu_{\tau_{H'|S}\circ H'\circ H^{-1}}(x) = \mu_{\tau_{H'|S}}(\mu'_H(\mu_{H^{-1}}(x))) = \mu_{undecided}(\mu'_H(\mu_{H^{-1}}(x))) = 1,$$

namely,

$$\tau_{H'|S} \circ H' \circ H^{-1} = \text{"}undecided\text{"}$$

thus, by Lemma 1, we have

$$\rho(\tau_{:H\to H} \circ H, \tau_{H'|S} \circ H' \circ H^{-1}) = 1.$$

Substituting the above result into (41), we have

$$\mu_{\tau_\rho}(x) = \mu_{RS}(\mu_{\tau_{:H\to H}}(x)).$$

Further substituting the above result into (40), we get (43).

In the case $\tau_{:H\to H} \circ H = \tau_{H'|S} \circ H' \circ H^{-1}$, by Lemma 1, we have

$$\rho(\tau_{:H\to H} \circ H, \tau_{H'|S} \circ H' \circ H^{-1}) = 1.$$

So, substituting the above result into (41) and (40), we get (43). □

The theorem below demonstrates how to use the fuzzy no-prerequisite rule to perform reasoning with uncertain information.

Theorem 3 *If* $\tau_{:H\to H} = \tau_{RS} =$ "*true*", *then*

$$\begin{aligned} & \infty(\tau_{H'|S}, \tau_{:H\to H}, \tau_{RS}, \varepsilon) \\ = & \begin{cases} \bigodot(\tau_{H'|S} \circ H' \circ H^{-1}) \\ \qquad \text{if } \rho(H, \tau_{H'|S} \circ H' \circ H^{-1}) < \varepsilon, \\ x^{\rho(H,\tau_{H'|S}\circ H'\circ H^{-1})} \\ \qquad \text{if } \rho(H, \tau_{H'|S} \circ H' \circ H^{-1}) \geq \varepsilon \text{ and } \bigodot(\tau_{RS} \circ \tau_{:H\to H}) \in TS, \\ x^{\frac{1}{\rho(H,\tau_{H'|S}\circ H'\circ H^{-1})}} \\ \qquad \text{if } \rho(H, \tau_{H'|S} \circ H' \circ H^{-1}) \geq \varepsilon \text{ and } \bigodot(\tau_{RS} \circ \tau_{:H\to H}) \in FS. \end{cases} \end{aligned} \tag{44}$$

Proof. From the assumption of this theorem, we have

$$\mu_{\tau:H\to H}(x) = \mu_{\tau_{RS}}(x) = x,$$

thus

$$\begin{aligned} \mu_{\tau:H\to H\circ H}(x) &= \mu_{\tau:H\to H}(\mu_H(x)) \\ &= \mu_H(x). \end{aligned}$$

Similarly, we have

$$\begin{aligned} \mu_{\tau_{RS}\circ\tau:H\to H}(x) &= \mu_{\tau_{RS}}(\mu_{\tau:H\to H}(x)) \\ &= \mu_{\tau:H\to H}(x) \\ &= x. \end{aligned}$$

Subsequently, substituting the above results into (40) and (41), we get (44). □

5.3. *Normal Subtype*

For the rule

$$E : E_\beta \to H,$$

suppose $um(E', S) = \tau_{E'|S} \in LTTS$, $um(E'_\beta, S) = \tau_{E'_\beta|S} \in LTTS$, and (20). We will find the value of $um(H, S)$.

1) Generally speaking, when the consistency degree between the justification and the available information is less than a predefined threshold ε, this is usually regarded as a contrary case. That is, if

$$\rho(\tau_{E_\beta} \circ E_\beta, \tau_{E'_\beta|S} \circ E'_\beta) < \varepsilon,$$

this rule should not go into effect. Thus, in this case, $um(H, S)$ should take the linguistic truth "*undecided*".

2) Otherwise, the following rule should go into effect. If the rule's justification and the available information are the same, the rule

$$((X \text{ is } E) \text{ is } \tau_E) : ((Y \text{ is } E_\beta) \text{ is } E_\beta) \xrightarrow{\tau_{RS}} ((Z \text{ is } H) \text{ is } \tau_H)$$

is equivalent to the fuzzy rule

$$((X \text{ is } E) \text{ is } \tau_E) \xrightarrow{\tau_{RS}} ((Z \text{ is } H) \text{ is } \tau_H).$$

In other cases where they are partially consistent, similarly to that in the discussion about the no-prerequisite type, the consistency degree $\rho(\tau_{E_\beta} \circ E_\beta, \tau_{E'_\beta|S} \circ E'_\beta)$ should be used to decrease the belief in H assessed by using the above equivalent rule. Moreover, if the result is not closed on $LTTS$, the linguistic approximation technique $\bigodot$ needs to be used.

Therefore, from 1) and 2), we have

$$um(H,S) = \begin{cases} \text{“\textit{undecided}”} & \text{if } \rho(\tau_{E_\beta} \circ E_\beta, \tau_{E'_\beta|S} \circ E'_\beta) < \varepsilon, \\ \bigodot(\tau_\rho) & \text{otherwise;} \end{cases} \tag{45}$$

$$\mu_{\tau_\rho}(x) = \begin{cases} (\mu_\tau(x))^{\rho(\tau_{E_\beta} \circ E_\beta, \tau_{E'_\beta|S} \circ E'_\beta)} & \text{if } \bigodot(\tau) \in TS, \\ (\mu_\tau(x))^{\frac{1}{\rho(\tau_{E_\beta} \circ E_\beta, \tau_{E'_\beta|S} \circ E'_\beta)}} & \text{if } \bigodot(\tau) \in FS; \end{cases} \tag{46}$$

$$\tau = \star\star(\tau_{E'|S}, \tau_E \xrightarrow{\tau_{RS}} \tau_H). \tag{47}$$

For convenience, we denote the operation, given by (45), (46) and (47), as $\bowtie_1$, that is,

$$um(H,S) = \bowtie_1 (\tau_{E'|S}, \tau_E, \tau_{E'_\beta|S}, \tau_{E_\beta}, \tau_H, \tau_{RS}, \varepsilon). \tag{48}$$

2′) When this rule goes into effect, we have another way of considering the influence of the consistency degree between $\tau_{E_\beta} \circ E_\beta$ and $\tau_{E'_\beta|S} \circ E'_\beta$. If they are the same, this rule is equivalent to the fuzzy rule

$$((X \text{ is } E) \text{ is } \tau_E) \xrightarrow{\tau_{RS}} ((Z \text{ is } H) \text{ is } \tau_H).$$

In other cases, the strength τ_{RS} of this rule should be reduced by their consistency degree $\rho(\tau_{E_\beta} \circ E_\beta, \tau_{E'_\beta|S} \circ E'_\beta)$, because when their consistency degree decreases, the degree to which E supports H should decrease correspondingly. If the decreased strength of this rule is not closed on $LTTS$, the linguistic approximation technique $\bigodot$ needs to be used. Then, we use the decreased strength of this rule to propagate uncertainty sequentially.

Therefore, from 1) and 2′), we have

$$um(H,S) = \begin{cases} \text{“\textit{undecided}”} & \text{if } \rho(\tau_{E_\beta} \circ E_\beta, \tau_{E'_\beta|S} \circ E'_\beta) < \varepsilon, \\ \star\star(\tau_{E'|S}, \tau_E \xrightarrow{\bigodot(\tau'_{RS})} \tau_H) & \text{otherwise;} \end{cases} \tag{49}$$

$$\mu_{\tau'_{RS}}(x) = \begin{cases} (\mu_{\tau_{RS}}(x))^{\rho(\tau_{E_\beta} \circ E_\beta, \tau_{E'_\beta|S} \circ E'_\beta)} & \text{if } \tau_{RS} \in TS, \\ (\mu_{\tau_{RS}}(x))^{\frac{1}{\rho(\tau_{E_\beta} \circ E_\beta, \tau_{E'_\beta|S} \circ E'_\beta)}} & \text{if } \tau_{RS} \in FS. \end{cases} \tag{50}$$

For convenience, we denote the operation, given by (49) and (50), as $\bowtie_2$, that is,

$$um(H,S) = \bowtie_2 (\tau_{E'|S}, \tau_E, \tau_{E'_\beta|S}, \tau_{E_\beta}, \tau_H, \tau_{RS}, \varepsilon). \tag{51}$$

In the subsequent theorem, we will examine the formulae developed above in a situation where we know nothing about the justification, and the situation where the available information is the same as the justification. Namely, we will examine the performance of the corresponding default reasoning in the absence of negative evidence, and the performance when the available information is the same as the justification.

Theorem 4 *If* $\tau_{E'_\beta|S} \circ E'_\beta =$ *"undecided", or* $\tau_{E'_\beta|S} \circ E'_\beta = \tau_{E_\beta} \circ E_\beta$, *then*

$$\begin{aligned} & \bowtie_1 (\tau_{E'|S}, \tau_E, \tau_{E'_\beta|S}, \tau_{E_\beta}, \tau_H, \tau_{RS}) \\ = \; & \bowtie_2 (\tau_{E'|S}, \tau_E, \tau_{E'_\beta|S}, \tau_{E_\beta}, \tau_H, \tau_{RS}) \\ = \; & \star\star(\tau_{E'|S}, \tau_E \xrightarrow{\tau_{RS}} \tau_H). \end{aligned} \tag{52}$$

Proof. In the case $\tau_{E'_\beta|S} \circ E'_\beta =$ "*undecided*", by Lemma 1, we have

$$\rho(\tau_{E_\beta} \circ E_\beta, \tau_{E'_\beta|S} \circ E'_\beta) = \rho(\tau_{E_\beta} \circ E_\beta, \text{“\textit{undecided}”}) = 1.$$

Substituting the above result into (45) and (49) respectively, we get

$$\mu_{\tau_\rho}(x) = \mu_\tau(x), \qquad \mu_{\tau'_{RS}}(x) = \mu_{\tau_{RS}}(x).$$

Further substituting them into (45) and (49) respectively, we get (52).

The proof in the case, $\tau_{E'_\beta|S} \circ E'_\beta = \tau_{E_\beta} \circ E_\beta$, is similar. □

The two theorems below give the inference mechanism for using the normal type of fuzzy default rules to perform reasoning under uncertainty.

Theorem 5 *If* $\tau_E = \tau_{E_\beta} = \tau_{RS} = \tau_H =$ *"true", then*

$$\bowtie_1 (\tau_{E'|S}, \tau_E, \tau_{E'_\beta|S}, \tau_{E_\beta}, \tau_H, \tau_{RS}, \varepsilon) = \begin{cases} \text{“\textit{undecided}”} & \textit{if } \rho(E_\beta, \tau_{E'_\beta|S} \circ E'_\beta) < \varepsilon, \\ \bigodot(\tau_\rho) & \textit{otherwise;} \end{cases} \tag{53}$$

$$\mu_{\tau_\rho}(x) = \begin{cases} (\mu_\tau(x))^{\rho(E_\beta, \tau_{E'_\beta|S} \circ E'_\beta)} & \textit{if } \bigodot(\tau) \in TS, \\ (\mu_\tau(x))^{\rho(E_\beta, \tau_{E'_\beta|S} \circ E'_\beta)} & \textit{if } \bigodot(\tau) \in FS; \end{cases} \tag{54}$$

$$\tau = **(\tau_{E'|S}, \text{“\textit{true}”}). \tag{55}$$

Proof. Firstly, because $\tau_{E_\beta} =$ "*true*" by (9) we have

$$\tau_{E_\beta} \circ E_\beta = E_\beta.$$

Secondly, noticing

$$\begin{aligned} f(x) = x &\Longrightarrow f^{-1}(x) = x, \\ f(x) = x &\Longrightarrow g(f(x)) = g(x), \end{aligned}$$

by the second branch of (29), we have

$$um(H,S) = \bigodot(**\,(\tau_{E'|S} \circ E' \circ E, \tau_{RS})).$$

Further, by (30), the above formula can be expressed as

$$um(H,S) = **\,(\tau_{E'|S}, \tau_{RS}),$$

or

$$um(H,S) = **\,(\tau_{E'|S}, \text{“}true\text{”}),$$

because $\tau_{RS} =$ “*true*”. Finally, substituting the above results into (45), (46) and (47), we obtain (53), (54) and (55). □

Similarly, we can prove the following theorem:

Theorem 6 *If* $\tau_E = \tau_{E_\beta} = \tau_{RS} = \tau_H =$ *“true”, then*

$$\bowtie_2 (\tau_{E'|S}, \tau_E, \tau_{E'_\beta|S}, \tau_{E_\beta}, \tau_H, \tau_{RS}, \varepsilon) = \begin{cases} \text{“}undecided\text{”} \\ \qquad \textit{if } \rho(E_\beta, \tau_{E'_\beta|S} \circ E'_\beta) < \varepsilon, \\ **\,(\tau_{E'|S}, \bigodot(\tau'_{RS})) \\ \qquad \textit{otherwise;} \end{cases} \tag{56}$$

$$\mu_{\tau'_{RS}}(x) = x^{\rho(E_\beta, \tau_{E'_\beta|S} \circ E'_\beta)}. \tag{57}$$

5.4. *Semi-Normal Subtype*

For the rule

$$E : E_\beta \wedge H \to H,$$

suppose $um(E', S) = \tau_{E'|S} \in LTTS$, $um(E'_\beta, S) = \tau_{E'_\beta|S} \in LTTS$, $um(H', S) = \tau_{H'|S} \in LTTS$, and (21). We will find the value of $um(H,S)$.

1) As before, when the consistency degree between the justification and the available information is less than a predefined threshold ε, we regard this as a contrary case. That is, if

$$\rho((\tau_{E_\beta} \circ E_\beta) \cap (\tau_H \circ H), (\tau_{E'_\beta|S} \circ E'_\beta) \cap (\tau_{H'|S} \circ H')) < \varepsilon,$$

this rule should not go into effect. So, in this case, $um(H,S)$ can be obtained only from the belief in H', being the assessment with the same effect upon H as that of $\tau_{H'|S}$ upon H'. In fact, in this case, we can obtain $um(H,S)$ by (39).

2) Otherwise, this rule should go into effect. In this case, our discussion is similar to that of the normal subtype. The difference is in the justifications, and thus their consistency degrees are distinctive.

Therefore, from 1) and 2), we have

$$um(H,S) = \begin{cases} \bigodot(\tau_{H'|S} \circ H' \circ H^{-1}) & \text{if } \delta < \varepsilon, \\ \bigodot(\tau_\rho) & \text{otherwise;} \end{cases} \tag{58}$$

$$\mu_{\tau_\rho}(x) = \begin{cases} (\mu_\tau(x))^\delta & \text{if } \bigodot(\tau) \in TS, \\ (\mu_\tau(x))^{\frac{1}{\delta}} & \text{if } \bigodot(\tau) \in FS; \end{cases} \tag{59}$$

$$\delta = \rho((\tau_{E_\beta} \circ E_\beta) \cap (\tau_H \circ H), (\tau_{E'_\beta|S} \circ E'_\beta) \cap (\tau_{H'|S} \circ H')); \tag{60}$$

$$\tau = \star\star\,(\tau_{E'|S}, \tau_E \xrightarrow{\tau_{RS}} \tau_H). \tag{61}$$

For convenience, we denote the operation, given by (58), (59), (60) and (61), as $\ddagger_1$, that is,

$$um(H,S) = \ddagger_1(\tau_{E'|S}, \tau_E, \tau_{E'_\beta|S}, \tau_{E_\beta}, \tau_{H'|S}, \tau_H, \tau_{RS}, \varepsilon). \tag{62}$$

Notice that, as we did in the normal subtype, here we can also consider using the consistency degree to modify the support degree, τ_{RS}, from the prerequisite to the conclusion. Thus, we have

$$um(H,S) = \begin{cases} \bigodot(\tau_{H'|S} \circ H' \circ H^{-1}) & \text{if } \delta < \varepsilon, \\ \star\star\,(\tau_{E'|S}, \tau_E \xrightarrow{\bigodot(\tau'_{RS})} \tau_H) & \text{otherwise}; \end{cases} \tag{63}$$

$$\mu_{\tau'_{RS}}(x) = \begin{cases} (\mu_{\tau_{RS}}(x))^\delta & \text{if } \tau_{RS} \in TS, \\ (\mu_{\tau_{RS}}(x))^{\frac{1}{\delta}} & \text{if } \tau_{RS} \in FS. \end{cases} \tag{64}$$

Here δ is given by (60). For convenience, we denote the operation, given by (63) and (64), as $\ddagger_2$, that is,

$$um(H,S) = \ddagger_2(\tau_{E'|S}, \tau_E, \tau_{E'_\beta|S}, \tau_{E_\beta}, \tau_{H'|S}, \tau_H, \tau_{RS}, \varepsilon). \tag{65}$$

In the following theorem, we will examine the performance of this type of default reasoning in these two situations. We will examine the formulae developed above in a situation where we know nothing about the justifications, and the situation where the available information is the same as the justification.

Theorem 7 *If $\tau_{E'_\beta|S} \circ E'_\beta =$ "undecided" and $\tau_{H'|S} \circ H' =$ "undecided", or $(\tau_{E_\beta} \circ E_\beta) \cap (\tau_H \circ H) = (\tau_{E'_\beta|S} \circ E'_\beta) \cap (\tau_{H'|S} \circ H')$, then*

$$\begin{aligned} & \ddagger_1(\tau_{E'|S}, \tau_E, \tau_{E'_\beta|S}, \tau_{E_\beta}, \tau_{H'|S}, \tau_H, \tau_{RS}, \varepsilon) \\ = \; & \ddagger_2(\tau_{E'|S}, \tau_E, \tau_{E'_\beta|S}, \tau_{E_\beta}, \tau_{H'|S}, \tau_H, \tau_{RS}, \varepsilon) \\ = \; & \star\star\,(\tau_{E'|S}, \tau_E \xrightarrow{\tau_{RS}} \tau_H). \end{aligned} \tag{66}$$

Proof. According to the assumption of the theorem, we have

$$\mu_{\tau_{E'_\beta|S} \circ E'_\beta}(x) \wedge \mu_{\tau_{H'|S} \circ H'}(x) = \mu_{undecided}(x) \wedge \mu_{undecided}(x) = 1 \wedge 1 = 1,$$

thus,

$$(\tau_{E'_\beta|S} \circ E'_\beta) \cap (\tau_{H'|S} \circ H')) = \text{"}undecided\text{"}.$$

So, by Lemma 1, we have

$$\begin{aligned}\delta &= \rho((\tau_{E_\beta} \circ E_\beta) \cap (\tau_H \circ H), (\tau_{E'_\beta|S} \circ E'_\beta) \cap (\tau_{H'|S} \circ H'))\\ &= \rho((\tau_{E_\beta} \circ E_\beta) \cap (\tau_H \circ H), \text{``}undecided\text{''})\\ &= 1.\end{aligned}$$

Substituting the above result into (58) and (63) respectively, we get

$$\mu_{\tau_\rho}(x) = \mu_\tau(x), \qquad \mu_{\tau'_{RS}}(x) = \mu_{\tau_{RS}}(x).$$

Further substituting them into (58) and (63) respectively, we get (66).

For the case $(\tau_{E_\beta} \circ E_\beta) \cap (\tau_H \circ H) = (\tau_{E'_\beta|S} \circ E'_\beta) \cap (\tau_{H'|S} \circ H')$, the proof is similar. □

The two theorems below give the inference mechanism for using the semi-normal type of fuzzy default rules to propagate uncertainty. Their proof is similar to that of Theorem 5.

Theorem 8 *If* $\tau_E = \tau_{E_\beta} = \tau_H = \tau_{RS} =$ *"true", then*

$$\ddagger_1(\tau_{E'|S}, \tau_E, \tau_{E'_\beta|S}, \tau_{E_\beta}, \tau_{H'|S}, \tau_H, \tau_{RS}, \varepsilon) = \begin{cases} \bigodot(\tau_{H'|S} \circ H' \circ H^{-1}) & \text{if } \delta < \varepsilon,\\ \bigodot(\tau_\rho) & \text{otherwise;}\end{cases} \tag{67}$$

$$\mu_{\tau_\rho}(x) = \begin{cases} (\mu_\tau(x))^\delta & \text{if } \bigodot(\tau) \in TS,\\ (\mu_\tau(x))^{\frac{1}{\delta}} & \text{if } \bigodot(\tau) \in FS;\end{cases} \tag{68}$$

$$\delta = \rho(E_\beta \cap H, (\tau_{E'_\beta|S} \circ E'_\beta) \cap (\tau_{H'|S} \circ H')); \tag{69}$$

$$\tau = **(\tau_{E'|S}, \text{``}true\text{''}). \tag{70}$$

Theorem 9 *If* $\tau_E = \tau_{E_\beta} = \tau_{RS} = \tau_H =$ *"true", then*

$$\ddagger_2(\tau_{E'|S}, \tau_E, \tau_{E'_\beta|S}, \tau_{E_\beta}, \tau_{H'|S}, \tau_H, \tau_{RS}) = \begin{cases} \bigodot(\tau_{H'|S} \circ H' \circ H^{-1}) & \text{if } \delta < \varepsilon,\\ **(\tau_{E'|S}, \bigodot(\tau'_{RS})) & \text{otherwise;}\end{cases} \tag{71}$$

$$\mu_{\tau'_{RS}}(x) = x^\delta, \tag{72}$$

where δ *is given by (69).*

6. Using Our Model

This section demonstrates prototypically how our model can deal with uncertainty in default reasoning, and, more importantly, makes it possible for fuzzy approximate reasoning models, uncertain reasoning models and fuzzy default reasoning models to share information in cooperative multi-agent systems. The value of the model can be assessed by its application in a multi-agent system (as shown in Table 6), which consists of six individual knowledge-based agents $\{KBA_1, KBA_2, KBA_3, KBA_4, KBA_5, KBA_6\}$, with different reasoning models.

KBA_1 uses Zadeh's fuzzy approximate reasoning model; KBA_2 employs a truth-qualification uncertain reasoning model; KBA_3 exploits a fuzzy default reasoning

Table 6. The knowledge bases of a knowledge-based multi-agent system.

KBA_1	$(X_{E_1} \text{ is } E_1) \to (X_{H_1} \text{ is } H_1)$ $(X_{H_1} \text{ is } H_1) \to (X_{H_2} \text{ is } H_2)$
KBA_2	$((X_{E_0} \text{ is } E_0) \text{ is } \tau_{E_0}) \to ((X_{E_1} \text{ is } E_1) \text{ is } \tau_{E_1}) \text{ is } \tau_{RS_{21}}$ $((X_{E_2} \text{ is } E_2) \text{ is } \tau_{E_2}) \to ((X_{E_3} \text{ is } E_3) \text{ is } \tau_{E_3})$
KBA_3	$:(X_{E_6} \text{ is } E_6) \to (X_{E_6} \text{ is } E_6)$ $(X_{E_4} \text{ is } E_4) : (X_{H_2} \text{ is } H_2) \to (X_{E_5} \text{ is } E_5)$ $(X_{E_6} \text{ is } E_6) : (X_{E_5} \text{ is } E_5) \wedge (X_{E_7} \text{ is } E_7) \to (X_{E_7} \text{ is } E_7)$
KBA_4	$:((X_{E_3} \text{ is } E_3) \text{ is } \tau_{:E_3 \to E_3})_{\varepsilon_{41}} \to ((X_{E_3} \text{ is } E_3) \text{ is } \tau_{:E_3 \to E_3})$ $((X_{E_3} \text{ is } E_3) \text{ is } \tau_{E_3}) : ((X_{E_{13}} is E_{13}) \text{ is } \tau_{E_{13}})_{\varepsilon_{42}} \to ((X_{E_5} \text{ is } E_5) \text{ is } \tau_{E_5})$ $((X_{E_7} \text{ is } E_7) \text{ is } \tau_{E_7}) : (((X_{E_5} \text{ is } E_5) \text{ is } \tau_{E_5}) \wedge ((X_{E_8} \text{ is } E_8) \text{ is } \tau_{E_8}))_{\varepsilon_{43}} \to ((X_{E_8} \text{ is } E_8) \text{ is } \tau_{E_8})$
KBA_5	$:((X_{E_{10}} \text{ is } E_{10}) \text{ is } \tau_{:E_{10} \to E_{10}})_{\varepsilon_{51}} \xrightarrow{\tau_{RS_{51}}} ((X_{E_{10}} \text{ is } E_{10}) \text{ is } \tau_{:E_{10} \to E_{10}})$ $((X_{E_5} \text{ is } E_5) \text{ is } \tau_{E_5}) : ((X_{E_{13}} \text{ is } E_{13}) \text{ is } \tau_{E_{13}})_{\varepsilon_{52}} \xrightarrow{\tau_{RS_{52}}} ((X_{E_9} \text{ is } E_9) \text{ is } \tau_{E_9})$ $((X_{E_9} \text{ is } E_9) \text{ is } \tau_{E_9}) : (((X_{H_2} \text{ is } H_2) \text{ is } \tau_{H_2}) \wedge ((X_{E_6} \text{ is } E_6) \text{ is } \tau_{E_6}))_{\varepsilon_{53}} \xrightarrow{\tau_{RS_{53}}} ((X_{E_6} \text{ is } E_6) \text{ is } \tau_{E_6})$
KBA_6	$(X_{E_{10}} \text{ is } E_{10}) \to ((X_{E_{11}} \text{ is } E_{11}) \text{ is } \tau_{E_{11}})$ $(((X_{E_9} \text{ is } E_9) \text{ is } \tau_{E_9}) \to (X_{E_{11}} \text{ is } E_{11})) \text{ is } \tau_{RS_{62}}$ $((X_{E_6} \text{ is } E_6) \text{ is } \tau_{E_6}) \to ((X_{E_{12}} \text{ is } E_{12}) \text{ is } \tau_{E_{12}})$ $:(X_{E_8} \text{ is } E_8)_{\varepsilon_{64}} \xrightarrow{\tau_{RS_{64}}} (X_{E_8} \text{ is } E_8)$ $((X_{E_8} \text{ is } E_8) \text{ is } \tau_{E_8}) : ((X_{E_{11}} \text{ is } E_{11}) \text{ is } \tau_{E_{11}})_{\varepsilon_{65}} \xrightarrow{\tau_{RS_{65}}} ((X_H \text{ is } H) \text{ is } \tau_H)$ $((X_{E_{12}} \text{ is } E_{12}) \text{ is } \tau_{E_{12}}) : ((X_{E_{13}} \text{ is } E_{13}) \text{ is } \tau_{E_{13}}) \wedge ((X_H \text{ is } H) \text{ is } \tau_H) \to ((X_H \text{ is } H) \text{ is } \tau_H)$

model; KBA_4 utilises a truth-qualification style fuzzy default reasoning model without strengths; KBA_5 makes use of a truth-qualification style fuzzy default reasoning model with strengths; KBA_6 is a hybrid model which includes all kinds of reasoning models used by the other individual knowledge-based agents.

The information exchanged among these six knowledge-based agents is as shown in Figure 4. From this figure, we can see, as a default reasoning system, KBA_3 utilises the result with uncertainty, from KBA_1 and KBA_5, to perform further reasoning. This is possible because of the fact that the default reasoning is actually a special case of our proposed approach which can accommodate uncertainty.

Suppose that the knowledge-based multi-agent system consisting of $KBA_1, \cdots, KBA_6$ obtains the following information from an end-user:¶

$$(X_{E_0} \text{ is } E_0') \text{ is } \tau_{E_0'|S}, \tag{73}$$

$$(X_{E_2} \text{ is } E_2') \text{ is } \tau_{E_2'|S}, \tag{74}$$

$$(X_{E_4} \text{ is } E_4') \text{ is } \tau_{E_4'|S}. \tag{75}$$

But from the end-user the system gets no idea about E_6, E_7, E_8, E_{10} and E_{13}, that is,

$$(X_{E_6} \text{ is } E_6) \text{ is } \mathit{undecided}, \tag{76}$$

$$(X_{E_7} \text{ is } E_7) \text{ is } \mathit{undecided}, \tag{77}$$

$$(X_{E_8} \text{ is } E_8) \text{ is } \mathit{undecided}, \tag{78}$$

$$(X_{E_{10}} \text{ is } E_{10}) \text{ is } \mathit{undecided}, \tag{79}$$

¶Actually, in a scenario of an automated group negotiation, the end-user becomes the negotiation opponent party (agent) of the negotiation multi-agent team.

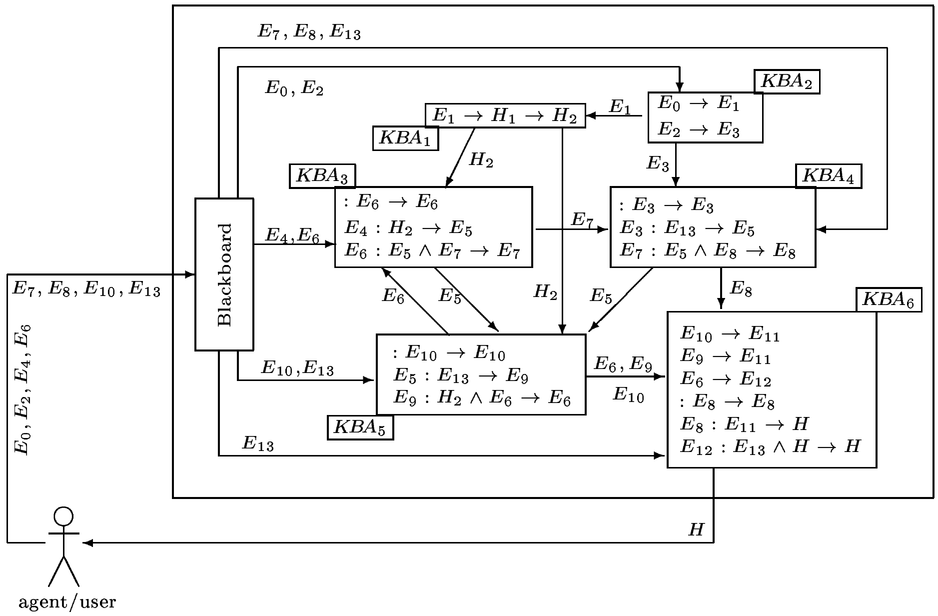

Fig. 4. Input, output and knowledge sharing of the multi-agent system of $KBA_1, \cdots, KBA_6$.

$$(X_{E_{13}} \text{ is } E_{13}) \text{ is } \textit{undecided}. \tag{80}$$

It is natural that sometimes the system cannot obtain all necessary information. For example, in the automated negotiation, participating agents often minimise the amount of their own private information they reveal during the encounter. The reason for minimising the amount of information that is revealed about an agent's preferences is that any such revelation may weaken the agent's bargaining position [86,90] in competitive scenarios.

In the following, we will show the procedure of reasoning and information exchange in the multi-agent system that comprises these six knowledge-based agents. For the sake of simplicity, we consider symbolic representation and propagation of uncertainty to illustrate all steps and corresponding code only in *Mathematics*.

1) From the end-user, KBA_2 gets some information about E_0 and E_2, *i.e.*, (73) and (74). Then it can perform reasoning and obtain some information about E_1 and E_3. In fact, from the first rule in KBA_2, by (32), we have

$$\tau_{E_1|S} = \star\star\,(\tau_{E_0'|S}, \tau_{E_0} \xrightarrow{\tau_{RS_{21}}} \tau_{E_1}). \tag{81}$$

That is, "$(X_{E_1}$ is $E_1)$ is $\tau_{E_1|S}$". By (9), the second rule in KBA_2, is equivalent to

$$(((X_{E_2} \text{ is } E_2) \text{ is } \tau_{E_2}) \to ((X_{E_3} \text{ is } E_3) \text{ is } \tau_{E_3})) \text{ is } \textit{true}.$$

Thus by (32), we have

$$\tau^{(0)}_{E_3|S} = \star\star\,(\tau_{E_2'|S}, \tau_{E_2} \xrightarrow{\text{“}true\text{”}} \tau_{E_3}). \tag{82}$$

That is, "(X_{E_3} is E_3) is $\tau^{(0)}_{E_3|S}$".

2) From *KBA*$_2$, *KBA*$_1$ gets some information about E_1, *i.e.*, (81). Then it can perform reasoning and obtain some information about H_2. In fact, by (9), the first rule in *KBA*$_1$ is equivalent to

$$((X_{E_1} \text{ is } E_1) \to (X_{H_1} \text{ is } H_1)) \text{ is } \mathit{true}.$$

Thus by (30), we have

$$\tau_{H_1|S} \quad = \quad **\,(\tau_{E_1'|S}, \tau_{RS}) = **\,(\tau_{E_1|S}, \text{“}\mathit{true}\text{”}), \tag{83}$$

where $\tau_{E_1|S}$ is given by (81). That is, "(X_{H_1} is H_1) is $\tau_{H_1|S}$". By (9), the second rule in *KBA*$_1$ is equivalent to

$$((X_{H_1} \text{ is } H_1) \to (X_{H_2} \text{ is } H_2)) \text{ is } \mathit{true}.$$

Thus by (30), we have

$$\tau_{H_2|S} = **\,(\tau_{H_1',S}, \tau_{RS}) = **\,(\tau_{H_1|S}, \text{“}\mathit{true}\text{”}), \tag{84}$$

where $\tau_{H_1|S}$ is given by (83). That is, "(X_{H_2} is H_2) is $\tau_{H_2|S}$".

3) From the end-user and *KBA*$_2$, *KBA*$_3$ gets, respectively, some information about E_4 and H_2, *i.e.*, (75) and (84). Then it can perform reasoning and obtain some information about E_5. In fact, by (9), the second rule in *KBA*$_3$ is equivalent to

$$((X_{E_4} \text{ is } E_4) \text{ is } \mathit{true})\colon ((X_{H_2} \text{ is } H_2) \text{ is } \mathit{true})_1 \xrightarrow{\mathit{true}} ((X_{E_5} \text{ is } E_5) \text{ is } \mathit{true}).$$

By (48), we have

$$\begin{aligned} \tau^{(1)}_{E_5|S} &= \bowtie_1 (\tau_{E_4'|S}, \tau_{E_4}, \tau_{H_2'|S}, \tau_{H_2}, \tau_{E_5}, \tau_{RS}, \varepsilon) \\ &= \bowtie_1 (\tau_{E_4'|S}, \text{“}\mathit{true}\text{”}, \tau_{H_2|S}, \text{“}\mathit{true}\text{”}, \text{“}\mathit{true}\text{”}, \text{“}\mathit{true}\text{”}, 1), \end{aligned} \tag{85}$$

where $\tau_{E_4'|S}$ is given by the end-user and $\tau_{H_2|S}$ is given by (84). That is, "(X_{E_5} is E_5) is $\tau^{(1)}_{E_5|S}$".

4) From *KBA*$_2$, *KBA*$_4$ gets information about E_3, *i.e.*, (82). Then it can perform reasoning and obtain some information about E_5. In fact, by (9), the first rule in *KBA*$_4$ is equivalent to

$$:((X_{E_3} \;\text{ is }\; E_3) \;\text{ is }\; \tau_{:E_3\to E_3})_{\varepsilon_{41}} \xrightarrow{\mathit{true}} ((X_{E_3} \;\text{ is }\; E_3) \;\text{ is }\; \tau_{:E_3\to E_3}).$$

Thus by (42), we have

$$\tau^{(1)}_{E_3|S} = \infty\,(\tau_{E_3'|S}, \tau_{E_3:\to E_3}, \tau_{RS}, \varepsilon) = \infty\,(\tau^{(0)}_{E_3|S}, \tau_{E_3:\to E_3}, \text{“}\mathit{true}\text{”}, \varepsilon_{41}), \tag{86}$$

where $\tau^{(0)}_{E_3|S}$ is given by (82). That is, "(X_{E_3} is E_3) is $\tau^{(1)}_{E_3|S}$". By (9), the second rule in *KBA*$_4$ is equivalent to

$$((X_{E_3} \text{ is } E_3) \text{ is } \tau_{E_3})\colon ((X_{E_{13}} \text{ is } E_{13}) \text{ is } \tau_{E_{13}})_{\varepsilon_{42}} \xrightarrow{\mathit{true}} ((X_{E_5} \text{ is } E_5) \text{ is } \tau_{E_5}).$$

Thus by (48) and (80), we have

$$\begin{aligned} \tau^{(2)}_{E_5|S_1} &= \bowtie_1 (\tau_{E'_3|S}, \tau_{E_3}, \tau_{E'_{13}|S}, \tau_{E_{13}}, \tau_{E_5}, \tau_{RS}, \varepsilon) \\ &= \bowtie_1 (\tau^{(1)}_{E_3|S}, \tau_{E_3}, \text{“}\textit{undecided}\text{”}, \tau_{E_{13}}, \tau_{E_5}, \text{“}\textit{true}\text{”}, \varepsilon_{42}), \end{aligned} \tag{87}$$

where $\tau^{(0)}_{E_3|S}$ is given by (85). That is, “(X_{E_5} is E_5) is $\tau^{(2)}_{E_5|S}$”.

5) From KBA_3 and KBA_4, KBA_5 gets some information about E_5, *i.e.*, (85) and (87), respectively. Besides, from KBA_1, it gets some information about H_2, *i.e.*, (84). In addition, from the end-user, it gets some information about E_6, E_{10} and E_{13}, *i.e.*, (76), (79) and (80). Then it can perform reasoning and obtain some information about E_{10}, E_9 and E_6.

a) From the first rule in KBA_5, by (42) and (79), we have

$$\tau^{(1)}_{E_{10}|S} = \infty (\tau_{E'_{10}|S}, \tau_{:E_{10}\to E_{10}}, \tau_{RS}, \varepsilon) = \infty (\text{“}\textit{undecided}\text{”}, \tau_{:E_{10}\to E_{10}}, \tau_{RS_{51}}, \varepsilon_{51}). \tag{88}$$

That is, “($X_{E_{10}}$ is E_{10}) is $\tau^{(1)}_{E_{10}|S}$”.

b) For the second rule in KBA_5, first we combine different information about E_5 from KBA_3 and KBA_4 by (27), that is,

$$\tau^{(12)}_{E_5|S} = ++ (\tau^{(1)}_{E_5|S}, \tau^{(2)}_{E_5|S}), \tag{89}$$

where $\tau^{(1)}_{E_5|S}$ and $\tau^{(2)}_{E_5|S}$ are given by (85) and (87), respectively. That is,

$$\text{“}(X_{E_5} \text{ is } E_5) \text{ is } \tau^{(12)}_{E_5|S}\text{”}$$

Second by (48) and (80), we have

$$\begin{aligned} \tau_{E_9|S} &= \bowtie_1 (\tau_{E'_5|S}, \tau_{E_5}, \tau_{E'_{13}|S}, \tau_{E_{13}}, \tau_{E_9}, \tau_{RS}, \varepsilon) \\ &= \bowtie_1 (\tau_{E^{(12)}_5|S}, \tau_{E_5}, \text{“}\textit{undecided}\text{”}, \tau_{E_{13}}, \tau_{E_9}, \tau_{RS_{52}}, \varepsilon_{52}), \end{aligned} \tag{90}$$

where $\tau^{(12)}_{E_5|S}$ is given by (89). That is, “(X_{E_9} is E_9) is $\tau_{E_9|S}$”.

c) From the third rule in KBA_5, by (62), we have

$$\begin{aligned} \tau_{E_6|S} &= \ddagger_1(\tau_{E'_9|S}, \tau_{E_9}, \tau_{H'_2|S}, \tau_{H_2}, \tau_{E_6|S}, \tau_{E_6}, \tau_{RS}, \varepsilon) \\ &= \ddagger_1(\tau_{E_9|S}, \tau_{E_9}, \tau_{H_2|S}, \tau_{H_2}, \tau_{E_6|S}, \tau_{E_6}, \tau_{RS_{53}}, \varepsilon_{53}), \end{aligned} \tag{91}$$

where $\tau_{E_9|S}$ and $\tau_{H_2|S}$ are given by (90) and (84), respectively. That is, (X_{E_9} is E_9) is $\tau_{E_9|S}$.

6) From KBA_5, KBA_3 gets some information about E_6, *i.e.*, (91). Moreover, from the end-user it gets some information about E_7, *i.e.*, (77). In addition, it has already had some information about E_5, *i.e.*, (85). Then it can perform reasoning and obtain some information about E_7. In fact, by (9), the third rule in KBA_3 is equivalent to

$$(X_{E_6} \text{ is } E_6) \text{ is } \textit{true}) \text{: } (((X_{E_5} \text{ is } E_5) \text{ is } \textit{true}) \wedge ((X_{E_7} \text{ is } E_7) \text{ is } \textit{true}))_1$$
$$\xrightarrow{\textit{true}} ((X_{E_7} \text{ is } E_7) \text{ is } \textit{true}).$$

Thus, by (62) and (77), we have

$$\begin{aligned}\tau_{E_7|S} &= \ddagger_1(\tau_{E_6'|S}, \tau_{E_6}, \tau_{E_5|S}, \tau_{E_5}, \tau_{E_7|S}, \tau_{E_7}, \tau_{RS}, \varepsilon)\\ &= \ddagger_1(\tau_{E_6|S}, \text{“}\mathit{true}\text{”}, \tau^{(1)}_{E_5|S}, \text{“}\mathit{true}\text{”}, \text{“}\mathit{undecided}\text{”}, \text{“}\mathit{true}\text{”}, \text{“}\mathit{true}\text{”}, 1),\end{aligned} \tag{92}$$

where $\tau_{E_6|S}$ and $\tau^{(1)}_{E_5|S}$ are given by (91) and (85), respectively. That is, “(X_{E_7} is E_7) is $\tau_{E_7|S}$”.

7) From KBA_3, KBA_4 gets some information about E_7, *i.e.*, (92). Besides, from the end-user it gets some information about E_8, *i.e.*, (78). In addition, it has already had some information about E_5, *i.e.*, (87). Then it can perform reasoning and obtain some information about E_8. In fact, by (9) the third rule in KBA_4 is equivalent to

$$((X_{E_7} \text{ is } E_7) \text{ is } \tau_{E_7})\text{: } (((X_{E_5} \text{ is } E_5) \text{ is } \tau'_{E_5}) \wedge ((X_{E_8} \text{ is } E_8) \text{ is } \tau_{E_8}))_{\varepsilon 43}$$
$$\xrightarrow{\mathit{true}} ((X_{E_8} \text{ is } E_8) \text{ is } \tau_{E_8}).$$

Thus, by (62) and (78), we have

$$\begin{aligned}\tau^{(0)}_{E_8|S} &= \ddagger_1(\tau_{E_7'|S}, \tau_{E_7}, \tau_{E_5|S}, \tau_{E_5}, \tau_{E_8|S}, \tau_{E_8}, \tau_{RS}, \varepsilon)\\ &= \ddagger_1(\tau_{E_7|S}, \tau_{E_7}, \tau^{(2)}_{E_5|S}, \tau_{E_5}, \text{“}\mathit{undecided}\text{”}, \tau_{E_8}, \text{“}\mathit{true}\text{”}, \varepsilon_{43}),\end{aligned} \tag{93}$$

where $\tau_{E_7|S}$ and $\tau^{(2)}_{E_5|S}$ are given by (92) and (87), respectively. That is, “(X_{E_8} is E_8) is $\tau^{(0)}_{E_8|S}$”.

8) From KBA_4, KBA_6 gets some information about E_8, *i.e.*, (93). Besides, from KBA_5, it gets some information about E_6, E_9, E_{10}, *i.e.*, (91), (90) and (88). Then it can perform reasoning and obtain some information about H.

a) By (9), the first rule in KBA_6 is equivalent to

$$((X_{E_{10}} \text{ is } E_{10}) \text{ is } \mathit{true} \xrightarrow{\mathit{true}} ((X_{E_{11}} \text{ is } E_{11}) \text{ is } \tau_{E_{11}}).$$

Thus, by (32) we have

$$\tau^{(1)}_{E_{11}|S} = \star\star(\tau_{E_{10}'|S}, \tau_{E_{10}} \xrightarrow{\tau_{RS}} \tau_{E_{11}}) = \star\star(\tau^{(1)}_{E_{10}|S}, \text{“}\mathit{true}\text{”} \xrightarrow{\text{“}\mathit{true}\text{”}} \tau_{E_{11}}), \tag{94}$$

where $\tau^{(1)}_{E_{10}|S}$ is given by (88). That is, “($X_{E_{11}}$ is E_{11}) is $\tau^{(1)}_{E_{11}|S}$”.

b) By (9), the second rule in KBA_6 is equivalent to

$$((X_{E_9} \text{ is } E_9) \text{ is } \tau_{E_9}) \xrightarrow{\tau_{RS_{62}}} ((X_{E_{11}} \text{ is } E_{11}) \text{ is } \tau_{E_{11}}).$$

Thus, by (32) we have

$$\tau^{(2)}_{E_{11}|S} = \star\star(\tau'_{E_9|S}, \tau_{E_9} \xrightarrow{\tau_{RS}} \tau_{E_{11}}) = \star\star(\tau_{E_9|S}, \tau_{E_9} \xrightarrow{\tau_{RS_{62}}} \text{“}\mathit{true}\text{”}), \tag{95}$$

where $\tau_{E_9|S}$ is given by (90). That is, “($X_{E_{11}}$ is E_{11}) is $\tau_{E^{(2)}_{11}|S}$”.

c) By (9), the third rule in KBA_6 is equivalent to

$$((X_{E_6} \text{ is } E_6) \text{ is } \tau_{E_6}) \xrightarrow{\text{"true"}} ((X_{E_{12}} \text{ is } E_{12}) \text{ is } \tau_{E_{12}}).$$

Thus, by (32) we have

$$\tau_{E_{12}|S} = \star\star(\tau_{E_6|S}, \tau_{E_6} \xrightarrow{\tau_{RS}} \tau_{E_{12}}) = \star\star(\tau_{E_6|S}, \tau_{E_6} \xrightarrow{\text{"true"}} \tau_{E_{12}}), \tag{96}$$

where $\tau_{E_6|S}$ is given by (91). That is, "$(X_{E_{12}}$ is $E_{12})$ is $\tau_{E_{12}|S}$".

d) By (9), the fourth rule in KBA_6 is equivalent to

$$:((X_{E_8} \text{ is } E_8) \text{ is } \mathit{true})_{\varepsilon 64} \xrightarrow{\tau_{RS_{64}}} ((X_{E_8} \text{ is } E_8)) \text{ is } \mathit{true})\ .$$

Thus by (42), we have

$$\tau^{(1)}_{E_8|S} = \infty(\tau_{E'_8|S}, \tau_{:E_8\to E_8}, \tau_{RS}, \varepsilon) = \infty(\tau^{(0)}_{E_8|S}, \text{"}\mathit{true}\text{"}, \tau_{RS_{64}}, \varepsilon_{64}), \tag{97}$$

where $\tau^{(0)}_{E_8|S}$ is given by (92). That is, "$(X_{E_8}$ is $E_8)$ is $\tau^{(1)}_{E_8|S}$".

e) For the fifth rule in KBA_6, first we combine information about E_{11} from its first and second rules by (27), namely,

$$\tau^{(12)}_{E_{11}|S} = ++(\tau^{(1)}_{E_{11}|S}, \tau^{(2)}_{E_{11}|S}), \tag{98}$$

where $\tau^{(1)}_{E_{11}|S}$ and $\tau^{(2)}_{E_{11}|S}$ are given by (94) and (95), respectively. That is, "$(X_{E_{11}}$ is $E_{11})$ is $\tau^{(12)}_{E_{11}|S}$". Thus, by (47)

$$\begin{aligned} \tau^{(0)}_{H|S} &= \bowtie_1 (\tau_{E'_8|S}, \tau_{E_8}, \tau_{E'_{11}|S}, \tau_{E_{11}}, \tau_H, \tau_{RS}, \varepsilon) \\ &= \bowtie_1 (\tau^{(1)}_{E_8|S}, \tau_{E_8}, \tau^{(12)}_{E_{11}|S}, \tau_{E_{11}}, \tau_H, \tau_{RS_{65}}, \varepsilon_{65}), \end{aligned} \tag{99}$$

where $\tau^{(1)}_{E_8|S}$ and $\tau^{(12)}_{E_{11}|S}$ are given by (97) and (98), respectively. That is, "$(X_H$ is $H)$ is $\tau^{(0)}_{H|S}$".

f) By (9), the sixth rule in KBA_6 is equivalent to

$$((X_{E_{12}} \text{ is } E_{12}) \text{ is } \tau_{E_{12}}){:}\ (((X_{E_{13}} \text{ is } E_{13}) \text{ is } \tau_{E_{13}}) \wedge ((X_H \text{ is } H) \text{ is } \tau_H))_1$$
$$\xrightarrow{\mathit{true}} (X_H \text{ is } H) \text{ is } \tau_H.$$

Thus, by (62) and (80) we have

$$\begin{aligned} \tau^{(1)}_{H|S} &= \ddagger_1(\tau_{E'_{12}|S}, \tau_{E_{12}}, \tau_{E'_{13}|S}, \tau_{E_{13}}, \tau_{H'|S}, \tau_H, \tau_{RS}, \varepsilon) \\ &= \ddagger_1(\tau_{E_{12}|S}, \tau_{E_{12}}, \text{"}\mathit{undecided}\text{"}, \tau_{E_{13}}, \tau^{(0)}_{H|S}, \tau_H, \text{"}\mathit{true}\text{"}, 1), \end{aligned} \tag{100}$$

where $\tau_{E_{12}|S}$ and $\tau^{(0)}_{H|S}$ are given by (96) and (99), respectively. That is, "$(X_H$ is $H)$ is $\tau^{(1)}_{H|S}$".

Finally, the multi-agent system outputs the conclusion "$(X_H$ is $H)$ is $\tau^{(1)}_{H|S}$" to the end user who inputs some information to this multi-agent system (see Figure 4).

To summarise, at the beginning an end user offers some information to the multi-agent system of $KBA_1, \cdots, KBA_6$, then the six agents draw individually some conclusions from their own knowledge according to the offered information and exchange these conclusions, and at the end a final conclusion is drawn out and sent back to the end user (see Figure 4). Actually, in negotiation scenario (see Figure 2 as well as Figure 4) the end user could be the negotiation opponent agent of the multi-agent negotiation team. Thus, the above procedure will be repeated until the opponent agent cannot offer any more information and the multi-agent system cannot draw any new conclusion.

7. Related Work

Through comparison with related work, this section explains how the model developed in this paper advances the state of the art in two research fields: uncertainty reasoning and heterogeneous uncertain information sharing in multi-agent systems.

7.1. *Uncertainty in Default Reasoning*

In everyday life, when estimating uncertainties, people are often reluctant to provide real numbers chosen from a predefined range, instead they prefer to give qualitative estimates. A first step in solving this problem has been suggested in several interval-valued approaches [84,85,96,2,3,4,10,100,125], in which *ignorance* can be explicitly represented (but not in single-valued approaches). Nevertheless, these approaches are still unable to deal with *imprecision*, a very important kind of uncertainty in artificial intelligence and, especially in knowledge-based systems. If we employ linguistic terms, such as *very true, true, fairly false, etc.*, to assess uncertainty, the problem can be solved more easily. This is because these terms allow for the integration of both the *vagueness* and the *uncertainty* caused by imprecise or vague information. Consequently, a second step in solving this problem is proposed in some linguistic-valued models (*e.g.* [60]). Accordingly, based upon fuzzy mathematical theory, this paper develops an alternative uncertainty scheme for default reasoning with linguistically expressed uncertainty. Actually, our model can be seen as a unified framework for fuzzy reasoning [120], truth-qualification uncertain reasoning [60], fuzzy default reasoning and truth-qualification uncertain default reasoning.

The work in the current paper is a sequel to a series of work [97,46,83,126,48] we have undertaken in this broad area. It is different from this work in a number of important ways. **First,** in our model, uncertainty, incompleteness and fuzziness can be handled simultaneously. This is not the case in other previous work. Previous work on uncertain reasoning is mainly concerned with single-valued uncertain reasoning models (such as the certainty factor model [99,65], the subjective Bayesian method [14], and Bayesian networks [78]), and two-valued uncertain reasoning models (such as the INFERNO model [84,85], the Fril model [2,3,4], Dempster and Shafer's evidence theory [96], the approach of possibility and necessity measures [10], the model suggested by Simoff [100] and the interval certainty factor model [125]). Unlike the model presented in this paper, these models cannot take into account commonsense knowledge in the form of default rules along with its uncertainty. In other words, in these models reasoning can be performed only under uncertainty rather than under incompleteness at the same time. **Second,** even if the issues related to both incompleteness and uncertainty in reasonings are involved in some papers, commonsense knowledge (especially with respect to fuzzy propositions) in the form of default rules with uncertainty has not been coped with in these papers. In fact, in Yager's work [113,114,115,116,117], the theory and approach for uncertainty are solely employed to represent and process nonmonotonic reasoning. In other words, uncertainties in nonmonotonic reasoning have not yet been adequately dealt with. In MILORD [21], so-called nonmonotonic reasoning is also considered, but simply as a mechanism to modify (decrease) the certainty of a fuzzy proposition. **Third**, our model is an explicit unification of uncertainty, incompleteness and fuzziness. Pearl [78] tried to encode uncertainty and incompleteness in a unified framework, but he did not explicitly present an approach for modelling uncertainty and incompleteness at the same time. Besides, facts can be uncertain in our proposed model, but not in Pearl's system. Alternatively, unlike the proposed model in our paper, Pearl's model cannot take the result *from an uncertain reasoning agent* to perform further reasoning in heterogeneous reasoning environments. There are also some other differences between our model and Pearl's system: 1) propositions are fuzzy in our model, but not in Pearl's system; 2) uncertainties are estimated by linguistic terms in our model, whereas in Pearl's system by single-values; and 3) the propagation mechanism for uncertainties in our model relies on fuzzy mathematics, while that in Pearl's system is based on probability theory.

7.2. *Heterogeneous Uncertain Information Sharing*

There are a number of papers which address this topic. Initially, Zhang and Orlowska [130] show that the sets of propositional uncertainties in several well-known uncertain reasoning models with appropriate operators are semi-groups with individual unit elements. Zhang's further work [123] uses this result to establish transformation criteria based on homomorphisms, and to define transformation functions approximately satisfying these criteria. These functions work well between any two of the uncertain reasoning models used in EMYCIN [65], PROSPECTOR [14] and MYCIN [99]. Hájek [28] also tries to build an isomorphism between the certainty factor model [99] and the subjective Bayesian method [14], but he implicitly assumed that in the subjective Bayesian method [14] the unit element is always 0.5. Unfortunately, the unit element is the prior probability of a proposition, and so it should be able to *vary* with different propositions.

Later on, Parsons and Saffiotti [73] try to use a qualitative method to deal with the issue of transformation between different uncertain reasoning models. They outline a kind of interlingua which is weak enough to be subsumed by different uncertainty representation languages. However, their interlingua is just qualitative and too weak to produce as accurate results as quantitative methods. In fact, their interlingua can only be used to express that a value increases, decreases or does not change.

Recently, we have also addressed this issue. Firstly, we solve the problem of information sharing among some existing models:

- *Probability-based model and pseudo-probability model.* We developed methods of finding the prior probabilities of nodes of an inference network so that a probability-based model and a pseudo-probability model, *i.e.*, the Bayesian network and the certainty factor model, can share information [128].

- *Default reasoning model and uncertain reasoning model.* We extend the certainty factor model to the interval certainty factor model [125] and embed [47] the interval certainty factor model into a conventional default model [89]. In this way, we solve the problem of information sharing among the certainty factor model, the interval certainty factor model and the default reasoning system.

- *One and two dimensional uncertain reasoning models.* We [124,129,54,58] discovered a class of perfect transformation functions between the certainty factor model [99,65] and the subjective Bayesian method [14], that enables these two models to share information. Then, we extend the subjective Bayesian method into the interval subjective Bayesian method [127,43]. Since the subjective Bayesian method is a special case of the interval subjective Bayesian method, they are able to share information. Finally, we provide reasonable transformation functions between the interval certainty factor model and the interval subjective Bayesian method, and make it possible for them to share information.

Therefore, these most popular heterogeneous reasoning models can be integrated in multi-agent systems, and cooperate to solve practical problems (*e.g.*, automated group negotiations). In addition, this part of our research is generalised, and a foundation for uncertain reasoning models in rule-based systems is constructed [55].

Secondly, we make it possible for some existing models to share information with potential uncertain reasoning models. Currently, probability theory is widely regarded as the most important foundation for uncertain reasoning. So most potential models for uncertain reasoning should be built on the probability framework. Thus, if we can rebuild some existing models on the probability framework, the problem of information sharing between existing models and potential models can be solved in this way. This is similar to unifying the currency units in Europe. Based on this idea, we

- normalise [48,49,50,56] the relationship between the certainty factor model and probability theory, and

- improve [44,45,51,52,53] the subjective Bayesian method so that the model is completely consistent with probability.

In our work mentioned above, for crisp positions, we investigate some issues related to information sharing among heterogeneous uncertain reasoning models in multi-agent systems, while Nakamatsu [69] tried to solve the problem of information sharing among different nonmonotonic reasoning models. In this paper, our hybrid model bridges Zadeh's fuzzy approximate reasoning [120], Mantaras' fuzzy uncertain reasoning [60], Reiter's fuzzy nonmonotonic reasoning, as well as their combination. In other words, this paper solves the problem of information sharing among the sorts of reasoning models with respect to fuzzy propositions. The proposed approach is not only of great practical importance for knowledge-based multi-agent systems, but also of great theoretical importance, since it actually reveals a relationship among these heterogeneous reasoning models.

8. Conclusions and Future Work

This paper makes two important contributions. The *first* contribution is that we have developed a hybrid model to deal with uncertainty in default reasoning, and from this hybrid model we have derived a fuzzy default reasoning system. The *second* is that we solve the problem of information sharing among a fuzzy approximate reasoning model, a kind of uncertain reasoning model, a fuzzy default reasoning model and a kind of uncertain default reasoning. This builds a valuable foundation for a wide rang of multiple knowledge-based agents' applications such as automated group negotiations.

This work also opens up a number of new possibilities for further work. For example:

- the study of the presented model from a logical point of view (*e.g.*, to build a logical system based on the presented model and study the system's logical properties);
- the investigation of a way to embed other uncertain reasoning models (*e.g.*, Pearl's Bayesian network [78]) into the default reasoning system;
- further research on the problem of information sharing among heterogeneous uncertain reasoning models, and finally to solve the problems of information sharing among various heterogeneous uncertain, nonmonotonic and fuzzy approximate reasoning models; and
- apply the proposed method into practical problems such as automated group negotiation in business domain.

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References

1. J. F. Baldwin, "A new approach to approximate reasoning using a fuzzy logic", *Fuzzy Sets and Systems* **2** (1979), 309–325.
2. J. F. Baldwin, "Support logic programming", in *Fuzzy Set Theory and Applications*, eds. A. Jones, H. Kaufman and H. Zimmermann (1986) pp. 133–171.
3. J. F. Baldwin, "Evidential support logic programming", *Fuzzy Sets and Systems* **14** (1987) 1–26.
4. J. F. Baldwin, "Evidential support logic in Fril and case based reasoning", *International Journal of Intelligent Systems* **8(9)** (1993) 939–960.
5. M. Bloemeke, "Agent encapsulated Bayesian networks", Ph.D. Thesis, Department of Computer Science, University of South Carolina, 1998.
6. P. P. Bonissone and K. S. Decker, "Selecting uncertainty calculi and granularity: an experiment in trading-off precision and complexity", in *Uncertainty in Artificial Intelligence*, eds. L. N. Kanal and J. F. Lemmer (Elsevier, 1986) pp. 217–247.
7. W. Brenner, R. Zarnekow and H. Wittig, *Intelligent Software Agents: Foundation and Applications* (Springer, 1998).
8. S. M. Brown, E. Santos Jr. and S. B. Banks, "A dynamic Bayesian intelligent interface agent", *Proceedings of the Sixth International Interfaces Conference*, 1997, pp. 118-120.
9. S. M. Brown, E. S. Jr. and S. B. Banks, "Utility theory-based user models for intelligent interface agents", in *Advances in Artificial Intelligence*, eds. R. E. Merler and E. Neufeld, **LNAI 1118** (Springer, 1998), pp. 378–392.
10. D. Dubois and H. Prade, *Possibility Theory* (Plenum Press, 1988).
11. D. Dubois and H. Prade, "Fuzzy sets in approximate reasonings, part 2: logical approaches", *Fuzzy sets and systems* **40** (1991) 203–244.
12. D. Dubois and H. Prade, "Possibility theory as a basis for qualitative decision theory", *Proceedings of the Fourteenth International Joint Conference on Artificial Intelligence*, 1995, pp. 1924–1930.
13. D. Dubois and H. Prade, "Qualitative possibility theory and its applications to constraint satisfaction and decision under uncertainty", *International Journal of Intelligent Systems* **14** (1999) pp. 45–61.
14. R. O. Duda, P. E. Hart and N. J. Nillson, "Subjective Bayesian methods for rule-based inference systems", in *AFIPS Conference Proceedings*, **45**, (AFIPS Press, 1976) pp. 1075–1082.
15. M. El-Nasr, J. Yen and T. R. Ioerger, "FLAME—Fuzzy logic adaptive model of emotions", *Autonomous Agents and Multi-Agents* **3** (2000) 217–257.
16. L. Foulloy and S. Galichet, "Fuzzy sensors for fuzzy control", *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems* **2(1)** (1994) 55–66.
17. P. Garcia, E. Gimenez, L. Godo and J. A. Rodr íguez-Aguilar, "Possibilistic-based design of bidding strategies in electronic auctions", *Proceedings of the Thirteenth European Conference on Artificial Intelligence*, 1998, pp. 575–579.
18. M. R. Genesereth, "Knowledge interchange format", *Proceedings of the Conference of the Principle of Knowledge Representation and Reasoning*, 1991, pp. 599–600.
19. M. R. Genesereth, R. E. Fikes, *et al.*, "Knowledge interchange format: version 3.1 reference manual", Technical Report Logic-92-1, Computer Science Department, Stanford University, 1992.
20. E. Giménez-Funes, L. Godo, J. A. Rodr íguez-Aguilar and P. Garcia-Calvés, "Designing bidding strategies for trading agents in electronic auctions", *Proceedings of the Third International Conference on Multi-Agent Systems*, 1998, pp. 136-143.
21. L. Godo *et al.*, "Fuzzy values in fuzzy logic", *Proceedings of the Third International Fuzzy Systems Association (IFSA) Congress* 1989, pp. 829–832,.

22. L. Godo, R. L. D. Mantaras, C. Sierro and A. Verdgauer, "MILORD: the architecture and the management of linguistically expressed uncertainty", *International Journal of Intelligent Systems* **4** (1989) 471–501.
23. A. Gómez-Pérez, "Knowledge sharing and reuse", in *The Handbook of Applied Expert Systems* (CRC Press, 1998) pp. 10-1–10-36.
24. T. Gruber, "Toward principles for the design of ontologies used for knowledge sharing", Technical Report KSL-93-04, Knowledge Systems Laboratory, Stanford University, 1993.
25. N. Guarino and P. Giaretta, "Ontologies and knowledge bases: towards a terminological clarification", in *Towards Very Large Knowledge Bases: Knowledge Building & Knowledge Sharing* (IOS Press, 1995) pp. 25–32.
26. E. H. Durfee and V. R. Lesser, "Negotiating task decomposition and allocation using partial global planning", *Distributed Artificial Intelligence* **2** (1989) 229–243.
27. N. Glaser and P. Morignot, "Societies of autonomous agents and their reorganisation", in *Nonlinear Systems Approaches to Cognitive Psychology and Cognitive Science—Dynamics, Synergetics, Autonomous Agents*, eds. Tschacher and Dauwalder (World Scientific, 1998).
28. P. Hájek, "Combining functions for certainty degrees in consulting systems", *International Journal of Man-Machines Studies* **22** (1985) 59–76.
29. R. A. Harrington, S. Banks and E. Santos Jr., "Development of an intelligent user interface for a generic expert systems", *Online Proceedings of the Seventh Midwest Artificial Intelligence and Cognitive Science Conference*, 1996, available at http://www.cs.indiana.edu/event/maics96/.
30. R. A. Harrington, S. Banks and E. S. Jr., "Uncertainty-based reasoning for a generic expert system intelligent user interface", *Proceedings of the Eighth IEEE International Conference on Tools with Artificial Intelligence*, 1996, pp. 52–55.
31. M. He and H. F. Lueng, "An agent bidding strategy based on fuzzy logic in a continuous double auction", *Proceedings of the Fifth International Conference on Autonomous Agents*, 2001, pp. 61–62.
32. M. He, H. F. Lueng and N. R. Jennings, "A fuzzy logic based bidding strategy in continuous double auctions", submitted to *IEEE Transactions on Knowledge and Data Engineering* (2001), to appear.
33. N. R. Jennings, "Joint intension as a model of multi-agent cooperation", Ph.D. Thesis, Queen Mary and Westfield College, The university of London, 1992.
34. N. R. Jennings, P. Faratin, A. R. Lomuscio, S. Parsons, C. Sierra and M. Wooldridge (2001) "Automated negotiation: prospects, methods and challenges", *International Journal of Group Decision and Negotiation* **10(2)** (2001) 199-215.
35. M. Knapik and J. Johnson, *Developing Intelligent Agents for Distributed Systems: Exploring Architecture, Technologies, and Applications* (McGraw-Hill, 1998).
36. S. Kraus and V. S. Subrahmanian, "Multiagent reasoning with probability, time, and beliefs", *International Journal of Intelligent Systems* **10** (1995) 459–499.
37. J. -J. Lee and R. McCartney, "Predicting user actions using interface agents with individual user models", in *Approaches to Intelligent Agents*, eds. H. Nakashima and C. Zhang, **LNAI 1733** (Springer, 1999), pp. 154–169.
38. V.R. Lesser and D.D. Corkill, "The distributed vehicle monitoring testbed: a tool for investigating distributed problem solving networks", *AI Magazine* **4(3)** (1983) 15–33.
39. Y. Li and C. Zhang, "Information fusion and decision making for utility-based agents", *Proceedings of the Third World Multiconference on Systemics, Cybernetics and Informatics and the Fifth International Conference on Information Systems Analysis and Synthesis*, 1999, pp. 377-384,
40. Y. Li and C. Zhang, "Information-based cooperation in multiple agent systems", in

Advanced Topics in Artificial Intelligence, ed. N. Foo, **LNAI 1747** (Springer, 1999), pp. 496–498.

41. R. Liscano, K. Baker and J. Meech, "The use of ontologies and meta-knowledge to facilitate the sharing of knowledge in a multi-agent personal communication system", in *Canadian AI2000*, eds. H. Hamilton and Q. Yang, **LNAI 1822** (Springer, 2000), pp. 187–200.
42. A. Lomuscio, M. Wooldridge and N. R. Jennings, "A classification scheme for negotiation in electronic commerce", in *Agent-Mediated Electronic Commerce: A European AgentLink Perspective*, eds. F. Dignum and C. Sierra, **LNAI 1991** (Springer, 2001), pp. 19–33.
43. X. Luo, C. Zhang and Y. Qiu, "Transformations of two dimensional uncertainty in distributed expert systems: a case study", in *ICCIMA'97—Proceedings of International Conference on Computational Intelligence and Multimedia Applications*, 1997, pp. 366–372.
44. X. Luo and C. Zhang, "The favoring-opposing cause model for uncertain reasoning", in *ICCIMA'97—Proceedings of International Conference on Computational Intelligence and Multimedia Applications*, 1997, pp. 373–379.
45. X. Luo and C. Zhang, "A solution to the problem of prior probabilities in the PROSPECTOR uncertain reasoning model", *Proceedings of the 1997 Joint Pacific Asian Conference on Expert Systems/Singapore International Conference on Intelligent Systems*, 1997, pp. 306-313.
46. X. Luo and C. Zhang, "A scheme for fuzzy default reasoning based on linguistic truth", *Proceedings of Australia-Pacific Forum on Intelligent Processing and Manufacture of Materials*, 1997, pp. 823–829
47. X. Luo and C. Zhang, "A method for simultaneously handling uncertainty associated with nonmonotonic reasoning and ignorance of knowledge", *Proceedings of the Second Australian Workshop on Commonsense Reasoning*, 1997, pp. 49–63.
48. X. Luo, C. Zhang and Y. Qiu, "Normalization of the relation between the certainty factor model and probability theory", *Proceedings of Australia-Pacific Forum on Intelligent Processing and Manufacture of Materials*, 1997, pp. 469-475.
49. X. Luo and C. Zhang, "The probability certainty factor model: an example-based approach", *Proceedings of the Sixth International Conference on Intelligent Systems*, 1997, pp. 203-208.
50. X. Luo, Y. Qiu and C. Zhang, "To make certainty factor model based on probability", *Proceedings of the Second Chinese World Congress on Intelligent Control and Intelligent Automation* , 1997, pp. 1453-1458.
51. X. Luo and C. Zhang, "ILSNCC: a PROSPECTOR-Like uncertain reasoning model", *Australia Journal of Intelligent Information Processing Systems* **4(1)** (1997) 1–11.
52. X. Luo and C. Zhang, "LSLNCF: a hybrid uncertain reasoning model based on probability", *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems* **6(4)** (1998) 401–422.
53. X. Luo and C. Zhang, "A hybrid uncertain reasoning model competitive with Bayesian networks", *Proceedings of the World Congress on Expert Systems*, 1998.
54. X. Luo, C. Zhang and H. F. Leung, "A class of isomorphic transformations for integrating EMYCIN-style and PROSPECTOR-style systems into a rule-based multi-agent system", in *Approaches to Intelligent Agent*, eds. H. Nakashima and C. Zhang, **LNAI 1733** (Springer, 1999) pp. 211–225.
55. X. Luo and C. Zhang, "An axiom foundation for uncertain reasonings in rule-based expert systems: NT-algebra", *Knowledge and Information Systems: An International Journal* **1(4)** (1999) 415–433.
56. X. Luo and C. Zhang, "Proof of the correctness of the EMYCIN sequential propaga-

tion", *IEEE Transaction on Knowledge and Data Engineering* **11(2)** (1999) 355–359.
57. X. Luo, H. F. Leung and J. H. M. Lee, "Theory and properties of a selfish protocol for multi-agent meeting scheduling using fuzzy constraints", *Proceedings of the 14th European Conference on Artificial Intelligence*, 2000, pp. 373-377.
58. X. Luo, C. Zhang and H. F. Leung, "Information sharing between heterogeneous uncertain reasoning models in a multi-agent environment: a case study", *International Journal of Approximate Reasoning* **27(1)** (2001) 27–59.
59. S. Maes, K. Tuyls and B. Manderick, "Combining influence diagrams to model agents", in *Multi-Agent Systems and Applications and Adaptability and Embodiment Using Multi-Agent Systems*, (Czech Technical University in Prague, 2001) pp. 34-39.
60. R. L. D. Mantars and L. Godo, "From intervals to fuzzy truth-values: adding flexibility to reasoning under uncertainty", *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems* **5(3)** (1997) 251–260.
61. N. Matos and C. Sierra, "Evolutionary computing and negotiating agents", *Agent-Mediated Electronic Commerce*, eds. P. Noriega and C. Sierra, **LNAI 1571** (Springer, 1998) pp. 126–150.
62. J. McCarthy, Situation, Actions, and Causal Laws, *Tech. Res. Memo 2* (Stanford Artificial Intelligence Project, 1963).
63. J. McCarthy and P. J. Hayes, "Some philosophical problems from the standpoint of artificial intelligence", in *Machine Intelligence 4*, eds. B. Meltzer and D. Michie (1996) pp. 463–502.
64. P. Mccauley-Bell, "Intelligent agent characterization and uncertainty management with fuzzy set theory: a tool to support early supplier integration", *Journal of Intelligent Manufacturing* **10** (1999) 135–147.
65. M. V. Melle, "A domain-independent system that aids in constructing knowledge-based consultation programs", Ph.D. Thesis, Report STAN-CS-8-820, Computer Science Department, Stanford University, CA, 1980.
66. C. Mudgal and J. Vassileva, "Bilateral negotiation with incomplete and uncertain information: a decision-theoretic approach using a model of the opponent", in *Cooperative Information Agents IV: The Future of Information Agents in Cyberspace*, eds. M. Klusch and L. Kerschberg, **LNAI 1860** (Springer, 2000) pp. 105-118.
67. R. Murch and T. Johson, *Intelligent Software Agents* (Prentice-Hall, 1999).
68. M. A. Musen, "Dimensions of knowledge sharing and reuse", *Computers and Biomedical Research* **25** (1992) 435–467.
69. K. Nakamatsu, "A logic characterization for non-monotonic truth maintenance systems based on an annotated logic program", *Proceedings of Australia-Pacific Forum on Intelligent Processing and Manufacture of Materials*, 1997.
70. R. Neches, R. Fikes, T. Finin, T. Gruber, R. Patil, T. Senator and W. R. Swartout, "Enabling technology for knowledge sharing", *AI Magazine* (Winter 1991) 36–56.
71. N. J. Nilsson, "Probabilistic logic", *Journal of Artificial Intelligence* **28** (1986) 71–87.
72. N. J. Nilsson, *Artificial Intelligence: A New Synthesis* (Morgan Kaufmann, 1998).
73. S. Parsons and A. Saffiotti, "Integrating uncertainty handling techniques in distributed artificial intelligence", in *Symbolic and Quantitative Approaches to Reasoning and Uncertainty*, eds. M. Clark, R. Kruse, and S. Moral (Springer, 1993) pp. 304–309.
74. S. Parsons and P. Giorigini, "On using degrees of belief in the BDI agents", *Proceedings of the International Conference on Information Processing and Management of Uncertainty in Knowledge-Based Systems*, 1998.
75. S. Parsons, C. Sierra, and N. R. Jennings, "Agents that reason and negotiate by arguing", *Journal of Logic and Computation* **8(3)** (1998) 261–292.
76. S. Parsons and P. Giorigini, "An approach to using degrees of belief in BDI agents", in *Information, Uncertainty and Fusion*, eds. B. Bouchon-Meunier, R. R. Yager and L.

A. Zadeh (Kluwer, 2000) pp. 81–92.

77. R. S. Patil, R. E. Fikes, P. F. Patel-Schneider, D. Mckay, T. Finin, T. Gruber and R. Neches, "The DARPA knowledge sharing effort: progress report", in *Readings in Agents*, eds. M. N. Huhns and M. P. Singh (Morgan Kaufmann, 1998) pp. 243–254.
78. J. Pearl, *Probabilistic Reasoning in Intelligent Systems: Networks of Plausible Inference* (Morgan Kaufmann, 1988).
79. C. D. C. Pereira, F. Carcia, J. Lang and R. Martin-Clouaire, "Planning with graded nondeterministic actions: a possibilistic approach", *International Journal of Intelligent systems* **12** (1997) 935–962.
80. J. Pinto, A. Sernadas, C. Sernadas and P. Mateus, "Non-determinism and uncertainty in the situation calculus", *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems* **8(2)** (2000) 127–149.
81. A. Poggi, "Integrating object-oriented and rule-based programming for building agent systems", *International Journal on Artificial Intelligence Tool* **7(2)** (1998) 221–238.
82. A. Preece, K. Hui, A. Gray, P. Marti, T. Bench-Capon, D. Jones, and Z. Cui, "The KRAFT architecture for knowledge fusion and transformation", *Knowledge-Based Systems* **13** (2000) 113–120.
83. Y. Qiu, Y. Tang and X. Luo, "Nonmonotonic linguistic certainty factor model and properties", *Proceedings of the Joint Pacific Asian Conference on Expert Systems/Singapore International Conference on Intelligent Systems*, 1997, pp. 758–763.
84. J. R. Quinlan, "INFERNO: a cautious approach to uncertain inference", *The Computer Journal* **26(3)** (1983) 255–269.
85. J. R. Quinlan, "Consistency and plausible reasoning", *Proceedings of the Eighth of International Joint Conference on Artificial Intelligence*, 1983, pp. 137–144.
86. H. Raiffa, *The Art and Science of Negotiation* (Harvard University Press, 1982).
87. A. S. Rao and M. P. Georgeff, "Modelling rational agents within a BDI-architecture", *Proceedings of the Second International Conference on Principles of Knowledge Representation and Reasoning*, 1991, pp. 473-484.
88. A. S. Rao and M. P. Georgeff, "BDI agents: from theory to practice", *Proceedings of the First International Conference on Multi-Agent Systems*, 1995, pp. 312-319.
89. R. Reiter, "On reasoning by default", *Proceeding of the Fifth International Joint Conference on Artificial Intelligence*, 1977, pp. 223–277.
90. D. Pruitt, *Negotiation Behavior* (Academic Press, 1981).
91. S. Russell and P. Novig, *Artificial Intelligence: A Modern Approach* (Prentice-Hall, 1995).
92. T. Schiex, H. Fargier and G. Verfaillie, "Valued constraint satisfaction problems: hard and easy problems", *Proceedings of the Fourteenth International Joint Conference on Artificial Intelligence*, 1995, pp. 631–637.
93. K. J. Schmucker, *Fuzzy Sets, Natural Language Computations, and Risk Analysis* (Computer Press, 1984).
94. M. Schroeder and G. Wagner, "Vivid agents: theory, architecture, and applications", *Applied Artificial Intelligence* **14** (2000) 645–675.
95. R. Shachter, "Probabilistic inference and influence diagrams", *Operations Research* **36(4)** (1988) 589–604.
96. G. Shafer, *A mathematical theory of evidence* (Princeton University Press, 1976).
97. Z. Shi and X. Luo, "NLPM: an approach for non-monotonic reasoning based on linguistic probability", in *Automated Reasoning*, ed. Z. Shi (North-Holland, 1992) pp. 107–121.
98. Y. Shoham, "What we talk about when talk about software agents", *IEEE Intelligent Systems* (March/April 1999) 28–31.
99. E. H. Shortliffe and B. G. Buchanan, "A model of inexact reasoning in medicine", *Mathematical Biosciences* **23** (1975) pp. 351–379.

100. S. J. Simoff, "Handling uncertainty in knowledge-based systems: an interval approach", *Proceedings of the Eighth Australian Joint Conference on Artificial Intelligence*, 1995, pp. 371–378.
101. K. Su, X. Luo, H. Wang, C. Zhang, S. Zhang and Q. Chen, "A logical framework for knowledge sharing in multi-agent systems", in *Computing and Combinatorics*, ed. J. Wang, **LNCS 2108** (Springer, 2001) pp. 561–570.
102. K. P. Sycara, "The many faces of agents", *AI Magazine* **19(2)** (1998) 11–12.
103. S. Thiébaux, J. Hertzberg, W. Shoaff and M. Schneider, "A stochastic model of actions and plans for anytime planning under uncertainty", *International Journal of Intelligent Systems* **10** (1995) 155–183.
104. C. -Y. Tyan, P. P. Wang and D. R. Bahler, "The design of an adaptive multiple agent fuzzy constraint-based controller (MAFCC) for a complex hydraulic system", *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems* **4(6)** (1996) 537–551.
105. M. Valtorta, Y. -G. Kim and J. Vomlel, "Soft evidential update for probabilistic multiagent systems", *International Journal Approximate Reasoning* **29(1)** (2002) 71–106.
106. L. Valverde and E. Trillas, "On modus ponens in fuzzy logic", *Proceedings of the Fifteenth International Symposium on Multi-Valued Logic*, 1985, pp. 294–301.
107. S. K. M. Wang and C. J. Butz, "Probabilistic reasoning in a distributed multi-agent environment", *Proceedings of the Third International Conference on Multi-Agent Systems*, 1998, pp. 341–348.
108. S. K. M. Wang and C. J. Butz, "Constructing the dependency structure of a multiagent probabilistic network", *IEEE Transactions of Knowledge and Data Engineering* **13(3)** (2001) 395–415.
109. M. Wooldridge, *Reasoning about Rational Agents* (The MIT Press, 2000).
110. M. Wooldridge, "Foundations of intelligent agents", in *Working Notes of the Second European Agent Systems Summer School*, eds. M. Klusch, K. Fischer, and M. Luck (2000).
111. Y. Xiang, "A probabilistic framework for cooperative multi-agent distributed interpretation and optimization of communication", *Artificial Intelligence* **87** (1996) 295–342.
112. P. Xuan and V. Lesser, "Incorporating uncertainty in agent commitments", in *Intelligent Agent VI: Agent Theories, Architectures, and Languages*, eds. N. R. Jennings and Y. Lespérance, **LNAI 1757** (Springer, 2000) pp. 57–70.
113. R. R. Yager, "Using approximate reasoning to represent default knowledge", *Artificial Intelligence* **31** (1987) 99–122.
114. R. R. Yager, "Probabilistic qualification and default rules", in *Uncertainty in Knowledge-Based Systems*, eds. B. Bouchon and R. R. Yager (Springer, 1987) pp. 41–57.
115. R. R. Yager, "Default and approximate reasoning", *Proceedings of the International Fuzzy Sets Association Conference*, 1987, pp. 690–692.
116. R. R. Yager, "On the representation of commonsense knowledge by possibilistic reasoning", *International Journal of Man-Machine Studies* **31** (1989) 587–610.
117. R. R. Yager, "On usual values in commonsense reasoning", *Fuzzy Sets and Systems* **30** (1989) 239–255.
118. R. R. Yager, "Penalizing strategic preference manipulation in multi-agent decision making", *IEEE Transactions on Fuzzy Systems* **9(3)** (2001) 393–403.
119. L. A. Zadeh, "Fuzzy Sets", *Information and Control* **8** (1965) 338–353.
120. L. A. Zadeh, "The concept of a linguistic variable and its application to approximate reasoning", Part I, *Information Science* **8** (1975) 199–249; Part II, *Information Science* **8** (1975) 301–357; Part III, *Information Science* **9** (1975) 43–80.
121. D. Zeng and K. Sycara, "Bayesian learning in negotiation", *International Journal of Human-Computer Studies* **48(1)** (1998) 125-141.

122. C. Zhang, "Cooperation under uncertainty in distributed expert systems", *Artificial Intelligence* **56** (1992) 21–69.
123. C. Zhang, "Heterogeneous transformation of uncertainties of propositions among inexact reasoning models", *IEEE Transactions on Knowledge and Data Engineering* **6(3)** (1994) 353–360.
124. C. Zhang and X. Luo, "Heterogeneous transformation of uncertainties of propositions among the EMYCIN and PROSPECTOR uncertain models", *Proceedings of the Second International Conference on Multi-Agent Systems*, 1996, p. 465.
125. C. Zhang and X. Luo, "The interval-based EMYCIN uncertain reasoning model", *Proceedings of Australia-Pacific Forum on Intelligent Processing and Manufacture of Materials*, 1997, pp. 462–468,
126. C. Zhang and X. Luo, "A framework for default reasonings based on linguistic terms", *Proceedings of the Sixth International Conference on Intelligent Systems*, 1997, pp. 148–152,
127. C. Zhang and X. Luo, "An issue on transformation of interval-based uncertainty in distributed expert systems", *Poster Papers Proceedings of the Tenth Australian Joint Conference on Artificial Intelligence*, 1997, pp. 38–43.
128. C. Zhang and X. Luo, "Transformation between the EMYCIN model and the Bayesian network", in *Agents and Multi-Agent Systems Formalisms, Methodologies and Applications*, eds. W. Wobcke, M. Pagnucco and C. Zhang, **LNAI 1441** (Springer, 1998) pp. 205–219.
129. C. Zhang and X. Luo, "Isomorphic transformations of uncertainties for incorporating EMYCIN-style and PROSPECTOR-style systems into a distributed expert system", *Journal of Computer Science and Technology* **14(4)** (1999) 368–392.
130. C. Zhang and M. Orlowska, "On algebraic structures of inexact reasoning models", in *Advances in Information Systems Research*, eds. G. E. Lasker, *et al.* (1991) pp. 58–77.