PERFORMANCE OF ERRORS-AND-ERASURES DECODED REED-SOLOMON CODES OVER FREQUENCY-SELECTIVE RAYLEIGH FADING CHANNELS USING M-ARY ORTHOGONAL SIGNALING

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ABSTRACT

The performance of Reed-Solomon (RS) codes is investigated over frequency-selective Rayleigh fading channels using M-ary orthogonal signaling schemes. 'Errors-and-erasures' decoding (E^2D) is considered, where erasures are judged based on Viterbi's ratio threshold test (RTT) and on the basis of the output likelihood ratio threshold test (LRT^2). The LRT^2 technique is compared with Viterbi's RTT, and both of these are compared to receivers using 'error-correction only' decoding (ECOD) over frequency-selective Rayleigh-fading channels. The numerical results show that upon using E^2D, RS codes of a given code rate can achieve higher coding gain, than that without erasure information, and that the LRT^2 technique outperforms the RTT, provided that both schemes are operated at the optimum decision thresholds.

1. INTRODUCTION

Forward error-correction (FEC) is often used for mitigating the channel effects in wireless communications. For so-called 'errors-and-erasures' decoding (E^2D) schemes [1], usually erasures are preferable to error correction, since typically more erasures than errors can be corrected. Hence, it is advantageous to determine the reliability of the received symbols and erase the low-reliability symbols prior to the decoding process. There are a number of methods for generating reliability-based information and their performance has been analyzed for example in [2]-[5]

In this contribution we consider the properties of the so-called ratio threshold test (RTT), which was originally proposed by Viterbi [3] and those of the likelihood ratio threshold test (LRT^2), which is defined during our further discourse. Both of them are then invoked in the context of M-ary orthogonal signaling, in order to generate channel-quality related information. Viterbi's RTT [3] was originally proposed for mitigating partial-band interference or multitone interference. Kim and Stark [5] have invoked it also for mitigating the effect of Rayleigh-fading and have analysed the performance of Reed-Solomon (RS) codes using E^2D. In this paper, we investigate the performance of RS codes [6], when M-ary orthogonal signaling is employed in conjunction with RTT or LRT^2 based detection over frequency-selective Rayleigh-fading channels. We study the probability density functions (PDF) of both the RTT and the LRT^2 at the demodulator's output conditioned on both the correct detection and erroneous detection of the M-ary signals. These PDFs are then used to derive the expressions of the codeword decoding error probability (CW-DEP). The CW-DEP of RS codes using E^2D employing RTT or LRT^2 is then estimated and compared with that of using 'error-correction only' decoding (ECOD) without side information. Furthermore, we also estimate and compare the optimum code rate for RS codes, upon employing different decoding schemes and different diversity combining arrangements.

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2. ERASURE INSERTION TEST

Let H_1 and H_0 represent the hypotheses that a received symbol is demodulated correctly and erroneously, respectively, according to a given optimum detection criterion, such as the *maximum a-posteriori probability* (MAP), *maximum likelihood* (ML) or *minimum error probability*, etc. We refer to this detection of data as the 1-st stage decision, as indicated in Fig.1. Let us denote the variable subjected to an erasure insertion decision by $\mathbf{Y}$. Given that H_i $(i=0,1)$ was stipulated, $\mathbf{Y}$ has a conditional PDF of $f(\mathbf{y}|H_i)$. Then, the erasure insertion strategy can be formulated as a 2-nd stage decision concerning erasure insertion, in order to distinguish between the hypotheses of:

H_0 : Erroneous demodulated symbol: insert an erasure

H_1 : Correct demodulated symbol: output an RS symbol.

Let the observation space be denoted by $\mathcal{R}$ and assume that $\mathcal{R}_0$ and $\mathcal{R}_1$ are the sets of values in $\mathcal{R}$ that map into the decisions H_0 and H_1, respectively, where $\mathcal{R}_0 \cup \mathcal{R}_1 = \mathcal{R}$. Let P_c, P_t, and P_e represent the correct RS symbol probability, symbol error probability and symbol erasure probability, respectively, after the 2-nd stage decision of Fig.1 but before RS decoding. Then these probabilities can be expressed as:

$$P_c = P(H_1) \int_{\mathcal{R}_1} f(\mathbf{y}|H_1) dy, \tag{1}$$

$$P_t = P(H_0) \int_{\mathcal{R}_1} f(\mathbf{y}|H_0) dy, \tag{2}$$

$$P_e = P(H_0)\int_{\mathcal{R}_0} f(\mathbf{y}|H_0)dy + P(H_1)\int_{\mathcal{R}_0} f(\mathbf{y}|H_1)dy, \quad (3)$$

which obey the relationship of

$$P_e = 1 - P_c - P_t. \quad (4)$$

According to Eq.(3) the RS-symbol erasure probability is constituted by two terms. The first term is based on the hypothesis of H_0, ie when a RS symbol was detected erroneously and hence erasure is required, while the second term accrues from the unintentional erasure of a RS symbol, which was detected correctly, due to its mapping into $\mathcal{R}_0$. Consequently, in order to minimize the RS symbol decoding error probability using E^2D, the optimum erasure insertion strategy to minimize the CW-DEP is that of maximizing the erasure probability under the hypothesis H_0 - which corresponds to the first term of Eq.(3) - and, simultaneously, minimizing the erasure probability under the hypothesis H_1, which corresponds to the second term of Eq.(3). The minimization of Bayes' risk can be invoked, in order to obtain the decision subspaces of $\mathcal{R}_0$ and $\mathcal{R}_1$. Bayes' risk is defined as [7]:

$$E(C) = \sum_{i=0}^{1}\sum_{j=0}^{1} C_{ij}P(H_i|H_j)P(H_j), \quad (5)$$

where C_{ij} is the so-called cost associated with erroneously deciding upon H_i, when H_j is true, $P(H_i|H_j)$ is the conditional probability that indicates the probability of erroneously deciding upon H_i, when H_j is true and, finally, $P(H_j)$ is the *a-priori probability* of H_j.

Let us assume that there is no performance cost, if the decision H_i, $(i = 0, 1)$ is correct, ie $C_{11} = C_{00} = 0$, and let $C_{10} = 1$ and $C_{01} = \theta$. Then the detector that minimizes Bayes' risk [7] opts for H_1, if

$$L_K = \frac{f(\mathbf{y}|H_1)}{f(\mathbf{y}|H_0)} \geq \frac{P(H_0)}{\theta P(H_1)} = \eta. \quad (6)$$

Otherwise, the detector decides H_0, if Eq.(6) is not obeyed, and hence we insert an erasure.

The decoding performance of RS codes can be quantified in terms of the CW-DEP, P_E. If we assume that the positions of RS symbol errors and symbol erasures within a codeword are independent, for example due to sufficiently long interleaving, then the CW-DEP, P_E of the RS(n,k) codes can be expressed in the form of [4]:

$$P_E = \sum_{i=0}^{n}\sum_{j=j_0(i)}^{n-i}\binom{n}{i}\binom{n-i}{j}P_t^iP_e^j(1-P_t-P_e)^{n-i-j}, \quad (7)$$

where $j_0(i) = \max\{0, n-k+1-2i\}$, and P_t, P_e represent the symbol error probability and symbol erasure probability before RS decoding, respectively, which are given by Eq.(2) and Eq.(3). Eq.(7) lends itself to the computation of the CW-DEP, if the code is not excessively long. However, if the RS codewords are long, the well-known Gaussian approximation can be invoked, and consequently, the CW-DEP of RS(n,k) codes can be approximated as [5]:

$$P_E = Q(\eta_1) - Q(\eta_2), \quad (8)$$

where $Q(x)$ is the Gaussian Q-function, which is defined as $Q(x) = \frac{1}{\sqrt{2\pi}}\int_x^\infty \exp\left(-\frac{t^2}{2}\right)dt$, and

$$\eta_1 \approx \frac{\sqrt{n}(1-R_c-P_e-2P_t)}{\sqrt{1-P_e-(1-P_e-2P_t)^2}}, \quad (9)$$

$$\eta_2 \approx \frac{\sqrt{n}(1-P_e-2P_t)}{\sqrt{1-P_e-(1-P_e-2P_t)^2}}, \quad (10)$$

where $R_c = k/n$ is the code rate.

Above we have developed the erasure insertion theory for the E^2D of RS codes. It was argued in [8] that the non-binary RS code symbols are amenable to transmission using M-ary orthogonal signaling schemes. For example, an M-ary orthogonal signaling scheme using $M = 64$, ie 6-bit symbols, has been proposed for the reverse link of IS-95 [9]. Hence, our following analysis will focus on studying the performance of RS codes using E^2D in the context of M-ary orthogonal signaling schemes.

3. ERASURE INSERTION USING VITERBI'S RATIO THRESHOLD TEST

Consider the wireless communication system of Fig.1 using M-ary orthogonal signaling over an independently and slowly fading dispersive Rayleigh channel, having L resolvable multipath components. Each signaling waveform in the symbol interval $[0, T)$ is equiproable and contains the same energy ξ. The received signal is corrupted by additive white Gaussian noise (AWGN) having double-sided power spectral density of $N_0/2$. The noise associated with each diversity component is assumed to be independent and identically distributed (iid). The optimum receiver for each diversity branch is a matched-filter followed by a square-law envelope detector [7] as shown in Fig.1.

Let U_{il}, $i = 1, 2, \ldots, M$, $l = 1, 2, \ldots, L$ be the output of the square-law envelope detector of Fig.1 for the ith symbol on the lth diversity channel. Assume that the first element of the symbol alphabet is sent. Then, the decision variables $(U_1, U_2, \ldots, U_M)$ after equal gain combining (EGC) can be expressed as [8] (pp.788):

$$U_1 = \sum_{l=1}^{L}|2\xi\alpha_l e^{-j\varphi_l} + N_{l1}|^2, \quad (11)$$

$$U_i = \sum_{l=1}^{L}|N_{li}|^2, \; i = 2, 3, \ldots, M, \quad (12)$$

where N_{li} is a complex zero-mean Gaussian random variable with variance $4\xi N_0$, and $\alpha_l e^{-j\varphi_l}$ represents the complex channel coefficient, which is also a complex zero-mean Gaussian random variable with variance $E[\alpha^2]$, where $E[\cdot]$ represents the expected value of the argument. Consequently, the PDF of the decision variables U_1 and U_i, $i = 2, 3, \ldots, M$ are chi-square distributed with $2L$ degree of freedom [8](pp.784). After normalization by $4\xi N_0$ the PDFs of U_1 and U_i, $i = 2, 3, \ldots, M$ can be expressed upon modifying Proakis' approach [8](pp.784) as:

$$f_{U_1}(x) = \frac{1}{(1+\overline{\gamma}_0)^L(L-1)!}x^{L-1} \cdot \exp\left(-\frac{x}{1+\overline{\gamma}_0}\right), \; x \geq 0, \quad (13)$$

$$f_{U_i}(x) = \frac{1}{(L-1)!} x^{L-1} \exp(-x), \ x \geq 0, \quad (14)$$

for $i = 2, 3, \ldots, M$, where $\overline{\gamma}_0 = \frac{\xi E[\alpha_l^2]}{N_0} = \frac{\xi E[\alpha^2]}{N_0}$ is the average signal-to-noise (SNR) ratio per diversity channel. The probability of error after MLD, ie the *a-priori probability* of the erroneous decision hypothesis H_0, as we discussed it in Section 2, is given by [8](pp.789, Eqs.(14-4-44)), which is expressed as:

$$P(H_0) = 1 - \int_0^\infty \frac{1}{(1+\overline{\gamma}_0)^L (L-1)!} y^{L-1} \cdot \exp\left(-\frac{y}{1+\overline{\gamma}_0}\right) \left[1 - \exp(-y) \sum_{k=0}^{L-1} \frac{y^k}{k!}\right]^{M-1} dy. \quad (15)$$

The *a-priori probability* of the correct decision hypothesis H_1 is given by:

$$P(H_1) = 1 - P(H_0). \quad (16)$$

Viterbi's RTT is defined as [3]:

$$\lambda = \frac{Y_1 =^1 \max\{U_1, U_2, \ldots, U_M\}}{Y_2 =^2 \max\{U_1, U_2, \ldots, U_M\}}, \quad (17)$$

where $Y_1 =^1 \max\{\cdot\}$ and $Y_2 =^2 \max\{\cdot\}$ represent the maximum and the 'second maximum' of the decision variables of $\{U_1, U_2, \ldots, U_M\}$, respectively. The PDFs of the RTT under the hypotheses of H_1 and H_0 can be derived as:

$$f_\lambda(y|H_1) = \frac{1}{[P(H_1)]^2 P(Y_2 < Y_1|H_1)} \cdot \frac{M-1}{(1+\overline{\gamma}_0)^L [(L-1)!]^2} y^{L-1} \cdot \int_0^\infty x^{2L-1} \exp\left(-x - \frac{xy}{1+\overline{\gamma}_0}\right) [1 - \Psi(xy)]^{M-1} [1-\Psi(x)]^{M-2} \Psi\left(\frac{x}{1+\overline{\gamma}_0}\right) dx, \ y \geq 1, \quad (18)$$

$$f_\lambda(y|H_0) = \frac{1}{[P(H_0)]^2 P(Y_2 < Y_1|H_0)} \cdot \frac{(M-1)^2}{[(L-1)!]^2} y^{L-1} \cdot \int_0^\infty x^{2L-1} \exp(-xy) [1-\Psi(xy)]^{M-2} \cdot \left[1 - \Psi\left(\frac{xy}{1+\overline{\gamma}_0}\right)\right] \Psi(x)[1-\Psi(x)]^{M-3} \cdot \left\{\frac{1}{(1+\overline{\gamma}_0)^L} \exp\left(-\frac{x}{1+\overline{\gamma}_0}\right) [1-\Psi(x)] + (M-2)\exp(-x)\left[1 - \Psi\left(\frac{x}{1+\overline{\gamma}_0}\right)\right]\right\} dx, \ y \geq 1, \quad (19)$$

where the short-hand $\Psi(t)$ was defined as:

$$\Psi(t) = \exp(-t) \sum_{k=0}^{L-1} \frac{t^k}{k!}, \quad (20)$$

and $P(Y_2 < Y_1|H_i)$, $i = 0, 1$ is the probability of $Y_2 < Y_1$ conditioned on the hypothesis H_i.

Let λ_T be a pre-set threshold invoked, in order to erase the low-reliability RS code symbols. Then for the RTT, P_c, P_t can be derived with the aid of Eqs. (1) and (2) as follows:

$$P_c = P(H_1) \cdot \int_{\lambda_T}^\infty f_\lambda(y|H_1) dy, \quad (21)$$

$$P_t = P(H_0) \cdot \int_{\lambda_T}^\infty f_\lambda(y|H_0) dy, \quad (22)$$

and the erasure probability P_e can be derived from Eq.(4). Finally, the CW-DEP P_E can be found by substituting P_t, and P_e into Eq.(7) or Eq.(8).

4. ERASURE INSERTION USING THE DEMODULATION OUTPUT LIKELIHOOD RATIO THRESHOLD TEST

While studying the characteristics of Viterbi's RTT over frequency-selective Rayleigh fading channels, we observed that the distributions of the maxima of the decision variables, ie the demodulator's output using MLD under the hypotheses of H_1 and H_0 also exhibit distinguishable characteristics and hence can be used in making Rs symbol erasure insertion decisions.

Let the demodulator's output in Fig.1 be denoted by Y, where $Y = \max\{U_1, U_2, \ldots, U_M\}$. Then the distributions of Y under the hypotheses H_1 of correct detection and H_0 of erroneous detection can be expressed as:

$$f_Y(y|H_1) = \frac{1}{P(H_1)} \cdot \frac{1}{(1+\overline{\gamma}_0)^L (L-1)!} y^{L-1} \exp\left(-\frac{y}{1+\overline{\gamma}_0}\right) [1-\Psi(y)]^{M-1}, \ y \geq 0, \quad (23)$$

$$f_Y(y|H_0) = \frac{1}{P(H_0)} \cdot \frac{M-1}{(L-1)!} y^{L-1} \exp(-y) [1-\Psi(y)]^{M-2} \left[1 - \Psi\left(\frac{y}{1+\overline{\gamma}_0}\right)\right], \ y \geq 0, \quad (24)$$

where $P(H_1)$ and $P(H_0)$ represent the correct and erroneous detection probabilities, or the *a-priori probabilities* of the 2nd stage detection of H_1 and H_0, which were given by Eq.(16) and Eq.(15), respectively, while $\Psi(y)$ is given by Eq.(20).

Consequently, for a given decision threshold Y_T, the correct RS symbol probability, P_c, and symbol error probability, P_t, after erasure insertion can be expressed as:

$$P_c = P(H_1) \cdot \int_{Y_T}^\infty f_Y(y|H_1) dy, \quad (25)$$

$$P_t = P(H_0) \cdot \int_{Y_T}^\infty f_Y(y|H_0) dy, \quad (26)$$

and the RS-symbol erasure probability P_e can be found from Eq.(4). Finally, the CW-DEP, P_E, after E^2D can be determined by substituting P_t, P_e into Eq.(7) or Eq.(8).

5. NUMERICAL RESULTS AND DISCUSSION

Fig.2 and Fig.3 show the CW-DEP of Eq.(8) over Rayleigh fading channels for the RS(32,20) code over the Galois field GF(32)=GF(2^5) corresponding to 5-bit symbols using E^2D. In the context of Fig.2, erasures were inserted according to Viterbi's RTT scheme, while in Fig.3, erasures were introduced according to the LRT^2 scheme. In these figures, the CW-DEP were computed for different values of SNR per bit and for different thresholds, in order to find the optimum thresholds for both erasure schemes. From the results we observe that for a constant SNR per bit, γ_b, and there exists an optimum threshold for both erasure insertion schemes, for which the E^2D achieves the minimum CW-DEP. Hence, an inappropriate threshold may lead to much higher CW-DEP than the minimum seen in the figures. Observe furthermore that for the erasure insertion scheme using Viterbi's RTT the optimum threshold assumes values around 1.5 to 2.0, even though the SNR per bit changes over a wide dynamic range from about 6 to 15dB. By contrast, for the erasure insertion scheme using the LRT^2, the optimum threshold value is more unpredictable, ranging from 6 to 11, when the SNR per bit changes from 6 to 15dB.

In Fig.4 we estimated the minimum SNR per bit required for achieving the CW-DEP of 1×10^{-6}, when using Eq.(7), for a given RS code rate $R_c = k/n$. The required SNR per bit was computed versus the RS code rate, R_c, for the diversity orders of $L = 1, 2, 3$ and for the 64 symbol long RS code family of RS(64, k) over the Galois field GF(64)=GF(2^6) using ECOD and E^2D employing both the RTT erasure insertion scheme and the LRT^2 scheme, respectively. The results imply that for all of the decoding schemes, the optimum RS code rate, ie the code rate that can achieve the required CW-DEP with the lowest SNR per bit, increases, when increasing the order of the diversity combining capability. For example, for 64-length RS codes using ECOD, the optimum code rate for $L = 1$ is about 0.4, for $L = 2$ is about 0.6, while for $L = 3$ it is somewhat higher than that for $L = 2$. The results also show that, for any given code rate, the minimum required SNR per bit for the ECOD in order to achieve the target codeword decoding error probability is higher than that for E^2D. Furthermore, the results of Fig.2-Fig.5 indicate that, for a given SNR per bit, in the case of optimum threshold setting for both RTT and LRT^2, the LRT^2 outperforms the RTT.

In Fig.5 the codeword decoding error probability performance of the RS(32,20) code was evaluated against the SNR per bit. From the results we observe that under frequency-selective Rayleigh fading, for a constant SNR per bit, for a constant number of diversity components, and also under the assumption that the receiver invoked the optimum threshold, the LRT^2 erasure insertion scheme outperforms the RTT scheme.

6. REFERENCES

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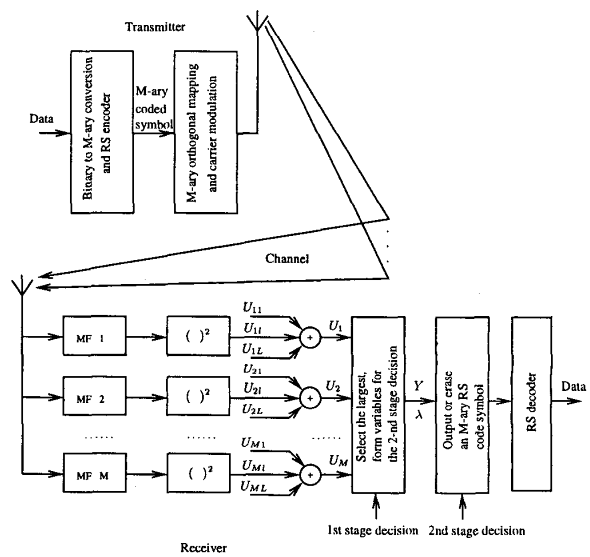

Figure 1: Transmitter and receiver schematic of an M-ary orthogonal signaling scheme using square-law detection, equal gain combining, 1st and 2nd stage decisions as well as RS channel coding.

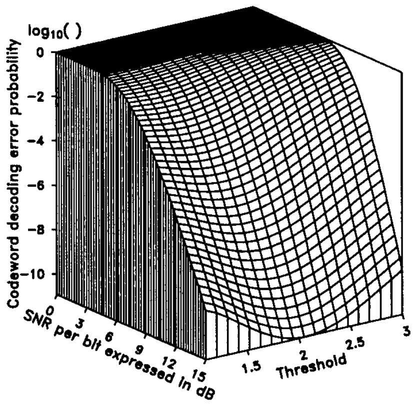

Figure 2: **RTT:** codeword decoding error probability (CW-DEP) versus the SNR per bit, γ_b and the threshold, λ_T for the erasure insertion scheme of RTT computed from Eq.(21), Eq.(22) and Eq.(8) using parameters of L = 2, M = 32 and the RS(32,20), GF(32) code over Rayleigh fading channels.

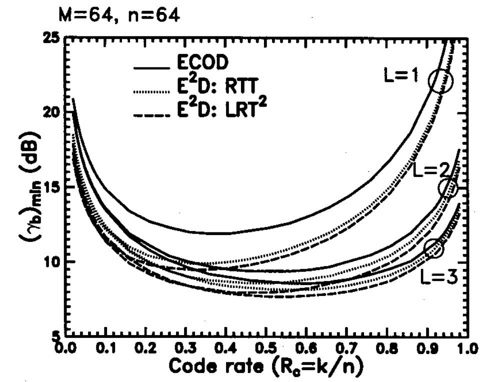

Figure 4: Minimum SNR per bit, γ_b, required to achieve the CW-DEP of 1×10^{-6} in Eq.(8) versus RS code rate $R_c = k/n$ performance comparison between 'error-correction only' decoding (ECOD) and 'errors-and-erasures' decoding (E^2D) using the parameters of $M = n = 64$ and $L = 1, 2, 3$ over Rayleigh fading channels.

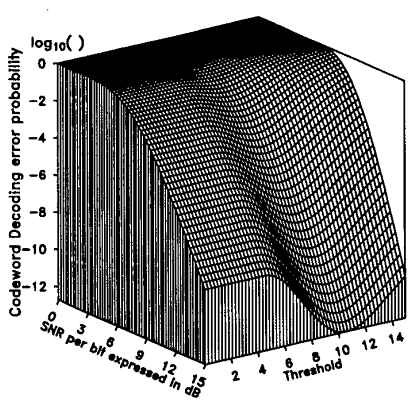

Figure 3: $\mathbf{LRT^2}$:CW-DEP versus the SNR per bit,γ_b and the threshold, λ_T for the erasure insertion scheme of OST computed from Eq.(25), Eq.(26) and Eq.(8) using parameters of L = 2, M = 32 and the RS(32,20), GF(32) code over Rayleigh fading channels.

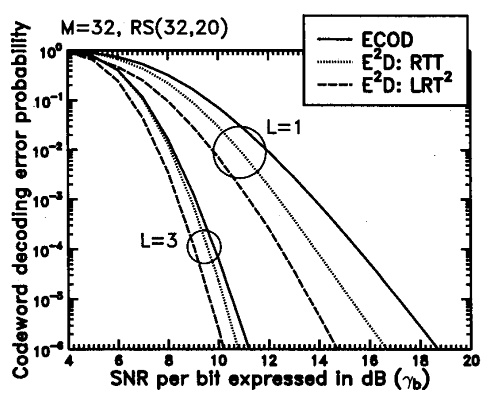

Figure 5: CW-DEP versus SNR per bit, γ_b for the RS(32,20) code using ECOD and E^2D with parameters of $M = n = 32$, $k = 20$ and $L = 1, 3$ over Rayleigh fading channels.