Global Optimal Realizations of Finite Precision Digital Controllers

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Motivations and Background

Finite precision controller implementation can seriously influence closed-loop performance.

- Two types of finite word length errors: roundoff errors in arithmetic operations – controller signal errors, and controller coefficient representation errors – controller parameter errors.

  This work is concerned with FWL controller parameter errors, which have critical influence on closed-loop stability.

- Two strategies: direct and indirect.

  This work adopts an indirect approach.

- We present a novel search algorithm for global solutions to an existing optimal finite precision controller realization problem.

Problem Definition

- Plant: $P(z) \sim (\mathbf{A}_P, \mathbf{B}_P, \mathbf{C}_P)$; $\mathbf{A}_P \in \mathcal{R}^{m\times m}$, $\mathbf{B}_P \in \mathcal{R}^{m\times l}$, $\mathbf{C}_P \in \mathcal{R}^{q\times m}$.

- Controller: $C(z) \sim (\mathbf{A}_C, \mathbf{B}_C, \mathbf{C}_C, \mathbf{D}_C)$; $\mathbf{A}_C \in \mathcal{R}^{n\times n}$, $\mathbf{B}_C \in \mathcal{R}^{n\times q}$, $\mathbf{C}_C \in \mathcal{R}^{l\times n}$, $\mathbf{D}_C \in \mathcal{R}^{l\times q}$.

  Denote an initially designed controller realization as $\mathbf{X}_0$ and a generic realization $\mathbf{X}$. Let $\overline{\mathbf{A}}(\mathbf{X})$ be the closed-loop transition matrix with $\mathbf{X}$.

- Controller realization set

$$\mathcal{S}_C \triangleq \{\mathbf{X} : \mathbf{A}_C = \mathbf{T}^{-1}\mathbf{A}_C^0\mathbf{T}, \mathbf{B}_C = \mathbf{T}^{-1}\mathbf{B}_C^0, \mathbf{C}_C = \mathbf{C}_C^0\mathbf{T}, \mathbf{D}_C = \mathbf{D}_C^0\}$$

  where $\mathbf{T} \in \mathcal{R}^{n\times n}$ is an arbitrary non-singular matrix

- All $\mathbf{X} \in \mathcal{S}_C$ are equivalent in infinite precision implementation: an identical set of closed-loop eigenvalues $\lambda_i(\overline{\mathbf{A}}(\mathbf{X}))$, $1 \le i \le m+n$, which are all within the unit disk.

Closed-Loop Stability Robustness Measure

- In FWL fixed-point implementation, $\mathbf{X} \to \mathbf{X} + \Delta\mathbf{X}$ and $\lambda_i(\overline{\mathbf{A}}(\mathbf{X})) \to \lambda_i(\overline{\mathbf{A}}(\mathbf{X} + \Delta\mathbf{X}))$.

- Closed-loop stability measure (Li 1998)

$$f(\mathbf{X}) \triangleq \min_{i\in\{1,\cdots,m+n\}} \frac{1 - |\lambda_i(\overline{\mathbf{A}}(\mathbf{X}))|}{\sqrt{N}\left\|\frac{\partial \lambda_i(\overline{\mathbf{A}}(\mathbf{X}))}{\partial \mathbf{X}}\right\|_F}$$

  where $N = (l+n)(q+n)$ and $\|\cdot\|_F$ the Frobenius norm.

- $f(\mathbf{X})$ quantifies the "robustness" of closed-loop stability for the realization $\mathbf{X}$ to FWL controller perturbations:

  Under some mild conditions, the larger $f(\mathbf{X})$, the larger the FWL error $\Delta\mathbf{X}$ that controller $\mathbf{X}$ can tolerate without causing closed-loop instability.

Optimal Realization Problem

- The optimal FWL controller realization problem (Li 1998)

$$v \triangleq \max_{\mathbf{X} \in \mathcal{S}_C} f(\mathbf{X})$$

- Closed-form solutions was attempted in (Li 1998), but ended with suboptimal solutions.
- Direct numerical optimization has to apply: computationally costly and no way to know if a solution obtained is a global optimal solution.
- Our two-stage approach:
  1. Construct a closed-form realization set that contains global optimal solutions under a mild condition;
  2. Search in this set for a global solution with a numerical algorithm that are much more efficient than usual numerical optimization.

Single-Pole Stability Functions and Their Peaks

- $\forall k \in \{1, \cdots, m+n\}$, define the single-pole FWL stability function of $\mathbf{X}$

$$g(\mathbf{X}, k) \triangleq \frac{1 - |\lambda_k(\overline{\mathbf{A}}(\mathbf{X}))|}{\sqrt{N} \left\| \frac{\partial \lambda_k(\overline{\mathbf{A}}(\mathbf{X}))}{\partial \mathbf{X}} \right\|_F}$$

and further define the single-pole peak of FWL stability as

$$\rho_k \triangleq \max_{\mathbf{X} \in \mathcal{S}_C} g(\mathbf{X}, k)$$

- **Theorem 1** $v = \rho_{k_1} \triangleq \min_{k \in \{1, \cdots, m+n\}} \rho_k$ if and only if there exists $\mathbf{X}_{\text{opt}} \in \mathcal{S}_C$ and $k_1 \in \{1, \cdots, m+n\}$ such that $g(\mathbf{X}_{\text{opt}}, k_1) = \rho_{k_1}$ and

$$\textit{Condition}\ \sharp: \quad g(\mathbf{X}_{\text{opt}}, k) \ge \rho_{k_1}, \ \forall k \in \{1, \cdots, m+n\} \setminus \{k_1\}$$

Obviously, such an $\mathbf{X}_{\text{opt}}$ is a global optimal solution.

Remarks

- The assumption that *Condition* $\sharp$ can be met is a reasonable one.
- It turns out that the set

$$\mathcal{X} \triangleq \{\mathbf{X} : g(\mathbf{X}, k_1) = \rho_{k_1}, \mathbf{X} \in \mathcal{S}_C\}$$

can be constructed in a closed-form. Since $\mathbf{X} = \mathbf{X}(\mathbf{T})$, $\mathcal{X}$ is defined on the transformation set

$$\mathcal{T} \triangleq \{\mathbf{T} : g(\mathbf{X}(\mathbf{T}), k_1) = \rho_{k_1}, \mathbf{T} \in \mathcal{R}^{n \times n}, \det \mathbf{T} \neq 0\}$$

- $\mathcal{T}$ can be searched for an $\mathbf{T}_{\text{opt}}$ meeting

$$g(\mathbf{X}(\mathbf{T}_{\text{opt}}), k) \ge \rho_{k_1}, \quad \forall k \in \{1, \cdots, m+n\} \setminus \{k_1\}$$

- This is much more efficient than minimizing $f(\mathbf{X})$ over $\mathcal{S}_C$. Moreover, if a solution can be found in this way it is a global optimum for sure.

Closed-Form Transformation Set

- Let $\mathbf{p}_k$ and $\mathbf{y}_k$ be the right and reciprocal left eigenvectors related to the closed-loop eigenvalue λ_k, respectively.
- **Theorem 2** The value of ρ_k is easily determined, and $g(\mathbf{X}(\mathbf{T}), k)$ achieves the maximum ρ_k if and only if

$$\mathbf{T} = \mathbf{Q} \begin{bmatrix} \mathbf{H}^{1/2} & \mathbf{0} \\ \mathbf{F}(\mathbf{H}^{1/2})^{-T} & \mathbf{\Omega} \end{bmatrix} \mathbf{V}$$

where $\mathbf{V} \in \mathcal{R}^{n \times n}$ is an arbitrary orthogonal matrix, the orthogonal matrix $\mathbf{Q}$ is know, and

1. complex $\mathbf{p}_k$ and $\mathbf{y}_k$: $\mathbf{\Omega} \in \mathcal{R}^{(n-2)\times(n-2)}$ is an arbitrary nonsingular matrix, the 2×2 matrix $\mathbf{H}$ and the $(n-2) \times 2$ matrix $\mathbf{F}$ are known;
2. real $\mathbf{p}_k$ and $\mathbf{y}_k$: $\mathbf{\Omega} \in \mathcal{R}^{(n-1)\times(n-1)}$ is an arbitrary nonsingular matrix, the scalar $\mathbf{H}^{1/2} = \sqrt{h}$ and the $(n-1) \times 1$ vector $\mathbf{F}$ are known.

Search Algorithm

- According to **Theorem 2** and setting $\mathbf{V} = \mathbf{I}$, $\mathcal{T}$ is given in the form:

$$\mathcal{T} = \left\{ \mathbf{T}(\mathbf{\Omega}) : \mathbf{T}(\mathbf{\Omega}) = \mathbf{Q} \left[\begin{array}{cc} \mathbf{H}^{1/2} & \mathbf{0} \\ \mathbf{F}(\mathbf{H}^{1/2})^{-T} & \mathbf{\Omega} \end{array} \right] \right\}$$

- The objective: search for a nonsingular $\mathbf{\Omega}_{\rm opt} \in \mathcal{R}^{(n-2)\times(n-2)}$ such that

$$g(\mathbf{X}(\mathbf{T}(\mathbf{\Omega}_{\rm opt})), k) \ge \rho_{k_1}, \ \forall k \in \{1, \cdots, m+n\} \setminus \{k_1\}$$

(If λ_{k_1} is real-valued, $\mathbf{\Omega} \in \mathcal{R}^{(n-1)\times(n-1)}$.)

- Notice that $g(\mathbf{X}(\mathbf{T}(\mathbf{\Omega})), k)$ is differentiable with respect to $\mathbf{\Omega}$.

- With the derivative, we know how to change $\mathbf{\Omega}$ so that $g(\mathbf{X}, k)$ increase for those $g(\mathbf{X}, k) < \rho_{k_1}$, and $g(\mathbf{X}, k)$ do not decrease for those $g(\mathbf{X}, k) \ge \rho_{k_1}$ $\Rightarrow$ $\mathbf{\Omega}$ is updated iteratively until all the $g(\mathbf{X}, k) \ge \rho_{k_1}$.

Design Example

- Li (1998): $m = 5$, $n = 6$, $l = q = 1$; a closed-loop system of order 11.

- The controller transfer function $C(z)$ has been given with the companion canonical form of $C(z)$ as the initial realization.

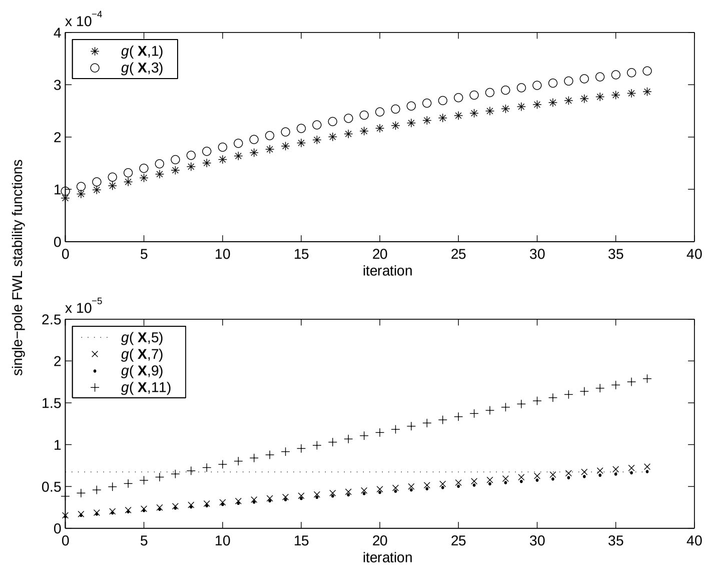
- The closed-loop system: five pairs conjugate complex-valued eigenvalues $\lambda_{1,2}$, $\lambda_{3,4}$, $\lambda_{5,6}$, $\lambda_{7,8}$ and $\lambda_{9,10}$, and one real-valued eigenvalue λ_{11}.

- The single-pole peaks of FWL stability are

$$\begin{array}{lll} \rho_{1,2} = 2.5072e-3, & \rho_{3,4} = 2.1295e-3, & \rho_{5,6} = 6.7344e-6, \\ \rho_{7,8} = 2.8586e-3, & \rho_{9,10} = 3.0832e-3, & \rho_{11} = 4.3181e-3. \end{array}$$

- The minimum value of all the ρ_ks is ρ_5 (or ρ_6) $\Rightarrow k_1 = 5$ and the matrices $\mathbf{Q}$, $\mathbf{H}$ and $\mathbf{F}$ are determined $\Rightarrow \mathcal{T} = \{\mathbf{T}(\mathbf{\Omega})\}$.

Iterative Search



Global Optimal Solution

- During each iteration, $\mathbf{X}(\mathbf{T}(\mathbf{\Omega}))$ meets: 1) $g(\mathbf{X}, k)$ increase for those $g(\mathbf{X}, k) < \rho_{k_1}$; 2) $g(\mathbf{X}, k)$ do not decrease for those $g(\mathbf{X}, k) \ge \rho_{k_1}$.

- At the 37th iteration, a global optimal realization $\mathbf{X}(\mathbf{T}(\mathbf{\Omega}_{\rm opt}))$ is found, since at this stage $Condition \ \sharp$ is met:

$$g(\mathbf{X}(\mathbf{T}(\mathbf{\Omega}_{\rm opt})), k) \ge \rho_{k_1}, \quad k \in \{1, 2, \cdots, 11\} \setminus \{5, 6\}$$

and the search algorithm terminated.

- Values of the closed-loop stability measure for the initial and global optimal realizations $\mathbf{X}_0$ and $\mathbf{X}(\mathbf{T}_{\rm opt})$ are:

$$f(\mathbf{X}_0) = 3.1797 \times 10^{-11} \qquad f(\mathbf{X}(\mathbf{T}_{\rm opt})) = 6.7344 \times 10^{-6}$$

a factor of 2×10^5 improvement in the closed-loop FWL stability measure.

Conclusions and Future Works

- We have developed an efficient method to solve the optimal controller realization problem based on maximizing a closed-loop FWL stability measure.

  1. This method does not suffer from the drawbacks associated with using direct numerical optimization methods to tackle the problem.
  2. under a reasonable and mild condition, our method can find global optimal controller realizations for most practical systems.

- The arbitrary orthogonal matrix $\mathbf{V} \in \mathcal{R}^{n \times n}$ in the closed-form transformation set $\mathcal{T}$ can be explored to design:

  1. global optimal (in closed-loop stability sense) realizations of fixed-point controller with the smallest dynamic range.
  2. sparse global optimal realizations of fixed-point controller.