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Positive real control of two-dimensional systems: Roesser models and linear repetitive processes

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This paper considers the problem of positive real control for two-dimensional (2-D) discrete systems described by the Roesser model and also discrete linear repetitive processes, which are another distinct sub-class of 2-D linear systems of both systems theoretic and applications interest. The purpose of this paper is to design a dynamic output feedback controller such that the resulting closed-loop system is asymptotically stable and the closed-loop system transfer function from the disturbance to the controlled output is extended strictly positive real. We first establish a version of positive realness for 2-D discrete systems described by the Roesser state space model, then a sufficient condition for the existence of the desired output feedback controllers is obtained in terms of four LMIs. When these LMIs are feasible, an explicit parameterization of the desired output feedback controllers is given. We then apply a similar approach to discrete linear repetitive processes represented in their equivalent 1-D state-space form. Finally, we provide numerical examples to demonstrate the applicability of the approach.

1. Introduction

Since the concept of positive realness was introduced, it has played an important role in control and system theory (Anderson and Vongpanitlerd 1973, Haddad and Bernstein 1991, Vidyasagar 1993). Applications of positive realness in stability analysis and robust stabilization of linear systems have been reported in, for example, Wen (1988), Haddad and Bernstein (1991, 1994) and references therein. In Agathoklis *et al.* (1991) an interesting application of positive realness for one-dimensional (1-D) systems to the stability analysis for two-dimensional (2-D) discrete systems has been reported. Recently, the positive real control problem has received considerable attention (Sun *et al.* 1994, Xie and Soh 1995). The study of this problem is motivated by robust and non-linear control in which a well-known fact is that the positive realness of a certain loop transfer function will guarantee the overall stability of a feedback system if uncertainty or non-linearity can be characterized by a positive real system (Vidyasagar 1993). It was shown in (Sun *et al.* 1994) that a solution to the positive real control problem for linear continuous systems involves solving a pair of Riccati inequalities. When parameter uncertainty is present, the problem was also solved by dynamic output feedback controllers in, for example, Xie and Soh (1995) and Mahmoud *et al.* (1999), respectively. The corresponding results for discrete time systems can be found in Haddad and Bernstein (1994) and Mahmoud and Xie (2000).

The systems related analysis of 2-D discrete systems has received much attention during the past years due to their theoretical importance as well as the extensive applications of these systems in many areas such as image processing, seismographic data processing, thermal processes, water stream heating, and so on (Kaczorek 1985). Different 2-D state-space models have been proposed and a great number of fundamental concepts and results based on 1-D discrete systems have been extended to 2-D systems (Roesser 1975, Fornasini and Marchesini 1978, Kaczorek 1985, Hinamoto 1993). Note also that 2-D (and, more generally, n-D ($n \geq 3$)) linear systems can pose systems theoretic questions which have no 1-D counterparts. Also in some cases there is a much weaker link between important concepts that are strongly related in the 1-D case. As an example in the latter case, the Smith form of an n-D linear system fails to provide much information about the system which it does supply in the 1-D case. To date, the concept of positive realness for 2-D systems has received much less attention than its 1-D counterpart and, to the best of our knowledge, no results on the problem of positive real control for 2-D systems have been reported. In this paper, we deal first with the positive real control problem for 2-D discrete systems described by the Roesser model. Attention is focused on the design of a dynamic output feedback controller such that the resulting closed-loop system is asymptotically stable and

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the closed-loop transfer function matrix from the disturbance to the controlled output has the so-called *extended strictly positive real* (ESPR) property. We first present a version of positive realness for 2-D discrete systems in terms of an LMI. It is shown that this result is an extension of the existing results of positive realness for 1-D discrete systems. Based on this, a sufficient condition for the existence of the desired output feedback controllers is given in terms of four LMIs, which define a convex set of solutions and can be solved easily. In addition, when these LMIs are feasible, an explicit parameterization of the desired output feedback controller is also given.

The essential unique characteristic of a repetitive, or multipass, process is a series of sweeps, termed passes, through a set of dynamics defined over a fixed finite duration known as the pass length. On each pass an output, termed the pass profile, is produced which acts as a forcing function on, and hence contributes to, the dynamics of the next pass profile. This, in turn, leads to the unique control problem for these processes in that the output sequence of pass profiles generated can contain oscillations that increase in amplitude in the pass to pass direction.

To introduce a formal definition, let $\alpha < +\infty$ denote the pass length (assumed constant). Then in a repetitive process the pass profile $y_k(p)$, $0 \le p \le \alpha$, generated on pass k acts as a forcing function on, and hence contributes to, the dynamics of the next pass profile $y_{k+1}(p)$, $0 \le p \le \alpha$, $k \ge 0$.

Physical examples of repetitive processes include long-wall coal cutting and metal rolling operations (Edwards 1974). Also in recent years applications have arisen where adopting a repetitive process setting for analysis has distinct advantages over alternatives. Examples of these so-called algorithmic applications of repetitive process theory include classes of iterative learning control schemes (Amann *et al.* 1998) and iterative algorithms for solving non-linear dynamic optimal control problems based on the maximum principle (Roberts 2000).

Attempts to control these processes using standard (or 1-D) systems theory/algorithms fail (except in a few very restrictive special cases) precisely because such an approach ignores their inherent 2-D systems structure, i.e. information propagation occurs from pass to pass and along a given pass, and the pass initial conditions are reset before the start of each new pass. In seeking a rigorous foundation on which to develop a control theory for these processes, it is natural to attempt to exploit structural links which exist between, in particular, the class of so-called discrete linear repetitive processes and 2-D linear systems described by the extensively studied Roesser (1975) or Fornasini–Marchesini (1978) state-space models. Discrete linear repetitive processes are distinct from such 2-D linear systems in the sense that information propagation in one of the two independent directions (along the pass) only occurs over a finite duration. In this paper, we produce the first significant results on the problem of positive real control for discrete linear repetitive processes.

The organization of the paper is as follows. A version of positive realness for 2-D discrete systems described by the Roesser model is given in § 2. Based on this, the solution of the positive real control problem for 2-D discrete systems described by the Roesser model is obtained in § 3. In § 4, positive realness for a linear discrete repetitive process is investigated. The positive real controller synthesis is given in § 5. Numerical examples are provided in § 6 to demonstrate the applicability of the proposed approach.

Notation: Throughout this paper, for symmetric matrices X and Y, the notation $X \ge Y$ (respectively, $X > Y$) means that the matrix $X - Y$ is positive semi-definite (respectively, positive definite). I is the identity matrix with appropriate dimensions. The superscripts 'T' and '*' denote the transpose and the complex conjugate transpose respectively. $\mathbf{Z}^+$ denotes the set of non-negative integers. For a matrix $M \in \mathbf{R}^{n\times m}$ with rank r, the orthogonal complement $M^\perp$ is defined as a (possibly non-unique) $(n-r)\times n$ matrix such that $M^\perp M = 0$ and $M^\perp M^{\perp \mathrm{T}} > 0$. M^+ is the Moore–Penrose inverse of M. Matrices, if not explicitly stated, are assumed to have compatible dimensions.

2. Positive realness of Roesser models

Consider a 2-D discrete-time system (Σ) described by the following Roesser state-space model (Roesser 1975)

$$(\Sigma):\begin{bmatrix} x^h(i+1,j)\\ x^v(i,j+1)\end{bmatrix} = A\begin{bmatrix} x^h(i,j)\\ x^v(i,j)\end{bmatrix} + Bw(i,j) \quad (1)$$

$$z(i,j) = C\begin{bmatrix} x^h(i,j)\\ x^v(i,j)\end{bmatrix} + Dw(i,j) \quad (2)$$

where $x^h(i,j) \in \mathbf{R}^{n_h}$ and $x^v(i,j) \in \mathbf{R}^{n_v}$ are the horizontal and vertical states, respectively, $w(i,j) \in \mathbf{R}^h$ is the exogenous input, $z(i,j) \in \mathbf{R}^s$ is the measured output, $i,j \in \mathbf{Z}^+$, A, B, C and D are known real constant matrices with appropriate dimensions. The boundary conditions of the system are

$$x_0 = [x^h(0,0)^\mathrm{T}, x^h(0,1)^\mathrm{T}, x^h(0,2)^\mathrm{T}\ldots;\; x^v(0,0)^\mathrm{T}, x^v(1,0)^\mathrm{T}, x^v(2,0)^\mathrm{T}\ldots]^\mathrm{T}$$

The transfer function matrix of the 2-D discrete-time system (Σ) under zero boundary conditions can be written as

$$G(z_1, z_2) = C(I(z_1, z_2) - A)^{-1}B + D \tag{3}$$

where

$$I(z_1, z_2) = \mathrm{diag}(z_1 I_{n_h}, z_2 I_{n_v}) \tag{4}$$

Definition 1 (Kaczorek 1985)**:** The 2-D linear discrete-time system (Σ) is said to be asymptotically stable if

$$\lim_{i,j\to\infty} \|x(i,j)\| = 0$$

under zero input $w(i,j) \equiv 0$ and boundary conditions such that $\sup_j \|x^h(0,j)\| < \infty$ and $\sup_i \|x^v(i,0)\| < \infty$, where $x(i,j) = [x^h(i,j)^{\mathrm{T}},\ x^v(i,j)^{\mathrm{T}}]^{\mathrm{T}}$.

We will also use the following result.

Lemma 1 (Anderson *et al.* 1986, Agathoklis 1988)**:** *The 2-D linear discrete-time system* (Σ) *is asymptotically stable is there exists a block-diagonal matrix* $P = \mathrm{diag}(P_h, P_v) > 0$ *with* $P_h \in \mathbf{R}^{n_h}$ *and* $P_v \in \mathbf{R}^{n_v}$ *such that*

$$A^{\mathrm{T}}PA - P < 0 \tag{5}$$

Motivated by the theory of positive realness for 1-D discrete systems (Anderson and Vongpanitlerd 1973), positive realness for 2-D systems can be defined as follows.

Definition 2:

(1) The 2-D discrete-time system (Σ) is said to be positive real (PR) if its transfer function matrix $G(z_1, z_2)$ is analytic in $|z_1| > 1$, $|z_2| > 1$ and satisfies $G(z_1, z_2) + G^*(z_1, z_2) \geq 0$ for $|z_1| > 1$, $|z_2| > 1$.

(2) The 2-D discrete-time system (Σ) is said to be strictly positive real (SPR) if its transfer function matrix $G(z_1, z_2)$ is analytic in $|z_1| \geq 1$, $|z_2| \geq 1$ and satisfies $G(\mathrm{e}^{j\theta_1}, \mathrm{e}^{j\theta_2}) + G^*(\mathrm{e}^{j\theta_1}, \mathrm{e}^{j\theta_2}) > 0$ for θ_1, $\theta_2 \in [0, 2\pi)$.

(3) The 2-D discrete-time system (Σ) is said to be extended strictly positive real (ESPR) if it is SPR and $G(\infty,\infty) + G(\infty,\infty)^{\mathrm{T}} > 0$.

The following theorem gives a sufficient condition for the 2-D discrete-time system (Σ) to be asymptotically stable and ESPR. This result will play a key role in solving the positive real control problem for 2-D systems defined in the following section.

Theorem 1: *The 2-D discrete-time system* (Σ) *is asymptotically stable and ESPR if there exists a block-diagonal matrix* $P = \mathrm{diag}(P_h, P_v) > 0$ *with* $P_h \in \mathbf{R}^{n_h}$ *and* $P_v \in \mathbf{R}^{n_v}$ *such that the following LMI holds*

$$\begin{bmatrix} A^{\mathrm{T}}PA - P & C^{\mathrm{T}} - A^{\mathrm{T}}PB \\ C - B^{\mathrm{T}}PA & -(D + D^{\mathrm{T}} - B^{\mathrm{T}}PB) \end{bmatrix} < 0 \tag{6}$$

Proof: From (6) it is easy to see that

$$A^{\mathrm{T}}PA - P < 0$$

By Lemma 1, it follows that system (Σ) is asymptotically stable. This implies that $G(z_1, z_2)$ is analytic in $|z_1| \geq 1, |z_2| \geq 1$. Next, we show that $G(\mathrm{e}^{j\theta_1}, \mathrm{e}^{j\theta_2}) + G^*(\mathrm{e}^{j\theta_1}, \mathrm{e}^{j\theta_2}) > 0$ for all $\theta_1, \theta_2 \in [1, 2\pi)$.

By Schur complements, it follows from (6) that

$$D = D^{\mathrm{T}} - B^{\mathrm{T}}PB > 0 \tag{7}$$

and

$$A^{\mathrm{T}}PA - P + (C^{\mathrm{T}} - A^{\mathrm{T}}PB)(D + D^{\mathrm{T}} - B^{\mathrm{T}}PB)^{-1} \times (C - B^{\mathrm{T}}PA) < 0 \tag{8}$$

Define

$$\Phi(\mathrm{e}^{j\theta_1}, \mathrm{e}^{j\theta_2}) = I(\mathrm{e}^{j\theta_1}, \mathrm{e}^{j\theta_2}) - A$$

where $I(z_1, z_2)$ is defined in (4). Now, writing

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

with compatible dimensions to $I(z_1, z_2)$ yields, after some (extensive but routine and hence the details are omitted here) calculations that for all $\theta_1, \theta_2 \in [0, 2\pi)$

$$\Phi(\mathrm{e}^{-j\theta_1}, \mathrm{e}^{-j\theta_2})^{\mathrm{T}}P\Phi(\mathrm{e}^{j\theta_1}, \mathrm{e}^{j\theta_2}) + \Phi(\mathrm{e}^{-j\theta_1}, \mathrm{e}^{-j\theta_2})^{\mathrm{T}}PA + A^{\mathrm{T}}P\Phi(\mathrm{e}^{j\theta_1}, \mathrm{e}^{j\theta_2}) = -(A^{\mathrm{T}}PA - P)$$

Noting that $\Phi(\mathrm{e}^{j\theta_1}, \mathrm{e}^{j\theta_2})$ is invertible for all $\theta_1, \theta_2 \in [0, 2\pi)$, it follows from the above equality that

$$B^{\mathrm{T}}PB + B^{\mathrm{T}}PA\Phi(\mathrm{e}^{j\theta_1}, \mathrm{e}^{j\theta_2})^{-1}B + B^{\mathrm{T}}\Phi(\mathrm{e}^{-j\theta_1}, \mathrm{e}^{-j\theta_2})^{-\mathrm{T}}A^{\mathrm{T}}PB = -B^{\mathrm{T}}\Phi(\mathrm{e}^{-j\theta_1}, \mathrm{e}^{-j\theta_2})^{-\mathrm{T}}(A^{\mathrm{T}}PA - P)\Phi(\mathrm{e}^{j\theta_1}, \mathrm{e}^{j\theta_2})^{-1}B \tag{9}$$

Conversely, equation (8) implies that there exists a matrix $Q > 0$ such that

$$Q + A^{\mathrm{T}}PA - P + (C^{\mathrm{T}} - A^{\mathrm{T}}PB)(D + D^{\mathrm{T}} - B^{\mathrm{T}}PB)^{-1} \times (C - B^{\mathrm{T}}PA) < 0 \tag{10}$$

Pre and post-multiplying (10) by $B^{\mathrm{T}}\Phi(\mathrm{e}^{-j\theta_1}, \mathrm{e}^{-j\theta_2})^{-\mathrm{T}}$ and $\Phi(\mathrm{e}^{j\theta_1}, \mathrm{e}^{j\theta_2})^{-1}B$ respectively, we now have that for all $\theta_1, \theta_2 \in [0, 2\pi)$

$$B^{\mathrm{T}}\Phi(\mathrm{e}^{-j\theta_1}, \mathrm{e}^{-j\theta_2})^{-\mathrm{T}}(A^{\mathrm{T}}PA - P)\Phi(\mathrm{e}^{j\theta_1}, \mathrm{e}^{j\theta_2})^{-1}B + B^{\mathrm{T}}\Phi(\mathrm{e}^{-j\theta_1}, \mathrm{e}^{-j\theta_2})^{-\mathrm{T}}\Psi\Phi(\mathrm{e}^{j\theta_2})^{-1}B \leq 0 \tag{11}$$

where

$$\Psi = Q + (C^{\mathrm{T}} - A^{\mathrm{T}}PB)(D + D^{\mathrm{T}} - B^{\mathrm{T}}PB)^{-1}(C - B^{\mathrm{T}}PA)$$

Now, substituting (9) into (11) gives

$$\begin{aligned} &-B^{\mathrm{T}}PB - B^{\mathrm{T}}PA\Phi(\mathrm{e}^{-j\theta_1}, \mathrm{e}^{-j\theta_2})^{-1}B \\ &\quad -B^{\mathrm{T}}\Phi(\mathrm{e}^{-j\theta_1}, \mathrm{e}^{-j\theta_2}, \mathrm{e}^{-j\theta_2})^{-\mathrm{T}}A^{\mathrm{T}}PB \\ &\quad +B^{\mathrm{T}}\Phi(\mathrm{e}^{-j\theta_1}, \mathrm{e}^{-j\theta_2})^{-\mathrm{T}}\Psi\Phi(\mathrm{e}^{j\theta_1}, \mathrm{e}^{j\theta_2})^{-1}B \le 0 \end{aligned}$$

for all $\theta_1, \theta_2 \in [0, 2\pi)$. Hence by this last inequality, we have that for all $\theta_1, \theta_2 \in [0, 2\pi)$

$$\begin{aligned} &G(\mathrm{e}^{j\theta_1}, \mathrm{e}^{j\theta_2}) + G^*(\mathrm{e}^{j\theta_1}, \mathrm{e}^{j\theta_2}) \\ &= D + D^{\mathrm{T}} + C\Phi(\mathrm{e}^{j\theta_1}, \mathrm{e}^{j\theta_2})^{-1}B + B^{\mathrm{T}}\Phi(\mathrm{e}^{-j\theta_1}, \mathrm{e}^{-j\theta_2})^{-\mathrm{T}}C^{\mathrm{T}} \\ &= (D + D^{\mathrm{T}} - B^{\mathrm{T}}PB) + C\Phi(\mathrm{e}^{j\theta_1}, \mathrm{e}^{j\theta_2})^{-1}B \\ &\quad +B^{\mathrm{T}}\Phi(\mathrm{e}^{-j\theta_1}, \mathrm{e}^{-j\theta_2})^{-\mathrm{T}}C^{\mathrm{T}} + B^{\mathrm{T}}PB \\ &\ge (D + D^{\mathrm{T}} - B^{\mathrm{T}}PB) + (C - B^{\mathrm{T}}PA)\Phi(\mathrm{e}^{j\theta_1}, \mathrm{e}^{j\theta_2})^{-1}B \\ &\quad +B^{\mathrm{T}}\Phi(\mathrm{e}^{-j\theta_1}, \mathrm{e}^{-j\theta_2})^{-\mathrm{T}}(C^{\mathrm{T}} - A^{\mathrm{T}}PB) \\ &\quad +B^{\mathrm{T}}\Phi(\mathrm{e}^{-j\theta_1}, \mathrm{e}^{-j\theta_2})^{-\mathrm{T}}\Psi\Phi(\mathrm{e}^{j\theta_1}, \mathrm{e}^{j\theta_2})^{-1}B \\ &= (D + D^{\mathrm{T}} - B^{\mathrm{T}}PB) - (C - B^{\mathrm{T}}PA)\Psi^{-1}(C^{\mathrm{T}} - A^{\mathrm{T}}PB) \\ &\quad +[B^{\mathrm{T}}\Phi(\mathrm{e}^{-j\theta_1}, \mathrm{e}^{-j\theta_2})^{-\mathrm{T}} \\ &\quad +(C - B^{\mathrm{T}}PA)\Psi^{-1}]\Psi[\Phi(\mathrm{e}^{j\theta_1}, \mathrm{e}^{j\theta_2})^{-1}B \\ &\quad +\Psi^{-1}(C^{\mathrm{T}} - A^{\mathrm{T}}PB)] \\ &\ge (D + D^{\mathrm{T}} - B^{\mathrm{T}}PB) - (C - B^{\mathrm{T}}PA)\Psi^{-1}(C^{\mathrm{T}} - A^{\mathrm{T}}PB) \end{aligned} \tag{12}$$

Noting

$$\begin{bmatrix} D + D^{\mathrm{T}} - B^{\mathrm{T}}PB & C - B^{\mathrm{T}}PA \\ C^{\mathrm{T}} - A^{\mathrm{T}}PB & \Psi \end{bmatrix} > 0$$

and using Schur complements, it follows that

$$(D + D^{\mathrm{T}} - B^{\mathrm{T}}PB) - (C - B^{\mathrm{T}}PA)\Psi^{-1}(C^{\mathrm{T}} - A^{\mathrm{T}}PB) > 0$$

This together with (12) shows that for all $\theta_1, \theta_2 \in [0, 2\pi)$

$$G(\mathrm{e}^{j\theta_1}, \mathrm{e}^{j\theta_2}) + G^*(\mathrm{e}^{j\theta_1}, \mathrm{e}^{j\theta_2}) > 0$$

Hence the 2-D discrete-time system (Σ) is ESPR. □

Remark 1: Theorem 1 provides an LMI condition for the 2-D discrete-time system (Σ) to be asymptotically stable and ESPR. In the case when the system (Σ) reduces to a 1-D discrete system, it is easy to show that Theorem 1 coincides with Lemma 4.2 in Haddad and Bernstein (1994). Therefore, Theorem 1 can be viewed as an extension of existing results on positive realness for 1-D discrete-time systems to 2-D linear systems described by the Roesser state space model.

3. Positive real control for Roesser models

The 2-D discrete-time linear systems to be considered in this section are described by the state-space model of the Roesser structure

$$(\Sigma_R): \begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix} = A \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} + Bw(i,j) + B_1u(i,j) \tag{13}$$

$$z(i,j) = C_1 \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} + D_{11}w(i,j) + D_{12}u(i,j) \tag{14}$$

$$y(i,j) = C_2 \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} + D_{21}w(i,j) + D_{22}u(i,j) \tag{15}$$

where $x^h(i,j) \in \mathbf{R}^{n_h}$, $x^v(i,j) \in \mathbf{R}^{n_v}$, $w(i,j) \in \mathbf{R}^s$, $u(i,j) \in \mathbf{R}^l$, $z(i,j) \in \mathbf{R}^s$ and $y(i,j) \in \mathbf{R}^p$ are the horizontal state, vertical state, exogenous input, control input, controlled output and measured output, respectively; A, B, B_1, C_1, C_2, D_{hk}, $h, k = 1, 2$ are known real constant matrices with compatible dimensions. Without loss of generality, we assume that $D_{22} = 0$.

In this section, we focus on the output feedback controller

$$(\bar{\Sigma}): \begin{bmatrix} \bar{x}^h(i+1,j) \\ \bar{x}^v(i,j+1) \end{bmatrix} = \bar{A} \begin{bmatrix} \bar{x}^h(i,j) \\ \bar{x}^v(i,j) \end{bmatrix} + \bar{B}y(i,j) \tag{16}$$

$$u(i,j) = \bar{C} \begin{bmatrix} \bar{x}^h(i,j) \\ \bar{x}^v(i,j) \end{bmatrix} + \bar{D}y(i,j) \tag{17}$$

where $\bar{x}^h(i,j) \in \mathbf{R}^{\bar{n}_h}$, $\bar{x}^v(i,j) \in \mathbf{R}^{\bar{n}_v}$ and $\bar{A}$, $\bar{B}$, $\bar{C}$ and $\bar{D}$ are the controller matrices to be selected. By introducing the augmented state vectors

$$\tilde{x}^h(i+1,j) = [x^h(i+1,j)^{\mathrm{T}} \quad \bar{x}^h(i+1,j)^{\mathrm{T}}]^{\mathrm{T}}$$

$$\tilde{x}^v(i,j+1) = [x^h(i,j+1)^{\mathrm{T}} \quad \bar{x}^h(i,j+1)^{\mathrm{T}}]^{\mathrm{T}}$$

we obtain the closed-loop system (Σ_c)

$$(\Sigma_c): \begin{bmatrix} \tilde{x}^h(i+1,j) \\ \tilde{x}^v(i,j+1) \end{bmatrix} = \tilde{A} \begin{bmatrix} \tilde{x}^h(i,j) \\ \tilde{x}^v(i,j) \end{bmatrix} + \tilde{B}w(i,j) \tag{18}$$

$$z(i,j) = \tilde{C} \begin{bmatrix} \tilde{x}^h(i,j) \\ \tilde{x}^v(i,j) \end{bmatrix} + \tilde{D}w(i,j) \tag{19}$$

where

$$\tilde{A} = \Theta(\hat{A} + \hat{F}\hat{G}\hat{H})\Theta^{-1}, \qquad \tilde{B} = \Theta(\hat{B} + \hat{F}\hat{G}\hat{N}),$$
$$\tilde{C} = (\hat{C} + \hat{S}\hat{G}\hat{H})\Theta^{-1}, \qquad \tilde{D} = \hat{D} + \hat{S}\hat{G}\hat{N} \tag{20}$$

$$\hat{A} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} B \\ 0 \end{bmatrix},$$

$$\hat{F} = \begin{bmatrix} B_1 & 0 \\ 0 & I \end{bmatrix}, \quad \hat{G} = \begin{bmatrix} \bar{D} & \bar{C} \\ \bar{B} & \bar{A} \end{bmatrix} \tag{21}$$

$$\hat{H} = \begin{bmatrix} C_2 & 0 \\ 0 & I \end{bmatrix}, \qquad \hat{N} = \begin{bmatrix} D_{21} \\ 0 \end{bmatrix}, \qquad \hat{C} = [C_1 \quad 0],$$

$$\hat{S} = [D_{12} \quad 0], \qquad \hat{D} = D_{11} \tag{22}$$

$$\Theta = \begin{bmatrix} I_{n_h} & 0 & 0 & 0 \\ 0 & 0 & I_{\bar{n}_h} & 0 \\ 0 & I_{n_v} & 0 & 0 \\ 0 & 0 & 0 & I_{\bar{n}_v} \end{bmatrix} \tag{23}$$

Then the closed-loop transfer function matrix $\bar{G}(z_1, z_2)$ is given by

$$\bar{G}(z_1, z_2) = \tilde{C}(I(z_1, z_2) - \tilde{A})^{-1}\tilde{B} + \tilde{D} \tag{24}$$

The positive real control we address in this section can now be formulated as determining the parameters $\bar{A}$, $\bar{B}$, $\bar{C}$ and $\bar{D}$ of the output feedback controller $(\bar{\Sigma})$ such that the resulting closed-loop system (Σ_c) is asymptotically stable and ESPR.

The following lemma will be used in the proof of our main result in this section.

Lemma 2 (Gahinet and Apkarian 1994, Iwasaki and Skelton 1994)**:** *Given a symmeric matrix $\mathcal{H}$ and two matrices Γ and Π, consider the problem of finding some matrix Δ such that*

$$\mathcal{H} + \Gamma\Delta\Pi + (\Gamma\Delta\Pi)^{\mathrm{T}} < 0 \tag{25}$$

Then (25) *is solvable for Δ if and only if*

$$\Gamma^{\perp}\mathcal{H}\Gamma^{\perp\mathrm{T}} < 0, \qquad \Pi^{\mathrm{T}\perp}\mathcal{H}\Pi^{\mathrm{T}\perp\mathrm{T}} < 0$$

Now we are in a position to give our main result on the positive real control problem.

Theorem 2: *Consider the 2-D discrete-time system (Σ_R). If there exists matrices $X = \mathrm{diag}(X_h, X_v) > 0$ and $Y = \mathrm{diag}(Y_h, Y_v) > 0$ with $X_h, Y_h \in \mathbf{R}^{n_h}$, X_v, $Y_v \in \mathbf{R}^{n_v}$ satisfying the LMIs*

$$\Gamma_X^{\perp} \begin{bmatrix} A^{\mathrm{T}}XA - X & A^{\mathrm{T}}XB - C_1^{\mathrm{T}} \\ B^{\mathrm{T}}XA - C_1 & B^{\mathrm{T}}XB - (D_{11} + D_{11}^{\mathrm{T}}) \end{bmatrix} \Gamma_X^{\perp\mathrm{T}} < 0 \tag{26}$$

$$\Gamma_Y^{\perp} \begin{bmatrix} AYA^{\mathrm{T}} - Y & AYC_1^{\mathrm{T}} - B \\ C_1YA^{\mathrm{T}} - B^{\mathrm{T}} & C_1YC_1^{\mathrm{T}} - (D_{11} + D_{11}^{\mathrm{T}}) \end{bmatrix} \Gamma_Y^{\perp\mathrm{T}} < 0 \tag{27}$$

$$\begin{bmatrix} X_h & I_{n_h} \\ I_{n_h} & Y_h \end{bmatrix} \geq 0, \qquad \begin{bmatrix} X_v & I_{n_v} \\ I_{n_v} & Y_v \end{bmatrix} \geq 0 \tag{28}$$

where

$$\Gamma_X = \begin{bmatrix} C_2^{\mathrm{T}} \\ D_{21}^{\mathrm{T}} \end{bmatrix}, \qquad \Gamma_Y = \begin{bmatrix} B_1 \\ D_{12} \end{bmatrix}$$

then there exists an output feedback controller $(\bar{\Sigma})$ such that the resulting closed-loop system (Σ_c) is asymptotically stable and ESPR. Moreover, if $\mathrm{rank}\,(I - X_hY_h) = k_h < n_h$ and $\mathrm{rank}\,(I - X_vY_v) = k_v < n_v$ for solution matrices (X, Y), then there exists a reduced order controller with order $k_h + k_v$. In this case, a desired output feedback controller corresponding to a feasible solution (X, Y) of (26)–(28) *is given by*

$$\begin{bmatrix} \bar{D} & \bar{C} \\ \bar{B} & \bar{A} \end{bmatrix} = \Upsilon_R^{+}K\Xi_L^{+} + Z - \Upsilon_R^{+}\Upsilon_R Z \Xi_L \Xi_L^{+} \tag{29}$$

$$K = -W^{-1}\Upsilon_L^{\mathrm{T}}\Lambda\Xi_R^{\mathrm{T}}(\Xi_R\Lambda\Xi^{|}T_R)^{-1} + W^{-1}S^{1/2}L(\Xi_R\Lambda\Xi_R^{\mathrm{T}})^{-1/2} \tag{30}$$

$$S = W - \Upsilon_L^{\mathrm{T}}[\Lambda - \Lambda\Xi_R^{\mathrm{T}}(\Xi_R\Lambda\Xi_R^{\mathrm{T}})^{-1}\Xi_R\Lambda]\Upsilon_L \tag{31}$$

$$\Lambda = (\Upsilon_L W^{-1}\Upsilon_L^{\mathrm{T}} - \Omega)^{-1} \tag{32}$$

$$\Omega = \begin{bmatrix} -X_{hv} & \hat{C}^{\mathrm{T}} & \hat{A}^{\mathrm{T}} \\ \hat{C} & -(\hat{D} + \hat{D}^{\mathrm{T}}) & -\hat{B}^{\mathrm{T}} \\ \hat{A} & -\hat{B} & -Y_{hv} \end{bmatrix} \tag{33}$$

$$X_{hv} = \begin{bmatrix} X & X_{12} \\ X_{12}^{\mathrm{T}} & X_{22} \end{bmatrix}, \qquad Y_{hv} = \begin{bmatrix} Y & Y_{12} \\ Y_{12}^{\mathrm{T}} & Y_{22} \end{bmatrix} \tag{34}$$

$$X_{12} = \begin{bmatrix} X_{h12} & 0 \\ 0 & X_{v12} \end{bmatrix}, \qquad X_{22} = \begin{bmatrix} X_{h22} & 0 \\ 0 & X_{v22} \end{bmatrix} \tag{35}$$

$$Y_{22} = \begin{bmatrix} (X_{h22} - X_{h12}^{\mathrm{T}}X_h^{-1}X_{h12})^{-1} & 0 \\ 0 & (X_{v22} - X_{v12}^{\mathrm{T}}X_v^{-1}X_{v12})^{-1} \end{bmatrix} \tag{36}$$

$$Y_{12} = \begin{bmatrix} -Y_hX_{h12}X_{h22}^{-1} & 0 \\ 0 & -Y_vX_{v12}X_{v22}^{-1} \end{bmatrix} \tag{37}$$

$$\Xi = [\hat{H} \quad -\hat{N} \quad 0], \qquad \Upsilon^{\mathrm{T}} = [0 \quad \hat{S}^{\mathrm{T}} \quad \hat{F}^{\mathrm{T}}] \tag{38}$$

where $\hat{A}$, $\hat{B}$, $\hat{C}$, $\hat{D}$, $\hat{H}$, $\hat{N}$, $\hat{S}$ and $\hat{F}$ are defined in (21) *and* (22). Z *and* L *are any matrices satisfying $\|L\| < 1$, where*

$\|\cdot\|$ denotes the spectral norm and $X_{h12} \in \mathbf{R}^{n_h \times k_h}$, $X_{h22} \in \mathbf{R}^{k_h \times k_h}$, $X_{v12} \in \mathbf{R}^{n_v \times k_v}$, $X_{v22} \in \mathbf{R}^{k_v \times k_v}$, $X_{h22} > 0$, $X_{v22} > 0$ and $W > 0$ satisfying

$$\Lambda > 0, \qquad X_h - Y_h^{-1} = X_{h12} X_{h22}^{-1} X_{h22}^{\mathrm{T}} \geq 0, \qquad X_v - Y_v^{-1} = X_{v12} X_{v22}^{-1} X_{v12}^{\mathrm{T}} \geq 0 \tag{39}$$

(Ξ_L, Ξ_R) and (Υ_L, Υ_R) are any full rank factors of Ξ and Υ, that is, $\Xi = \Xi_L \Xi_R$, $\Upsilon = \Upsilon_L \Upsilon_R$.

Proof: It follows from (28) that there always exist matrices $\bar{X}_{h12}$, $\bar{X}_{h22} \in \mathbf{R}^{n_h \times h_h}$, $\bar{X}_{v12}$, $\bar{X}_{v22} \in \mathbf{R}^{n_v \times n_v}$ and $\bar{X}_{h22} > 0$, $\bar{X}_{v22} > 0$ such that

$$Y_h^{-1} - X_h = -\bar{X}_{h12} \bar{X}_{h22}^{-1} \bar{X}_{h12}^{\mathrm{T}} \leq 0$$

$$Y_v^{-1} - X_v = -\bar{X}_{v12} \bar{X}_{v22}^{-1} \bar{X}_{v12}^{\mathrm{T}} \leq 0$$

Define

$$P_h = \begin{bmatrix} X_h & \bar{X}_{h12} \\ \bar{X}_{h12}^{\mathrm{T}} & \bar{X}_{h22} \end{bmatrix}, \qquad P_v = \begin{bmatrix} X_v & \bar{X}_{v12} \\ \bar{X}_{v12}^{\mathrm{T}} & \bar{X}_{v22} \end{bmatrix} \tag{40}$$

Then

$$P_h^{-1} = \begin{bmatrix} Y_h & Z_{h12} \\ Z_{h12}^{\mathrm{T}} & Z_{h22} \end{bmatrix}, \qquad P_v^{-1} = \begin{bmatrix} Y_v & Z_{v12} \\ Z_{v12}^{\mathrm{T}} & Z_{v22} \end{bmatrix} \tag{41}$$

where

$$Z_{h12} = -Y_h \bar{X}_{h12} \bar{X}_{h22}^{-1}, \qquad Z_{v12} = -Y_v \bar{X}_{v12} \bar{X}_{v22}^{-1}$$

$$Z_{h22} = (\bar{X}_{h22} - \bar{X}_{h12}^{\mathrm{T}} X_h^{-1} \bar{X}_{h12})^{-1},$$

$$Z_{v22} = (\bar{X}_{v22} - \bar{X}_{v12}^{\mathrm{T}} X_v^{-1} \bar{X}_{v12})^{-1}$$

Let

$$[W_1 \quad W_2] = \begin{bmatrix} B_1 \\ D_{12} \end{bmatrix}^{\perp}, \qquad [v_1 \quad V_2] = \begin{bmatrix} C_2^{\mathrm{T}} \\ D_{21}^{\mathrm{T}} \end{bmatrix}^{\perp}$$

Then it is easy to show that

$$\Upsilon^{\perp} = \begin{bmatrix} [V_1 \quad 0] & -V_2 & 0 \\ [0 \quad 0] & 0 & I \end{bmatrix}, \quad \Xi^{\mathrm{T}\perp} = \begin{bmatrix} 0 & W_2 & [W_1 \quad 0] \\ I & 0 & [0 \quad 0] \end{bmatrix}$$

Define

$$\Omega_1 = \begin{bmatrix} -\bar{X}_{hv} & \hat{C}^{\mathrm{T}} & \hat{A}^{\mathrm{T}} \\ \hat{C} & -(\hat{D} + \hat{D}^{\mathrm{T}}) & -\hat{B}^{\mathrm{T}} \\ \hat{A} & -\hat{B} & -\hat{Y}_{hv} \end{bmatrix}$$

where

$$\bar{X}_{hv} = \begin{bmatrix} X & \bar{X}_{12} \\ \bar{X}_{12}^{\mathrm{T}} & \bar{X}_{22} \end{bmatrix}, \qquad \bar{Y}_{hv} = \begin{bmatrix} Y & \bar{Y}_{12} \\ \bar{Y}_{12}^{\mathrm{T}} & Y_{22} \end{bmatrix}$$

$$\bar{X}_{12} = \begin{bmatrix} \bar{X}_{h12} & 0 \\ 0 & \bar{X}_{v12} \end{bmatrix}, \qquad \bar{X}_{22} = \begin{bmatrix} \bar{X}_{h22} & 0 \\ 0 & \bar{X}_{v22} \end{bmatrix}$$

$$\bar{Y}_{12} = \begin{bmatrix} Z_{h12} & 0 \\ 0 & Z_{v12} \end{bmatrix}, \qquad \bar{Y}_{22} = \begin{bmatrix} Z_{h22} & 0 \\ 0 & Z_{v22} \end{bmatrix}$$

From (40) and (41), we have $\bar{X}_{hv}^{-1} = \bar{Y}_{hv}$. Then, using (26) and (27), we can verify that

$$\Xi^{\mathrm{T}\perp} \Omega_1 \Xi^{\mathrm{T}\perp\mathrm{T}} < 0, \qquad \Upsilon^{\perp} \Omega_1 \Upsilon^{\perp\mathrm{T}} < 0 \tag{42}$$

Therefore, by Lemma 2 it follows that there exists a matrix $\hat{G}$ such that

$$\Omega_1 + \Upsilon \hat{G} \Xi + (\Upsilon \hat{G} \Xi)^{\mathrm{T}} < 0 \tag{43}$$

Pre- and post-multiplying (43) by $\mathrm{diag}(\Theta, I, I)$ and $\mathrm{diag}(\Theta^{\mathrm{T}}, I, I)$ respectively, we now have that

$$\begin{bmatrix} -\tilde{P} & \tilde{C}^{\mathrm{T}} & \tilde{A}^{\mathrm{T}} \\ \tilde{C} & -(\tilde{D} + \tilde{D}^{\mathrm{T}}) & -\tilde{B}^{\mathrm{T}} \\ \tilde{A} & -\tilde{B} & -\tilde{P}^{-1} \end{bmatrix} < 0 \tag{44}$$

where the relationship $\Theta^{-1} = \Theta^{\mathrm{T}}$ has been used, and $\tilde{P}$ is given by

$$\tilde{P} = \mathrm{diag}\,(P_h, P_v)$$

By Schur complements, equation (33) implies that

$$\begin{bmatrix} \tilde{A}^{\mathrm{T}} \tilde{P} \tilde{A} - \tilde{P} & \tilde{C}^{\mathrm{T}} - \tilde{A}^{\mathrm{T}} \tilde{P} \tilde{B} \\ \tilde{C} - \tilde{B}^{\mathrm{T}} \tilde{P} \tilde{A} & -(\tilde{D} + \tilde{D}^{\mathrm{T}} - \tilde{B}^{\mathrm{T}} \tilde{P} \tilde{B}) \end{bmatrix} < 0 \tag{45}$$

Noting this and applying Theorem 1, we have that there exists an output feedback controller $(\bar{\Sigma})$ such that the resulting closed-loop system Σ_c is asymptotically stable and ESPR, i.e. the positive real control problem is solvable. Furthermore, when (26)–(28) are satisfied, the parameterization of all desired output feedback controllers satisfying the LMI (43) can be obtained by using the results in Gahinet and Apkarian (1994) and Iwasaki and Skelton (1994). This completes the proof.

Remark 2: Theorem 2 provides a sufficient condition for designing an output feedback controller which stabilizes a 2-D discrete linear system described by the Roesser state-space model and achieves the extended strictly positive realness property of the closed-loop system. It is worth pointing out that the LMIs (26)–(28) in Theorem 2 can be solved efficiently, and no tuning of parameters is required (Boyd *et al.* 1994).

4. Positive realness of linear repetitive processes

The state-space model of a discrete linear repetitive process has the following form over $0 \le p \le \alpha$, $k \ge 0$

$$\left.\begin{aligned} x_{k+1}(p+1) &= \widehat{A}x_{k+1}(p) + \widehat{B}u_{k+1}(p) + \widehat{B}_0 y_k(p) \\ y_{k+1}(p) &= \widehat{C}x_{k+1}(p) + \widehat{D}u_{k+1}(p) + \widehat{D}_0 y_k(p) \end{aligned}\right\} \quad (46)$$

Here on pass $x_k(p) \in \mathbf{R}^n$ is the current pass state vector, $y_k(p) \in \mathbf{R}^m$ is the current pass profile vector, and $u_k(p) \in \mathbf{R}^m$ is the vector of current pass inputs. To complete the process description, it is necessary to specify the initial, or boundary conditions, i.e. the state initial vector on each pass $x_{k+1}(0)$, $k \ge 0$, and the initial pass profile $y_0(p)$. Here these are taken to be of the simplest possible form, i.e. $x_{k+1}(0) = d_{k+1}$, $k \ge 0$, and $y_0(p) = y(p)$, $0 \le p \le \alpha$, where d_{k+1} is an $n \times 1$ vector with constant entries and the entries in the $m \times 1$ vector $y(p)$ are known functions of p. Note, however, that the structure of the boundary conditions alone can cause instability in these processes. (See Owens and Rogers (1999) where this fact is established for the differential counterparts of the processes considered here using a pass state initial vector sequence which is an explicit function of points on the previous pass profile.)

Recall that the unique control problem for repetitive processes is that the output sequence of pass profiles generated can contain oscillations that increase in the pass-to-pass direction (i.e. in the k direction in the notation for variables used here). Also this problem cannot be solved (in all but a few very restrictive cases) by 1-D systems based control action. This fact has led to the development of a rigorous stability theory as the first essential step in the development of a mature systems theory for onward translation (where possible/appropriate) into computationally and implementable control laws.

The stability theory for repetitive processes with linear dynamics and a constant pass length is based on an abstract model of the underlying dynamics in a Banach space setting which includes all such processes as special cases—for a full treatment see Rogers and Owens (1992). This theory consists of two distinct concepts termed asymptotic stability and stability along the pass respectively where the former is a necessary condition for the latter. In effect, asymptotic stability demands that bounded input sequences produce bounded sequences of pass profiles (where here bounded is defined in terms of the norm on the underlying function space) over the (finite and constant) pass length.

If this asymptotic stability property holds then the output sequence of pass profiles converges (in the k direction) to the so-called limit profile but the fact that the pass length is finite does not ensure that this limit profile has acceptable along the pass dynamics. For example, in the case of processes described by (46) asymptotic stability (with the assumed boundary conditions) holds if, and only if, all eigenvalues of the matrix $\widehat{D}_0$ have modulus strictly less than unity and the resulting limit profile is given by a 1-D linear systems state space model where in the case when $\widehat{D} = 0$ (which incurs no loss of generality) the state matrix in this model is $\widehat{A} + \widehat{B}_0(I_m - \widehat{D}_0)^{-1}\widehat{C}$. Hence it is possible for asymptotic stability to hold but this last matrix has at least one eigenvalue with modulus greater than unity, e.g. the case when $\widehat{A} = -0.5$, $\widehat{B}_0 = 0.5 + \beta$, $\widehat{C} = 1$, $\widehat{D}_0 = 0$, where β is a real scalar.

In a case such as this last example, the resulting limit profile is unstable along the pass. The stronger property of stability along the pass prevents such a case from occurring by demanding the bounded input bounded output stability property uniformly, i.e. independent of the pass length. For applications, therefore, a critical task is to specify the structure for a control law to ensure asymptotic stability and also to obtain computationally tractable algorithms for designing the control law parameters. This is still an essentially open problem and the remainder of this section addresses this problem for discrete linear repetitive processes described by (46) via positive realness analysis based on the 1-D equivalent model of the dynamics of such processes which, in turn, can be constructed from a Roesser model interpretation of the repetitive dynamics in this case.

In Roesser model terms, the pass profile vector here $y_k(p)$ plays the role of the vertical state vector and the pass state vector $x_k(p)$ plays the role of the horizontal state vector. Also the pass profile vector is simultaneously the output vector in Roesser model terms and hence we can write for $k \ge 1$

$$\left.\begin{aligned} y_k(p) &= \widehat{C}\begin{bmatrix} x_k(p) \\ y_k(p) \end{bmatrix} + \widehat{D}u_k(p) \\ &= \begin{bmatrix} 0 & I \end{bmatrix}\begin{bmatrix} x_k(p) \\ y_k(p) \end{bmatrix} + 0u_k(p) \end{aligned}\right\} \quad (47)$$

The corresponding 2D z transfer function matrix is

$$G(z_1, z_2) = \begin{bmatrix} 0 & I \end{bmatrix}\begin{bmatrix} z_1 I - \widehat{A} & -\widehat{B}_0 \\ -\widehat{C} & z_2 I - \widehat{D}_0 \end{bmatrix}^{-1}\begin{bmatrix} \widehat{B} \\ \widehat{D} \end{bmatrix} \quad (48)$$

Hence, it follows immediately that no discrete linear repetitive process of the form considered here can ever by asymptotically stable and ESPR since $\widehat{D} = 0$ and hence $D + \widehat{D}^{\mathrm{T}} > 0$, which is necessary for ESPR, see (7), can never hold.

To apply PR theory to discrete linear repetitive processes, we propose a route via the 1-D equivalent state space model description of the underlying dynamics. This 1-D equivalent model has been developed in, for example, Galkowski *et al.* (1998) and here we need only give the final construction.

The starting point is to make the substitutions $l = k+1$ and $y_{k-1}(p) = v_k(p)$, $0 \le p \le \alpha - 1$, $l = 1, 2, \ldots$. Now define the so-called global pass profile, state and input vectors respectively for (46) as

$$\left.\begin{aligned} Y(l) &:= [v_l^{\mathrm{T}}(0),\ v_l^{\mathrm{T}}(1),\ \ldots,\ v_l^{\mathrm{T}}(\alpha-1)]^{\mathrm{T}} \\ X(l) &:= [x_l^{\mathrm{T}}(1),\ x_l^{\mathrm{T}}(2),\ \ldots,\ x_l^{\mathrm{T}}(\alpha)]^{\mathrm{T}} \\ U(l) &:= [u_l^{\mathrm{T}}(0),\ u_l^{\mathrm{T}}(1),\ \ldots,\ u_l^{\mathrm{T}}(\alpha-1)]^{\mathrm{T}} \end{aligned}\right\} \tag{49}$$

Then, assuming without loss of generality that the state initial vector on each pass is zero, i.e. $d_{k+1} = 0$, $k \ge 0$, the 1-D equivalent state space model of the dynamics of (46) has the form

$$\begin{aligned} Y(l+1) &= \Phi Y(l) + \Delta U(l) \\ X(l) &= \Gamma Y(l) + \Sigma U(l) \end{aligned} \tag{50}$$

where

$$\left.\begin{aligned} \Phi &= \begin{bmatrix} \widehat{D}_0 & 0 & \cdots & 0 \\ \widehat{C}\widehat{B}_0 & \widehat{D}_0 & \cdots & 0 \\ \widehat{C}\widehat{A}\widehat{B}_0 & \widehat{C}\widehat{B}_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \widehat{C}\widehat{A}^{\alpha-2}\widehat{B}_0 & \widehat{C}\widehat{A}^{\alpha-3}\widehat{B}_0 & \cdots & \widehat{D}_0 \end{bmatrix}, \\ \Delta &= \begin{bmatrix} \widehat{D} & 0 & 0 & \cdots & 0 \\ \widehat{C}\widehat{B} & D & 0 & \cdots & 0 \\ \widehat{C}\widehat{A}\widehat{B} & \widehat{C}\widehat{B} & \widehat{D} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \widehat{C}\widehat{A}^{\alpha-2}\widehat{B} & \widehat{C}\widehat{A}^{\alpha-3}\widehat{B} & \widehat{C}\widehat{A}^{\alpha-4}\widehat{B} & \cdots & \widehat{D} \end{bmatrix} \\ \Gamma &= \begin{bmatrix} \widehat{B}_0 & 0 & 0 & \cdots & 0 \\ \widehat{A}\widehat{B}_0 & \widehat{B}_0 & 0 & \cdots & 0 \\ \widehat{A}^2\widehat{B}_0 & \widehat{A}\widehat{B}_0 & \widehat{B}_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \widehat{A}^{\alpha-1}\widehat{B}_0 & \widehat{A}^{\alpha-2}\widehat{B}_0 & \widehat{A}^{\alpha-3}\widehat{B}_0 & \cdots & \widehat{B}_0 \end{bmatrix}, \\ \Sigma &= \begin{bmatrix} \widehat{B} & 0 & 0 & \cdots & 0 \\ \widehat{A}\widehat{B} & \widehat{B} & 0 & \cdots & 0 \\ \widehat{A}^2\widehat{B} & \widehat{A}\widehat{B} & \widehat{B} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A^{\alpha-1}B & \widehat{A}^{\alpha-2}\widehat{B} & A^{\alpha-3}\widehat{B} & \cdots & \widehat{B} \end{bmatrix} \end{aligned}\right\} \tag{51}$$

Given this 1-D equivalent model, we can now establish one of the main results in this paper which requires the additional assumption that the dimension of $x_k(p)$ is equal to that of $u_k(p)$. This assumption arises from the fact that in the 1-D equivalent model the pass profile, which in the 2-D linear systems interpretation of the dynamics of these processes, is the subject of dynamic updating and the pass profile vector (horizontally transmitted information in the 2-D setting) is embedded in a static (or purely algebraic) equation. The proof of this result follows immediately from the known result for 1-D discrete linear systems (Sun *et al.* 1994) and the structure of the 1-D equivalent model. Hence it is omitted here.

Theorem 3: *Discrete repetitive processes of the form* (46) *with 1-D equivalent state space model defined by* (50) *and* (51) *are asymptotically stable and ESPR if, and only, if there exists an* $m\alpha \times m\alpha$ *real matrix* $P > 0$ *such that the following LMI is satisfied*

$$\begin{bmatrix} \Phi^{\mathrm{T}} P \Phi - P & \Gamma^{\mathrm{T}} - \Phi^{\mathrm{T}} P \Delta \\ \Gamma - \Delta^{\mathrm{T}} P \Phi & -(\Sigma + \Sigma^{\mathrm{T}} - \Delta^{\mathrm{T}} P \Delta) \end{bmatrix} < 0 \tag{52}$$

The only major difficulty with Theorem 3 is that the (potentially) large dimension of the matrix P may cause numerical difficulties. In what follows, we develop a feasible way of avoiding such problems by assuming that P has a block diagonal form, i.e.

$$P = \mathrm{diag}(P_1, P_2, \cdots, P_\alpha) \tag{53}$$

Under the assumption of (53), the block sub-matrices of (52) can be expressed as

$$\Phi^{\mathrm{T}} P \Phi - P = [\Omega^1_{ij}]_{\alpha\times\alpha} \tag{54}$$

where

$$\left.\begin{aligned} \Omega^1_{ii} &= \widehat{D}_0^{\mathrm{T}} P_i \widehat{D}_0 + \sum_{k=0}^{\alpha-1-i} \widehat{B}_0^{\mathrm{T}} (\widehat{A}^{\mathrm{T}})^k \widehat{C}^{\mathrm{T}} P_{k+i+1} \widehat{C}\widehat{A}^k \widehat{B}_0 - P_i \\ \Omega^1_{i+q,i} &= \widehat{D}_0^{\mathrm{T}} P_{i+q} \widehat{C}\widehat{A}^{q-1}\widehat{B}_0 \\ &\quad + \sum_{k=q}^{\alpha-1-i} \widehat{B}_0^{\mathrm{T}} (\widehat{A}^{\mathrm{T}})^{k-q} \widehat{C}^{\mathrm{T}} P_{k+i+1} \widehat{C}\widehat{A}^k \widehat{B}_0 \\ \Omega^1_{i,i+q} &= (\Omega^1_{i+q,i})^{\mathrm{T}} \end{aligned}\right\} \tag{55}$$

$i = 1, 2, \ldots, \alpha$; $q = 1, 2, \ldots, \alpha - i$

$$\Gamma^{\mathrm{T}} - \Phi^{\mathrm{T}} P \Delta = [\Omega^2_{ij}]_{\alpha\times\alpha} \tag{56}$$

with

$$\left.\begin{aligned}\Omega^2_{ii} &= \widehat{B}_0^{\mathrm{T}} - \widehat{D}_0^{\mathrm{T}} P_i \widehat{D} - \sum_{k=0}^{\alpha-1-i} \widehat{B}_0^{\mathrm{T}} (A^{\mathrm{T}})^k \widehat{C}^{\mathrm{T}} P_{k+i+1} \widehat{C}\widehat{A}^k \widehat{B} \\ \Omega^2_{i+q,i} &= -\widehat{D}_0^{\mathrm{T}} P_{i+q} \widehat{C}\widehat{A}^{q-1}\widehat{B} \\ &\quad - \sum_{k=q}^{\alpha-1-i} \widehat{B}_0^{\mathrm{T}} (\widehat{A}^{\mathrm{T}})^{k-q} \widehat{C}^{\mathrm{T}} P_{k+i+1} \widehat{C}\widehat{A}^k \widehat{B} \\ \Omega^2_{i,i+q} &= \widehat{B}_0^{\mathrm{T}} \widehat{A}^{q\mathrm{T}} - \widehat{B}_0^{\mathrm{T}} (\widehat{A}^{\mathrm{T}})^{q-1} \widehat{C}^{\mathrm{T}} P_{i+q} \widehat{D} \\ &\quad - \sum_{k=q}^{\alpha-1-i} \widehat{B}_0^{\mathrm{T}} \widehat{A}^{k\mathrm{T}} C^{\mathrm{T}} P_{k+i+1} \widehat{C}\widehat{A}^{k-q}\widehat{B}\end{aligned}\right\} \tag{57}$$

and

$$-(\Sigma + \Sigma^{\mathrm{T}} - \Delta^{\mathrm{T}} P \Delta) = [\Omega^3_{ij}]_{\alpha\times\alpha} \tag{58}$$

with

$$\left.\begin{aligned}\Omega^3_{ii} &= -\widehat{B} - \widehat{B}^{\mathrm{T}} + \widehat{D}^{\mathrm{T}} P_i \widehat{D} \\ &\quad + \sum_{k=0}^{\alpha-1-i} \widehat{B}^{\mathrm{T}} (\widehat{A}^{\mathrm{T}})^k \widehat{C}^{\mathrm{T}} P_{k+i=1} \widehat{C}\widehat{A}^k \widehat{B} \\ \Omega^3_{i+q,i} &= -A^q \widehat{B} + \widehat{D}^{\mathrm{T}} P_{i+q} \widehat{C}\widehat{A}^{q-1}\widehat{B} \\ &\quad + \sum_{k=q}^{\alpha-1-i} \widehat{B}^{\mathrm{T}} \widehat{A}^{k-q,\mathrm{T}} \widehat{C}^{\mathrm{T}} P_{k+i+1} \widehat{C}\widehat{A}^k \widehat{B} \\ \Omega^3_{i,i+q} &= (\Omega^3_{i+q,i})^{\mathrm{T}}\end{aligned}\right\} \tag{59}$$

and $i = 1, 2, \ldots, \alpha; q = 1, 2, \ldots, \alpha - i$.

Hence, all blocks in (52) are of the form

$$K_0 + \sum_{i=1}^{\alpha} K_i P_i L_i \tag{60}$$

where the matrices K_i and L_i have constant entries, which are defined by the matrices in the original process state space model, and the positive definite P_i, $1 \le i \le \alpha$, are the problem solution matrices to be searched for in the LMI computation. Note also that the underlying assumption here, i.e. that P has a block diagonal structure, will make the stability condition more conservative. Also this would be increased further if it were to be assumed that $P_j = P, j = 1, 2, \ldots, \alpha$.

5. Positive real control for linear repetitive processes

Consider the following repetitive process

$$\left.\begin{aligned}x_{k+1}(p+1) &= \widehat{A}x_{k+1}(p) + \widehat{B}u_{k+1}(p) \\ &\quad + \widehat{B}_0 y_k(p) + \widehat{E}w_{k+1}(p) \\ y_{k+1}(p) &= \widehat{C}x_{k+1}(p) + \widehat{D}u_{k+1}(p) \\ &\quad + \widehat{D}_0 y_k(p) + \widehat{R}w_{k+1}(p)\end{aligned}\right\} \tag{61}$$

where $w_{k+1}(p)$ is an exogenous input vector. Then the 1-D equivalent state space model of the dynamics of (61) (with the pass state initial vector sequence set equal to zero) has the form

$$\left.\begin{aligned}Y(l+1) &= \Phi Y(l) + \Delta U(l) + \Pi W(l) \\ X(l) &= \Gamma Y(l) + \Sigma U(l) + \Upsilon W(l)\end{aligned}\right\} \tag{62}$$

where Φ, Δ, Γ and Σ are given in (51)

$$W(l) := \begin{bmatrix} w_l(0) \\ w_l(1) \\ \vdots \\ w_l(\alpha-1) \end{bmatrix} \tag{63}$$

and

$$\Pi = \begin{bmatrix} \widehat{R} & 0 & 0 & \cdots & 0 \\ \widehat{C}\widehat{E} & \widehat{R} & 0 & \cdots & 0 \\ \widehat{C}\widehat{A}\widehat{E} & \widehat{C}\widehat{E} & \widehat{R} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \widehat{C}\widehat{A}^{\alpha-2}\widehat{E} & \widehat{C}\widehat{A}^{\alpha-3}\widehat{E} & \widehat{C}\widehat{A}^{\alpha-4}\widehat{E} & \cdots & \widehat{R} \end{bmatrix}$$

$$\Upsilon = \begin{bmatrix} \widehat{E} & 0 & 0 & \cdots & 0 \\ \widehat{A}\widehat{E} & \widehat{E} & 0 & \cdots & 0 \\ \widehat{A}^2\widehat{E} & \widehat{A}\widehat{E} & \widehat{E} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \widehat{A}^{\alpha-1}\widehat{E} & \widehat{A}^{\alpha-2}\widehat{E} & \widehat{A}^{\alpha-3}\widehat{E} & \cdots & \widehat{E} \end{bmatrix}$$

Then we have the following synthesis result whose proof is immediate from Theorem 3 and a simple application of the Schur complement's formula. Hence it is omitted here.

Theorem 4: *Consider the discrete repetitive processes described by the 1-D equivalent state space model of* (62). *Then if there exists an* $m\alpha \times m\alpha$ *real matrix* $P > 0$ *and a matrix* Z *such that the following LMI is satisfied*

$$\begin{bmatrix} -P & (\Gamma P + \Sigma Z)^{\mathrm{T}} & (\Phi P + \Delta Z)^{\mathrm{T}} \\ \Gamma P + \Sigma Z & -(\Upsilon + \Upsilon^{\mathrm{T}}) & -\Pi^{\mathrm{T}} \\ \Phi P + \Delta Z & -\Pi & -P \end{bmatrix} < 0 \tag{64}$$

the state feedback control law

$$U(l) = KY(l) \tag{65}$$

where $K = ZP^{-1}$ *will be such that the resulting closed-loop system formed by* (62) *and* (65) *is asymptotically stable and ESPR.*

Now we give two examples to illustrate the effectiveness of the proposed method—one for each of the two model classes considered.

Example 1: Consider the 2-D discrete linear system (Σ_R) defined by

$$A = \left[\begin{array}{cc|cc} 0.2 & 0.3 & 0.2 & -0.1 \\ 0.2 & 0.1 & 0 & 0.5 \\ \hline 0.8 & 0.2 & -0.3 & -0.1 \\ 0.2 & 0 & 0.3 & 0.1 \end{array}\right], \quad B = \left[\begin{array}{cc} 0.2 & 0.2 \\ 0.5 & 0 \\ \hline 0.5 & 0 \\ 0.2 & 0.3 \end{array}\right]$$

$$B_1 = \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \\ \hline 1 & 1 \\ 0 & 0 \end{array}\right], \quad C_1 = \left[\begin{array}{cc|cc} 0.2 & 0 & 0.1 & 0.2 \\ 0 & -0.3 & 0.1 & 0 \end{array}\right]$$

$$D_{11} = \begin{bmatrix} 2 & 0.5 \\ 0.1 & 2.5 \end{bmatrix}, \quad D_{12} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$C_2 = \left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \end{array}\right], \quad D_{21} = [1 \quad 1]$$

Then, it is easy to see that

$$\Gamma_X^{\perp} = \begin{bmatrix} C_2^{\mathrm{T}} \\ D_{21}^{\mathrm{T}} \end{bmatrix}^{\perp} = \left[\begin{array}{cccc|cc} -1 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Gamma_Y^{\perp} = \begin{bmatrix} B_1 \\ D_{12} \end{bmatrix}^{\perp} = \left[\begin{array}{cccc|cc} -1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \end{array}\right]$$

Noting this, we can verify that the pair (X, Y) with $X = \operatorname{diag}(X_h, X_v) > 0$ and $Y = \operatorname{diag}(Y_h, Y_v) > 0$ satisfies (26)–(28) with

$$X_h = \begin{bmatrix} 2.5290 & 0.3576 \\ 0.3576 & 3.0691 \end{bmatrix}, \quad X_v = \begin{bmatrix} 2.5374 & 0.0126 \\ 0.0126 & 3.7856 \end{bmatrix} \tag{66}$$

$$Y_h = \begin{bmatrix} 0.4470 & -0.0521 \\ -0.0521 & 0.3319 \end{bmatrix}, \quad Y_v = \begin{bmatrix} 0.4908 & -0.0019 \\ -0.0019 & 0.3044 \end{bmatrix} \tag{67}$$

Therefore, from Theorem 2, there exists an output feedback controller $(\bar{\Sigma})$ such that the resulting closed-loop system Σ_c is asymptotically stable and ESPR. To construct such a controller, we can choose

$$X_{h12} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad X_{h22} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}, \quad X_{v12} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$X_{v22} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \tag{68}$$

$$W = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$L = \begin{bmatrix} 0.8 & 0 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 & 0 \\ 0 & 0 & 0.8 & 0 & 0 \\ 0 & 0 & 0 & 0.8 & 0 \\ 0 & 0 & 0 & 0 & 0.8 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{69}$$

It can be shown that (68) and (69) satisfy (39). Thus, by (29) we can obtain a desired output controller

$$\left[\begin{array}{c|c} \bar{D} & \bar{C} \\ \hline \bar{B} & \bar{A} \end{array}\right] = \left[\begin{array}{c|cccc} 0.0151 & -0.1054 & 0.0267 & -0.0729 & -0.0032 \\ -0.1459 & 0.6637 & -0.0688 & -0.0555 & -0.1214 \\ \hline -0.0130 & -0.0829 & 0.7245 & 0.0117 & 0.0150 \\ 0.0273 & 0.0091 & -0.0001 & 0.2289 & -0.0021 \\ -0.0465 & -0.0655 & 0.0147 & -0.0603 & 0.5843 \\ -0.0804 & -0.0023 & -0.0006 & 0.0106 & 0.0143 \end{array}\right]$$

That is

$$\left[\begin{array}{c} \bar{x}^h(i+1,j) \\ \hline \bar{x}^v(i,j+1) \end{array}\right] = \left[\begin{array}{cc|cc} -0.0829 & 0.7245 & 0.0117 & 0.0150 \\ 0.0091 & -0.0001 & 0.2289 & -0.0021 \\ \hline -0.0655 & 0.0147 & -0.0603 & 0.5843 \\ -0.0023 & -0.0006 & 0.0106 & 0.0143 \end{array}\right] \left[\begin{array}{c} \bar{x}^h(i,j) \\ \hline \bar{x}^v(i,j) \end{array}\right]$$

$$+\left[\begin{array}{c} -0.0130 \\ 0.0273 \\ \hline -0.0465 \\ -0.0804 \end{array}\right] y(i,j)$$

$$u(i,j) = \left[\begin{array}{cc|cc} -0.1054 & 0.0267 & -0.0729 & -0.0032 \\ 0.6637 & -0.0688 & -0.0555 & -0.1214 \end{array}\right] \times \left[\begin{array}{c} \bar{x}^h(i,j) \\ \hline \bar{x}^v(i,j) \end{array}\right] + \begin{bmatrix} 0.0151 \\ -0.1459 \end{bmatrix} y(i,j)$$

Example 2: Consider the discrete linear repetitive process defined by (61) with

$$\widehat{A} = 0.6, \quad \widehat{B} = 0.2, \quad \widehat{B}_0 = 0.1, \quad \widehat{C} = 0.1, \quad \widehat{D} = 0.2, \quad \widehat{D}_0 = 0.99, \quad \widehat{R} = 0.5, \quad \widehat{E} = 0.3$$

This process is not ESPR stable since (52) does not hold. The LMI of (64) is, however, feasible and one solution is the positive definite 10×10 matrix $P = 1.7530 I_{10}$, where I_{10} is the 10×10 identity matrix and

$$Z = \begin{bmatrix} -8.6775 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.7801 & -8.6775 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.3900 & 0.7801 & -8.6775 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.1950 & 0.3900 & 0.7801 & -8.6775 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.0975 & 0.1950 & 0.3900 & 0.7801 & -8.6775 & 0 & 0 & 0 & 0 & 0 \\ 0.0488 & 0.0975 & 0.1950 & 0.3900 & 0.7801 & -8.6775 & 0 & 0 & 0 & 0 \\ 0.0244 & 0.0488 & 0.0975 & 0.1950 & 0.3900 & 0.7801 & -8.6775 & 0 & 0 & 0 \\ 0.0122 & 0.0244 & 0.0488 & 0.0975 & 0.1950 & 0.3900 & 0.7801 & -8.6775 & -0 & 0 \\ 0.0061 & 0.0122 & 0.0244 & 0.0488 & 0.0975 & 0.1950 & 0.3900 & 0.7801 & -8.6775 & 0 \\ 0.0030 & 0.0061 & 0.0122 & 0.0244 & 0.0488 & 0.0975 & 0.1950 & 0.3900 & 0.7801 & -8.6775 \end{bmatrix}$$

Hence state feedback control law (65) with

$$K = \begin{bmatrix} -4.9500 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.4450 & -4.9500 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.2225 & 0.4450 & -4.9500 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.1112 & 0.2225 & 0.4450 & -4.9500 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.0556 & 0.1113 & 0.2225 & 0.4450 & -4.9500 & 0 & 0 & 0 & 0 & 0 \\ 0.0278 & 0.0556 & 0.1113 & 0.2225 & 0.4450 & -4.9500 & 0 & 0 & 0 & 0 \\ 0.0139 & 0.0278 & 0.0556 & 0.1112 & 0.2225 & 0.4450 & -4.9500 & 0 & 0 & 0 \\ 0.0070 & 0.0139 & 0.0278 & 0.0556 & 0.1112 & 0.2225 & 0.4450 & -4.9500 & 0 & 0 \\ 0.0035 & 0.0070 & 0.0139 & 0.0278 & 0.0556 & 0.1112 & 0.2225 & 0.4450 & -4.9500 & 0 \\ 0.0017 & 0.0035 & 0.0070 & 0.0139 & 0.0278 & 0.0556 & 0.1112 & 0.2225 & 0.4450 & -4.9500 \end{bmatrix}$$

will ensure that the resulting closed loop system is asymptotically stable and ESPR.

6. Conclusions

In this paper we have studied the problem of positive real control for 2-D discrete linear systems described by the Roesser model. A version of positive realness for such systems has been established and an LMI approach has been developed to construct a dynamic output feedback controller, which guarantees not only the asymptotic stability of the closed-loop system but also the extended strictly positive realness property of a certain closed-loop transfer function matrix. A similar problem has been considered for discrete linear repetitive processes based on the application of their 1-D equivalent state-space model representation. Analogous results to those for the Roesser model have also been developed for this case. Finally, numerical examples have been included which demonstrate the application of the design procedure for each case.

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