LDPC Assisted Block Coded Modulation for Transmission Over Rayleigh Fading Channels

F. Guo and S. X. Ng and L. Hanzo[1]

Dept. of ECS, University of Southampton, SO17 1BJ, UK.
Tel: +44-23-8059-3125, Fax: +44-23-8059-4508
Email: lh[1]@ecs.soton.ac.uk, http://www-mobile.ecs.soton.ac.uk

Abstract – **Coded modulation is a bandwidth efficient scheme that combines the functions of coding and modulation. Low Density Parity Check (LDPC) codes have attracted wide research interests owing to their relatively low decoding complexity, while maintaining a comparable performance to that of the best known channel codes, namely turbo codes. In this contribution we proposed an LDPC assisted Block Coded Modulation (LDPC-BCM) scheme for transmission over Rayleigh fading channels, which requires no bandwidth expansion. The performance of the proposed LDPC-BCM scheme will be compared to that of similar throughput Turbo Trellis Coded Modulation (TTCM) schemes. Specifically, 1, 2 and 3 bit per symbol effective throughput Quadrature Phase Shift Keying (QPSK), 8-level Phase Shift Keying (8PSK) and 16-level Quadrature Amplitude Modulation (16QAM) are studied in comparison to the corresponding TTCM benchmarkers. Furthermore, we comparatively study the associated decoding complexity. It is shown that owing to its block-based structure which is capable of overbridging deep channel fades, LDPC-BCM constitutes a better solution for transmission over hostile Rayleigh fading channels than TTCM, in particular when the system's affordable delay is limited.**

1. INTRODUCTION

Low Density Parity Check (LDPC) codes were devised by Gallager [1] in 1962. During the early evolutionary phase of channel coding, LDPC schemes made limited impact on the research of the channel coding community, despite their impressive performance, which was unprecedented prior to the turbo coding era. This was a consequence of its relatively high storage requirement and complexity. However, owing to their capability of approaching Shannon's predicted performance limits, in recent years research interests in LDPC codes have been rekindled. LDPC codes have been applied in conjunction with BPSK for transmission over both Additive White Gaussian Noise (AWGN) and Rayleigh fading channels [2]. It has also been shown that LDPC codes outperform convolutional codes by a significant margin in the context of large transmission packet sizes [3].

Turbo Trellis Coded Modulation (TTCM) [4] is an amalgamated derivative of Trellis Coded Modulation (TCM) [5] and turbo coding. TCM was originally proposed for transmission over Gaussian channels and later it was further developed for applications in mobile communications. TTCM [6] is a joint coding and modulation scheme that exhibits a structure similar to that of the family of power efficient binary turbo codes [6], but employs TCM schemes [6] as component codes. Specifically, TTCM invoked Set Partitioning (SP) based signal labelling, in order to achieve an increased free Euclidean distance between the unprotected bits of the modulated signal constellation.

The basic philosophy of Coded Modulation (CM) is that instead of sending a symbol formed by m information bits, for example two information bits for QPSK, we introduce a parity bit, while maintaining the same effective throughput of 2 bits/symbol by doubling the number of constellation points in the original constellation to eight, i.e. by extending the modulation scheme to 8PSK. As a consequence, the redundant bit can be absorbed by the expansion of the signal constellation, instead of accepting a 50% increase in the signalling rate, i.e. bandwidth. A positive coding gain is achieved, when the detrimental effect of decreasing the Euclidean distance of the neighbouring phasors is outweighed by the coding gain of the channel coding scheme incorporated.

In this contribution, we proposed an LDPC coding assisted BCM (LDPC-BCM) scheme for transmission over Rayleigh fading channels, which requires no bandwidth expansion. We evaluate the Bit Error Rate (BER) performance of the LDPC-BCM scheme and a set of similar throughput TTCM schemes are used as benchmarker. Specifically, QPSK, 8PSK and 16QAM will be utilised for achieving effective system throughputs of 1, 2 and 3 bits/symbol.

2. LDPC-BCM SCHEME

LDPC codes [1] belong to the family of linear block codes, which are defined by a parity check matrix having M rows and N columns. The column weight and row weight is low compared to the dimension M and N of the parity check matrix. The construction of the parity check matrix is referred to as regular or irregular, depending on whether the Hamming weight per column or row is identical. It has been shown [7] that carefully designed irregular LDPC codes may perform better than their regular counterparts. Furthermore, when the blocklength is increased, irregular LDPC codes may become capable of outperforming turbo codes [8]. However, the high blocklength required for outperforming turbo codes may be excessive for employment in interactive speech communications, for example, where a low delay is required. Thus in mobile communication, the main advantage of employing LDPC codes over turbo codes is their reduced complexity. The number of columns N is the number of coded bits hosted by an LDPC codeword. By contrast, the number of rows M corresponds to the number of parity check constraints imposed by the design of the LDPC code. The number of information bits encoded by an LDPC codeword is denoted by $K = N - M$, yielding a coding rate of K/N.

The LDPC decoder may invoke the probability propagation algorithm [1], also referred to as the sum-product algorithm [7] or belief propagation algorithm. The concept of the algorithm may be conveniently augmented using the bipartite graph shown in Fig 1, as suggested by Tanner [9] in 1981. This graph is constituted by two types of nodes, namely the *message nodes*, each of which corresponds to a column of the parity check matrix, and the *check nodes*, each of which corresponds to a row of the matrix. There are lines connecting these two types of nodes, and each connection corresponds to a non-zero entry in the parity check matrix of Fig 1. For example, the non-zero entry at the bottom right corner of the parity check matrix in Fig 1 corresponds to the connection between the 6^{th} node on the left and the 3^{rd} node on the right.

The decoding process is carried out as follows. The demodulator

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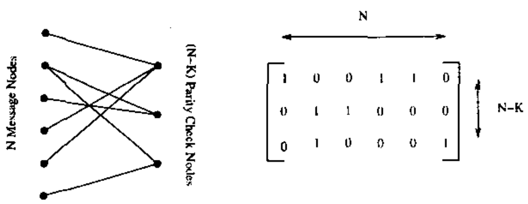

Figure 1: Graph Representation of the Parity Check Matrix

passes the N number of soft outputs received to the LDPC decoder. Based on the channel's soft output, the likelihood ratio of a binary one or zero associated with of each of the N message nodes can be calculated and stored as $intrinsic\,information$. Then the intrinsic information will be passed to the parity check nodes seen at the right of the left hand side illustration of Fig 1 according to the connection lines for determining the likelihood ratio associated with each parity check. Afterwards, the decoder's output *extrinsic information* is fed back to its input for recalculating the likelihood ratio. The decoder's final decision will be made based on the *a-posteriori information* which is calculated as [1]:

$$\frac{P[x_d = 1|\{y\}, S]}{P[x_d = 0|\{y\}, S]} = \frac{P_d}{1 - P_d} \prod_{i=1}^{j} L_r\,[S|\{x_i\}]. \qquad (1)$$

Afterwards, the associated decoded result will be verified by the parity check matrix. If a valid codeword is detected, the decoder outputs the codeword. If not, the above operation will be continued iteratively, until a valid codeword is detected or the maximum affordable number of iterations is reached.

The left hand side of the equation is the likelihood ratio of bit x_d at index d being a 1, given the values of y and S, where y represents the soft channel output and S stands for the probability that the parity check equations associated with bit x_d are satisfied. On the right, P_d denotes the probability that the bit x_d at index d is a logical 1, based only on the demodulator's soft output, thus the first multiplicative term is the *intrinsic information*, while the second is the likelihood ratio of the parity checks being satisfied for the information bit x_i, which is iteratively updated, as the decoding process iterates. Furthermore, $\{x_i\}$ is used for representing the outputs of the decision device at row i. More explicitly, when the likelihood ratio of each message node has converged to its correct value, i.e. it becomes positive for a logical 1 and negative for a logical 0, a valid codeword will be decoded after decision making. Thus the parity check will be satisfied and the decoding process of this codeword will be terminated.

Gray coding based signal labelling was employed in the LDPC-BCM scheme and a coding rate of $m/(m+1)$ is used in conjunction with a modulation scheme having 2^{m+1} number of modulation levels, similar to the TTCM benchmarker. Since the coding rate of the LDPC code is determined by $K/N = (N-M)/N$, while N and M constitute the number of columns and rows in the parity check matrix, respectively, the coding rate can be readily adjusted by changing the dimension of the parity check matrix. Specifically, we have $K = (N-M) = L \cdot m$ and $N{=}L \cdot (m+1)$, where L is the number of $(m+1)$-bit modulated symbols in a transmission burst and we fixed the column weight denoted by j to three. A lower column weight will render the information exchanged between the nodes in Fig 1 insufficient, while a higher column weight will increase the density of the parity check matrix. Therefore the associated complexity increases as well. Hence, for a fixed transmission burst length L, the binary LDPC code's codeword length N will increase with m and the resultant parity check matrix will have an increased number of non-zero entries as N increases, since the column weight was set to three, while the number of columns grows, as N increases.

Tx. Burst Length, L	1000 Symbols
Coding Rate, R	1/2, 2/3, 3/4
Modulation Mode	QPSK, 8PSK, 16QAM
Channel	Uncorrelated Rayleigh Fading Correlated Rayleigh Fading
Channel Interleaver Length	3000 Symbols
LDPC-BCM Column Weight	3
Maximum No. of Iterations for LDPC-BCM	50
TTCM Iteration	4
No. of LDPC-BCM Iterations	QPSK:15, 8PSK:10, 16QAM:10

Table 1: LDPC-BCM System Parameters

The decoding complexity of LDPC-BCM is proportional to the number of non-zero entries in the parity check matrix and to the average number of iterations invoked by the probability propagation decoding process [10]. When the channel SNR is high, the information obtained from the demodulator is typically reliable, and the parity-related information provided by the high-reliability bits will assist the erroneous bits in converging to their correct value significantly more rapidly, than in a low-SNR scenario. Thus the average number of iterations is significantly decreased, when the BER is below 10^{-3}. As a result, the decoding complexity imposed by the LDPC-BCM decoder decreases significantly, when the SNR approaches the so-called $cliff\ region$ of the BER curve, where the BER curve drops dramatically. Hence, the complexity imposed by the LDPC-BCM decoder may be reduced to a value below that of the TTCM decoder in the high-SNR region.

3. SIMULATION RESULTS

In this section, the results of our computer simulations are presented. Regular LDPC codes were used and again, the column weight was set to three. Other simulation parameters are summarised in Table 1. The maximum number of LDPC-BCM iterations was adjusted for the different modulation modes for ensuring that the maximum decoding complexity of the LDPC-BCM scheme did not exceed that of the TTCM benchmarker. The LDPC-BCM was constrained to have a maximum of 50 iterations. The number of iterations used was also given in the associated diagrams for demonstrating as to how much performance gain the LDPC-BCM scheme was capable of achieving at the cost of increasing the decoding complexity.

3.1. Performance in Uncorrelated Rayleigh Fading Channels

The performance of TTCM and LDPC-BCM was studied comparatively when communicating over uncorrelated Rayleigh fading channels. For the sake of fair comparisons, the codeword length of LDPC-BCM as well as the turbo interleaver length of TTCM was set to 1000 symbols. The transmission burst length was also 1000 symbols. Both TTCM and LDPC-BCM had the same coding rate, accommodating the associated parity information without bandwidth expansion.

Fig 2,3 and 4 demonstrated that in the context of uncorrelated Rayleigh channels LDPC-BCM outperforms the TTCM benchmarker scheme by almost 3dB, 1.5dB and 3dB, when using QPSK, 8PSK and 16QAM modulation, respectively at the BER of 10^{-5}. The Frame Error Rate (FER) is portrayed in Fig 5, where LDPC-BCM demonstrated a significantly better FER performance than the TTCM bench-

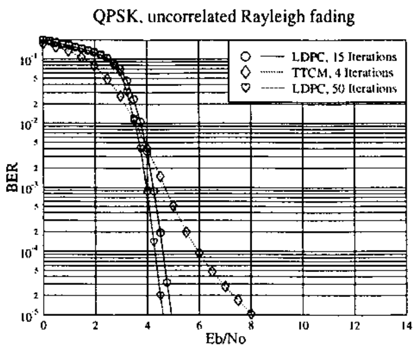

Figure 2: BER performance of LDPC-BCM and TTCM using the parameters of Table 1 and utilising QPSK modulation, when communicating over uncorrelated Rayleigh fading channels

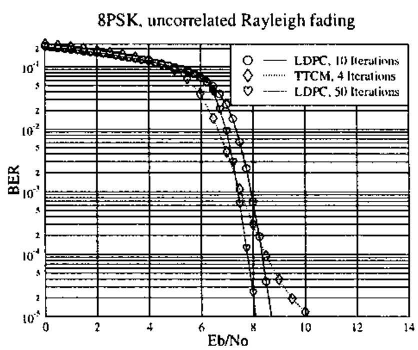

Figure 3: BER performance of LDPC-BCM and TTCM using the parameters of Table 1 and utilising 8PSK modulation, when communicating over uncorrelated Rayleigh fading channels

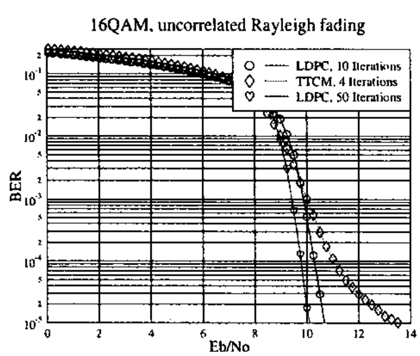

Figure 4: BER performance of LDPC-BCM and TTCM using the parameters of Table 1 and utilising 16QAM modulation, when communicating over uncorrelated Rayleigh fading channels

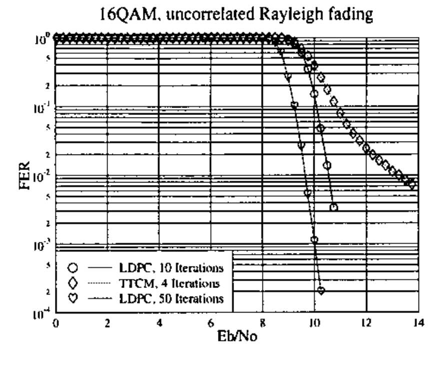

Figure 5: FER performance of LDPC-BCM and TTCM using the parameters of Table 1 and utilising 16QAM modulation, when communicating over uncorrelated Rayleigh fading channels

marker scheme. In Fig 2, 3 and 4, it was shown that increasing the number of iterations from 10 to 50 results in an approximately 0.5dB further coding gain for the LDPC-BCM scheme.

3.2. Performance in Correlated Rayleigh Fading Channel

In this subsection, we will study the achievable performance of TTCM and LDPC-BCM, when communicating over correlated Rayleigh fading channels. The normalised Doppler frequency of the non-dispersive Rayleigh fading channel was set to 3.25×10^{-5}. We considered a Time Division Multiple Access (TDMA) system, where there are 16 time-slots in the transmission frame and each user is assigned to one time-slot having a transmission burst duration of 1000 symbols. In order to disperse the bursty channel errors, a symbol-based channel interleaver depth of three TDMA bursts was utilised by the various TTCM schemes. Again for the sake of fair comparisons, we considered a total delay of of 3000 symbols for both the TTCM and for the LDPC-BCM schemes. A turbo interleaver length of 3000 information symbols was also utilised by the TTCM schemes.

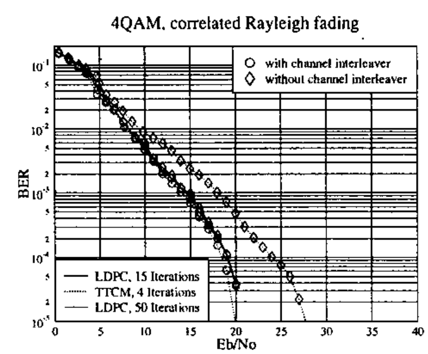

Figure 6: BER performance of LDPC-BCM and TTCM using the parameters of Table 1and utilising QPSK modulation, when communicating over correlated Rayleigh fading channels

On the other hand, LDPC-BCM does not require the assistance of either turbo or channel interleavers, which is an advantage over

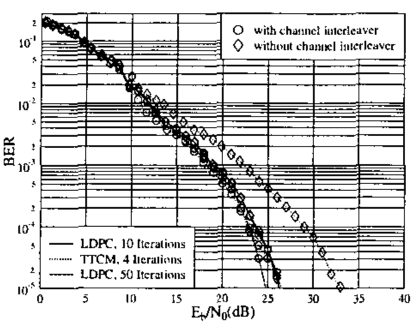

Figure 7: BER performance of LDPC-BCM and TTCM using the parameters of Table 1 and utilising 8PSK modulation, when communicating over correlated Rayleigh fading channels

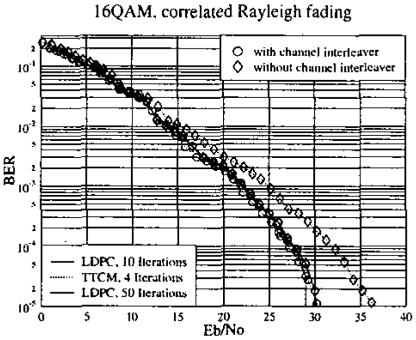

Figure 8: BER performance of LDPC-BCM and TTCM using the parameters of Table 1 and utilising 16QAM modulation, when communicating over correlated Rayleigh fading channels

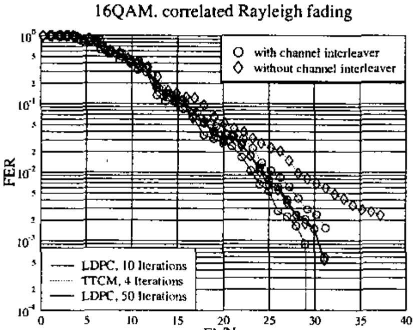

Figure 9: FER performance of LDPC-BCM and TTCM using the parameters of Table 1 and utilising 16QAM modulation, when communicating over correlated Rayleigh fading channels

the TTCM scheme in terms of reducing the associated complexity. The corresponding BER performance results are shown in Fig 6, 7 and 8 both with and without channel interleaving, demonstrating that for the block-coding based LDPC scheme no interleaving is necessary. Similar findings are valid also in terms of achievable FER, as it is demonstrated in Fig 9. This advantageous property of LDPC is achieved with the advent of the randomly constructed parity check matrix of the LDPC-BCM scheme, which is capable of separating the bits suffering from deep channel fades with the aid of different parity check equations. Thus, again, no additional channel interleaver is required for separating the bursty channel errors by the LDPC-BCM scheme.

4. DECODING COMPLEXITY

In Section 3, it has been demonstrated that the BER and FER performance of LDPC-BCM is superior to that of the TTCM scheme under the experimental conditions used. However, the associated decoding complexity is another important issue when comparing various schemes. In this section, the decoding complexity of LDPC-BCM and TTCM will be compared in terms of the number of mathematical operations required.

TTCM Complexity: Let us now consider the decoding complexity of the TTCM scheme utilising the symbol-based MAP decoder [6]. Briefly, the a posteriori probability of an m-bit information symbol u_t given the received sequence $\mathbf{y} = \{y_0, \dots, y_{N-1}\}$ of N number of transmitted symbols can be computed as follows [6]:

$$\mathbf{Pr}\{u_t = a|\mathbf{y}\}(t) \quad = \sum_{\substack{(\grave{s},s)\Rightarrow \\ u_t=a}} \beta_t(s)\cdot\alpha_{t-1}(\grave{s})\cdot\gamma_t(\grave{s},s), \quad (2)$$

where $(\grave{s}, s) \Rightarrow u_t = a$ indicates the specific set of trellis transitions emerging from the previous trellis state $S_{t-1} = \grave{s}$ to the present state $S_t = s$ that can be encountered, when the m-bit information symbol is $u_t = a$, where a is one of the legitimate 2^m-ary information symbols. Furthermore, we have $\alpha_t(s) = \sum_{\text{all } \grave{s}} \gamma_t(\grave{s}, s)\cdot\alpha_{t-1}(\grave{s})$ which is the results of the forward recursion [6], $\beta_{t-1}(\grave{s}) = \sum_{\text{all } s} \beta_t(s)\cdot\gamma_t(\grave{s}, s)$ is that of the backward recursion [6] and $\gamma_t(\grave{s}, s) = \Pi_{t,a}\cdot\eta_t(\grave{s}, s)$ is the branch transition metric [6]. Finally, $\Pi_{t,a}$ is the a priori information regarding the information symbol a and η_t is obtained at the demodulator, where we have $\eta_t(\grave{s}, s) = exp(-\frac{|y_t-x|^2}{2\sigma^2})$ for AWGN channels, and x is the legitimate 2^{m+1}-ary transmitted symbol corresponding to the information symbol a, while y_t is the received signal at time t and σ^2 is the noise's variance.

Let us compute the complexity of the MAP decoder for an information symbol decoded at time instant t. In order to compute the term γ_t given $\Pi_{t,a}$ and η_t, we need $M = 2^m$ multiplications. As for α_t and β_t, each term requires MS number of multiplication and $M(S-1)$ number of additions, where we have $S = 2^\nu$ and ν is the code memory. Finally, the evaluation of the term $\mathbf{Pr}(t)$ requires $2MS$ number of multiplications and $M(S-1)$ number of additions. Therefore, a total of $M(1+4S)$ number of multiplications and $3M(S-1)$ number of additions are required by one MAP decoder for decoding an m-bit symbol. Since two MAP decoders are required in the TTCM decoder, which performs T number of turbo iterations for decoding a block of N symbols, the total complexity of the TTCM scheme imposed by decoding mN number of information bits is $2TNM(1+4S)$ multiplications plus $6TNM(S-1)$ number of additions.

However, we utilised the log-MAP decoder for the sake of reducing the computational complexity imposed. Explicitly, the multiplication and addition operations are substituted by additions and by the Jacobian sum operations [11] carried out in the logarithmic domain, respectively. More specifically, each Jacobian sum consists of an addition, a subtraction, a table look-up and a maximum evaluation operation [11]. However, we can ignore the table lookup and max operations due to their comparably low complexity. As a result, the total complexity of the TTCM scheme utilising two log-MAP decoders for decoding mN number of information bits is $2TNM(1+4S) + 12TNM(S-1) = 10TNM(2S-1)$ additions/subtractions. For example, the total complexity imposed by decoding a block of $N = 3000$ symbols using the 8PSK (m=2) based TTCM scheme associated with $\nu = 3$ and $T = 4$ amount to $10TNM(2S-1) = 10 \cdot 4 \cdot 3000 \cdot 2^2(2 \cdot 2^3 - 1) = 7.2 \times 10^6$ number of additions/subtractions. The corresponding complexity per bit per iteration of this TTCM scheme is: $\frac{10M(2S-1)}{m} = \frac{10 \cdot 2^2(2 \cdot 2^3-1)}{2} = 300$ additions/subtractions.

LDPC Complexity: Let us consider the decoding complexity of LDPC-BCM in conjunction with a parity check matrix having column and row weights of j and k, respectively. Firstly, the likelihood ratio of all the parity check sets being satisfied, which is quantified by the producton the right hand side of the $a\,posteriori$ information formula of Equation 1, is calculated in [1] as follows:

$$\prod_{i=1}^{j} L_r\left[S|\{x_i\}\right] = \frac{P(S|x_d = 1, y)}{P(S|x_d = 0, y)} = \prod_{i=1}^{j}\left[\frac{1 - \prod_{l=1}^{k-1}(1-2P_{il})}{1 + \prod_{l=1}^{k-1}(1-2P_{il})}\right] \quad (3)$$

P_{ij} is the probability the lth digit in the ith parity check set being a 1. k here in LDPC refers to the row weight, calculated as $j/(1-R)$ [1], which is 6, 9 and 12 for the scenarios of 4QAM, 8PSK and 16QAM respectively. An intermediate variable T_i where $i \in \{1, \ldots, j\}$ may be used for representing $\prod_{l=1}^{k-1}(1-2P_{il})$. For each message node, calculating T_i will incur $(2k-3)$ number of multiplications and $(k-1)$ additions. Note that there are j number of different T_i values to be determined and hence the total number of multiplications and additions are $(2k-3) \times j$ and $(k-1) \times j$, respectively. When all T_i are determined, the evaluation of the likelihood ratio at the left hand side of Eq 3, will require another $2 \times j$ additions and $j + j - 1$ multiplications. Having carried out the above operation, the likelihood ratio will have to be multiplied with the $intrinsic\ information$ of the considering message node, and a decision will have to be made based on the resultant product, which requires a comparison. Then all the non-zero entries belonging to the message node considered have to be updated with the likelihood ratio of the other $j-1$ number of parity check sets leading to $j \times (j-1)$ number of multiplications. Hence a total of $(k+1) \times j$ number of additions and $(2k - 2 + j) \times j$ multiplications are required. However, these operation are normally implemented in the logarithmic domain, which reduces the complexity by converting multiplications to additions, as argued in the context of the previous TTCM decoding complexity calculations. Thus in the logarithmic domain, the total complexity associated with each message node is $(4k+j) \times j$ number of additions per iteration. Total decoding complexity associated with a codeword hence becomes $(4k+j) \times j \times N$ number of additions, which is consistent with the estimates of [10], since we stated in Section 2 that $j \times N$ number of non-zero entries can be found in the parity check matrix. Thus for the rate 1/2, 2/3 and 3/4 code used in conjunction with the QPSK, 8PSK and 16QAM modulation modes, the associated complexity per bit per iteration becomes 81, 117 and 153 additions,respectively. Thus the maximum number of iterations used by the LDPC-BCM schemes was set to 15, 10 and 10, respectively for the QPSK, 8PSK and 16QAM modes, for the sake of approximately matching the TTCM decoding complexity.

5. CONCLUSION

In this contribution, the BER performance of LDPC-BCM and TTCM has been studied comparatively. The FER performance of the two schemes was also studied in the context of 16QAM scenario. When communicating over uncorrelated Rayleigh fading channels, LDPC-BCM performs several dBs better than TTCM beyond a given channel SNR value. When communicating over correlated Rayleigh fading channels, the LDPC-BCM scheme employing no channel interleaver is capable of achieving a similar or better performance to that of the TTCM scheme requiring a channel interleaver. This was observed both in terms of the achievable BER and FER respectively.

In conclusion, depending on the channel conditions encountered, the proposed LDPC-BCM scheme exhibits a better or similar performance as our TTCM benchmarker scheme at a similar decoding complexity. However, the iterations of LDPC-BCM scheme may be terminated before reaching the maximum number of iterations, when a valid codeword is detected. In particular, at high SNRs, owing to the reduced number of LDPC decoding iteration required, the average complexity of the LDPC-BCM scheme was reduced. When communicating over the uncorrelated Rayleigh fading channel, the LDPC-BCM scheme performs better, than the TTCM benchmarker scheme. By contrast, when transmitting over a correlated Rayleigh fading channel, the two schemes perform comparably to each other in terms of the achievable BER, although the LDPC-BCM scheme requires no interleaving. Finally, LDPC-BCM scheme exhibits a better FER than the TTCM benchmarker.

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