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A behavioral approach to the control of discrete linear repetitive processes

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Abstract

This paper formulates the theory of linear discrete time repetitive processes in the setting of behavioral systems theory. A behavioral, *latent variable* model for repetitive processes is developed and for the physically defined inputs and outputs as manifest variables, a *kernel representation* of their behavior is determined. Conditions for external stability and controllability of the behavior are then obtained. A sufficient condition for stabilizability is also developed for the behavior and it is shown under a mild restriction that, whenever the repetitive system is stabilizable, a regular constant output feedback stabilizing controller exists. Next, a notion of eigenvalues is defined for the repetitive process under an action of a closed-loop controller. It is then shown how under controllability of the original repetitive process, an arbitrary assignment of eigenvalues for the closed-loop response can be achieved by a constant gain output feedback controller under the above restriction. These results on the existence of constant gain output feedback controllers are among the most striking properties enjoyed by repetitive systems, discovered in this paper. Results of this paper utilize the behavioral model of the repetitive process which is an analogue of the 1D equivalent model of the dynamics studied in earlier work on these processes.

Keywords: Repetitive dynamics; Behavioural theory; Controllability; Stabilization

1. Introduction

Repetitive processes are a distinct class of dynamical systems of interest to both theory and applications which involve two distinct time histories. Hence such systems exhibit an inherent 2D character. The essential characteristic of such a process is a series of sweeps termed passes, through a set of dynamical laws defined over a finite duration known as the pass length. On each pass an output, termed the pass profile, is produced which acts as a forcing function on, and hence contributes to, the dynamics of the new pass profile. This, in turn, leads to the unique control problem for these processes in that the output sequence of pass profiles generated can contain oscillations that increase in amplitude in the pass to pass direction.

To introduce a formal definition, let $m < +\infty$ be an integer denoting the pass length (assumed constant). Then in a repetitive process the pass profile $y_k(p)$, $0 \leqslant p < m$ generated on pass k acts as a forcing function on, and hence contributes

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to, the dynamics of the new pass profile $y_{k+1}(p)$, $0 \leqslant p < m$, $k \geqslant 0$. The fact that the pass length is finite (and hence information in this direction only occurs over a finite duration) is the key difference with other classes of 2D (discrete) linear systems and, in particular, those described by the well known and extensively studied Roesser [9] and Fornasini Marchesini [3] state-space models.

Physical examples of repetitive processes include long-wall coal cutting and metal rolling operations (see, for example, [2]). Also in recent years applications have arisen where adopting a repetitive process setting for analysis has distinct advantages over alternatives. Examples of these so-called algorithmic applications include classes of iterative learning control (ILC) schemes [1] and iterative algorithms for solving non-linear dynamic optimal control problems based on the maximum principle [8]. In the case of ILC for the linear dynamics case, the stability theory for differential and discrete linear repetitive processes is the essential basis for a rigorous stability and convergence analysis of a powerful class of such algorithms.

Attempts to control these processes using standard (or 1D) systems theory and algorithms fail in general precisely because such an approach ignores their inherent 2D systems structure, i.e. information propagation occurs from pass to pass and along a given pass, and also the resetting of the pass initial conditions before the start of each new pass. A rigorous stability theory for linear repetitive processes has been developed. This theory [11] is based on an abstract model in a Banach space setting which includes all such processes as special cases. Also the results of applying this theory to a wide range of such cases have been reported, including the one considered here, which has resulted in stability tests that can be implemented by direct application of well known 1D linear system tests.

The purpose of this paper is to reconsider some of the key problems of repetitive process theory within the framework of behavioral systems theory [7,13]. In this framework an analysis of the dynamic system is developed using the most natural model of the system rather than in terms of models in which inputs, outputs and states are pre-specified. This offers several advantages and allows transformations of all the variables of the system rather than being restricted to these three separate classes inputs, outputs or states.

The behavioral approach is inherently suitable for modelling repetitive processes. We show how the well-known model of a discrete time repetitive process leads to a hybrid (or a latent variable) representation of its behavior. In this discrete time case, considering the time history of variables at all the intermediate instances of a pass, the behavior turns out to have a finite number of manifest variables and has also a 1D character. Hence its kernel representation can be determined by standard methods of behavioral theory. We show in this paper that this kernel representation can be computed by simple linear algebraic procedures without recourse to polynomial methods (see, for example, [5]). This further enables an analysis of asymptotic stability and controllability properties of these systems to be undertaken in much simpler ways than reported previously (see, for example, [10]).

We then derive the conditions for stability and controllability of the repetitive processes considered and also develop a procedure for obtaining the stabilizing controller, which is a counterpart of the well-known constant output feedback controller in standard systems. We prove a remarkable property of repetitive process that under a mild restriction such as non-singularity of a coefficient matrix (see Theorem 5) a stabilizable repetitive process can be actually stabilized by a constant output feedback controller and similarly the closed-loop eigenvalues can be placed by a constant output feedback controller if the process is controllable.

1.1. Basic model and summary of results

A discrete time linear repetitive processes having a K-pass memory, i.e. when the previous K pass profiles explicitly contribute to the current one, and having a constant pass length of m is given by the dynamical equations,

$$\begin{aligned} x_{k+1}(p+1) &= Ax_{k+1}(p) + Bu_{k+1}(p) + \sum_{j=0}^{j=K-1} B_{0j}\, y_{k-j}(p), \\ y_{k+1}(p) &= Cx_{k+1}(p) + Du_{k+1}(p) + \sum_{j=0}^{j=K-1} D_{0j}\, y_{k-j}(p), \end{aligned} \qquad (1)$$

together with boundary conditions which incorporate the history of variables of the previous pass given by

$$x_{k+1}(0) = d_{k+1} + \sum_{j=0}^{m-1} K_j y_k(j), \qquad (2)$$

where d_k is the disturbance, k denotes the pass number or index, and p, $0 \leqslant p \leqslant (m-1)$ denotes an instant during a pass. The *unit pass* repetitive process has $K = 1$ hence for this case we denote $B_{00} = B_0$ and $D_{00} = D_0$. In this paper we shall consider the unit memory case only since the results for the K-pass memory case follow as natural generalizations.

1.1.1. Stability analysis

A fundamental problem in the analysis of repetitive processes is to determine the conditions for stability. In this paper we approach this problem by developing a hybrid model representing the behavior of the above system (with $d_k = 0$) in which $y_k(p)$ and $u_k(p)$ are manifest variables while $x_k(p)$ are latent variables. In order to eliminate the independent variable p in every pass we consider the behavior of all these variables for $0 \leqslant p \leqslant (m-1)$. Hence we consider variables $Y(k)$, $U(k)$ denoting the collection of all outputs $y_k(p)$ and inputs $u_k(p)$ respectively. Due to finiteness of m the resulting behavior has finite number of manifest variables $W = \text{col}(Y, U)$ and whose dynamical behavior evolves with k as the independent variable.

Once such a representation is available, the stability of the repetitive process can be defined as the stability of the autonomous behavior of all variables $Y(k)$ with respect to k when the free variables $U(k)$ are zero. This behavioral way of defining the stability results in a much simpler approach to deriving the conditions under which this property holds. In the earlier approach [10] these conditions were derived by developing a discrete system for evolution of $y_k(p)$ along the pass k involving an initial condition and then by showing norm bounds on these variables. The behavioral approach allows a direct elimination of variables to get the representation of the behavior of $Y(k)$ variables as a discrete time system. These results compare with the 1D equivalent model approach given in [10].

1.1.2. Controllability analysis

Next, we treat the question of controllability of the repetitive process in the behavioral framework. For this purpose we consider the kernel representation of the behavior of $W(k)$ which can be obtained by the use of the elimination theorem on the hybrid model referred to above. In the behavioral approach the variables $U(k)$ are just treated as a subset of manifest variables $W(k)$ and the controllability of the behavior is the existence of a patching trajectory $W(k)$ for two arbitrary trajectories in the behavior. We refer the reader to [7,13] for details and the criteria for controllability in terms of a kernel representation. We show that the criterion for controllability can be obtained much easily in terms of a transformation of the manifest variables to obtain an isomorphic behavior.

The behavioral approach to controllability subsumes the classical approach in which $u_k(p)$ are inputs and controllability amounts to driving the states (or outputs) to a desired pass profile [10]. We show that the criterion for controllability can be obtained using matrix computational methods as in the classical approach.

1.1.3. Closed-loop stabilization and eigenvalue assignment

Having undertaken the controllability analysis of the behavior, we now consider the problems of closed-loop stabilization and eigenvalue assignment. In the case of repetitive processes the meaning of eigenvalues (or poles) as well as the concept of assignment of eigenvalues has to be redefined from first principles. This difficulty again stems from the inherent 2D nature of the process. However, the 1D equivalent model of the behavior of $W(k)$ referred above is a discrete time linear time invariant behavior in which the variables $Y(k)$ and $U(k)$ serve physically as collections of outputs and inputs, respectively, but are not differentiated mathematically. Hence obtaining a constant gain controller of the form $Y(k) = KU(k)$ is analogous to the classical constant gain output feedback and hence the exponents in the exponential solutions in the finite dimensional (or autonomous) behavior of variables $Y(k)$ after such a controller is incorporated, are an analogue of the eigenvalues of the closed-loop repetitive process. We show under a mild restriction that, if the repetitive process is stabilizable, then a controller of this form exists such that the resulting closed-loop process is stable. Moreover such a constant output feedback gain also exists for arbitrary placement of closed-loop eigenvalues (permitted within this restriction).

2. The behavioral models

In this section we develop the hybrid and kernel forms of models of the behavior of the repetitive processes considered in this work. A representation of the behavior of the unit memory repetitive process (1), (2) can be obtained by specifying the manifest variables. The resultant set of equations then gives a hybrid representation of the behavior. To define the manifest and latent variables we shall first consider the collections of variables at all instances of a pass to obtain a 1D behavior. Consider the following notations:

$$Y(k) = \mathrm{col}(y_k(0), y_k(1) \dots y_k(m-1)),$$

$$U(k) = \mathrm{col}(u_k(0), u_k(1) \dots u_k(m-1)),$$

$$X(k) = \mathrm{col}(x_k(0), x_k(1) \dots x_k(m-1)).$$

Thus $Y(k)$, $U(k)$, $X(k)$ are vector valued variables of m vector components of the variables $y_k(p)$, $u_k(p)$ and $x_k(p)$, respectively, along the instances of the kth pass. Let $W(k)$ denote the following vector:

$$W(k) = \begin{bmatrix} Y(k) \\ U(k) \end{bmatrix}$$

which we shall consider as the vector valued manifest variable. Then Eqs. (1) and (2) can be rewritten as (taking $d_k = 0$)

$$\left(\begin{bmatrix} 0 & \hat{B} \\ I & -\hat{D} \end{bmatrix} \sigma + \begin{bmatrix} \hat{K} & 0 \\ -\hat{D_0} & 0 \end{bmatrix} \right) W(k) = \sigma \begin{bmatrix} \hat{Q} \\ \hat{C} \end{bmatrix} X(k), \tag{3}$$

where σ denotes the difference operator $\sigma f(k) = f(k+1)$ on sequences $f(k)$.

The matrices in the above equation are given by

$$\hat{D} = \mathrm{diag}\{D, D, \dots D\},$$

$$\hat{D_0} = \mathrm{diag}\{D_0, D_0, \dots D_0\},$$

$$\hat{C} = \mathrm{diag}\{C, C, \dots C\},$$

$$\hat{B} = \begin{bmatrix} 0 & & & & \\ B & 0 & & & \\ & B & 0 & & \\ & & \ddots & \ddots & \\ & & & B & 0 \end{bmatrix},$$

$$\hat{K} = \begin{bmatrix} K_0 & K_1 & \dots & \dots & K_{(m-1)} \\ B_0 & & & & \\ & B_0 & & & \\ & & \ddots & & \vdots \\ & & & B_0 & 0 \end{bmatrix},$$

$$\hat{Q} = \begin{bmatrix} I & & & & \\ -A & I & & & \\ & -A & & & \\ & & \ddots & \ddots & \\ & & & -A & I \end{bmatrix}.$$

This completes the description of the latent variable (or hybrid) representation (3) of the behavior of the repetitive process.

This is a linear time invariant discrete time behavior with finite number of manifest variables $W(k)$. To denote such a behavior in standard notation, observe that the manifest variables W now take values in $\mathbb{R}^q$, where $q = m(n_y + n_u)$, where n_y, n_u are respectively, number of y and u variables, respectively and m the pass length. Hence this behavior is denoted by $\mathscr{L}(\mathbb{Z}_+)$ or by the triple $(\mathbb{Z}_+, \mathbb{R}^q, \mathscr{B})$. We shall simply denote such behaviors by $\mathscr{B}$.

2.1. Fundamental theorems

We now discuss two of the important mathematical facts concerned with the above representation, the elimination theorem and the module behavior correspondence [6] which form fundamental pillars of the behavioral theory. As observed in the above model the matrices of representation (3) are defined over the polynomial ring $\mathbb{R}[\sigma]$. Let the collection of all sequences $f(k), k = 0, 1, \dots, f(k) \in \mathbb{R}$ be denoted as $\mathscr{V}$.

Then $\mathscr{V}$ is a module over the commutative ring $\mathbb{R}[\sigma]$ under the operation $\sigma f(k) = f(k+1)$. The solution trajectories of variables $W(k)$ and $X(k)$ are q-tuples whose entries are also such sequences. Hence the behavior of W is also defined as a module over this ring. This allows all the techniques of behavioral theory to be applicable for our system since the manifest and latent variables are defined over a finite Cartesian product of the module $\mathscr{V}$. The elimination theorem is stated next.

Theorem 1. *Let $\mathscr{B}$ be a behavior whose trajectories belong to $\mathscr{V}^q$ and is given by a latent variable representation*

$$R(\sigma)w(n) = M(\sigma)l(n).$$

If $Q(\sigma)$ is a matrix whose rows generate the $\mathbb{R}[\sigma]$-module of relations of rows of the matrix $M(\sigma)$ then a kernel representation of $\mathscr{B}$ is given by

$$Q(\sigma)R(\sigma)w(n) = 0.$$

The elimination theorem shows that a behavior over $\mathscr{V}^q$ represented by a latent variable representation has a kernel representation. We omit the proof of this theorem as it can be developed on the lines similar to the well-known discrete time case of behavioral systems over the ring $\mathbb{R}[\sigma, \sigma^{-1}]$ and trajectories defined over two sided infinite sequences [13]. Next we state the theorem on module behavior correspondence for behaviors in terms of matrices of kernel representations. (The notation $\langle R(\sigma)\rangle_r$ denotes the $\mathbb{R}[\sigma]$-module generated by the rows of the matrix $R(\sigma)$.)

Theorem 2. *Let $R_i(\sigma)w_i(n)=0$ be kernel representations of behaviors $\mathscr{B}_i$, respectively, where R_i are full row rank polynomial matrices. Then $\mathscr{B}_1 = \mathscr{B}_2$ if, and only if, there exists a unimodular matrix $U(\sigma)$ such that $R_1(\sigma) = U(\sigma)R_2(\sigma)$ while $\mathscr{B}_1 \subset \mathscr{B}_2$ if, and only if, $\langle R_2(\sigma)\rangle_r \subset \langle R_1(\sigma)\rangle_r$.*

We shall again omit the proof which can be developed from the discrete time results of [6]. Using these we can compute the kernel representation of the behavior of $W(k)$ with the help of following lemma. We denote by $\operatorname{rel} P(\sigma)$ the module of relations of the rows of a matrix $P(\sigma)$ over the polynomial ring $\mathbb{R}[\sigma]$. When P is a constant matrix we denote the vector spaces generated by rows of P (respectively, the vector space of relations of rows of P) also as $\langle P\rangle_r$ (respectively, $\operatorname{rel} P$).

Lemma 1.

$$\operatorname{rel}\begin{bmatrix}\hat{Q}\\ \hat{C}\end{bmatrix} = \langle [Q_0 \;\; I]\rangle_r,$$

where

$$Q_0 = \begin{bmatrix} -C & & & \\ -CA & -C & & \\ \vdots & \vdots & & \\ -CA^{m-1} & -CA^{m-2} & \dots & -C \end{bmatrix}.$$

Proof. Follows by computing the relations for $m=2$ and then by induction. The details are mainly computational and are omitted. □

Proposition 1. *The behavior of $W(k)$ has the kernel representation*

$$([\sigma I + (Q_0\hat{K} - \hat{D_0}) \;\; \sigma(Q_0\hat{B} - \hat{D})])W(k) = 0 \tag{4}$$

and the variables $U(k)$ are maximally free.

Proof. The module of relations of rows of the polynomial matrix $\sigma[\hat{Q}^{\mathrm{T}} \; \hat{C}^{\mathrm{T}}]^{\mathrm{T}}$ is given by that of the relations of the constant matrix $[\hat{Q}^{\mathrm{T}} \; \hat{C}^{\mathrm{T}}]^{\mathrm{T}}$. This linear algebraic computation is given by the above lemma. The result now follows on using the elimination theorem on the latent variable model (3). Note that $W(k)^{\mathrm{T}} = (Y(k)^{\mathrm{T}}, U(k)^{\mathrm{T}})$. Since $\det(\sigma I + (Q_0\hat{K} - \hat{D_0})) \neq 0$ it follows that $U(k)$ are maximally free variables. □

We have thus obtained the hybrid and kernel representations of the behavior of the manifest variables W. In the next section we shall employ these to obtain the criteria for stability and controllability.

3. Stability and controllability analysis

The derivation of the kernel representation of the behavior of the repetitive process in terms of the manifest variable W was carried out in the last section.

This computation turned out to be greatly simplified due to the fact that the module of relations of the polynomial matrix on the right-hand side of (3) turned out to have the same generators as that of the vector space of relations of rows of the constant matrix col $[\hat{Q}\ \hat{C}]$. Hence the cumbersome computation of relations of a polynomial matrix usually required in elimination of latent variables is completely avoided. We now develop the stability and controllability analysis of the repetitive process (1), (2).

3.1. Stability criterion

From a behavioral point of view, stability (asymptotic stability) of a behavior is characterized by uniform boundedness (asymptotic decay) of all of its trajectories. Hence such a notion is applicable only to behaviors which are autonomous i.e. those which do not have free (or input) variables since such variables can always be chosen unbounded. Hence it is necessary to define the notions of stability and asymptotic stability of the repetitive process (1), (2) in terms of variables whose laws of motion determined by these equations are autonomous. A physically meaningful notion of stability for this system is given as follows.

Definition 1. The repetitive process (1), (2) is *externally stable* (*asymptotically stable*) if for $u_k(p) = 0$ for all $0 \leqslant p \leqslant (m-1)$ and $k = 1, 2, \ldots$ the solutions $y_k(p)$ are uniformly bounded (tend to zero) for $k \to \infty$ under arbitrary initial conditions $y_0(p)$ of these variables for all $0 \leqslant p \leqslant (m-1)$.

Thus it is necessary to verify from model (4) that the behavior of the variables $Y(k)$ is autonomous when $U(k) = 0$ and then determine the stability of the behavior under the conditions of zero $U(k)$. The external stability of the repetitive process (1), (2) under zero external input is given by the following result.

Theorem 3. *The repetitive process* (1), (2) *is externally asymptotically stable if, and only if, the matrix* $(\hat{D}_0 - \hat{Q}_0\hat{K})$ *has all its eigenvalues inside the open unit disc of the complex plane.*

Proof. The kernel representation (4) shows that when $U(k) = 0$ the resultant behavior of $Y(k)$ is represented by

$$(\sigma I + Q_0\hat{K} - \hat{D_0})Y(k) = 0. \tag{5}$$

Hence the presence of σI term shows that the matrix of kernel representation of $Y(k)$ under the condition $U(k) = 0$ is non-singular making this behavior autonomous. Hence from the kernel representation (4) of the behavior it follows that the laws of the behavior of $Y(k)$ under zero inputs $U(k) = 0$ are given by (5). The solutions of this system under arbitrary initial conditions $Y(0)$ are given by

$$Y(k) = (-1)^k(Q_0\hat{K} - \hat{D_0})^k Y(0).$$

The theorem now follows readily from the above expression. □

The above stability criterion was also obtained in [10], however, the above proof is much simpler due to the direct approach facilitated by notions of behavioral theory.

3.2. Controllability analysis

We now take up investigation of controllability of repetitive processes from a behavioral point of view. For completeness of exposition we shall recall the notion of controllability of a behavior which appears quite distinct from the classical notion of controllability in state space models. Let $\mathscr{B}$ denote a linear time invariant behavior in the space $\mathscr{V}^q$ with manifest variables W. When such a behavior is complete [12] there exists a kernel representation. The behavioral notion of controllability for such a system is given by the following definition.

Definition 2. The behavior $\mathscr{B}$ is said to be *controllable* if for any two trajectories $W_1(n)$, $W_2(n)$ in $\mathscr{B}$ there exists a time n_1 and a trajectory $W(n)$ in $\mathscr{B}$ such that

$$W(k) = \begin{cases} W_1(k) & \text{for } k = 0, 1, \ldots, n, \\ W_2(k - n_1) & \text{for } k \geqslant n \geqslant n_1. \end{cases}$$

From the above definition it follows that the behavioral notion of controllability does not depend on any notion of state. It is due to this reason that the behavioral concept of controllability is useful even when there is no state space representation available.

In terms of a kernel representation a criterion of controllability of a behavior is given by the following.

Proposition 2. *Consider a behavior* $\mathscr{B}$ *given by a kernel representation* $R(\sigma)W(k)=0$ *in which the matrix* $R(\sigma)$ *has r rows all of which are linearly independent over the ring* $\mathbb{R}[\sigma]$. *Then* $\mathscr{B}$ *is controllable if, and only if, there is no complex number* λ *such that* $\operatorname{rank} R(\lambda) < r$.

We shall omit proof of this result as it follows from well-known results [12]. It is worthwhile to note, however, that from a computational viewpoint the above criterion can pose a numerical hurdle since the polynomial matrix computations are exhaustive from point of view of numerical stability and floating point error accumulation, especially for large sizes and degrees of matrices $R(\sigma)$. Hence constant matrix (or linear algebraic) computational procedures are preferred in practice.

We can now immediately apply the above criterion to the kernel representation (4) of our repetitive system. In our problem we can in fact derive a criterion for controllability for the repetitive process which involves purely linear algebraic computations instead of polynomial computations due to the first order nature of our kernel representation (4).

Introduce now the notation $M_0 = \hat{D_0} - Q_0\hat{K}$ and $N_0 = Q_0\hat{B} - \hat{D}$. Then we use the classical state space terminology of calling a pair of matrices (A, B) controllable if they satisfy the following.

Definition 3. A pair of real matrices $(A\ B)$, A $n \times n$ and B with n rows is said to be controllable if rank $[\lambda I - A\ B] = n$.

Theorem 4. *The behavior of* $W(k)$ *represented by* (4) *is controllable if, and only if, the pair* (M_0, M_0N_0) *is controllable.*

Proof. Note that

$$[\lambda I - M_0\ \ M_0N_0]\begin{bmatrix} I & N_0 \\ 0 & 1 \end{bmatrix} = [\lambda I - M_0\ \ \lambda N_0].$$

Hence the rank of (M_0, M_0N_0) is equal to the rank of the kernel representation (4) for all λ. □

An advantage of the above result is now clear. The criterion of controllability of (4) is stated in terms of the controllability of a pair of matrices in its representation instead of in terms of the roots of the polynomial matrix in the kernel representation. Hence the above criterion can be computed using matrix computations. The above criterion is equivalent to that of Theorem 1 of [10].

4. Stabilization and eigenvalue assignment

The problem of stabilizing the repetitive process (1), (2) is now considered. We attempt the solution in the output feedback form as this is most desirable from a practical point of view.

Consider the behavior of the manifest variables $W(k)$ which consists of the variables $Y(k)$ and $U(k)$ which are traditionally output and input variables of the repetitive process. This behavior is given by the kernel representation (4). A stabilizing controller for this behavior is defined as follows.

Definition 4. Let $\mathscr{B}$ be the behavior of W represented by (4). A behavior $\mathscr{B}_c$ represented by $R_c(\sigma)W_c(k)=0$ is said to be a stabilizing controller of $\mathscr{B}$ if (1) the number of variables W_c are same as that of W and (2) the behavior $\mathscr{B} \cap \mathscr{B}_c$ is asymptotically stable. If such a stabilizing controller $\mathscr{B}_c$ exists we call $\mathscr{B}$ stabilizable.

These conditions imply that W_c can be partitioned in the variables Y_c, U_c of same sizes as that of Y, U respectively and that the behavior represented by

$$\begin{bmatrix} R(\sigma) \\ R_c(\sigma) \end{bmatrix} W(k) = 0$$

is autonomous and asymptotically stable. We moreover have the following.

Definition 5. Let W be partitioned as col$(Y\ U)$ in which U is a maximal family of free variables (or inputs) and let W_c be manifest variables of the stabilizing controller above with the partition col$(Y_c\ U_c)$. Then $\mathscr{B}_c$ is said to be a *regular stabilizing controller* relative to the choice of free variables U (inputs) if Y_c are free variables in the behavior $\mathscr{B}_c$.

In the notation of Theorem 4 of the last section we can first establish the stabilizability of $\mathcal{B}$ as follows. Recall that in the classical notion of stabilizability, a pair of matrices $(A\ B)$ (A square and B with same number of rows as in A) is said to be stabilizable relative to the unit disc if, there exists a matrix F such that $A + BF$ has all its eigenvalues inside the unit circle.

Proposition 3. *$\mathcal{B}$ is stabilizable if the pair $(\tilde{M}_0, \tilde{M}_0 N_0)$ is stabilizable, where $\tilde{M}_0 = Q_0\hat{K} - \hat{D}_0$,*

Proof. The control law here is

$$(I - FN_0)U(k) = FY(k)$$

and the closed-loop system is

$$\begin{bmatrix} \sigma I + \tilde{M}_0 & \sigma N_0 \\ -F & I - FN_0 \end{bmatrix} W(k) = 0.$$

Also

$$\begin{bmatrix} \sigma I + \tilde{M}_0 & \sigma N_0 \\ -F & I - FN_0 \end{bmatrix} \begin{bmatrix} I & -N_0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} \sigma I + \tilde{M}_0 & -\tilde{M}_0 N_0 \\ -F & I \end{bmatrix}$$

and, since the determinant of the matrix on the right-hand side here equals

$$\det(\sigma I + \tilde{M}_0 - \tilde{M}_0 N_0 F)$$

the closed-loop is autonomous and asymptotically stable. □

Although the above proposition gives a sufficient condition for stabilizability of the behavior of W, it is as yet not clear whether there is a regular stabilizing controller. In the next section however, regularity of the controller is established by construction of a class of controllers.

4.1. Output feedback stabilization

Output feedback stabilization is one of the most sought after method of stabilization in control engineering since this does not involve estimation of the state. This method of stabilization is also among the more difficult methods particularly in multiple input output systems. It may in fact be noted that so far no necessary and sufficient conditions are known, to determine whether or not an LTI multiple input output system is stabilizable by a constant output feedback. In the case of single input single output systems, the root locus method can determine whether a constant output feedback can stabilize the system. It is well known from the root locus theory that not all LTI systems can be stabilized by constant output feedback.

We shall first consider the problem of output feedback stabilization of the repetitive process (1), (2) using the behavioral approach discussed above. Proposition 3 gives a sufficient condition for stabilizability of the behavior of the manifest variables W of the repetitive process. However, this does not guarantee that there exists a regular stabilizing controller. This requires additional conditions on the system as shown in the following result.

Theorem 5. *Let the matrix $\tilde{M}_0$ in the kernel representation (4) of the behavior of W be non-singular and the pair $(\tilde{M}_0\ \tilde{M}_0 N_0)$ stabilizable. Then there exists a constant gain feedback controller K such that the controller $U(k)=KY(k)$ is a stabilizing controller for the behavior of $W(k)$.*

Proof. Given the structure of the control law in the proof of Proposition 3, all we need to show is that $I - FN_0$ is invertible. This is true if, and only if, $I - N_0F$ is invertible, which follows immediately as the assumptions ensure that $\tilde{M}_0(I - N_0F)$ is non-singular. □

The above result establishes a constant gain output feedback stabilizing control law for the repetitive process. Such a controller is likely to be of good practical utility. However the procedure of computation of the matrix F above needed to determine such a controller is outside the scope of the present paper and shall be discussed elsewhere. Clearly, it is now important to determine the usefulness of such a controller in achieving a desired closed-loop response. This is the subject of the next section.

4.2. Eigenvalue assignment

We showed above that under the stabilizability condition of Proposition 3 and the non-singularity of M_0 there exists an output feedback stabilizing control law. A natural question then is to achieve a closed-loop eigenvalue assignment. However in the context of repetitive processes such an objective has to be clearly defined since the original model (1), (2) cannot be used to define the eigenvalues of the closed-loop system under any control law. However under any closed-loop control law there is a well defined behavior of the manifest variables W which becomes autonomous whenever the controller is a stabilizing controller. We can thus define the characteristic exponents in the exponential solutions of this autonomous behavior as the representative closed-loop eigenvalues. Using such a concept we now show that under the condition of controllability of the behavior (4) it is possible to assign arbitrarily specified closed-loop eigenvalues of the dynamics of the Y variables except for placing any one of them at the origin. This is the topic of the following result.

Corollary 1. *Let the behavior of W represented in* (4) *be controllable. Then for every conjugate symmetric set of distinct complex numbers Λ inside the unit disc but not containing the origin*, *there exists a constant gain controller of the form $U(k) = KY(k)$ such that the behavior of Y under this control law is autonomous and has a basis of exponential functions characteristic of complex numbers of Λ.*

Proof. From the controllability Theorem 4 it follows that if the behavior of W variables is controllable then the pair $(\tilde{M}_0\ \tilde{M}_0 N_0)$ is controllable. Hence there exists a matrix F such that $(\tilde{M}_0(I - N_0F)$ has arbitrarily specified eigenvalues in the unit circle. These eigenvalues thus form a set of complex numbers which is conjugate symmetric due to the matrices involved being real. Let this set be distinct and not include the origin. Then as shown in Theorem 3 the behavior of Z^2 variables is autonomous and has fundamental exponential solutions whose characteristics are the elements of Λ. Now since zero is not in Λ the matrix $(I - N_0F)$ is non-singular. Hence every trajectory in the behavior of Y is in one to one correspondence with a trajectory of the behavior of Z^2 as can be easily observed from the pencil representation of the behavior of Y in the proof of Theorem 5 above. Hence a basis of exponential solutions of Z^2 under this isomorphism is also a basis of behavior of Y. Hence these solutions in the behavior of Y have the same characteristic exponents. Hence the control law gives the required closed-loop eigenvalue assignment. This proves the corollary. □

Remark 1. Theorem 5 shows that in repetitive systems the eigenvalue assignment can be done by constant output feedback control which is in fact a much stronger property than stabilizability by constant output feedback control shown in Theorem 5.

5. Conclusion

It is shown in this paper that the class of time invariant linear repetitive processes can be fruitfully investigated using the ideas of behavioral systems theory. Criteria for stability and controllability are derived in a direct manner using behavioral concepts. These conditions moreover turn out to be expressible in terms of algebraic conditions reminiscent of the classical state space theory. An important structural property of repetitive process which we have discovered is that whenever the original model is stabilizable, the process is stabilizable by a constant output feedback. Moreover, whenever the original model is controllable, an arbitrary assignment of eigenvalues in closed-loop can be achieved by constant output feedback. These are remarkable properties of repetitive systems not enjoyed by even the LTI systems. Further analysis of this property and a solution by a causal feedback controller shall be a subject of a future article.

Our major conclusion is that the control of repetitive processes in the behavioral setting is a potentially very powerful approach. Specifications and control theory of these systems are still very much in infancy and are to a large extent open. In this paper we could define and solve an eigenvalue assignment problem for these systems. Although a substantial progress has been made in the iterative learning control as well as in design schemes for repetitive processes in the LMI setting, see, for example, [4], a more general theory is certainly very much desirable. This paper is a first such attempt using the concepts of behavioral theory.

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