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Cyclotron acceleration of radiation belt electrons by whistlers

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[1] After reviewing briefly the theory of the gyroresonant interactions between a quasimonochromatic whistler-mode wave and energetic electrons trapped in the magnetosphere, we extend this theory to consider such interactions for a natural whistler arising from a lightning discharge in the Earth's atmosphere. It is shown that, near the equatorial plane of the magnetosphere, whistler components above the nose frequency can accelerate energetic electrons. This acceleration takes place when the gyroresonant electrons are trapped by the wave field. The acceleration rate in this regime is much greater than is stochastic acceleration in the untrapped regime. It is not accompanied by pitch angle scattering which characterizes the untrapped regime. For example, at $L = 3$, a gyroresonant electron with an energy of ~6 keV and a pitch angle of 45° could have its energy increased by ~24% to 7.4 keV and its pitch angle changed to 70° after a single interaction with a whistler whose frequency changes from 1/3 to 1/2 the equatorial gyrofrequency. Highly anisotropic distributions of van Allen radiation belt electrons with "pancake" pitch angle distributions can result from such an acceleration. *INDEX TERMS:* 7807 Space Plasma Physics: Charged particle motion and acceleration; 2730 Magnetospheric Physics: Magnetosphere—inner; 2720 Magnetospheric Physics: Energetic particles, trapped; 7867 Space Plasma Physics: Wave/particle interactions; *KEYWORDS:* nonlinear wave-particle interaction, energetic electron acceleration, whistlers, radiation belts

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1. Introduction

[2] *Friedel et al.* [2002] has recently reviewed ten different mechanisms by which electrons may be accelerated to large energies in the magnetosphere. Amongst these are electromagnetic ion cyclotron waves and whistler-mode waves. *Summers and Ma* [2000] have shown that electrons can be accelerated to >1 M, stochastically, by enhanced whistler-mode chorus at $3 < L < 6$, beyond the plasmapause, during the recovery phase of a magnetic storm. *Meredith et al.* [2001] have followed this up, and suggested that the process is most effective when there are periods of prolonged substorm activity following the main phase of a magnetic storm. They have suggested that, within the plasmasphere, the dominant signals are independent of substorms, being whistlers or signals from VLF radio transmitters on the ground.

[3] Here, we consider whistler signals, generated by lightning discharges, which could contribute significantly to the energetics of the van Allen radiation belts, especially to the precipitation [*Dungey*, 1963] and acceleration [*Helliwell and Bell*, 1960] of radiation belt electrons. The main reason for this is the very high efficiency of whistler-electron interactions associated with the Doppler-shifted cyclotron resonance. Whistler-triggered signals observed on the ground provide important evidence of the occurrence of such an interaction [*Helliwell*, 1965; *Smith et al.*, 1985; *Helliwell*, 1993].

[4] Theoretical investigations of triggered VLF emissions generated by quasimonochromatic signals from ground-based VLF transmitters [*Sudan and Ott*, 1971; *Nunn*, 1974, 1993; *Nunn et al.*, 1999; *Karpman et al.*, 1974; *Istomin et al.*, 1976; *Vomvoridis et al.*, 1982; *Molvig et al.*, 1988; *Carlson et al.*, 1990; *Omura et al.*, 1991; *Hobara et al.*, 2000; *Trakhtengerts et al.*, 2001] have shown that generation of triggered whistler-mode signals is due to the acceleration of electrons trapped in the potential well of a pump-wave. These electrons form an electron beam with a small spread of geomagnetic field-aligned velocities, and serve as a source of secondary whistler-mode radiation [*Sudan and Ott*, 1971; *Istomin et al.*, 1976; *Hobara et al.*, 2000; *Trakhtengerts et al.*, 2001].

[5] Similar effects should take place for the wave packet of a whistler, generated by a lightning discharge. However, the change of the whistler's frequency in space and time due to dispersion prevents the direct application of the theory developed for quasimonochromatic signals. Some very important modifications to the theory are necessary, and these are discussed here for the first time.

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[6] In section 2, we review the basic features of the theory of cyclotron interactions between energetic electrons and a quasimonochromatic whistler wave in an inhomogeneous (dipolar) geomagnetic field. The generalization of this for the case of a whistler whose frequency changes appreciably with time is given in section 3. We then consider the acceleration of electrons by a lightning-generated whistler wave packet with changing frequency in section 4. Discussion of the results obtained and some conclusions are given in section 5.

2. Basic Equations: Acceleration of Energetic Electrons by a Quasimonochromatic Whistler Packet

[7] As is well known, whistler waves can propagate in the Earth's magnetosphere in ducted and unducted modes. We consider the first case, when a whistler-wave packet is trapped within a geomagnetic-field aligned column of enhanced plasma density, which serves as a waveguide directing the wave energy along the geomagnetic field. If the relative density enhancement is small, and the duct width is much larger than the whistler wavelength in the plasma ($\lambda \sim 1$ km), this ducted mode can be treated as a plane wave with $\vec{k} \parallel \vec{B}$.

[8] The equations of an electron's motion in an inhomogeneous magnetic field $\vec{B}$, in the presence of a whistler wave whose wave vector $\vec{k}$ is antiparallel to $\vec{B}$, can be written in their simplest form if we use as variables the kinetic energy W and the first adiabatic invariant μ of the electron, and if we take the coordinate z along the magnetic field line instead of time t:

$$W = \frac{m}{2}\left(v_\parallel^2 + v_\perp^2\right), \qquad \mu = \frac{mv_\perp^2}{2B}, \qquad v_\parallel = \sqrt{\frac{2}{m}(W - \mu B)}, \tag{1}$$

where $v_\parallel$ and $v_\perp$ are the electron's velocity components along and across the geomagnetic field, $B = |\vec{B}|$, and m is the electron mass. The relation between z and t for a test electron in terms of their initial values z_0 and t_0 is

$$t - t_0 = \int_{z_0}^{z} \frac{dz'}{\sqrt{\frac{2}{m}(W - \mu B(z'))}} \tag{2}$$

where z' is the variable of integration along the field line.

[9] The equations of a test electron's motion in these variables are written [*Sudan and Ott*, 1971; *Nunn*, 1974, 1993; *Karpman et al.*, 1974] in the form:

$$\frac{dW}{dz} = -e\left(\frac{\mu B}{W - \mu B}\right)^{1/2} \mathrm{Re}(A\exp(i\psi)) \tag{3}$$

$$B\frac{d\mu}{dz} = -e\left(\frac{\mu B}{W - \mu B}\right)^{1/2}\left(1 - n\beta_\parallel\right)\mathrm{Re}(A\exp(i\psi)) \tag{4}$$

$$\frac{d\psi}{dz} = \left(\frac{2}{m}(W - \mu B)\right)^{-1/2}\left(\omega - \omega_B - kv_\parallel\right), \tag{5}$$

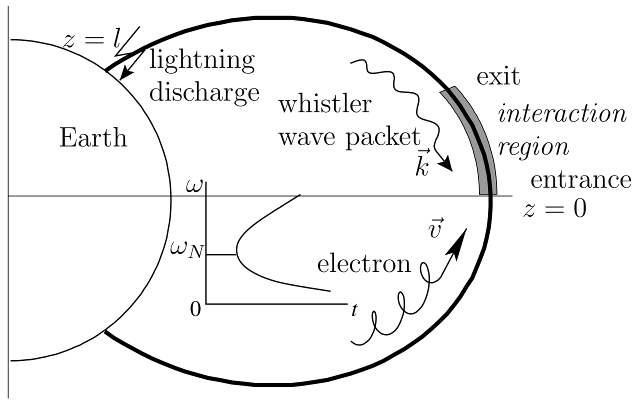

Figure 1. Diagram indicating the interaction occurring near the equatorial plane (L = 3) between a whistler and an energetic electron traveling in the opposite direction. The electron enters the interaction region at the entrance, at the equator where $z = 0$. It leaves the interaction region at the exit, about 5000 km (using the numbers given in section 5) away from the equator, with a higher energy, having interacted with the ducted whistler above its nose frequency ω_N.

where $\psi = \Theta - \varphi$ is the phase difference between the wave electric field and the electron's perpendicular velocity. Here, by definition, $\frac{\partial\Theta}{\partial t} = \omega$, $\frac{\partial\Theta}{\partial z} = -k$, and $\frac{\partial\varphi}{\partial t} = \omega_B$, A is the amplitude of the whistler-mode wave electric field, $n = |k|c/\omega$ is the whistler-mode refractive index, $\beta_\parallel = v_\parallel/c$, and c is the velocity of light in free space. In equation (5) we have omitted the term which is proportional to the wave amplitude A and which gives a small change to the solution of equation (3)–(5). We consider an electron moving in the $+z$ direction, which interacts via cyclotron resonance

$$\omega - \omega_B - kv_\parallel \approx 0 \tag{6}$$

with a whistler-mode wave with $k < 0$ (i.e., propagating in the $-z$ direction) as shown in Figure 1. We assume that the whistler wave amplitude is given and changes only slowly along the trajectory of the energetic electrons. For the gyroresonant electrons, when the equality (6) is satisfied, the system of equations (3)–(5) has an integral of motion, which can be written as

$$W - \frac{mc}{e}\omega\mu = \mathrm{const.} \tag{7}$$

This can be demonstrated by differentiation of (7) with respect to z, and using (3) and (4). This integral exists if the wave frequency ω is constant.

[10] For the case of a small-amplitude whistler $b \ll B$, where b is the wave magnetic field amplitude, and a slowly varying geomagnetic field, we use the adiabatic approach to solve the system of equations (3)–(5) following *Laval and Pellat* [1970] and *Karpman et al.* [1974]. The right hand side of (5) is presented in the form of a Taylor series expansion over the small parameters near the point of the exact resonance (6) (see below). Differentiating both parts of equation (5) with respect to z, and using (3)–(4), we obtain:

$$\frac{d^2\psi}{dz^2} - l_{\mathrm{tr}}^{-2}\cos\psi + \alpha = 0 \tag{8}$$

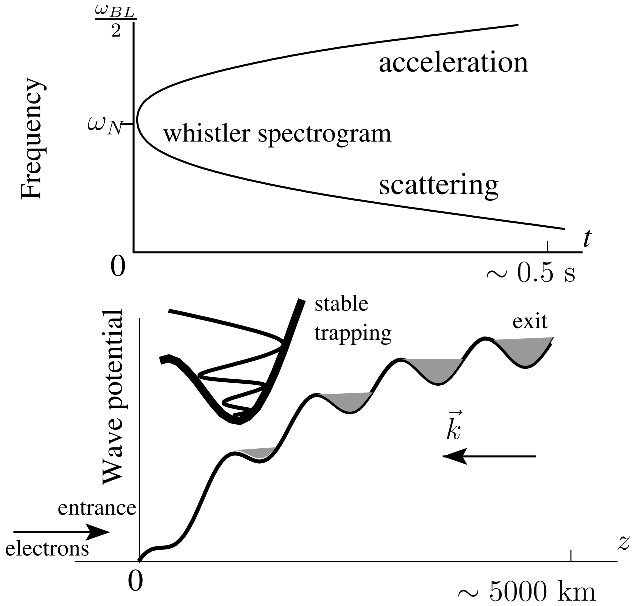

Figure 2. Variation of the effective potential of the whistler wave above its nose frequency ω_N, with distance z from the equator (where $z = 0$); the shaded regions show where gyroresonant electrons are phase trapped by the wave. In reality, there are far more oscillations than shown, because the wavelength in the medium is much less than ~1000 km.

where the trapping length l_{tr} is equal to

$$l_{tr} = \frac{v_R}{\Omega_B}, \tag{9}$$

$$\Omega_B = (kv_\perp \omega_b)^{1/2} \tag{10}$$

is the frequency of electron oscillations in the potential well of the wave (see Figure 2), and $\omega_b = (e/mc)b$ is the electron gyrofrequency in the wave magnetic field b. The resonance velocity for the electrons is

$$v_R = \frac{\omega - \omega_B}{k} > 0. \tag{11}$$

The inhomogeneity factor α is

$$\alpha = k\frac{d}{dz}\left(\frac{v_R}{\sqrt{\frac{2}{m}(W - \mu B)}}\right)_{z_R} \tag{12}$$

where z_R is the point of exact resonance (6). In the differentiation in (12), W and μ are assumed to be independent of z. Equation (8) is valid for the electrons with $v_\parallel$ close to the resonance velocity v_R, and includes only first order terms of the small parameters $\varepsilon_1 = b/B$ and $\varepsilon_2 = l_{tr}/l$; here, l is the characteristic scale length of B and k changes. To have the solution of (8) in a standard form we introduce a new variable ξ, which is connected to ψ by the relation

$$2\xi = \psi - 3\pi/2 \tag{13}$$

For l_{tr} and α constant over z, equation (8) has the energy integral [*Shklyar et al.*, 1992]

$$\dot{\xi}^2 l_{tr}^2 + \sin^2\xi + 2\alpha l_{tr}^2 \xi = \text{const} \tag{14}$$

where $\dot{\xi} = d\xi/dz$. It follows from (14) that there is a trapping threshold, which is determined by the condition that the effective potential $U_{\text{eff}} = \sin^2\xi + 2\alpha l_{tr}^2\xi$ has a minimum (see Figure 2), i.e.,

$$\left|2\alpha l_{tr}^2\right| \leq 1. \tag{15}$$

For $|2\alpha l_{tr}^2| \ll 1$, trapped particles satisfy the condition $\kappa^2 > 1$, where

$$\kappa^2 = \left(\dot{\xi}^2 l_{tr}^2 + \sin^2\xi\right)^{-1}. \tag{16}$$

[11] If α and l_{tr} are slowly changing functions of z, the integral relation (14) is not valid. However, in the case $|2\alpha l_{tr}{}^2| \ll 1$, κ can be found using the adiabatic approach [see *Karpman et al.*, 1974]. For trapped particles ($\kappa^2 > 1$), this approach gives the following relation:

$$\frac{d}{dz}\left[\frac{E(1/|\kappa|) - (1 - \kappa^{-2})K(1/|\kappa|)}{l_{tr}}\right] = 0 \tag{17}$$

where $|\kappa|$ is the modulus of κ.

[12] This equality permits us to find the condition for stable trapping: $d\kappa^2/dz > 0$ ($\kappa^2 > 1$). In (17), E and K are the full elliptic integrals of the first and second type, respectively. Using a Taylor series expansion of E and K for $|\kappa| > 1$, we find that the inequality $d\kappa^2/dz > 0$ is fulfilled [*Hobara et al.*, 2000] when

$$\frac{d}{dz} l_{tr} < 0 \tag{18}$$

[13] The acceleration of trapped electrons (i.e., the net growth of energy W) for a whistler-mode wave of constant frequency ω occurs when the value of the resonance velocity $|v_R|$ is decreasing along the electron trajectory [*Istomin et al.*, 1976; *Hobara et al.*, 2000]. In this case the inequality (18) is obeyed. Changes of W and μ are obtained from the relation (7), and using $v_\parallel{}^2 \equiv \frac{2}{m}(W - \mu B) = v_R^2$ [*Istomin et al.*, 1976; *Hobara et al.*, 2000].

3. Acceleration of Electrons by a Whistler Wave Packet With Changing Frequency

[14] The cyclotron interaction of an energetic electron with a whistler of changing frequency $\omega(z, t)$ is described by the same system of equations (3)–(5). Now, $\omega(z, t)$ and $k(\omega(z, t), z)$ are known functions of z and t. As a result, an additional term appears in the expression for the inhomogeneity factor α in comparison with (12). The expression given in (7) is not the motion integral in this case. Thus, it is necessary to generalize the results obtained above for the case of a changing frequency $\omega(z, t)$.

[15] The change of frequency of a wave packet along its ray path is given by the standard equation of geometrical

optics in a medium whose properties do not change with time [*Bernstein and Fridland*, 1984]:

$$\frac{\partial\omega}{\partial t} - v_g \frac{\partial\omega}{\partial z} = 0. \tag{19}$$

Here v_g is the group velocity, which in general depends on ω and z; the fact that $\vec{k}$ is parallel to $\vec{v}_g$ and antiparallel to $\vec{v}_\parallel$ and the z direction is taken into account (see Figure 1). Thus, ω is a function of the argument $t + \int^z \frac{dz'}{v_g}$. In accordance with relation (2), we should take ω(z, t) for the frequency experienced by the test electron at z, t in the form

$$\omega(z,t) = \omega\left(t + \int^z \frac{dz'}{v_g}\right)$$

$$= \omega\left(t_0 + \int_{z_0}^{z} \frac{dz'}{\sqrt{\frac{2}{m}(W - \mu B(z'))}} + \int^z \frac{dz'}{v_g}\right) \tag{20}$$

[16] In this case, the inhomogeneity factor α is defined by

$$\alpha = k\frac{d}{dz}\left[\frac{v_R(\omega,z)}{\sqrt{\frac{2}{m}(W-\mu B)}}\right] = \alpha_1 + \alpha_2. \tag{21}$$

Here, as earlier,

$$\alpha_1 = \frac{k}{2}\left(\frac{2\omega_B + \omega}{\omega_B - \omega} + \frac{v_\perp^2}{v_R^2}\right)\frac{d\ln\omega_B}{dz} \tag{22}$$

is the value of α for the case of ω = const defined by (12), and the new term α_2 results from differentiation over ω which, in our case, is itself a function of z, as determined by the expression (19):

$$\alpha_2 = k\frac{\partial v_R\omega/\partial\omega}{\sqrt{\frac{2}{m}(W-\mu B)}}\frac{d\omega}{dz} = -\frac{k}{2}\frac{2\omega + \omega_B}{\omega_B - \omega}\frac{d\ln\omega}{dz} \tag{23}$$

To obtain (21)–(23) we took into account that, for whistler waves in the Earth's magnetosphere [*Helliwell*, 1965],

$$k \simeq \left(\frac{\omega\omega_B}{\omega_B - \omega}\right)^{1/2} p \tag{24}$$

where $p = \omega_p/(\omega_B^{1/2}c) \approx$ const and ω_p is the electron plasma frequency. It is assumed that along a particular flux tube (i.e., along the duct axis) the electron density is proportional to the geomagnetic field.

[17] Trapping of electrons by the wave takes place under the same condition (15), but with α defined by (21). The trapping is stable if the condition (18) is obeyed, where the expression (19) for ω(z, t(z)) is taken into account. For stable trapping, the field-aligned velocity component $v_\parallel$ of the electron when it exits from the whistler packet is equal to

$$v_\parallel(z_{\text{ext}}) = |v_R| = \left[\frac{\omega_B - \omega(z,t)}{k}\right]_{z_{\text{ext}}}. \tag{25}$$

[18] The system of equations (3)–(5) enables us to find the change of the first adiabatic invariant μ. From equations (3)–(4) we have, similarly to (7):

$$\frac{d\mu}{dz} = \frac{e}{mc\omega(z,t)}\frac{dW}{dz} \tag{26}$$

Putting the relation $W = \mu B + \frac{mv_\parallel^2}{2} \approx \mu B + \frac{mv_R^2}{2}$ into (26), after some algebraic manipulations we obtain

$$\frac{d\mu}{dz} + \frac{d\omega_B/dz}{\omega_B - \omega}\mu + \frac{e}{2c(\omega_B - \omega)}\frac{dv_R^2}{dz} = 0 \tag{27}$$

where

$$\frac{dv_R^2}{dz} = \frac{(\omega_B - \omega)^2}{\omega_B\omega p^2}\left[\left(2 + \frac{\omega}{\omega_B}\right)\frac{d\omega_B}{dz} - \left(2 + \frac{\omega_B}{\omega}\right)\frac{d\omega}{dz}\right]. \tag{28}$$

[19] The solution of the differential equation (27) for μ has the form

$$\mu = \mu_{\text{ent}} - \frac{e}{2p^2c}\int_{z_{\text{ent}}}^{z_{\text{ext}}} dz \exp\left[\int_{\omega_B(z_{\text{ent}})}^{\omega_B(z)} (\omega_B' - \omega)^{-1} d\omega_B'\right]$$

$$\times\left[\left(\frac{2\omega_B}{\omega} - 1 - \frac{\omega}{\omega_B}\right)\frac{d\ln\omega_B}{dz} + \left(\frac{\omega_B}{\omega} + 1 - \frac{2\omega}{\omega_B}\right)\frac{d\ln\omega}{dz}\right] \tag{29}$$

where z_{ent} and z_{ext} are the coordinates of the forward and trailing edges of the wave packet at the times of electron entry and exit from the packet, respectively, and μ_{ent} is the value of μ when the test electron enters the wave packet. In the case of a constant wave frequency, equation (28) is consistent with the results published earlier [*Istomin et al.*, 1976; *Hobara et al.*, 2000]. When the rate of change of wave frequency is higher than that of ω_B (for example, near the equator, where $\omega_B \simeq$ const), the change of magnetic moment and hence the electron acceleration is determined by the following expression:

$$\mu_{\text{ext}} - \mu_{\text{ent}} = \frac{e}{2p^2c}\left[\frac{\omega_B}{\omega_{\text{ent}}} - \frac{\omega_B}{\omega_{\text{ext}}} + \ln\frac{\omega_{\text{ext}}}{\omega_{\text{ent}}} - 2\frac{\omega_{\text{ext}} - \omega_{\text{ent}}}{\omega_B}\right] \tag{30}$$

Here ω_{ent} and ω_{ext} refer to the wave frequency at the point of particle entrance and exit, respectively. Electron acceleration occurs if $\omega_{\text{ext}} > \omega_{\text{ent}}$.

[20] Expression (30) does not contain the whistler amplitude b. However, it is borne in mind that (30) is valid if the inequality (15), which includes the wave amplitude, is satisfied along the entire acceleration path. In the following section, we analyze this criterion for a particular case of a whistler generated by lightning.

4. Case of Acceleration by a Whistler Wave Packet, Generated by a Lightning Discharge

[21] Now we consider an important case of acceleration when a whistler wave packet with changing frequency is generated by a lightning discharge in which all frequencies are emitted instantaneously. As is well-known [*Helliwell*,

1965], equation (19) expresses the propagation and dispersion of the whistler wave $\omega(z, t)$ everywhere, where the condition for geometrical optics is valid, and gives rise to the nose whistler phenomenon. Let us represent the frequency dependence of an electromagnetic signal generated by lightning at the point $z = l$ (ground level of a particular magnetic field line) using the model

$$\omega = \omega_0 + \delta \cdot t\,, \quad 0 \le t \le \Delta \sim \delta^{-1} \tag{31}$$

The parameter δ has a very large value for real lightning discharges (i.e., the causative atmospheric is approximated here by an extremely steep ramp in frequency). Integrating equation (19), we obtain the frequency $\omega(z, t)$ at any arbitrary point z:

$$\omega = \omega_0 + \delta \cdot t + \delta \int_l^z \frac{dz'}{v_g(z', \omega)} \tag{32a}$$

or, using (2),

$$\omega = \omega_0 + \delta \cdot t_0 + \delta \int_{z_0}^{z} \frac{dz'}{\sqrt{\frac{2}{m}(W - \mu B(z'))}} + \delta \int_l^z \frac{dz'}{v_g(z', \omega)} \tag{32b}$$

[22] From (32b) we find:

$$\frac{d\omega}{dz} = \left(\frac{1}{v_R} + \frac{1}{v_g}\right) \Big/ \left(\int_l^z \frac{dz'}{v_g^2} \frac{\partial v_g}{\partial \omega} + \delta^{-1} \right) \tag{33}$$

[23] For a whistler wave, we define q as

$$q \equiv \int_z^l \frac{dz'}{v_g^2} \frac{\partial v_g}{\partial \omega} = \frac{p}{4} \int_z^l \frac{y^{3/2}(y-4)}{\omega^{3/2}(y-1)^{5/2}} dz' \approx \frac{T_g}{2\omega} \frac{\omega_{BL} - 4\omega}{\omega_{BL} - \omega} \tag{34}$$

where $y = \omega_B/\omega$ is the magnetoionic variable and $T_g(\omega) = \int_0^l dz'/v_g(\omega, z')$ is the group delay from the lightning flash to the equator. Strictly speaking, the relation (33) supplemented by (34) serves as a differential equation for $\omega(z)$ (in (34), ω depends on z, not z'). However, to estimate $d\omega/dz$ near the equator ($z = 0$) we can simplify the expression for q and obtain the last equation in (34). Herewith, the subscript L refers to values taken at the equatorial plane. Substituting (33)–(34) into (23), and assuming that $\delta \to \infty$ so that $\delta^{-1} \to 0$, we obtain

$$\alpha_2 = \left(1 + \frac{\omega_B}{2\omega}\right)^2 \left(v_R^2 q\right)^{-1} \approx \left(1 + \frac{\omega_{BL}}{2\omega}\right)^2 \frac{2\omega}{v_{RL}^2 T_g} \frac{\omega_{BL} - \omega}{\omega_{BL} - 4\omega}. \tag{35}$$

[24] To study electron acceleration due to the whistler near the equator ($\omega_B \simeq \omega_{BL}$) we use the expression (9) and write the condition for trapping (15) in the form

$$\left|2\alpha_2 l_{\mathrm{tr}}^2\right| = \frac{4 v_{RL} v_{gL}}{v_\perp l \omega_{bL}} \left(1 + \frac{\omega_{BL}}{2\omega}\right)^2 \left|4 - \frac{\omega_{BL}}{\omega}\right|^{-1} \le 1. \tag{36}$$

Considering the case $\vec{k} \| \vec{B}$ we actually assumed that, as a whistler wave propagates in a duct, the wave amplitude b changes along this duct in accordance with the law for the conservation of energy flux [*Hobara et al.*, 2000]. Neglecting absorption, this law relates the wave amplitude at the entrance to the duct in the ionosphere (I) to that in the equatorial plane (L):

$$b_L = \left(\frac{\omega_{BL}}{\omega_{BI}} \frac{v_{R0}}{v_{RL}}\right)^{1/2} b_I \approx \left(\frac{\omega_{BL}}{\omega_{BI}}\right)^{1/2} \left(1 - \frac{\omega}{\omega_{BL}}\right)^{-3/4} b_I. \tag{37}$$

Substituting the expressions for v_{R0}, v_{RL}, and b_L, and putting $l \simeq 1.38\, L R_E$ for a dipolar geomagnetic field line [*Lyons and Williams*, 1984] into (36), we obtain the trapping condition in the form

$$\frac{4\omega_{BL}^2 c^2 L^{1/2}}{1.38\, \omega_{pL}^2 v_\perp R_E \omega_{bI}} \varphi(\omega) \le 1. \tag{38}$$

Here

$$\varphi(\omega) = \left(1 - \frac{\omega}{\omega_{BL}}\right)^{15/4} \left(1 + \frac{\omega_{BL}}{2\omega}\right)^2 \left(4 - \frac{\omega_{BL}}{\omega}\right)^{-1}, \tag{39}$$

L is McIlwain's parameter for the flux tube, and R_E is the Earth's radius. The dependence of the left hand side of (38) on L is determined by the multiplier $\omega_{BL}^2 L^{1/2}/\omega_{pL}^2$. If it is also assumed that on different geomagnetic flux tubes the electron density in the plasmasphere is proportional to B, this multiplier becomes $L^{-5/2}$.

[25] We can see from (36) that the conditions for trapping (17) and for the validity of the adiabatic approach are broken at the nose frequency $\omega_N \simeq \omega_{BL}/4$. This is a very important front point of a whistler wave packet, which plays similar role as the sharp leading edge of a constant-frequency wave packet. Resonant electrons crossing this edge get trapped in the wave potential well. For a whistler, the situation is more complicated, since an electron near the packet front moves in the field of two frequencies with a frequency gap increasing along the particle trajectory. Therefore, a special consideration, which is beyond the scope of this paper, is needed to say by which frequency branch of a whistler, $\omega < \omega_N$ or $\omega > \omega_N$, an electron is trapped.

5. Discussion and Conclusions

[26] The relations (30), (36), and (38) permit us to estimate the electron acceleration by a whistler for the conditions occurring in the Earth's magnetosphere. As follows from (28)–(30), acceleration near the equatorial plane takes place if $\frac{d\omega}{dz} > 0$, or $\omega(z_{\mathrm{ext}}) > \omega(z_{\mathrm{ent}})$. Therefore, electron acceleration in this case is effective only for the higher-frequency components of a whistler, above the nose frequency ω_N, where $\omega_N \approx 0.25\omega_{BL}$ in our approximation. From the inequality (18) it is easy to show that the condition $\frac{d\omega}{dz} > 0$ is also necessary for stable particle trapping. Trapping occurs when the condition (36) is met. For that it is necessary to move from the nose frequency to higher frequencies.

Table 1. Parameters of Electron Acceleration by Whistlers in the Regime of Nonlinear Trapping

L	N_{cL}, cm^{-3}	ω_{BL}, s^{-1}	$W_{R\mathrm{ent}}$, keV	$\Delta W_\perp$, keV	$\Delta W_\parallel$, keV	ΔW, keV	α_{ext}, deg
3	10^3	$2 \cdot 10^5$	2.95	3.56	−2.12	1.44	70.35
4	30	$8.6 \cdot 10^4$	17.5	21.13	−12.59	8.54	70.35
5	3	$4.4 \cdot 10^4$	45.9	55.39	−33.01	22.38	70.35

[27] Let us consider a quantitative example. According to (24) and (25), an electron is in gyroresonance with the wave at the equator if its parallel energy is

$$W_{\parallel L} = W_{RL} = \frac{(1-\omega/\omega_{BL})^3}{(\omega/\omega_{BL})} W_0 , \qquad (40)$$

where $W_0 = 0.5\ mc^2\ (\omega_{BL}/\omega_{pL})^2$. Putting, as *Rycroft* [1976] did, $L = 3$, the density of cold plasma $N_{cL} = 10^3$ cm^{-3}, and $\omega = \omega_{BL}/3$ (which is equivalent to about 10 kHz), we obtain $W_0 \simeq 3.32$ keV and $W_{\parallel L} \simeq 2.95$ keV. If the electron pitch angle is 45°, $W_{\perp L}$ is also $\simeq$2.95 keV, and the total energy of the gyroresonant electron is $W \simeq 5.9$ keV. These parameters correspond to $v_{RL} \simeq 0.11\ c$. Substituting these values in (38), we find the whistler amplitude necessary to trap the gyroresonant electrons at the entrance to the whistler-wave packet. Taking into account that, according to (39), $\varphi(\omega = \omega_{BL}/3) \sim 1.36$, we obtain $b_L \simeq 115$ pT.

[28] This amplitude is rather high, but consistent with values extrapolated to the equatorial plane from observations of whistlers in the ionosphere above a thunderstorm [*Kelley et al.*, 1985]. Note that the influence of the magnetic field inhomogeneity and the wave frequency variation on trapping is opposite for electrons moving off the equator, if $\omega > \omega_B/4$. Therefore, the parameter α decreases its absolute value as the distance from the equator increases, until it vanishes at some distance. For the parameters chosen, this distance is about $z \simeq 0.36\ L\ R_E$. If the wave magnetic field at equator is $b_L = 20$ pT, then the trapping conditions for $\omega = 0.3\omega_{BL}$ are satisfied from $z = 0.29LR_E$ to $z = 0.41LR_E$. For higher frequencies, the trapping criterion at the equator is satisfied much more easily, since the frequency sweep rate is lower: for $\omega = 0.5\omega_{BL}$ and with the other parameters the same as those above, we obtain $b_L \simeq 29$ pT and $b_I \simeq 158$ pT.

[29] The transverse energy increase of an electron $\Delta W_\perp$ due to a single interaction with a whistler wave packet can be estimated from (30). Putting $\omega_{\mathrm{ent}} = \omega_{BL}/3$ and $\omega_{\mathrm{ext}} = \omega_{BL}/2$ (the latter corresponds to about 15 kHz), we obtain $\Delta W_\perp \simeq 1.07$ keV and $W_0 \simeq 3.56$ keV. Under these conditions, $m{v_{R\mathrm{ext}}}^2/2 \approx W_0/4 \approx 0.83$ keV. Now we can estimate the change of the total energy:

$$\Delta W = \Delta W_\perp + \Delta W_\parallel = \Delta W_\perp + W_{R\,\mathrm{ext}} - W_{R\,\mathrm{ent}} \simeq 3.56\mathrm{keV}$$
$$+0.83\mathrm{keV} - 2.95\mathrm{keV} \simeq 1.44\mathrm{keV}.$$

In other words, the total energy increases by about 24%, from 5.9 keV to 7.35 keV. The pitch angle also increases, from 45° to $\arctan(v_{\perp\mathrm{ext}}/v_{\parallel\mathrm{ext}}) \simeq 70°$.

[30] Table 1 compares the change of energy components and the resulting electron pitch angle for three typical values of L and N_{cL}, the cold plasma density in the equatorial plane. It is not surprising that the final pitch angle α_{ext} does not depend on these values. Indeed, for the near-equatorial acceleration and a given initial pitch angle α_{ent}, α_{ext} is solely determined by the frequencies at the entrance to and exit from the acceleration region, normalized to ω_{BL}.

[31] On the plasmapause ($L = 4$), the energy increase during a single interaction with a whistler is from 35 keV to 43.5 keV. For a whistler propagating outside the plasmapause (at $L = 5$), a rather unusual occurrence, the energy increase is from 92 keV to 114 keV.

[32] We can also estimate the maximum flux density of accelerated electrons, using the conservation of total energy flux of waves and particles. In our case, the conservation law is written in the form

$$P_{\mathrm{wext}} = P_e \sim v_{\parallel\mathrm{ext}}\, n_h\, \Delta W , \qquad (41)$$

where $P_{\mathrm{w}} \simeq v_{\mathrm{g}} b_L^2/(8\pi)$ is the whistler-wave energy flux, and n_h is the number density of accelerated electrons. The resulting flux density of electrons is equal to

$$S_e \sim v_{\parallel\mathrm{ext}}\, n_h \simeq \frac{v_{\mathrm{g}} b_L^2}{8\pi\Delta W}. \qquad (42)$$

For the parameters used above for $L = 3$, $\Delta W \sim 1.44$ keV and $b_L \sim 10^2$ pT, and we obtain $S_e \sim 10^5$ cm^{-2} s^{-1}.

[33] It is important to note that the threshold whistler amplitude for trapping is proportional to the ratio $v_\parallel/v_\perp$(36), which decreases during the acceleration process. This means that the high frequency components of whistlers can assist in the formation of highly anisotropic energetic tails in the distribution function of the radiation belt electrons ("pancake" pitch angle distributions). The experimental data presented by *Bell et al.* [2000] could be considered as some experimental evidence for such acceleration.

[34] We can compare this acceleration of gyroresonant electrons in the trapping regime with the stochastic acceleration of electrons by whistlers, when trapping is absent. In the latter case, acceleration takes place in multiple interactions of an electron, oscillating between the magnetic mirror points, with many whistlers; it is accompanied by pitch angle scattering. The maximum change of electron energy in this case is comparable to the energy gain in the trappng regime (30) and is described by the relation (7) [see, e.g., *Brice*, 1964]. However, the efficiency of such acceleration (i.e., the acceleration rate) is much smaller; moreover, this process is accompanied by the loss of gyroresonant electrons due to pitch angle scattering [*Smith et al.*, 2001].

[35] The interaction of gyroresonant electrons with whistlers at frequencies below the nose frequency $f_N \simeq f_{BL}/4$ is not so simple. According to the general result (30), an electron should lose its energy during such an interaction, but the trapping is unstable in this case. As a result, an electron leaves the whistler packet before it reaches the wave packet's trailing edge.

[36] If the interaction takes place far from the equator and the whistler wave packet with $\omega < \omega_N$ propagates towards a decreasing geomagnetic field, the magnetic field inhomogeneity and the frequency variation have opposite contributions to the total inhomogeneity factor α ($\alpha_1 < 0$ and $\alpha_2 > 0$) and can compensate for each other. This case requires a special consideration in a future paper.

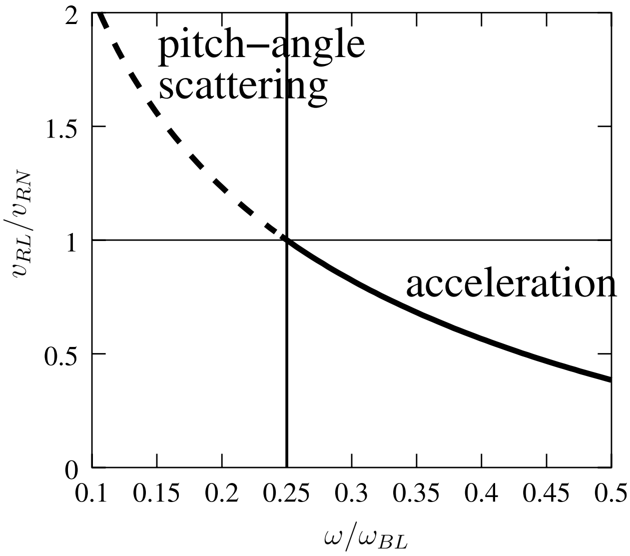

Figure 3. The minimum gyroresonance velocity (at the equator), normalized to the value at the nose frequency, is plotted here against the ratio of the whistler wave frequency to the equatorial gyrofrequency. At frequencies above the nose frequency (taken as $0.25\omega_{BL}$), gyroresonant electrons are accelerated and their pitch angles increase; below the nose frequency, gyroresonant electrons suffer pitch angle diffusion but their energy is not changed appreciably.

[37] The general conclusions of this paper are as follows.

- The velocity space for electrons can be divided into two parts along the $v_{\|L}$ axis, the boundary between the parts corresponding to the resonance velocity v_R^* for the nose frequency f_N(40). This separation is shown in Figure 3.
- Electrons with $v_{\|L} < v_R^*$ can be stably trapped by a whistler wave with frequency $f > f_N$ and accelerated according to (30).
- Estimates show the very high efficiency of this acceleration; the increase of the gyroresonant electron energy reaches ~20% of its energy in one interaction (see Table 1).
- Acceleration in such a regime leads to the formation of highly anisotropic electron distributions, whereas the interaction with the low-frequency components of whistlers ($f < f_N$) results mainly in pitch angle scattering and the loss of van Allen belt electrons.
- It would be not only interesting but also extremely important to investigate experimentally the relationship between thunderstorm activity and the appearance of highly anisotropic energetic electrons, with larger fluxes well away from the loss cone, in the van Allen radiation belts.

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