# Control Theory for a Class of 2D Continuous-Discrete Linear Systems

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**Abstract**

This paper considers a general class of 2D continuous-discrete linear systems of both systems theoretic and applications interest. The focus is on the development of a comprehensive control systems theory for members of this class in a unified manner based on analysis in an appropriate algebraic and operator setting. In particular, important new results are developed on stability, controllability, stabilization, and optimal control.

## 1 Introduction

The past two to three decades, in particular, have seen a continually growing interest in so-called two-dimensional (2D) systems or, more generally, (nD ($n > 2$)) systems. This interest is clearly related to the wide variety of applications of both practical and/or theoretical interest. The key unique feature of an nD system is that the plant or process dynamics depend on more than one indeterminate and hence information is propagated in many independent directions.

Many physical processes have a clear nD structure. Also the nD approach is frequently used as an analysis tool to assist, or in some cases enable, the solution of a wide variety of problems. A key point is that the applications areas for nD systems theory can be found within the general disciplines of circuits, control and signal processing (and many others). For a representative cross-section of these see, for example, the edited text [1]. In this paper, however, it is a particular class of 2D systems which is considered.

Some classes of 2D/nD linear systems share strong structural links with, in particular, standard (1D) linear systems, e.g. so-called differential and discrete linear repetitive processes (see, for example, [8]) where the common structure assertion arises from structural similarities between the state space models which describe the underlying dynamics. This immediately suggests that nD systems can be studied by direction extension of existing/emerging 1D systems theory. Experience has shown, however, that there are a great many problems in generalizing 1D systems theory to the nD case. Some of these problem are fundamentally algebraic in nature, e.g. the distinction in the nD case between factor primeness, minor primeness, and zero primeness, or the lack of an Euclidean algorithm.

Other problems concern the apparent absence of relationships between important concepts that are strongly related in the 1D case. For example, there are many generalizations of the concepts of (state) controllability, observability and minimality to the nD linear systems, but for none of the accepted definitions is it the case that controllability plus observability is equivalent to minimality. As another example, the Smith form of an nD linear system fails to provide much information about the system which it does supply in the 1D case.

At an abstract level, 2D/nD systems theory sets out to examine the same basic questions as 1D theory, such as controllability, observability, causality, construction of state space models (realization

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theory), stability and stabilization, feedback control, filtering. Given the problems with extending 1D systems theory, it is clear that much of this development must start from a basic level. The task is made even more diverse by the "rich" variety of dynamics which can be encountered in terms of the indeterminates. For example, there are 2D systems where the propagation of the dynamics in the two independent directions is either a function of (i) two discrete variables, (ii) two continuous variables, or (iii) a continuous variable in one direction and a discrete variable in the other leading to so-called 2D continuous-discrete systems.

Of these combinations, a very large volume of work has been reported on case (i) based, in the main, on the Roesser [2] and Fornasini Marchesini [3, 5] state space models and also there has been work reported on (ii) — see, for example, [6, 7]. It is the case of (iii) which is considered here where other work, see, for example, [8, 12] has focused on the special case of so-called differential linear repetitive processes whose state space model has dynamics in one direction (along the pass) governed by a linear matrix differential equation and along the other (pass-to-pass) by a linear matrix difference equation.

In this paper we use a general operator setting (see, for example, [13, 14] for related results) to study key control theoretic properties of a class of 2D continuous-discrete linear systems which allows for a significant generalization and extension of previous results.

A key fact about the model structure used here is that it allows a general algebraic and operator setting to be used as a setting for analysis. This leads to major new results on controllability, stability, stabilization by feedback action and optimal control in this general setting. We begin in the next section by giving a summary of the necessary background mathematical tools and then introduce the model to be studied.

## 2 Preliminaries and Background Results

In this section, we give the necessary background and results required for the analysis in this paper. To start with, let $E$, $V$ and $W$ be finite dimensional normed spaces over the complex field $\mathbb{C}$ with norm denoted by $|\cdot|$. Also let $\mathbb{Z}_+$ denote the set of nonnegative integers and denote the set of all linear operators acting from $E$ to $V$ by $\mathcal{L}(E, V)$. We use $BC^\infty(E)$ to denote the set of all infinitely differentiable functions $\psi : \mathbb{R} \to E$ such that for each $\psi \in BC^\infty(E)$, $\exists$ a constant $c_\psi > 0$ such that $|\psi^{(j)}(s)| \leq c_\psi$, $\forall\, s \in \mathbb{R}$, and $j = 0, 1, 2, \ldots$, where $\psi^{(j)}(s)$ denotes the derivative $\frac{d^j\psi}{ds^j}$, This set becomes a normed linear space under the sup norm defined as $||\psi||_E = \sup_{s\in\mathbb{R},\, j\in\mathbb{Z}_+} |\psi^{(j)}(s)|_E$. Also $BC^\infty(E)$ is a Banach space. (To simplify notation, we drop the explicit representation of the space from $||\cdot||$ throughout the paper.)

Suppose now that $A_t$, $t \in \mathbb{Z}_+$ is a set of linear operators in $\mathcal{L}(E, E)$. Also let $B \in \mathcal{L}(W, E)$. Then the class of 2D continuous-discrete linear systems considered in this paper is defined as follows

$$x(t+1, s) = \sum_{j\in\mathbb{Z}_+} A_j \frac{d^j x(t, s)}{ds^j} + Bu(t, s),\ s \in \mathbb{R},\ t \in \mathbb{Z}_+ \tag{1}$$

where $x : \mathbb{Z}_+ \times \mathbb{R} \to E$, and $u : \mathbb{Z}_+ \times \mathbb{R} \to W$ is the control input function. We assume that the mapping $s \to u(t, s)$ belongs to the space $BC^\infty(W)$ for each fixed $t \in \mathbb{Z}_+$, and such functions are termed admissible controls here.

Suppose now that the convergence condition for the series in (1) detailed below holds. Then it is easy to verify that for any function $\alpha \in BC^\infty(E)$, which represents the initial conditions, there is a unique solution $x(t, s) = x(t, s, \alpha, u)$ of (1) satisfying

$$x(0, s) = \alpha(s),\ s \in \mathbb{R} \tag{2}$$

Here the notation $x(t, s, \alpha, u)$ is used to emphasize the fact that the solution $x(t, s)$ depends on $(\alpha, u)$.

By way of motivation for the work reported here, first note that linear equations of the form considered here arise in the modelling of various physical processes. For example, the two-dimensional

Schrödinger equation used in quantum theory is

$$\imath \frac{\partial u}{\partial t} = \frac{c}{\omega}\sqrt{1-\omega^2 \Delta u} \tag{3}$$

where $\Delta$ is the known differential operator. Expanding the square root on the right-hand side of this equation in a power series in $\Delta$ leads to the model

$$\imath \frac{\partial u}{\partial t} = \frac{c}{\omega}\left(1+\sum_{j=1}^{\infty} c_j \omega^{2j} \frac{\partial^{2j} u}{\partial x^{2j}}\right), \quad \text{with} \quad c_j = (-1)^j \frac{1}{2}\cdot \ldots \cdot \frac{3-2j}{2}$$

where the differential operator on the right-hand side is of infinite degree (see, for example, [10]). Also it is known that some differential-difference equations can be represented by such an equation where again the differential operator involved is of infinite degree operator — see, for example, [14] for a bibliography of the literature on such cases.

Suppose that the operators $A_i$ satisfy

$$\sum_{j\in\mathbb{Z}_+} (1+\epsilon)^j ||A_j|| < \infty \tag{4}$$

for a real number $\epsilon > 0$, which guarantees the convergence of the series $\sum\limits_{j\in\mathbb{Z}_+} A_j z^j$ in a domain including the unit disc (where $||\cdot||$ is also used to denote the induced norm). Then the mapping $s \to x(t,s)$ belongs to $BC^\infty(E)$ for each fixed $t \in \mathbb{Z}_+$.

The right-hand side of (1) generates a differential operator $D : BC^\infty(E) \to BC^\infty(V)$ of the form

$$(D\psi)(s) = \sum_{j\in\mathbb{Z}_+} A_j \psi^{(j)}(s),\ s \in \mathbb{R} \tag{5}$$

where for generality and the analysis of next section, we now assume that $A_j,\ j \in \mathbb{Z}_+$ is a set of operators from $\mathcal{L}(E,V)$. Properties of the operator $D$ play a significant role in the analysis to follow in this paper.

Associate with the differential operator $D$ its representation $\Upsilon(z)$ in the ring of power series defined by

$$\Upsilon(z) = \sum_{j\in\mathbb{Z}_+} A_j z^j,\ z \in \mathbb{C} \tag{6}$$

Suppose also that the operators $A_j$ are such that the power series (6) converges in some domain containing the unit disk in the complex plane (which is certainly true if, for example, (4) holds). Also let $\mathcal{X}(E,V)$ denote the set of all power series of the form of (6) which satisfy (4). Next, to each element $\Upsilon(z) \in \mathcal{X}(E,V)$ associate a bounded linear operator $D$ defined by (5) and let $\mathcal{D}(E,V)$ denote the image of $\mathcal{X}(E,V)$ under the mapping $\Upsilon \to D$. Then it follows immediately that $\mathcal{D}(E,V)$ and $\mathcal{X}(E,V)$ are linear spaces with respect to the usual pointwise addition and multiplication by scalars.

It is easy to prove that the mapping $\Upsilon \to D$ is injective since the equalities $\Upsilon = 0,\ D = 0$, hold if, and only if, $A_j = 0,\ j \in \mathbb{Z}_+$. Hence the pre-image of the trivial element of $\mathcal{D}(E,V)$ is only the trivial element of $\mathcal{X}(E,V)$. The fact that this mapping is also surjective follows immediately from the definition of the set $\mathcal{D}(E,V)$. Consequently the map $\Upsilon \leftrightarrow D$ is a bijection.

In the case when $E = V$, the sets $\mathcal{X}(E,E)$ and $\mathcal{D}(E,E)$ (denoted here by $\mathcal{D}(E)$ and $\mathcal{X}(E)$ for brevity ) are algebras with respect to the standard addition and multiplication operations. Also, $\mathcal{D}(E)$ and $\mathcal{X}(E)$ are Banach algebras when the norm on $\mathcal{D}(E)$ is the usual operator norm. In the case of $\mathcal{X}(E)$ this is defined by $||\Upsilon|| = \max_{|z|\le 1} |\Upsilon(z)|$, and the mapping $\Upsilon \to D$ is an isomorphism from $\mathcal{X}(E)$ to $\mathcal{D}(E)$.

This last fact leads immediately to the following result.

**Lemma 1.** *Let $E = V$. Then the operator $D \in \mathcal{D}(E)$ has an inverse in $\mathcal{D}(E)$ if, and only if,*

$$det(\Upsilon(z)) \neq 0,\ |z| \le 1 \tag{7}$$

In the remainder of the section, some basic results from the general theory of operators defined on topological semigroups [15], [16] required for the analysis in this paper are summarized. First, let $X$ denote a Banach space over the complex field $\mathbb{C}$ and $\mathcal{L}(X)$ be the set of all linear bounded operators from $X$ to $X$ equipped with the standard operator norm. Next, let $\mathcal{U}(n,r)$ denote the set of linear maps of $A : X^n \to X^r$, where $n$ and $r$ are integers, defined by

$$Ay = \begin{pmatrix} a_{11}y_1 + a_{12}y_2 + \ldots + a_{1n}y_n \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{r1}y_1 + a_{r2}y_2 + \ldots + a_{rn}y_n \end{pmatrix}, \; y = (y_1, \ldots, y_n) \in X^n$$

where $a_{ij},\ i = 1, ..., r,\ j = 1, ..., n$ are elements of $\mathbb{C}$, (i.e. the mapping is defined by matrix $A = (a_{ij})$ and denoted by the same symbol $A$.) In $X^n$ define the norm as $\|y\| = \max\{\|y_1\|, ..., \|y_n\|\}$. Then, since $\|Ay\| \le \|y\| \max\limits_{1\le j\le r} \sum\limits_{k=1}^{n} |a_{jk}|$, $A$ is a bounded linear operator.

Let $G$ be a commutative topological semigroup with unit $e$ and the group operation $G^2 \ni (g, h) \to gh \in G$. Also let $d$ be a continuous representation of $G$ in the Banach space $X$, i.e. $d$ is a homomorphism $g \to d(g)$ of the semigroup $G$ into the semigroup $\mathcal{L}(X)$ which satisfies

$$d(g_1g_2) = d(g_1)d(g_2) \tag{8}$$

and the mapping $g \to d(g)$ is continuous. Also we assume that $\|d(e)\| = 1,\ \|d(g)\| \le 1,\ g \in G$.

Let $H$ be a discrete sub-semigroup of $G$ and let $A_g,\ g \in H$ denote a summable set of operators from $\mathcal{U}(n,r)$, i.e. $\sum\limits_{g\in H} \|A_g\| < \infty$. Then, using the given representation $d$ in $X$ and the collection of $A_g,\ g \in H$ of operators from $\mathcal{U}(n,r)$, we can define the linear operator $T : X^n \to X^r$ by the formula

$$Ty = \sum_{g\in H} A_g d(g)y,\ y = (y_1, ..., y_n) \in X^n \tag{9}$$

where $d(g)$ for every $y = (y_1, ..., y_n)$ is defined by $d(g)y = (d(g)y_1, ..., d(g)y_n)$. It is also easy to show that $T$ is a bounded operator and $\|T\| \le \sum\limits_{g\in H} \|A_g\|$.

**Remark 1.** *Choosing a suitable space $X$, and a semigroup $G$ with a representation $d$ in $X$ recovers a wide class of the operators used in many areas of systems theory. In particular, let $G$ be a local compact topological group and $X = C(G, \mathbb{C})$ be the space of continuous bounded functions $f : G \to \mathbb{C}$ with norm $\|f\| = \sup\limits_{x\in G} |f(x)|$. Then the collection of linear mappings $d(g) : C(G, \mathbb{C}) \to C(G, \mathbb{C})$ given by the formula $d(g)f = f_g$, where $f_g(x) = f(gx),\ x \in G$, is termed the representation of $G$ in $C(G, \mathbb{C})$. In this case, the operator $T$ of (9) is the shift operator [16]. Also we show below that another special case of (9) leads to the differential operator $D$ of (5) acting on the space of infinitely differentiable functions.*

We say that the span for $v \in X$, denoted by $\{span\}(v)$, characterizes the representation $d$ if $\exists$ a complex-valued function $\lambda : G \to \mathbb{C}$ which satisfies

$$d(g)v = \lambda(g)v,\ \forall\, g \in G \tag{10}$$

Also $\lambda(g)$ is continuous and satisfies the relationship $\lambda(g_1g_2) = \lambda(g_1)\lambda(g_2)$, i.e. $\lambda(g)$ is a continuous semi-character of the semi-group $G$. Note that if $G$ is a group then $\lambda(g)$ is a continuous character of $G$ (see, for example, [17]). Here we use $\mathcal{R}(d)$ to denote the set of all such elements $v \in X$ which characterizes $d$ and by $\Gamma(d)$ the set of all functions $\lambda$ satisfying (10) for $v \in \mathcal{R}(d),\ v \ne 0$.

Now let the operator $T$ be described by its representation [16], i.e. by the matrix $\hat{T}$ defined for $\lambda \in \Gamma(d)$ as

$$\hat{T}(\lambda) = \sum_{g\in H} A_g\lambda(g),\ \lambda \in \Gamma(d) \tag{11}$$

Then we have the following result.

**Lemma 2.** *Let $n = r$. Then each eigenvalue $\rho$ of the matrix $\hat{T}(\lambda_0)$ belongs to the spectrum of the operator $T$, $\forall\, \lambda_0 \in \Gamma(d)$.*

**Proof.** If (11) holds, there exists a non-trivial $n \times 1$ vector $c_0 \in \mathbb{C}^n$ such that $\hat{T}(\lambda_0)c_0 = \rho c_0$. Since $\lambda_0 \in \Gamma(d)$ then there exists a nontrivial element $r_0 \in X$ for which $d(g)r_0 = \lambda_0(g)r_0,\ g \in G$. Finally, set $y_0 = (c_1 r_0, ..., c_n r_0) \in X^n$, where $c_j,\ j = 1, ..., n$ are the entries in the vector $c_0$, and it is easy to check that $Ty_0 = \rho y_0$. □

In general, the evaluation of the spectrum of $T$ is a non-trivial task but for some cases this set corresponds to the set of eigenvalues of the matrix $\hat{T}(\lambda)$, $\lambda \in \Gamma(d)$. In particular, for the shift operator this fact has been established in [16].

Return now to the differential operator $D : BC^\infty(E) \to BC^\infty(V)$ of (5) and suppose that the semigroup $G$ coincides with the set $\mathbb{Z}_+$. Consider also the space $BC^\infty$ of infinitely differentiable bounded complex-valued functions $\phi : \mathbb{R} \to \mathbb{C}$ and the set of mappings $d(k) : BC^\infty \to BC^\infty$ defined by

$$d(k)\phi = \phi^{(k)},\ k \in \mathbb{Z}_+ \tag{12}$$

Then this map is a representation of the semigroup $\mathbb{Z}_+$ in the space $BC^\infty$, and it is clear that the functions $d(k)$ satisfy (8) since $d(k_1 + k_2) = d(k_1)d(k_2)$ (by definition, addition is the group operation in $\mathbb{Z}_+$). If we fix the bases in $E$ and $V$ ($n = dimE,\ r = dimV$) then each $T$ of the form (9) is a differential operator $T : \psi \to \sum_{k=0}^{\infty} A_k \psi^{(k)}$, acting from the space $(BC^\infty)^n$ of the $n \times 1$ vector functions $\psi : \mathbb{R} \to \mathbb{C}^n$ to the space $(BC^\infty)^r$, where $A_k,\ k \in \mathbb{Z}_+$ are given $r \times n$ matrices over $\mathbb{C}$ which also satisfy the inequality (4).

In this last case, $T$ is the differential operator $D$ defined by (5) and it is easy to see that in this case

$$\mathcal{R}(d) = \{z_\omega(s) = e^{\omega s},\ \omega \in [-\imath, \imath],\ \imath^2 = -1\} \tag{13}$$

Also since $d(k)z_\omega = \omega^k z_\omega$, we have that

$$\Gamma(d) = \{\lambda_\omega : k \to \omega^k,\ \omega \in [-\imath, \imath]\} \tag{14}$$

and it now follows that the representation (11) in this case is

$$\hat{T}(\lambda_\omega) = \sum_{k=0}^{\infty} A_k \lambda_\omega(k) = \sum_{k=0}^{\infty} A_k \omega^k = \Upsilon(\omega) \tag{15}$$

The inequality (4) guarantees the convergence of $\Upsilon(z)$ in a domain in the complex plane containing the interval or segment $[-\imath, \imath]$ of the complex plane $\mathbb{C}$. Hence the representation $\hat{T}$ for $D$ is the function $\Upsilon(z)$ of (6). This fact leads to the following result.

**Theorem 1.** *Let $E = V$. Then the spectrum $\Sigma(D)$ of the differential operator $D$ is pointwise and is given by*

$$\Sigma(D) = \bigcup_{z \in [-\imath, \imath]} \sigma(\Upsilon(z)) \tag{16}$$

*where $\sigma(\Upsilon(z))$ denotes the set of eigenvalues of the matrix $\Upsilon(z)$.*

**Proof.** Let $\rho \in \bigcup_{z \in [-\imath, \imath]} \sigma(\Upsilon(z))$. Hence $\exists \omega \in [-\imath, \imath]$ such that $\rho \in \sigma(\Upsilon(\omega))$. Therefore $\exists c_\rho \in \mathbb{C}^n$ such that $\Upsilon(\omega)c_\rho = \rho c_\rho$. Next, define the function $f_\rho(t) = c_\rho e^{\omega t},\ t \in \mathbb{R}$. Then it is straightforward to show that $f_\rho \in (BC^\infty)^n$ and

$$(Df_\rho)(t) = \sum_{k=0}^{\infty} A_k f_\rho^{(k)}(t) = \sum_{k=0}^{\infty} A_k \omega^k c_\rho e^{\omega t} = \Upsilon(\omega) c_\rho e^{\omega t} = \rho f_\rho, \quad t \in \mathbb{R}.$$

Hence, we have shown that the inclusion

$$\bigcup_{z \in [-\imath, \imath]} \sigma(\Upsilon(z)) \subset \Sigma(D) \tag{17}$$

is valid.

Now, let $\hat{\lambda}$ be a complex number such that $\hat{\lambda} \notin \bigcup_{z\in[-\imath,\imath]} \sigma(\Upsilon(z))$. Consider also the operator $U = D - \hat{\lambda}I$, mapping the space $BC^\infty(E)$ into itself, where $I$ is the identity operator. Then it is easy to check that the representation $\hat{U}$ of the operator $U$ is given by the formula $\hat{U} = \Upsilon(z) - \hat{\lambda}I$. Since $\hat{\lambda} \notin \bigcup_{z\in[-\imath,\imath]} \sigma(\Upsilon(z))$ then $det(\Upsilon(z) - \hat{\lambda}I) \neq 0$, $\forall z \in [-\imath, \imath]$ and, hence, $det(\Upsilon(z) - \hat{\lambda}I) \neq 0$, $\forall\ |z| \leq 1$. By Lemma 1 the operator $U = D - \hat{\lambda}I$ has an inverse in $\mathcal{D}(E)$ and this means that $\hat{\lambda}$ is not a member of the spectrum of the operator $D$, which together with the inclusion (17) completes the proof.

$\square$

It is essential for further study to establish that the differential operator $D : BC^\infty(E) \to BC^\infty(V)$ is surjective. This is the subject of the next result.

**Theorem 2.** *The operator $D \in \mathcal{D}(E, V)$ is surjective if, and only if,*

$$rank\, \Upsilon(\omega) = dim\, V,\ \omega \in [-\imath, \imath] \tag{18}$$

**Proof.** First note that if $E = V$ and $\det(\Upsilon(\omega)) \neq 0, \omega \in [-\imath, \imath]$, then the operator $D$ is invertible. This follows immediately since in this case Theorem 1 shows that $z = 0$ does not belong to the spectrum of the operator $D \in \mathcal{D}(E)$.

Suppose now that $\Upsilon(z)$ is an analytic extension of the function $\Upsilon(\omega)$ to some domain $\Omega$ of the complex plane which includes the segment $[-\imath, \imath]$. The existence of such an extension is guaranteed by the assumption (4) and, in fact, applying elementary operations yields the following factorization of the matrix function $\Upsilon(z)$

$$\Upsilon(z) = Q_1(z)P(z)Q_2(z) \tag{19}$$

where $Q_1(z)$ and $Q_2(z)$ are square matrices of appropriate dimensions which are analytic in $\Omega$ and have nonzero determinants in the segment $[-\imath, \imath]$. The matrix $P(z)$, which has the same dimensions as $\Upsilon(z)$, has elements which are zero except possibly on the leading diagonal where entries which are monic polynomials with roots in the segment $[-\imath, \imath]$ can occur.

No loss of generality arises from assuming that any non-zero diagonal elements $p_1(z), \cdots, p_r(z)$ of $P(z)$ occur in the first $r$ rows of this matrix. These polynomials also have the property that $p_i(z)$ divides $p_{i+1}(z)$, $1 \leq i \leq r-1$. The elementary operations used to obtain (19) are (i) interchanging two rows (columns), (ii) multiplication of a row (column) by a function which is analytic in $\Omega$ and is non-zero in the segment $[-\imath, \imath]$, and (iii) multiplication of a row (column) by an analytic function in $\Omega$ and adding the result to another row (column).

As noted above, the mapping $\Upsilon \leftrightarrow D$ is a bijection and the composition of differential operators $D_1, D_2$ is equivalent to multiplication of the corresponding matrix functions $Q_1(z)$ and $Q_2(z)$. Hence, (19) and the fact that the matrices $Q_1(z)$, $Q_2(z)$ are nonsingular in the segment $[-\imath, \imath]$, yield that the surjective property of $D$ is equivalent to this same property for those operators whose representations coincide with the matrix $P(\omega)$. Hence, by the structure of the matrix $P(z)$, the map $D$ can be decomposed into a system of $r = \dim V$ scalar differential operators on the space $BC^\infty$ each of which has the form

$$T_0 : \phi \to \phi^{(p)} + a_1\phi^{(p-1)} + \cdots + a_{p-1}\phi^{(1)} + a_p\phi \tag{20}$$

where $a_j \in \mathbb{C}$, and $p$ is an integer.

Each of the operators in (20) is surjective in the space $BC^\infty$ if, and only if,

$$\omega^p + a_1\omega^{p-1} + \cdots + a_{p-1}\omega + a_p \neq 0,\ \omega \in [-\imath, \imath]. \tag{21}$$

To show this, suppose that $\omega_0^p + a_1\omega_0^{p-1} + \cdots + a_{p-1}\omega_0 + a_p = 0$, for some $\omega_0 \in [-\imath, \imath]$. Then choosing a function of the form $\alpha(t) = e^{\omega_0 t},\ t \in \mathbb{R}$, from the space $BC^\infty(\mathbb{R})$ yields that the equation $T_0\phi = \alpha$ has no solution in this space. Hence (21) is necessary for operators of the form $T_0$ to be surjective. The fact that this condition is sufficient follows immediately from (16). Consequently $D : BC^\infty(E) \to BC^\infty(V)$ is surjective if, and only if, (18) holds and the proof is complete.

$\square$

**Example 1.** Let $E := \mathbb{C}$, and $X := BC^{\infty}(\mathbb{C})$ be the Banach space of the infinitely differentiable functions $f : \mathbb{R} \to \mathbb{C}$, and consider the following differential operator $D : X^2 \longrightarrow X^2$

$$D\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}(t) = \begin{pmatrix} \frac{3}{2}\phi_1(t) + \phi_1^{(1)}(t) \\ \phi_2(t) + \phi_2^{(1)}(t) \end{pmatrix} \tag{22}$$

where the task is to interpret the spectral properties of Section 2 for this case.

Let $G := \mathbb{Z}_+$ with semigroup addition operation $G^2 \ni (g, h) \longrightarrow g + h \in G$. The representation $d(g) : G \longrightarrow \mathcal{L}(X)$ of the semigroup $G$ in the space $X$ is defined by (12), i.e.

$$d(k)\phi = \phi^{(k)},\ k \in \mathbb{Z}_+ \tag{23}$$

The semi-character $\lambda : \mathbb{Z}_+ \to \mathbb{C}$ is defined by

$$\phi^{(g)} = \lambda(g)\phi, \quad \forall g \in \mathbb{Z}_+ \tag{24}$$

It now is straightforward to show that the function $\phi(t) = e^{\omega t},\ t \in \mathbb{R}$ and the set of functions $\lambda_\omega(g) = \omega^g,\ g \in \mathbb{Z}_+$ with parameter $\omega \in \mathbb{C}$ satisfy (24). To guarantee the inclusion $\phi \in BC^{\infty}(\mathbb{C})$ it is necessary that $\omega$ is purely imaginary and $\omega \in [-i, i]$. Hence

$$\Gamma(d) = \{\ \lambda_\omega : g \longrightarrow \omega^g, \quad \omega \in [-i, i]\ \} \tag{25}$$

To define the operator $T : X^2 \longrightarrow X^2$ of (9), choose the discrete sub-semigroup $H = \mathbb{Z}_+$ and the collection $A_g,\ g \in H$ of matrices from $\mathcal{U}(n, n)$ with $n = 2$ as

$$A_0 = \begin{pmatrix} 3/2 & 0 \\ 0 & 1 \end{pmatrix},\ A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ A_i = 0,\ i \in H,\ i > 1$$

The representation $d$ of the group $G$ in the space $X^2$ is defined by

$$d(g)\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} d(g)\phi_1 \\ d(g)\phi_2 \end{pmatrix} = \begin{pmatrix} \phi_1^{(g)} \\ \phi_2^{(g)} \end{pmatrix}$$

and the operator $T$ is

$$T\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = A_0\begin{pmatrix} d(0)\phi_1 \\ d(0)\phi_2 \end{pmatrix} + A_1\begin{pmatrix} d(1)\phi_1 \\ d(1)\phi_2 \end{pmatrix} = \begin{pmatrix} 3/2\phi_1 + \phi_1^{(1)} \\ \phi_2 + \phi_2^{(1)} \end{pmatrix} \tag{26}$$

which coincides with the differential operator $D$. The representation $\hat{T}$ of $T$ is

$$\hat{T}(\lambda_\omega) = \sum_{g \in H} A_g \lambda_\omega(g) = \begin{pmatrix} 3/2 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\omega = \begin{pmatrix} 3/2 + \omega & 0 \\ 0 & 1 + \omega \end{pmatrix}, \quad \forall\ \lambda_\omega \in \Gamma(d) \tag{27}$$

and the eigenvalues of the matrix $\hat{T}(\lambda_\omega)$ are $\sigma_1 = 3/2 + \omega$ and $\sigma_2 = 1 + \omega$ for $\forall \omega \in [-i, i]$. Hence by Theorem 1 these values are the elements of the spectrum of $D$.

This last fact can be also established by direct calculation. In particular, consider

$$(D - \mu I)\phi = 0, \quad \mu \in \mathbb{C},\ \phi \in X^2 \tag{28}$$

which can be rewritten as

$$\begin{aligned} (\frac{3}{2} - \mu)\phi_1(t) + \phi_1^{(1)}(t) &= 0, \\ (1 - \mu)\phi_2(t) + \phi_2^{(1)}(t) &= 0 \quad \mu \in \mathbb{C} \end{aligned} \tag{29}$$

in terms of the unknown functions $\phi_1$ and $\phi_2$ from the space $BC^{\infty}(\mathbb{C})$. Hence it follows that the functions

$$\phi_(t) = e^{-(3/2-\mu)t}, \quad \phi_2(t) = e^{-(1-\mu)t}, \quad t \in \mathbb{R},\ \mu \in \mathbb{C} \tag{30}$$

satisfy (29). These functions belong to $BC^{\infty}(\mathbb{C})$ if, and only if, the numbers $(\frac{3}{2} - \mu)$ and $(1 - \mu)$ are purely imaginary and have modulus less than or equal to unity, i.e. if, and only if, $\mu = \frac{3}{2} + i\nu,\ \mu = 1 + i\nu,\ \nu \in [-1, 1]$. Hence the spectrum of the operator $D$ coincides with the set $\{\sigma : \sigma = \frac{3}{2} + \omega,\ \sigma = 1 + \omega, \quad \omega \in [-i, i]\}$.

## 3 Stability, Controllability and Stabilization

In this section, we first use properties of the representation $\Upsilon(z)$ of the differential operator $D$ to characterize stability of systems described by (1) for the case $E = V$. The formal definition of this property is as follows.

**Definition 1.** *Systems described by (1) with $u = 0$ are said to be stable if $\exists$ a real scalar $q \in (0, 1)$ such that*

$$|x(t, s)|_E \le \hat{C} q^t \tag{31}$$

*holds for initial condition $\alpha \in BC^\infty(E)$, where $\hat{C}$ is a positive constant (independent of $s$ and $t$).*

**Note:** In order to highlight the role of $q$, systems satisfying this last definition are termed $q$ stable.

The following result gives the necessary and sufficient condition for $q$ stability of the systems under consideration here.

**Theorem 3.** *Systems described by (1) are $q$ stable if, and only if,*

$$det(\Upsilon(z) - \lambda I) \ne 0, \; z \in [-\imath, \imath], \; |\lambda| \ge 1 \tag{32}$$

**Proof.** It is a routine to show that (31) holds if, and only if, the spectrum of the operator $D$ in this case lies in the unit disk in the complex plane. Theorem 1 here states that, under the assumptions invoked, this holds if the spectrum of $D$ generated by the right-hand side of (1) lies in the unit disk in the complex plane and the proof is complete. □

Stable systems are closely linked to the existence of bounded solutions to the equations which describe their dynamics in the presence of inputs (or disturbances). In particular, consider non-homogeneous systems of the form

$$x(t+1, s) = \sum_{j \in \mathbb{Z}_+} A_j \frac{d^j x(t, s)}{ds^j} + f(t, s), x(0, s) = \alpha(s), \; s \in \mathbb{R} \tag{33}$$

where the vector valued function $f(t, s)$ represents inputs and/or disturbances acting on the system. Then we have the following result.

**Theorem 4.** *For each initial condition $\alpha \in BC^\infty(E)$ and each bounded input function $f(t, s)$, $\exists$ a bounded solution $x(t, s, \alpha, u)$ of (33) if, and only if, this system is $q$ stable.*

**Proof.** To prove sufficiency, first note that (33) can be rewritten in the operator form

$$\omega(t+1) = D\omega(t) + v(t) \tag{34}$$

where the differential operator $D$ is defined by (5) and

$$\omega(t)(s) = x(t, s), \; \omega(0) = \alpha, \; v(t)(s) = f(t, s), \; t \in \mathbb{Z}_+, \; s \in \mathbb{R} \tag{35}$$

The solution of (33) can now be written as

$$x(t+1, s) = (D^t \alpha)(s) + \sum_{k=0}^{t-1} (D^{t-k-1} v(k))(s) \tag{36}$$

Since the system is stable, $r(D) < 1$, and choose $\epsilon > 0 : q = r(D) + \epsilon < 1$. Then it follows immediately that $\exists$ a positive constant $d(\epsilon)$ such that $||D^t|| \le d(\epsilon) q^t$. Hence

$$|x(t, s)| \le d(\epsilon)(||\alpha|| q^t + \sum_{k=0}^{t-1} q^{t-k-1} ||v(k)||) \tag{37}$$

which yields immediately that the solution $x(t, s)$ is bounded. (where now $|| \cdot ||$ denotes the norm on $BC^\infty(E)$ introduced in Section 2).

To establish necessity, suppose that (33) is not $q$ stable. Then by Theorem 3, $\exists$ an element $\lambda_0 \in \Sigma(D)$ such that $|\lambda_0| \geq 1$. Also since the spectrum of $D$ is pointwise, there is a nontrivial element $\beta \in BC^\infty(E)$ such that $D\beta = \lambda_0\beta$.

Now consider the case of $|\lambda_0| > 1$, and set $f(t,s) = 0$, $x(0,s) = \beta(s)$. Then the corresponding solution of (33) is $x(t,s) = \lambda_0^t\beta(s)$ and clearly this solution is not bounded. When $|\lambda_0| = 1$, we can write $\lambda_0 = e^{\imath\hat{r}}$ for some real number $\hat{r}$. Then if $f(t,s) = \beta(s)e^{\imath\hat{r}t}$, $x(0,s) = 0$

$$x(t+1,s) = \sum_{k=0}^{t-1}(D^{t-k-1}\beta)(s)e^{\imath k\hat{r}} = te^{\imath\hat{r}(t-1)}\beta(s) \tag{38}$$

which is obviously not bounded and the proof is complete. □

Next we study controllability for these systems where this property is defined as follows.

**Definition 2.** *The system (1) is said to be controllable if $\exists$ an integer $t_0 \in \mathbb{Z}_+$ such that for any pair of functions $\alpha \in BC^\infty(E)$ and $\beta \in BC^\infty(E)$ there is an admissible control function $u_{\alpha,\beta}(t,\cdot) \in BC^\infty(W)$, $t = 0,\cdots,t^0-1$, such that*

$$x(t_0,s,\alpha,u_{\alpha,\,\beta}) = \beta(s)\ s \in \mathbb{R} \tag{39}$$

*where $x(t,s,\alpha,u_{\alpha,\,\beta})$ denotes the solution of (1) corresponding to the given $\alpha$ and $u_{\alpha,\,\beta}$.*

**Theorem 5.** *The system (1) is controllable if, and only if,*

$$rank\{B, \Upsilon(z)B, \cdots, \Upsilon^{n-1}(z)B\} = n,\ z \in [-\imath,\imath],\ (n = dimE) \tag{40}$$

**Proof.** Definition 2 states that controllability of (1) is equivalent to solvability of the following equation

$$D^{t_0}\alpha + \sum_{j=0}^{t_0-1} D^jBu_{t_0-j-1} = \beta \tag{41}$$

with respect to unknown functions $u_t = u(t,\cdot) \in BC^\infty(W)$, $t = 0,\cdots,t^0-1$, for some $t_0 \in \mathbb{Z}_+$ and arbitrary $\alpha$ and $\beta$. Also the mapping $\mathcal{F} : (u_0,\cdots,u_{t_0-1}) \to \sum_{j=0}^{t_0-1} D^jBu_{t_0-1-j}$ is a differential operator acting from the space $(BC^\infty(W))^{t_0}$ to the space $BC^\infty(E)$ whose representation in the ring of power series is $\mathcal{F}(z) = \{\Upsilon(z)^{t_0-1}B,\cdots,\Upsilon(z)B,B\}$. Application of Theorem 2 now yields that (41) is solvable if, and only if,

$$\text{rank}\mathcal{F}(z) = \text{dim}E,\ z \in [-\imath,\imath] \tag{42}$$

The Cayley-Hamilton theorem now shows that this last condition is equivalent to (40). □

Having characterized controllability, we can move on (as in the 1D linear time invariant case) to consider the pole (or spectrum) assignment problem and hence the existence of stabilizing control laws for systems described by (1) which, as one example, can be used to ensure $q$ stability closed loop. Let $P(n)$ be the set of all polynomials of degree $n$ of the form

$$p(z,\lambda) = \lambda^n + a_{n-1}(z)\lambda^{n-1} + \cdots + a_0(z) \tag{43}$$

where the coefficients $a_0(z),\cdots,a_{n-1}(z)$ are analytic functions in some domain $\Omega$ of the complex plane which contains the segment $[-\imath,\imath]$. Suppose also that the control function in (1) has the following feedback form

$$u(t,s) = \sum_{j\in\mathbb{Z}_+} F_j\frac{dx^j(t,s)}{ds^j} \tag{44}$$

Here $F_0,F_1,\cdots$ are linear operators from $E$ to $W$ such that the power series $\sum_{j\in\mathbb{Z}_+} F_jz^j$ converges in some domain which contains the unit disk in the complex plane where, for example, this condition holds if

$$\sum_{j\in\mathbb{Z}_+}(1+\eta)^j||F_j|| < \infty \tag{45}$$

where $\eta > 0$ is a real number.

Now we can formulate the pole assignment problem as: for each element $p(\lambda)$ from $P(n)$ choose the operators $F_0, F_1, \cdots$, defining the control law of (44) such that the polynomial

$$\det\left(\sum_{j\in\mathbb{Z}_+}(A_j + BF_j)z^j - \lambda I\right) = 0 \tag{46}$$

is equal to a prescribed, i.e. pre-specified, polynomial $p^*(z,\lambda)$ of the form (43), where, in particular, this polynomial can be chosen such that $q$ stability holds. By Theorem 1, (46) completely determines the spectrum of the differential operator defined by the right-hand side of (1) under the action of (44). Hence the pole assignment problem is equivalent to determining the conditions under which the spectrum of the closed loop system is precisely the prescribed one.

To solve this problem, we need the following preliminary result.

**Lemma 3.** *Let $\mathcal{X}(z)$ and $\eta(z)$ be analytic $n\times n$ and $n\times r$ matrices respectively in some domain $H$ containing the $l$ connected set $\Omega$ such that*

$$rank\{\eta(z), \mathcal{X}(z)\eta(z), \cdots, \mathcal{X}(z)^{n-1}\eta(z)\} = n \tag{47}$$

*holds $\forall\, z \in \Omega$. Then for any $r$ vector $\delta(z)$ analytic in $H$ which satisfies the condition*

$$\eta(z)\delta(z) \neq 0,\ z \in \Omega \tag{48}$$

*$\exists$ an $r\times n$ matrix $\psi(z)$ analytic in $H$ such that*

$$rank\{\delta_0(z), \mathcal{X}_0(z)\delta_0(z), \cdots, \mathcal{X}_0^{n-1}(z)\delta_0(z)\} = n \tag{49}$$

*holds $\forall\, z \in \Omega$, where $\delta_0(z) = \eta(z)\delta(z)$ and $\mathcal{X}_0(z) = \mathcal{X}(z) + \eta(z)\psi(z)$*

The proof of this result follows from slight modifications of that given in [4] and is hence omitted here.

**Theorem 6.** *The pole assignment problem for systems described by (1) has a solution if, and only if, the system is controllable.*

**Proof.** To prove necessity, first note that for each point $\omega \in [-\imath, \imath]$ and any collection $\lambda_1, \cdots, \lambda_n$ of $n$ complex numbers, it is easy to see that $\exists$ some element $p \in P(n)$ such that $p(\omega,\ \lambda_i) = 0, i = 1, 2, \cdots, n,$ holds. Since the pole assignment problem is solvable, we can choose the feedback control function (44) such that the polynomial (46) coincides with the given polynomial $p(z,\lambda)$. Hence it follows that the numbers $\lambda_i$, $i = 1, \cdots, n$, belong to the spectrum of the matrix $\Upsilon(\omega) + BF(\omega)$, i.e. by standard theory [18], the pair $\{\Upsilon(\omega), B\}$ is controllable.

To prove sufficiency, first assume that $r = 1$ and write $\det(\Upsilon(z) - \lambda I) = \lambda^n - (a_{n-1}(z)\lambda^{n-1} + a_{n-2}(z)\lambda^{n-2} + \cdots + a_0(z))$. Also let $X(z)$ denote the $n\times n$ matrix whose $j$ *th* column has the form $\Upsilon(z)^{n-j}B - (a_j(z)B + a_{j+1}(z)\Upsilon(z)B + \cdots + a_{n-1}(z)\Upsilon(z)^{n-j-1}B)$, $j = 1, \cdots, n$. Then since (1) is controllable $X(z) \neq 0$, $z \in [-\imath, \imath]$ and also the matrix $X^{-1}\Upsilon(z)X(z)$ and the vector $X^{-1}(z)B$ have the well known canonical forms

$$X^{-1}\Upsilon(z)X(z) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ a_0(z) & a_1(z) & a_2(z) & \cdots & a_{n-1}(z) \end{bmatrix},\ X^{-1}(z)B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \tag{50}$$

Now select an arbitrary element $g(z,\lambda) = \lambda^n - b_{n-1}(z)\lambda^{n-1} - b_{n-2}(z)\lambda^{n-2} - \cdots - b_0(z)$ from the set $P(n)$ and let $f(z)$ denote the row vector for which $f(z)X(z) = (b_0(z) - a_0(z), \cdots, b_{n-1}(z) - a_{n-1}(z))$. Then $\det(\Upsilon(z) + Bf(z) - \lambda I) = \det(X^{-1}\Upsilon(z)X(z) + X^{-1}f(z)X(z) - \lambda I) = g(z,\lambda)$ and $f(z) =$

$\sum_{j\in\mathbb{Z}_+} F_j z^j$, where $F_j : E \to \mathbb{C}$ are linear mappings. Hence the control law (44), where the $F_k$ are the coefficients of the power series expansion of $f(z)$ guarantees that the polynomial (46) coincides with $g(z,\lambda)$ and the theorem is proved for the case of $r = 1$.

In the case when $r \neq 1$, Lemma 3 can be used to reduce this case to that for $r = 1$. In particular, choose an element $b \in V$ such that $Bb \neq 0$. Since (40) holds it follows from Lemma 3 that $\exists$ an $r \times n$ matrix of the form $\psi(z) = \sum_{j\in\mathbb{Z}_+} \Phi_j z^j$ such that

$$\operatorname{rank}\{Bb,\ \hat{\Upsilon}(z)Bb, \cdots, \hat{\Upsilon}^{n-1}(z)Bb\} = n,\ (z \in [-\imath, \imath]) \tag{51}$$

where $\hat{\Upsilon}(z) = \Upsilon(z) + B\psi(z)$. Hence the system

$$x(t+1,s) = \sum_{j\in\mathbb{Z}_+} (A_j + B\Phi_j)\frac{d^j x(t,s)}{ds^j} + Bbv(t,s) \tag{52}$$

is controllable in the class of scalar functions $v(t,s) \in BC^\infty(\mathbb{R})$. This means that for each polynomial $p(n) \in P(n)$ $\exists$ a feedback control law

$$v(t,s) = \sum_{j\in\mathbb{Z}_+} g_j \frac{d^j x(t,s)}{ds^j} \tag{53}$$

such that the polynomial $p(n)$ coincides with the polynomial

$$\det(\hat{\Upsilon}(z) + Bb\sum_{j\in\mathbb{Z}_+} g_j z^j - \lambda I) = 0 \tag{54}$$

Hence the control law

$$u(t,s) = \sum_{j\in\mathbb{Z}_+} F_j \frac{d^j x(t,s)}{ds^j} \tag{55}$$

where $F_j = \Phi_j + bg_j$ shows that the given polynomial $p(n)$ satisfies (46), i.e. the pole assignment problem has a solution and the proof is complete. □

A particular case of the pole assignment problem is the $q$ stabilization problem.

**Definition 3.** *The system (1) is said to be $q$ stabilizable if there exists a feedback law of the form (44) such that the resulting closed loop system is $q$ stable.*

**Theorem 7.** *Suppose that the system (1) is controllable. Then (1) is $q$ stabilizable* $\forall\ q \in (0,1)$.

The proof here follows immediately from Theorem 6.

## 4 Systems with Differential Operator of Finite Degree

In this section we consider the very important case when the system (1) only has a finite number of non-zero operators $A_j$ and $E = V$, and, in particular, we consider systems of the form

$$x(t+1,s) = \sum_{j=0}^{N} A_j \frac{d^j x(t,s)}{ds^j} + Bu(t,s) \tag{56}$$

As such a system is a special case of (1), many of the results given in the previous section also generalize to this case but the stabilization problem is more complicated. In particular, it is not generally true that there exists a feedback law of the form

$$u(t,s) = \sum_{j=0}^{M} F_j \frac{d^j x(t,s)}{ds^j} \tag{57}$$

with $M \leq N$ which stabilizes (56). To confirm this fact, the following preliminary result is required.

**Lemma 4.** *Let*

$$p_n(z) = a_0^{(n)} + a_1^{(n)} z + \cdots + a_{m(n)}^{(n)} z^{m(n)},\ n = 0, 1, 2, \cdots,\ z \in \mathbb{C}, \tag{58}$$

*be a sequence of polynomials (here the coefficients $a_j^{(n)}$ and the exponent $m(n)$ depend on $n$, in general) such that for some compact set $\mathcal{M} \in \mathbb{C}$ which does not only consist of a finite number of points from $\mathbb{C}$, these polynomials are uniformly bounded on $\mathcal{M}$, i.e.*

$$|p_n(z)| \leq M,\ z \in \mathcal{M},\ n = 0, 1, 2, \cdots \textit{ for some } M > 0 \tag{59}$$

*Then if $\exists$ at least one point $v^0 \in \mathbb{C}$, $v^0 \notin \mathcal{M}$ such that*

$$\lim_{n \to \infty} \sup |p_n(v^0)| = \infty \tag{60}$$

*then*

$$\lim_{n \to \infty} m(n) = \infty \tag{61}$$

*i.e. the set of exponents $\{m(n) :\ n = 0, 1, 2, \cdots\}$ is unbounded.*

**Proof.** Suppose to the contrary that $\exists$ an integer $N > 0$ such that $m(n) < N$, and in this case also suppose that

$$p_n(z) = a_0^{(n)} + a_1^{(n)} z + \cdots + a_N^{(n)} z^N,\ n = 0, 1, 2, \cdots,\ z \in \mathbb{C} \tag{62}$$

Now choose $(N+1)$ points $z_0, z_1, \cdots, z_N$ from the set $\mathcal{M}$, such that $z_i \neq z_j$, $\forall\, i, j = 1, 2, \cdots, N$. Then

$$\begin{bmatrix} p_n(z_0) \\ p_n(z_1) \\ \vdots \\ p_n(z_N) \end{bmatrix} = Q_N \begin{bmatrix} a_0^{(N)} \\ a_1^{(N)} \\ \vdots \\ a_n^{(N)} \end{bmatrix} \tag{63}$$

where

$$Q_N = \begin{bmatrix} 1 & z_0 & z_0^2 & \cdots & z_0^N \\ 1 & z_1 & z_1^2 & \cdots & z_1^N \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & z_N & z_N^2 & \cdots & z_N^N \end{bmatrix} \tag{64}$$

and also $\det(Q_N) \neq 0$. It now follows from (59) and (63) that the coefficients $a_j^{(n)}$ are uniformly bounded, i.e. $\exists$ a positive constant $L$ such that $|a_j^{(n)}| \leq L$, $j = 1, 2, \cdots, N$, $n = 0, 1, 2, \cdots$. Hence

$$|p_n(v^0)| \leq L(1 + |v^0| + \cdots + |v^0|^N) \tag{65}$$

which contradicts (60) and the proof is complete. □

The following example shows that there can be cases when no feedback control law of the form (57) can stabilize processes described by (56).

**Example 2.** Consider the process described by

$$x(t+1, s) = A_1 x^{(1)}(t, s) + A_{0n} x(t, s) + B u(t, s) \tag{66}$$

where

$$A_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\ A_{0n} = \begin{bmatrix} 10 & a_n \\ 0 & 0 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{67}$$

and $a_n = \imath + (n+1)^{-1}$, $n = 1, 2, \cdots$.

Since

$$\det\{(A_1 z + A_{0n})B,\ B\} \neq 0,\ z \in [-\imath, \imath] \tag{68}$$

this example is controllable in the class $BC^{\infty}$ for each $n = 0, 1, 2, \cdots$. Now consider a feedback control law of the form

$$u(t,s) = F_0 x(t,s) + F_1 x^{(1)}(t,s) + \cdots + F_M x^{(M)}(t,s) \tag{69}$$

where $M$ is an integer, and $F_j$ is an operator mapping $E$ into itself and is, in general, a function of $n$. Suppose also that the closed loop system in this case, i.e.

$$x(t+1,s) = A_{0n} x(t,s) + A_1 \frac{d\,x(t,s)}{ds} + \sum_{j=0}^{M} BF_j \frac{d^j x(t,s)}{ds^j} \tag{70}$$

is stable and hence the solutions $(\lambda_1(z), \lambda_2(z))$ of

$$\det(A_{0n} + A_1 z + \sum_{j=0}^{M} BF_j z^j - \lambda I) = 0 \tag{71}$$

satisfy $|\lambda_j(z)| < 1,\ j = 1, 2,\ z \in [-\imath, \imath]$.

As a particular case of (66)-(69), we can take $F(z)$ to be of the form $F(z) = [\psi_{1n}(z), \psi_{2n}(z)]$, where $\psi_{1n}(z)$ and $\psi_{2n}(z)$ are polynomials of the form (58). Also

$$A_{0n} + A_1 z + BF(z) = \begin{bmatrix} 10 & z + a_n \\ \psi_{1n}(z) & \psi_{2n}(z) \end{bmatrix} \tag{72}$$

and hence

$$\begin{aligned} \lambda_1(z) + \lambda_2(z) &= 10 + \psi_{2n}(z) \\ \lambda_1(z)\lambda_2(z) &= 10\psi_{2n}(z) - \psi_{1n}(z)(z + a_n) \end{aligned} \tag{73}$$

This gives

$$\psi_{1n}(z) = \frac{10(\lambda_1(z) + \lambda_2(z)) - 100 - \lambda_1(z)\lambda_2(z)}{(z + a_n)} \tag{74}$$

Next, introduce $\mathcal{M} = \{z \in \mathbb{C} : |z| = 1\} \setminus \{z \in \mathbb{C} : |z + \imath| \leq \delta\}$, where $\delta$ is a positive number whose value does not exceed $\frac{1}{2}$. Then

$$\begin{aligned} |\psi_{1n}(z)| &\leq \frac{20 + 100 + 1}{|z + a_n|} \leq 121 \\ |\psi_{1n}(-\imath)| &\geq \frac{-20 + 100 - 1}{(\frac{1}{n+1})} \to \infty,\ n \to \infty \end{aligned} \tag{75}$$

Applying Lemma 4 now yields that $M$ of (69) is not bounded in this case. Hence we conclude that if $n$ is enough large then the system (66) is not stabilizable by feedback control action of the form

$$u(t,s) = F_0 x(t,s) + F_1 x^{(1)}(t,s) \tag{76}$$

If the system (66) is stabilizable by (69) then $M \to \infty$ as $n \to \infty$.

Noting the above example, there are at least two ways to stabilize the system (56). The first of them is to extend the structure of (56) by allowing growth in the order of the differential operator on the right-hand side of the system model until $M > N$ in (57). In the limit, this will lead to systems with infinite degree operator of the form (1).

The second option here is to relax the definition of stability. Of these, we consider the first here for which the following result can be established.

**Theorem 8.** *If the system (56) is controllable then for each $q \in (0,1)$ $\exists$ an integer $M \geq 0$ such that (56) is $q$ stabilizable by a feedback control law of the form (57).*

**Proof.** By Theorem 7, $\exists$ a stabilizing feedback control law of the form

$$u(t,s)=\sum_{j\in\mathbb{Z}_+}F_j\frac{dx^j(t,s)}{ds^j} \tag{77}$$

Hence the roots $\lambda_i(z)$ of

$$\det\Bigg(\sum_{j\in\mathbb{Z}_+}(A_j+BF_j)z^j-\lambda I\Bigg)=0 \tag{78}$$

satisfy

$$|\lambda_i(z)|<1,\ j=1,2,\cdots,n,\ z\in[-\imath,\imath] \tag{79}$$

Also it can be shown [20] that the spectrum of an operator is an upper semi-continuous function and hence the inequalities of (79) are also valid for small perturbations of the matrix $\hat{\Upsilon}(z)=\sum\limits_{j\in\mathbb{Z}_+}(A_j+BF_j)z^j$ and the power series $\sum\limits_{j\in\mathbb{Z}_+}F_jz^j$ is uniformly convergent $\forall\ z\in[-\imath,\imath]$. Hence the elements of the matrices $\hat{\Upsilon}(z)$ and $\hat{\Upsilon}_F(z)=\hat{\Upsilon}(z)+\sum\limits_{j=1}^{M}F_jz^j$ are infinitesimally close to each other $\forall\ z\in[-\imath,\imath]$ and for some integer $M$. This leads to the conclusion that $\exists$ an integer $M>0$ such that all eigenvalues of the matrix $\hat{\Upsilon}_F(z)$ lie in the interior of the unit disk $\forall\ z\in[-\imath,\imath]$. Hence for the given $F_j$ and this $M$ the feedback control law (57) stabilizes the system (56) and the proof is complete.

$\square$

# 5 Optimization Theory

The previous analysis has shown that infinitely differentiable functions provide an essential tool for the control related analysis of the class of 2D continuous-discrete linear systems considered here. In this section, we apply the methodology of entire functions, a sub-class of infinitely differentiable functions, to solve the linear optimization problem for these systems. The aim is to show that this problem can be reduced to the extremal problem in an appropriate Hilbert space of entire functions. Next we give some necessary definitions and basic results.

**Note:** Let $G$ be a finite dimensional vector space over the complex field $\mathbb{C}$. Then a complex function $f:\mathbb{C}\to G$ is an entire function of exponential type and finite degree $\sigma$ if $f$ is regular on $\mathbb{C}$ and for any $\epsilon>0$ $\exists$ a constant $M=M(\epsilon)$ such that $Me^{(\sigma-\epsilon)}|z_s|<||f(z)||<Me^{(\sigma+\epsilon)}|z_s|$ holds $\forall\ z\in\mathbb{C}$ and some $z_s\in\mathbb{C},\ z_s\to\infty,\ s\to\infty$.

Consider a complex function $f:\mathbb{C}\to E$, where $E$ is a finite dimensional normed linear space, which is an entire function of exponential type with finite degree $\sigma\leq\pi$ ( see, e. g. [21, 22, 23]). Also let $\mathcal{W}_E$ denote the set of entire functions of exponential type and finite degree $\sigma$, such that their restrictions to $\mathbb{R}$ are functions from the space $L_2(\mathbb{R},E)$. Then it is known [22] that $\mathcal{W}_E$ is a Hilbert space (often termed the generalized Wiener-Paley space in the Russian literature).

An inner product on $\mathcal{W}$ can be defined as $(f,g)_{\mathcal{W}}=\int\limits_{\mathbb{R}}Re(f(x),\overline{g(x)})_E\ dx$, where $(\cdot,\cdot)_E$ denotes an inner product in $E$ and the over-bar denotes the complex conjugate operation. Denote also by $\ell_2(\mathcal{W}_E)$ the Hilbert space of square summable sequences from $\mathcal{W}_E$ with the usual inner product $(\varphi,\psi)_{\ell_2(\mathcal{W})}=\sum_{i\in\mathbb{Z}_+}(\varphi_i,\psi_i)_{\mathcal{W}}$.

Consider again systems described by (1). Then the optimization problem we consider is to minimize over the solutions of this system the quadratic cost function

$$J(u)=\sum_{t\in\mathbb{Z}_+}\int\limits_{\mathbb{R}}\Big[(Qx(t,s),\overline{x(t,s)})_E+(Ru(t,s),\overline{u(t,s)})_W\Big]\ ds \tag{80}$$

Here $u:\mathbb{Z}_+\times\mathbb{C}\to W$ is the control input to be determined and $Q:E\to E$ and $R:W\to W$ are linear self-adjoint operators which are positive semi-definite and positive definite respectively, i.e. $Q\geq 0,\ R>0$.

Assume that the control variables belong to the admissible set $u \in \mathcal{U}$, where $\mathcal{U}$ is a given closed convex set in $\ell_2(\mathcal{W}_E)$. Note here that when $\mathcal{U} = \ell_2(\mathcal{W}_W)$, the optimal control problem to be solved has no constraints on the control inputs.

**Definition 4.** *For given functions $u(t,s)$, $\alpha(s)$, we say that a function $x \in \ell_2(\mathcal{W}_E)$ is a solution of the system (1) if $x$ satisfies (1) $\forall\, t \in \mathbb{Z}_+,\ s \in \mathbb{R}$. The control function $u : \mathbb{Z}_+ \times \mathbb{C} \to W$ is said to be admissible for (1) if $u \in \mathcal{U}$, and an admissible control $u^0$ is said to be optimal for (1) if $J(u^0) = \min\limits_{u \in \mathcal{U}} J(u)$.*

Now we have the following result which characterizes the optimal solution of the problem defined above.

**Theorem 9.** *Suppose that $\gamma = \sum\limits_{k \in \mathbb{Z}_+} \sigma^k ||A_k|| < 1$. Then the optimization problem (80) for (1) has a unique optimal solution in the space $\ell_2(\mathcal{W}_E)$ for any initial data $\alpha \in \mathcal{W}_E$.*

**Proof.** We first establish the existence solutions of (1) in the space $\ell_2(\mathcal{W}_E)$ for the given admissible data. Introduce first the linear differential operator $a : \mathcal{W}_E \to \mathcal{W}_E$ by the formula

$$(a\psi)(z) = \sum_{k \in \mathbb{Z}_+} A_k \frac{d^k \psi(z)}{dz^k},\ z \in \mathbb{C}$$

Then it is known [22] that if $\psi \in \mathcal{W}_E$ then $\frac{d\psi}{dz} \in \mathcal{W}_E$ and the inequality $||\frac{d\psi}{dz}||_{\mathcal{W}_E} \le \sigma ||\psi||_{\mathcal{W}_E}$ holds. Hence the operator $a$ is bounded and $||a|| < 1$ .

Next, note that the system (1) can be written as

$$\omega(t+1) = a\omega(t) + \psi(t),\ \omega(0) = \alpha,\ \omega(t) \in \mathcal{W}_E,\ t \in \mathbb{Z}_+ \tag{81}$$

where $\alpha$ is the given element from $\mathcal{W}_E$, $\psi(t) = \mathcal{B}u(t)$, $t \in \mathbb{Z}_+$, and the operator $\mathcal{B} : \mathcal{W}_W \to \mathcal{W}_E$ is defined in an obvious way as $(\mathcal{B}u)(t,z) = Bu(t,z)$, $z \in \mathbb{C}$, $t \in \mathbb{Z}_+$. Then the solution of (81) can be written as

$$\omega(t+1) = \mathcal{B}u(t) + a\mathcal{B}u(t-1) + \cdots + a^t \mathcal{B}u(0) + a^{t+1}\alpha,\ t \in \mathbb{Z}_+ \tag{82}$$

Now define the linear operator $\mathcal{L} : \ell_2(\mathcal{W}_W) \to \ell_2^0(\mathcal{W}_E)$ as

$$(\mathcal{L}f)(t+1) = \mathcal{B}f(t) + a\mathcal{B}f(t-1) + \cdots + a^t \mathcal{B}f(0),\ (\mathcal{L}f)(0) = 0,\ t \in \mathbb{Z}_+ \tag{83}$$

where $\ell_2^0(\mathcal{W}_E)$ denotes the space of sequences in $\ell_2(\mathcal{W}_E)$ with zero first element. Then it is easy to verify that in the case when $\gamma = \sum\limits_{k \in \mathbb{Z}_+} \sigma^k ||A_k|| < 1$, the operator $\mathcal{L}$ is bounded. Moreover, we have that (leaving out the details of some obvious intermediate manipulations)

$$\begin{aligned}
||\mathcal{L}f||_{\ell_2(\mathcal{W})} &= \left[\sum_{t \in \mathbb{Z}_+} ||(\mathcal{L}f)(t+1)||^2_{\mathcal{W}}\right]^{\frac{1}{2}} \\
&\le \Big(\sum_{t \in \mathbb{Z}_+} ||\mathcal{B}||^2 (||f(t)||^2 + \cdots + ||a(t)||^t ||f(0)||^2)\Big)^{\frac{1}{2}} \\
&\le ||f|| \frac{||\mathcal{B}||}{1 - ||a||}
\end{aligned} \tag{84}$$

Hence for a given $u \in \ell_2(\mathcal{W}_W)$, $\alpha \in \mathcal{W}_E$, the solution of (82) can be written in the form

$$\omega = \mathcal{L}u + \psi,\ \psi = (\alpha, a\alpha, a^2\alpha, \cdots) \tag{85}$$

where $\mathcal{L}$ is the bounded linear operator defined by (83). This proves that an admissible solution exists.

At this stage, it is convenient to represent the original problem as an extremal problem in the Hilbert space setting since such an approach enables us to prove the existence of optimal solutions. Consequently, define the bounded linear operators $\mathcal{Q} : \ell_2(\mathcal{W}_E) \to \ell_2(\mathcal{W}_E)$, $\mathcal{H} : \ell_2(\mathcal{W}_W) \to \ell_2(\mathcal{W}_W)$ as follows

$$(\mathcal{Q}x)(t,z) = Qx(t,z),\ (\mathcal{H}u)(t,z) = Ru(t,z),\ z \in \mathbb{C},\ t \in \mathbb{Z}_+ \tag{86}$$

Then using (85), the cost function (80) can be written in the form

$$\begin{aligned} J(u) &= Re(\mathcal{Q}(\mathcal{L}u+\psi),(\mathcal{L}u+\psi))_{\ell_2(\mathcal{W})} + Re(\mathcal{H}u,u)_{\ell_2(\mathcal{W})} \\ &= Re((\mathcal{H}+\mathcal{L}^*\mathcal{Q}\mathcal{L})u,u)_{\ell_2(\mathcal{W})} + 2Re(\mathcal{L}^*\mathcal{Q}\psi,u)_{\ell_2(\mathcal{W})} + Re(\mathcal{Q}\psi,\psi)_{\ell_2(\mathcal{W})} \end{aligned} \tag{87}$$

where (for the rest of this paper) $*$ denotes the adjoint operator. Since the operators $\mathcal{H}$ and $\mathcal{Q}$ are positive and nonnegative definite respectively, the Hermitian form

$$\mathcal{G}(u,v) = ((\mathcal{H}+\mathcal{L}^*\mathcal{Q}\mathcal{L})u,v)_{\ell_2(\mathcal{W})}, \tag{88}$$

is coercive on $\mathcal{U}$, i.e. the inequality $\mathcal{G}(u,v) \geq c||v||$ holds for any $v \in \mathcal{U}$, where $c > 0$ is a constant. Hence the cost function $J(u)$ can be written as

$$J(u) = Re\mathcal{G}(u,u) - 2L(u) + (\mathcal{Q}\psi,\psi)_{\ell_2(\mathcal{W})} \tag{89}$$

where $L(u) = -Re(\mathcal{L}^*\mathcal{Q}\psi,u)_{\ell_2(\mathcal{W})}$ is a linear form on $\mathcal{U}$. Also it is well known that in this case there is a unique element $u^0$ from the closed convex set $\mathcal{U}$ such that $J(u^0) = \inf_{u\in\mathcal{U}} J(u)$, and the proof is complete. □

It now follows from (89) that $u^0$ is optimal for $J(u)$ if, and only if,

$$Re(\mathcal{G}(u^0, v-u^0)) \geq L(v-u^0),\ \forall\, v \in \mathcal{U} \tag{90}$$

and this inequality is the basis for obtaining explicit optimality conditions.

**Corollary 1.** *Let $\mathcal{U} = \ell_2(\mathcal{W}_W)$, which corresponds to the case when there are no constraints on the control inputs. Then substituting $v = u^0 \pm \mu$, where $\mu$ is an arbitrary element from $\ell_2(\mathcal{W}_W)$, yields*

$$Re(\mathcal{G}(u^0,v)) = L(v),\ \forall\, v \in \mathcal{U} \tag{91}$$

**Corollary 2.** *Let $\mathcal{U}$ be a closed convex cone in $\ell_2(\mathcal{W}_W)$. Then (90) in this case is equivalent to the following conditions*

$$\begin{aligned} Re(\mathcal{G}(u^0,v)) &\geq L(v),\ \forall\, v \in \mathcal{U} \\ Re(\mathcal{G}(u^0,v^0)) &= L(u^0) \end{aligned} \tag{92}$$

The first condition in this last result follows immediately from (90) on replacing $v$ by $v+u^0$. Also if $v = 0$ in (90) then $Re(\mathcal{G}(u^0,v^0)) \leq L(u^0)$ and hence the second formula holds.

In what follows, we develop modified optimality conditions based on using so-called adjoint variables.

**Theorem 10.** *Consider the problem of minimizing the cost function (80) for systems described by (1) and suppose that $\gamma = \sum_{k\in\mathbb{Z}_+} \sigma^k ||A_k|| < 1$. Then the admissible control $u^0 \in \mathcal{U}$ is optimal if, and only if,*

$$Re\left[\sum_{t\in\mathbb{Z}_+} \int_{\mathbb{R}} \left((B^*y^0(t,s) + Ru^0(t,s)), \overline{(v(t,s)-u^0(t,s))}\right)_W ds\right] \leq 0 \tag{93}$$

*holds $\forall\, v \in \mathcal{U}$, where $y^0$ is the solution of the adjoint system*

$$y^0(t,s) = \sum_{k\in\mathbb{Z}_+} (-1)^k A_k^* \frac{d^k y^0(t+1,s)}{ds^k} + Qx^0(t+1,s),\ t \in \mathbb{Z}_+,\ s \in \mathbb{R} \tag{94}$$

**Proof.** The condition of (90) can be re-written in the form

$$Re\left[((\mathcal{H}+\mathcal{L}^*\mathcal{Q}\mathcal{L})u^0+\mathcal{L}^*\mathcal{Q}\psi),(v-u^0))_{\ell_2(\mathcal{W})}\right]\geq 0,\ \forall\, v\in\mathcal{U} \tag{95}$$

Using (85), the solution of (81) (and hence of (1)) can be written as $\omega^0=\mathcal{L}u^0+\psi$. Hence

$$Re\left[((\mathcal{H}+\mathcal{L}^*\mathcal{Q}\omega^0),(v-u^0))_{\ell_2(\mathcal{W})}\right]\geq 0,\ \forall\, v\in\mathcal{U} \tag{96}$$

Since we are considering Hilbert spaces, the corresponding conjugates are also Hilbert spaces, and it is easy to verify that the adjoint operator $\mathcal{L}^*$ of $\mathcal{L}$ is given by $\mathcal{L}^*\beta=\mathcal{B}^*\Lambda\beta$, where $\mathcal{B}^*:\ell_2(\mathcal{W}_E)\to\ell_2(\mathcal{W}_W)$ is the adjoint operator of $\mathcal{B}$, and $\Lambda:\ell_2^0(\mathcal{W}_E)\to\ell_2(\mathcal{W}_E)$ is given by

$$(\Lambda\beta)(t)=\beta(t+1)+a^*\beta(t+2)+(a^*)^2\beta(t+3)+\cdots,\ \beta(0)=0,\ t\in\mathbb{Z}_+ \tag{97}$$

and the adjoint operator $a^*:\mathcal{W}_E\to\mathcal{W}_E$ is given by

$$(a^*\psi)(z)=\sum_{k\in\mathbb{Z}_+}(-1)^kA_k^*\psi^{(k)}(z),\ \psi\in\mathcal{W}_E,\ z\in\mathbb{C} \tag{98}$$

To obtain this last expression, note that we are using functions from $\mathcal{W}_E$ which are entire and vanish at infinity, and therefore

$$\begin{aligned}(a\pi,\psi)_{\mathcal{W}_E}&=\sum_{k\in\mathbb{Z}_+}(A_k\pi^{(k)},\psi)_{\mathcal{W}_E}=\sum_{k\in\mathbb{Z}_+}(\pi^{(k)},A_k^*\psi)_{\mathcal{W}_E}=\sum_{k\in\mathbb{Z}_+}\int_{\mathbb{R}}(\pi^{(k)}(x),\overline{A_k^*\psi(x)})_E dx=\\&=\sum_{k\in\mathbb{Z}_+}\int_{\mathbb{R}}((-1)^k\pi(x),\overline{A_k^*\psi^{(k)}(x)})_E dx=\sum_{k\in\mathbb{Z}_+}(\pi,(-1)^kA_k^*\psi^{(k)})_{\mathcal{W}_E},\ \psi\in\mathcal{W}_E,\ \pi\in\mathcal{W}_E\end{aligned} \tag{99}$$

This establishes the required representation of the adjoint operator $a^*$ and it is easy to show that the condition $\gamma=\sum_{k\in\mathbb{Z}_+}\sigma^k||A_k||<1$ guarantees that $||a^*||<1$.

For the given control $u\in\mathcal{U}$, we define the element $y\in\ell_2(\mathcal{W}_E)$ as $y=\Lambda\mathcal{Q}\omega$, where $\omega$ is the solution of (85) corresponding to the control $u$. Hence using (97), the elements of $y$ satisfy

$$y(t)=a^*y(t+1)+\mathcal{Q}\omega(t+1),\ y(t)\in\mathcal{W}_E,\ t\in\mathbb{Z}_+ \tag{100}$$

Since $y\in\ell_2(\mathcal{W}_E)$, then $||y(t)||_E\to 0,\ t\to\infty$, and this last condition can be taken as a boundary condition for the system (100).

Note now that the system of equations (100) is the adjoint system for (81) and recall that we have already shown that both (1) and its equivalent system (81) are solvable for any right-hand side function in $\ell_2(\mathcal{W}_E)$ and initial data $\alpha\in\mathcal{W}_E$ when $\gamma=\sum_{k\in\mathbb{Z}_+}\sigma^k||A_k||<1$. Also it is straightforward to show that under these assumptions, the adjoint system (100) has a unique solution $y\in\ell_2(\mathcal{W}_E)$ for any right-hand side function that belongs to $\ell_2(\mathcal{W}_E)$ which is given by

$$y(t)=\sum_{s\in\mathbb{Z}_+}(a^*)^s\mathcal{Q}\omega(t+s+1),\ t\in\mathbb{Z}_+ \tag{101}$$

Also at $t,\ t+1,\ t+2,\ldots$ it follows from (100) that

$$\begin{aligned}y(t)&=a^*y(t+1)+\mathcal{Q}\omega(t+1),\\y(t+1)&=a^*y(t+2)+\mathcal{Q}\omega(t+2),\\y(t+2)&=a^*y(t+3)+\mathcal{Q}\omega(t+3),\\&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\end{aligned} \tag{102}$$

Now apply the operator $a^*$ to the first of these equations, the operator $(a^*)^2$ to the second and so on. Then summing the resulting equalities, and noting that $||a^*|| < 1$ and $\lim_{t\to\infty} ||y(t)||_{\mathcal{W}_E} = 0$, yields (101). Also, due to the representation (98) for $a^*$, (100) in the space $\mathcal{W}_E$ can be transformed to (94) as required. The inequality (93) follows directly from (96) and the definition of the inner product in $\ell_2(\mathcal{W}_E)$, which completes the proof. $\square$

**Remark 2.** *Note that the solutions $\omega$ and $y$ of the following equations*

$$\begin{aligned} \omega(t+1) =& a\omega(t) + f(t) \\ y(t) =& a^* y(t+1) + g(t),\ t \in \mathbb{Z}_+ \end{aligned} \tag{103}$$

*have the property that $(\omega(t+1), y(t))_{\mathcal{W}_E}$ for the homogeneous systems ($f = 0,\ g = 0$) is constant $\forall\, t \in \mathbb{Z}_+$. To show this, let $\tau$ be an arbitrary integer and then (after some routine manipulations)*

$$(\omega(\tau+1), y(\tau))_{\mathcal{W}_E} = (\omega(1), y(0))_{\mathcal{W}_E} + \sum_{s=1}^{\tau} \left[(\omega(s), g(s))_{\mathcal{W}_E} + (f(s), y(s))_{\mathcal{W}_E}\right] \tag{104}$$

*The result now follows immediately on setting $f = 0,\ g = 0$.*

The following result solves the quadratic optimization problem in the absence of constraints on the control variables and yields an optimal control vector which can be expressed as a linear function of the adjoint variables.

**Theorem 11.** *Consider the problem of minimizing the cost function (80) for systems described by (1). Also let $\mathcal{U} = \ell_2(\mathcal{W}_W)$ and $\gamma = \sum_{k\in\mathbb{Z}_+} \sigma^k ||A_k|| < 1$. Then if the pair $u^0$, $x^0$ is the optimal solution of the problem defined by (1) and (80), $\exists$ a unique solution $y \in \ell_2(\mathcal{W}_E)$ of the adjoint system*

$$y(t,s) = \sum_{k\in\mathbb{Z}_+} (-1)^k A_k^* \frac{d^k y(t+1,s)}{ds^k} + Qx^0(t+1,s),\ t \in \mathbb{Z}_+,\ s \in \mathbb{R} \tag{105}$$

*such that*

$$u^0(t,s) = -R^{-1}B^* y(t,s),\ t \in \mathbb{Z}_+,\ s \in \mathbb{R} \tag{106}$$

*Also the optimal trajectory $x^0(t,s)$ satisfies*

$$x^0(t+1,s) = \sum_{j\in\mathbb{Z}_+} A_j \frac{d^j x^0(t,s)}{ds^j} - BR^{-1}B^* y(t,s),\ x^0(0,s) = \alpha(s) \tag{107}$$

**Proof.** By Corollary 1, the optimal control $u^0$ for this problem satisfies

$$Re(\mathcal{G}(u^0, v)) = L(v),\ \forall\, v \in \ell_2(\mathcal{W}_W) \tag{108}$$

i.e.

$$Re\left[(\mathcal{H} + \mathcal{L}^*\mathcal{Q}\mathcal{L})u^0, v)_{\ell_2(\mathcal{W})}\right] = -Re\left[\mathcal{L}^*\mathcal{Q}\psi, v)_{\ell_2(\mathcal{W})}\right],\ \forall\, v \in \ell_2(\mathcal{W}_W) \tag{109}$$

where $\psi = (\alpha, a\alpha, a^2\alpha, \cdots)$. Next, it follows immediately that

$$u^0 = (\mathcal{H} + \mathcal{L}^*\mathcal{Q}\mathcal{L})^{-1}\mathcal{L}^*\mathcal{Q}\psi \tag{110}$$

where the inverse involved exists under the assumptions invoked. Also since $\omega^0 = \mathcal{L}u^0 + \psi$, then by (110) we have that $u^0 = -\mathcal{L}^*\mathcal{Q}\omega^0$ or $u^0 = -\mathcal{H}^{-1}\mathcal{B}^* y$, where the adjoint variable $y \in \ell_2(\mathcal{W}_E)$ is defined as in the proof of Theorem 9, i.e. as the solution of

$$y(t) = a^* y(t+1) + \mathcal{Q}\omega(t+1),\ y(t) \in \mathcal{W}_E,\ t \in \mathbb{Z}_+ \tag{111}$$

Using (98) for $a^*$ yields (105) as required. The solvability of (105) under the assumption that $\gamma = \sum_{k\in\mathbb{Z}_+} \sigma^k ||A_k|| < 1$, can be established as in Theorem 9. Finally, since $u^0 = -\mathcal{H}^{-1}\mathcal{B}y$, (106) follows immediately and the proof is complete.

$\square$

## 6 Conclusions

In this paper key elements of a control oriented systems theory for a class of 2D continuous-discrete linear systems of both theoretical and practical interest has been developed. The analysis is based on a general model setting of the form defined by (1). The first set of results relate to stability, controllability and stabilization, where it has been shown the existence of feedback stabilizing control laws is linked to controllability in a similar manner to the 1D linear system case.

In the final section of this paper, an optimization/optimal control theory has been developed. The optimization problem (minimization of the quadratic cost function) has been reduced to an extremal problem in an appropriate Hilbert space. This fact has been developed using the theory of entire functions, i.e. a sub-class of infinitely differentiable functions. Finally, the solution has been developed into a constructive form using adjoint variables.

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