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Nonlinear Output Feedback Control:
An Analysis of Performance and Robustness

by

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ABSTRACT

FACULTY OF ENGINEERING, SCIENCE AND MATHEMATICS

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By considering a non-singular performance cost functional, observer backstepping designs and adaptive observer backstepping designs are compared to high-gain observer designs for an output feedback system and a parametric output feedback system. For the output feedback system, if the initial error between the initial condition of the state and the initial condition of the observer is large, the high-gain observer design has better performance than the observer backstepping design. Whilst, for the parametric output feedback system, if the a-priori estimate for the bound of the uncertain parameter is conservative, the adaptive observer backstepping design has better performance than the high-gain observer design.

In the sense of gap metric robustness, by a backstepping procedure, a robust state feedback controller is developed for the nominal plant in strick-feedback form. For the closed-loop, the controller achieves gain-function stability, and stability if the initial condition is zero. By the gap metric robustness theory, the controller achieves robustness to plant perturbations which are small in gap sense. In this way, it is shown that for any perturbed plant the controller stabilizes the closed-loop in the presence of input and measurement disturbances if the gap metric distance between the nominal and perturbed plant is less than a computable constant.

For output feedback control, a nominal plant in output-feedback form is considered, and the observer backstepping procedure is amended to design a robust controller and an observer in the presence of input and measurement disturbances. The closed-loop is shown to be gain-function stable, and stable if the initial condition is zero. If the nonlinearities are only locally Lipschitz continuous, the results are only local to input and measurement disturbances; if the nonlinearities are globally Lipschitz continuous, then results are global to input and measurement disturbances. By gap metric robustness theory, if the initial condition is zero the controller is shown to be robust to plant perturbations in a gap metric sense. As an application, the theory is applied to a system with time delay, and it is shown that if the time delay is suitably small, the controller is able to achieve stability of the closed-loop.

To investigate the robustness of high-gain designs to loop disturbances and plant perturbations, a restricted class of nonlinear nominal plant in normal form are considered. An amended high-gain observer control design is shown to be robust to loop disturbances and has a non-zero plant perturbation margin, which is independent of the high-gain factor.

Contents

Nomenclature

Symbol	Description
x	state vector
$\hat{x}$	state estimate vector
$\tilde{x}$	error vector
x_1	state vector of plant
x_2	perturbed state vector
$\hat{x}_2$	observer vector
$x_0,\ x_1^0$	initial condition of a system
$\hat{x}_0,\ \hat{x}_2^0$	initial condition of an observer
$\tilde{x}_0,\ \tilde{x}^0$	initial error
y_1	measured state vector
y_2	perturbed measured state vector
y_0	measurement disturbance
u_0	input disturbance
θ	uncertain parameter
η	small positive number
$\mathcal{T}_\eta$	time set
k	vector
$A,\ B,\ C,\ K$	matrices
A^T	transpose of matrix A
H	high-gain vector
$\underline{\lambda}(A)$	minimum eigenvalue of matrix A
$\bar{\lambda}(A)$	maximum eigenvalue of matrix A
$\mathbf{T}$, $\mathbf{T}_i$	transformation
$\mathbb{R}$	real number set
$\mathbb{R}_+$	the interval $[0,\infty)$
$\Sigma,\ \Sigma_1$	systems or plants
$\Sigma(s),\ \Sigma$	transfer function (linear case)
$\eta(\Sigma)$	number of open right-half plane poles of $\Sigma(s)$
Ξ	controller
$[\Sigma,\Xi]$	system, controller interconnection
$P,\ P[\Sigma,\Xi]$	performance

$\mathcal{U},\ \mathcal{Y}$	signal spaces
$\mathcal{L}_\infty$	space of transfer functions essentially bounded on the imaginary axis
$\mathcal{H}_\infty$	space of transfer functions of stable linear, time-invariant, continuous time, systems
$\mathcal{RH}_\infty$	space of rational $\mathcal{H}_\infty$ functions
$\mathcal{K}_\infty$	space of continuous functions $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ which are strictly increasing, and satisfy $\gamma(0) = 0$ and $\gamma(r) \to \infty$ as $r \to \infty$
$\mathcal{U}_\Sigma$	Dom(Σ)
$\mathcal{U}_\Xi$	Dom(Ξ)
$\mathcal{W}$	$\mathcal{U}_\Sigma \times \mathcal{U}_\Xi$
$\mathcal{G}_\Sigma,\ \mathcal{M}$	graph of plant Σ
$\mathcal{G}_{\Sigma_1},\ \mathcal{M}_1$	graph of plant Σ_1
$\mathcal{G}_\Xi,\ \mathcal{N}$	graph of controller Ξ
$\mathcal{O}$	set of causal, bijective mappings from $\mathcal{M}$ to $\mathcal{M}_1$
$H_{\Sigma,\Xi}$	operator from $\mathcal{W}$ to $\mathcal{W} \times \mathcal{W}$
Π_i	natural projection onto the ith component of $\mathcal{W} \times \mathcal{W}$
$\Pi_{\mathcal{M}//\mathcal{N}},\ \Pi_{\mathcal{N}//\mathcal{M}}$	operators $\Pi_1 H_{\Sigma,\Xi}$ and $\Pi_2 H_{\Sigma,\Xi}$ from $\mathcal{W}$ to $\mathcal{W}$
$g[\Pi_{\mathcal{M}//\mathcal{N}}](\alpha)$	gain function
$\vec{\delta}(\Sigma, \Sigma_1)$	directed gap metric between plant Σ and Σ_1
$\delta(\Sigma, \Sigma_1)$	non-directed gap metric between plant Σ and Σ_1
$\lVert \cdot \rVert$	Euclidian norn
$\lVert \cdot \rVert_2,\ \lVert \cdot \rVert_{L^2(\Omega)}$	L^2 norm over Ω
$\lVert \cdot \rVert_\infty,\ \lVert \cdot \rVert_{L^\infty(\Omega)}$	essential bound over Ω

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Chapter 1

Introduction

Control theory and engineering is the study of techniques that allow humans to achieve a desired behaviour of a plant. To manipulate the behaviour of the plant, a controller is designed to realize this purpose. The connection of the plant and the controller is called a control system. To design a controller and put the controller into practice, a mathematical model which describes the physical plant must be built, which is called the nominal plant. So, in general, a nominal control system is in the form of mathematical equations. In the term of mathematics, the following paragraph characterizes the purpose of control.

> Generally speaking, the objective in a control system is to make some output, say y, behave in a desired way by manipulating some input, say u. The simplest objective might be to keep y small (or close to some equilibrium point)$-$a regulator problem$-$or to keep $y - r$ small for r, a reference or command signal, in some set$-$a servomechanism or servo problem.

John Doyle, Bruce Francis, Allen Tannenbaum [13]

A control system can be open-loop or closed-loop. In a open-loop control system, the controller is designed without using measurable information, whereas, in a closed-loop control system, the controller uses measurable information for feedback comparison, that is feedback control. The purpose of feedback is to reduce the effect of uncertainties in the system, such uncertainties are from uncertainties in the dynamics (i.e., the mismatch between the nominal and the real plant) or external disturbances. If all the states of a system are measurable and can be used for feedback, the control is referred to as state feedback, if only some of the states or a combination of some states is measurable for feedback, the control is referred to as output feedback. If the mathematical model for a plant is linear, the system is called linear system, otherwise, the system is termed nonlinear. Furthermore, a mathematical model only approximates the physical plant: uncertainties or plant perturbations arise from the mismatch. Loop disturbances also arise from the imprecise measurement of the output and the inaccurate implementation of the control input. A controller is required to be ‘robust’ to these perturbations and disturbances.

Control engineering is a very wide discipline, and has a long history. It has been developed with the advance of technology. Over the last forty years, with great industrial demand, the field of control has been greatly advanced and widely used. Nowadays, control systems play a crucial roles in many areas such as manufacturing, aerospace and transportation, and military weapon systems (see, for example, Hurray [63]). To solve increasing control problems, improve control performance and robustness, many new control principles and methods are being developed.

In the past two decades, many of control techniques have been developed for nonlinear systems using feedback control. Most of the results, however, assume full state feedback. Efforts to extend some of these results to output feedback have naturally included the idea of designing an observer to estimate the state of the system from its output, see, e.g., [52, 30, 60, 32].

In recent years, a number of techniques have been developed for controlling nonlinear systems using output feedback control. Among them, high-gain observer and observer backstepping are two classes of important designs. The first class of controllers are based on high-gain observers with saturated controls, see, e.g., [15, 31, 48, 32, 81, 82, 45, 46, 3, 6]. We refer to this class of control designs as *Khalil* designs. The second class of controllers are based on backstepping techniques, see, e.g., [50, 51, 69, 40, 41, 42, 59, 61, 60, 55], and we refer to this class of controllers as *KKK* designs.

Despite their status as two important design types, their performance theory and robustness to loop disturbances and plant perturbations, in most cases, are still open questions. In this thesis, first, we are interested in introducing performance measurement and comparing the two kinds of output feedback control designs analytically; second, we study the robustness of the two kinds of designs in the framework of gap metric. The thesis is divided into two parts. The first part is about performance comparison, see Xie and French [87, 89, 90]. The second part is robust backstepping and high-gain observer designs, see Xie and French [88].

1.1 Backstepping Designs

Backstepping design is being developed with the need to cope with the presence of unknown parameters and breaking matching condition in models.

In 1980s, researchers, e.g., Isidori [36] introduced differential-geometric theory of nonlinear feedback control to linearize nonlinear systems. Nonlinear control theory made great progress. But this class of designs require the matching condition for systems, and are restricted to the systems without unknown parameters. To cope with the unknown parameters motivated the study of adaptive control. In the early research, the matching condition was still required. Craig [12] first designed a robotic adaptive controller in 1988, but his design needed to measure joint accelerations, which was impractical. Afterwards, other researchers, e.g., Slotine and Li [75, 76], Middleton and Goodwin [62], and Ortega and Spong [64], designed controllers without this condition. Taylor, Kokotović, Marino and Kanellakopoulos [79] further developed the adaptive control designs, and got a general design idea. On the other hand, researchers, e.g., Kanellakopoulos, Kokotović and Marino [40] and Campion and Bastin [5, 11] tried to remove the matching condition, and generalized nonlinear adaptive control to systems which satisfy the extended matching condition.

To overcome the requirement of matching condition, many researchers made contributions, e.g., Tsinias [83], Byrnes and Isidori [10], Kokotović and Sussmann [51], and Saberi, Kokotović, and Sussmann [69]. Finally, Kanellakopoulos, Kokotović and Morse [42] gave the backstepping design scheme. Backstepping was a new recursive procedure. The assumption of matching condition was not required anymore.

Backstepping was first used in adaptive nonlinear control, and further developed into adaptive observer backstepping for output feedback control. Kanellakopoulos, Kokotović, and Marino [39] addressed the problem under restrictive structural and growth conditions on the nonlinearities. Afterwards, Kanellakopoulos, Kokotović, and Morse [43] removed the growth restrictions, but the output nonlinearities were not allowed to precede the control input. By developing a new adaptive scheme, Marino and Tomei [59, 60] achieved global boundedness and tracking of trajectories for systems in output feedback form. Praly and Jiang [65] solved the stabilization for a class of systems broader than the output-feedback form. Teel and Praly [80, 82] extended the result to the systems with uncertain nonlinearities.

Overparametrization was thought as a disadvantage of the backstepping. Jiang and Praly [38] partially reduced its overparametrization, and with tuning functions, Kristić [56, 54] removed the overparametrization. Recently, however, Beleznay and French [7] have shown that in some cases, the overparametrization can reduce control cost, and has an advantageous aspect.

An extensive discussion of the development of these ideas can be found in [55].

1.2 High-gain Observer Designs

In linear control theory, the separation principle is a very convenient design tool for design output feedback control. When a system is completely certain, the separation principle enables a designer to separate a output feedback design into two steps, namely, a state feedback controller design and an observer design. For bilinear systems with dissipative drift, Gauthier and Kupka [31] also developed a separation principle.

However, in the presence of unknown parameters in a system, a design through the separation principle cannot satisfy the control requirement, and may even result in instability. Moreover, when nonlinearity is present in a system, the controller and observer cannot in general be designed separately. Thus, for systems with uncertainty or nonlinearity, it is advantageous to develop conditions under which a similar separation principle can be utilized for designing output feedback controllers.

High-gain feedback is a classical tool for desensitization and stabilization of minimum-phase systems. From the middle of 1980s, Marino [58], Isidori and Krener [53], Isidori [36], Saberi and Sannuti [70], and Khalil [49] used the high-gain feedback to stabilize input-output linearizable systems. This comprises the early work on using high-gain observers to design output feedback schemes for nonlinear systems with uncertainty. In 1992, Esfandiari and Khalil [15] used the high-gain observer design to obtain the output feedback stabilization of fully linearizable systems. In this paper a theory for the design was developed, the peaking phenomenon of the design was studied, and recovery of the state feedback control was achieved by Tikhonov theorem, a separation principle for nonlinear systems was obtained by the high-gain observer. In 1993 Khalil and Esfandiari [48] generalized this design to the systems depending on uncertain parameters with non-zero dynamics. Other researchers such as Teel and Praly [81, 82] used this technique to achieve semi-global stabilization. In 1996 Khalil [45] developed this design idea to adaptive output feedback control of nonlinear systems.

Because of the innate peaking phenomenon of the high-gain observer design, the state feedback control is required to be globally bounded. In [15, 45], saturation was introduced to obtain a globally bounded control, overcoming the peaking phenomenon. Khalil [45] summarized these developments. Atassi and Khalil [3] greatly generalized the design to generic systems and the principal idea of this design. The design procedure is as follows. First, a globally bounded state feedback control (generally achieved by saturation) is designed to meet the design objective. Second, a high-gain observer, designed to be fast enough, recovers the performance achieved under state feedback. This is the so called separation principle for nonlinear systems.

An early separation principle developed by Teel and Praly [81] did not guarantee performance, it only guaranteed preservation of stability. Atassi and Khalil in [4] further developed other high-gain observers.

1.3 Performance of Backstepping and High-gain Designs

The performance theory in output feedback control is still an open field. For the adaptive state feedback control, in the past few years, French [25] initiated the work in the area of control comparison by performance. French, Szepesvári and Rogers [27] introduced a performance measure for approximate adaptive nonlinear control and obtained an upper bound of performance. As to the comparison of performance for controllers, French [26] introduced a cost functional to measure performance of control designs, comparing robust to adaptive backstepping. Sanei and French [71, 72] compared two robust adaptive control designs. Beleznay and French [7] compared the performances of adaptive backstepping and tuning functions designs.

For output feedback control, it is only possible to measure the output. Hence, the designs are more complicated, and it is harder to handle the resulting closed-loop systems. So, it is more difficult to develop the corresponding performance theory. Khalil [47] initiated work about comparison of controllers for output feedback. He used numerical simulation tools to compare output feedback control designs. French, Szepesvári and Rogers [28] first obtained the bounds for the performance of designs for output-feedback control analytically. It should be observed that whilst there are many results concerning the transient performance of the output, see, e.g., [55], there is little work in the literature on non-singular costs for non-optimal designs, see however [27, 71, 26, 7, 72] for related results and techniques.

The results in [47] are purely numerical, and give rise to many interesting questions, such as

- When do the *KKK* designs require greater control effort than the *Khalil* designs, and vice versa?
- When do the *Khalil* designs have superior output transients to the *KKK* designs, and vice versa?
- Are the *Khalil* and *KKK* designs sensitive to disturbances and plant perturbations?

In this thesis we will study these problems. In the first part, we will compare the *KKK* and *Khalil* designs in two situations; in the second part, we will design robust *KKK* and *Khalil* controllers.

The *Khalil* designs are applicable to affine systems of full relative degree, whilst the *KKK* designs are applicable to an alternative class of systems, namely those which possess an output feedback normal form. By considering systems which are both full relative degree and have a output feedback normal form, we can compare the behaviour of the controllers on common systems,[1] as initiated in [47].

We introduce the measure of performance

$$P(\Sigma, \Xi) = \|y\|^2_{L^2(\mathcal{T}_\eta)} + \|u\|_{L^\infty(\mathbb{R}_+)}$$

where y is the output, and u is the input, and the time set $\mathcal{T}_\eta$ is defined by

$$\mathcal{T}_\eta = \left\{ t \geq 0 \;\middle|\; |y(t)| > \eta \right\}$$

and η is a small positive number. By comparing the performance of controllers, we would like to be able to characterize situations in which one design is preferable to another. Such characterizations have obvious consequences for design choices, and also should lead to insight into the dynamics and trade-offs inherent in these controllers.

It is impossible to compare two designs generally. So, we will characterize situations in which one design is preferable to another.

The notion of stability in this part is the Lyapunov stability.

[1] Note also that such systems are characterized in a coordinate free manner, [55].

1.3.1 Poor Information on Initial Conditions

Firstly, we will consider the system which can be written in output-feedback form

$$\begin{aligned}\dot{x}_i &= x_{i+1} + \varphi_i(y), \quad 1 \le i \le n-1 \\ \dot{x}_n &= u + \varphi_n(y), \quad x_i(0) = x_{0i}, \quad 1 \le i \le n \\ y &= x_1\end{aligned}$$

where u is the control input, y is the measured output, and

$$x_0 = (x_{01}, \cdots, x_{0n})^T$$

is the initial condition of the state, and φ_i are sufficiently smooth.

Let us consider a generic observer based controller $\Xi(\hat{x}_0)$, where $\hat{x}_0$ is the initial condition for the observer. The performance of the closed-loop $[\Sigma(x_0), \Xi(\hat{x}_0)]$ is dependent on both the initial state x_0 and the initial condition for the observer $\hat{x}_0$. Whilst the initial state x_0 is the property of a system, the control designer has the freedom to chose the initial condition $\hat{x}_0$ for the observer.

It is intuitive that good performance results from initializing the observer state $\hat{x}_0$ to be close to the actual initial state x_0. Of course, in practice, the initial state is often unknown, so it can be hard to initialize in this manner. Nevertheless standard practice is to try to minimize

$$\|\tilde{x}_0\| = \|x_0 - \hat{x}_0\|$$

according to the best information available. However, we may well not possess complete information concerning the value of the initial condition of the state, that is we do not exactly know x_0, and hence we have to take $\hat{x}_0$ to be the best estimate to x_0. Then we are interested in studying the situation in which our estimate of x_0 is not accurate and $\|\tilde{x}_0\|$ is large, in particular how does poor information on x_0, (which causes 'bad' choices of $\hat{x}_0$), affect the performance of the controllers?

We first consider an observer backstepping design [55] which achieves global regulation of the output. Although the observer backstepping design has a global region of attraction (in $(x_0, \hat{x}_0)$), we will prove that the performance of the controller may become worse as the initial error $\|\tilde{x}_0\|$ becomes large for any fixed initial condition of the state vector x_0.

Next, by a suitable coordinate transformation the system can also be written as integrator chain with a matched nonlinearity.

$$\begin{aligned}\dot{z}_i &= z_{i+1}, \quad 1 \le i \le n-1 \\ \dot{z}_n &= u + \psi(z), \quad z_i(0) = z_{0i}, \quad 1 \le i \le n \\ y &= z_1\end{aligned}$$

where $\psi(z)$ is to be specified later, and

$$z_0 = (z_{01}, \cdots, z_{0n})^T$$

is the initial condition. For the system, the high-gain designs treated on [15, 48, 3], can be applied and semi-global regulation of the output can be achieved. For this system, if φ_i and its higher derivatives are globally bounded, through the high-gain observer, for fixed initial condition of the state z_0 and any initial condition of the observer $\hat{z}_0$, we can design a globally bounded controller, achieving bounded performance. That is, if the initial error

$$\|\tilde{z}_0\| = \|z_0 - \hat{z}_0\|$$

becomes large, this design still achieves bounded performance.

1.3.2 Poor Information for Unknown Parameter

Secondly, we will consider a system in output-feedback normal form with an uncertain parameter

$$\begin{aligned}
\dot{x}_i &= x_{i+1}, \quad 1 \le i \le n-1 \\
\dot{x}_n &= u + \theta\varphi(y), \quad x_i(0) = x_{0i}, \quad 1 \le i \le n \\
y &= x_1
\end{aligned}$$

where u is the control input, y is the measured output,

$$x_0 = (x_{01}, \cdots, x_{0n})^T$$

is the initial condition of the state, and $\varphi(y)$ is a locally Lipschitz continuous function.

This is a parametric output feedback system, for which both *KKK* and *Khalil* controllers can be designed to achieve regulation of the output and bounded performance.

To design a *Khalil* -type output feedback controller with a high-gain observer, we need first to design a globally bounded state feedback controller. Generally, this is achieved by saturation of the state feedback controller. But we need the saturated controller still to stabilize the system. For this purpose, we need to determine suitable saturation levels. However, the required saturation levels are typically dependent on θ, the unknown constant. Therefore, we have to first quantify an a-priori estimate for the magnitude of θ. Since θ is assumed to be unknown our knowledge of it is typically poor. Hence we have to estimate θ conservatively. But when our a-priori upper bound for $|\theta|$ is conservative, we will show that the performance of the *Khalil* design becomes poor.

For a *KKK* design, the performance is independent of any a-priori upper bound for $|\theta|$. Therefore, the performance keeps uniformly bounded as the a-priori upper bound for $|\theta|$ becomes conservative. Hence, for this system we will establish a result with the contrary performance relationship to that in above section.

1.4 Gap Metric Robust Designs

Next we consider the third problem-the robustness of backstepping and high-gain designs. The definition for stability in this part is robust stability.

To design a controller for a plant, a mathematical model (called the nominal plant) for the plant is necessary. But, in practice, the nominal mathematical model for the plant cannot completely describe the actual plant–there always exists a difference between the nominal plant and the ‘true’ plant. On the other hand, when we measure a signal, what we measured is not exactly the real signal, namely, there is a measurement disturbance. When we use the measured signals for feedback control, another disturbance, the input disturbance, is typically present.

A closed-loop could become unstable if a controller cannot tolerate these kinds of uncertainties. EI-Sakkary [14] gave an example that a small uncertainty changed the stability of the closed-loop, which is described as follows. For a single-input and single-output linear system represented by the transfer function

$$K(s) = \frac{2}{s-1}$$

the closed-loop is stabilized by unity feedback to give

$$\frac{1}{1+K(s)} = \frac{s-1}{s+1}$$

If K is perturbed to

$$K_1(s) = K(s) + \frac{\epsilon}{s-2}$$

the additive uncertainty $\epsilon/(s-2)$ results in a pole-zero pair close to the point $s = 2$, and makes $1/(1+K_1(s))$ unstable for small ϵ. Rohrs in [68] gave examples, where existing adaptive control designs became unstable in the presence of small plant perturbations, input and measurement disturbances. These examples show that modelled or unmodelled uncertainties in plants and loop systems are challenges to control designs, especially to nonlinear systems. For control purposes, a basic requirement is that a controller designed for the nominal plant tolerates plant perturbations, measurement disturbances and input disturbances, that is the controller is robust to these kinds of uncertainties. Hence, the study of robust control is an important area in control engineering.

Although the study of robustness for control designs is as old as feedback control, even for linear systems effective systematic tools for robust control have only been developed since 1980’s. An appropriate topological structure for studying the robustness of linear systems is the gap metric (the graph topology) introduced by Zames and EI-Sakkary [91, 14]. The gap metric between two linear systems is defined as the gap of their graphs, which originated from the notion of the distance between two sets (see [44]). The tolerable uncertainties are constrained in the gap. The theory of robustness for linear systems is then well established. Vidyasagar [84, 85] defined an alternative metric-the graph metric, which is topologically equivalent to the gap metric. In contrast, other frameworks for studying robustness have restrictions; e.g., if there exists an additive uncertainty it is impossible to compare a stable closed-loop with an unstable one, the order of parametric uncertainty cannot be changed, a small time delay is not an allowable uncertainty, etc. However, it is pertinent to observe that the gap or graph notion of distance corresponds naturally to the notion of coprime factor uncertainty.

For nonlinear systems, it had been a target to build up a corresponding gap metric theory. But, it is difficult to cope with the complexity of nonlinear phenomena even in the absence of disturbances and other uncertainties. The robustness study of nonlinear systems is far less developed than for linear systems. In 1997, in a fundamental paper [35], Georgiou and Smith established a theory of gap metric for nonlinear case, and a series of applicable robust stability theorems were obtained.

As we introduced previously, the backstepping (see [55]) is a well established constructive design procedure, which can be applied to models without the matching condition. But, ordinary backstepping designs do not guarantee robustness. In 1992 Freeman and Kokotović [19] initiated the study of robust backstepping designs. Marino and Tomei [61], Qu [66], Slotine and Hedreick [74] independently obtain robust backstepping results in 1993. In successive papers [20, 22], robust backstepping designs were developed. The established results were summarized in [23].

In the above work robust control Lyapunov functions were introduced as a design tool. Hence, the uncertainties allowed in plants are only modelled dynamics. Un-modelled dynamics or plant perturbations are not allowed. Another restriction is that the measurement disturbances are required to enter system equations multiplied by a class $\mathcal{K}_\infty$ function[2] of the state magnitude. That is, the measurement disturbances are in the set

$$Y(x) = x + \rho(x)B$$

where ρ is a class $\mathcal{K}_\infty$ function, and B is the closed unit ball. This means that the effects of measurement disturbances decrease to zero as the states are regulated to zero. But, in practice, actual measurement disturbances do not satisfy this assumption.

Recently, many researchers further developed robust backstepping designs on some restrict conditions. The results can be found in [24, 18, 1, 16, 2, 37]. The work of Freeman and Kokotović, and other researchers, is only concerned with state feedback control. So far, the area of robust backstepping designs for output feedback control is still open.

[2] A continuous function $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ is said to belong to $\mathcal{K}_\infty$ if it is strictly increasing, and $\gamma(0) = 0$, and $\gamma(r) \to \infty$ as $r \to \infty$.

We have mentioned before that the high-gain observer was alternative design. Similarly, the standard high-gain design does not guarantee the robustness. When a high-gain observer is used to output feedback, it is required that the high-gain factor $1/\epsilon$ is large. Consequently, it is believed that the high-gain observer designs are sensitive to loop disturbances and plant perturbations. But it is surprising that the simulation results in [47] show that a high-gain observer design exhibits almost the same level of degradation with the other designs in the presence of disturbances. So far, there are no results about the robustness of high-gain designs except the above simulation result. Therefore, it is important to investigate the robustness of high-gain designs to loop disturbances and plant perturbations.

In this thesis, in the framework of gap metric we will consider robust backstepping for state feedback and output feedback designs, and robust high-gain observer designs. Since standard backstepping and high-gain designs do not guarantee robustness, we amend the backstepping and high-gain designs to achieve the robustness of controllers to input and output disturbances. Then, we use gap metric robustness framework of [35] to obtain the robustness of the controller to plant perturbations.

The critical steps are designs of controllers and the construction of stable operators between the external disturbances and the internal signals of a closed-loop.

1.4.1 Robust Backstepping Designs

In the framework of gap metric robustness, we will study robust backstepping design procedures. The plant uncertainties can be modelled or unmodelled dynamics, that is plant perturbations are also included. There is no restriction on input and measurement disturbances. The results can even be global to disturbances. All the restrictions for plant uncertainties, input and measurement disturbances are removed.

State Feedback Control

For state feedback control, we will consider a nominal plant in strict-feedback form

$$\begin{aligned}\dot{x}_{1i} &= x_{1(i+1)} + \varphi_i(x_{11}, \cdots, x_{1i}), \quad 1 \le i \le n-1 \\ \dot{x}_{1n} &= u_1 + \varphi_n\left(x_{11}, \cdots, x_{1(n-1)}, x_{1n}\right), \quad x_{1i}(0) = x_{1i}^0, \quad 1 \le i \le n\end{aligned}$$

where we assume that φ_i, $1 \le i \le n$ satisfy

$$\varphi_i(0) = 0, \quad 1 \le i \le n$$

and are Lipschitz continuous, and

$$x_1^0 = (x_{11}^0, \cdots, x_{1n}^0)^T$$

is the initial condition.

Since ordinary backstepping designs do not guarantee robustness, the designs in [55] cannot be directly used to achieve our purpose. By an amended backstepping procedure, we design a robust controller for the nominal plant. The controller achieves gain-function stability, and if the initial condition is zero then the controller achieves stability, that is, the controller is robust to input and measurement disturbances of the closed-loop. Then we make use of the gap metric robustness results in [35] to obtain robustness of the closed-loop to plant perturbations which are small in some sense. In this way, we show that for any perturbed plants the controller stabilizes the closed-loop with input and measurement disturbances if the gap metric distance between the nominal and a perturbed plant is less than a computable constant (also see [88]).

Output Feedback Control

For output feedback control, we will consider a nominal plant in output-feedback form, in which nonlinearities only depend on the output

$$\begin{aligned} \dot{x}_{1i} &= x_{1(i+1)} + \varphi_i(y_1), \quad 1 \le i \le n-1 \\ \dot{x}_{1n} &= u_1 + \varphi_n(y_1), \quad x_{1i}(0) = x_{1i}^0, \quad 1 \le i \le n \\ y_1 &= x_{11} \end{aligned}$$

where $\varphi_i,\ i = 1, 2, \cdots, n$ are either locally or global Lipschitz continuous, and satisfy

$$\varphi_i(0) = 0, \quad 1 \le i \le n$$

and

$$x_1^0 = (x_{11}^0, \cdots, x_{1n}^0)^T$$

is the initial condition.

Again, we amend the backstepping method, design a controller and an observer in the presence of input and measurement disturbances, proving it's robustness. The closed-loop is gain-function stable, and stable if the initial condition is zero.

If the nonlinearities are only locally Lipschitz continuous, the results are local to input and measurement disturbances; if the nonlinearities are globally Lipschitz continuous, then the results are global to input and measurement disturbances.

By the robustness results in [35], if the initial condition is zero we obtain the robustness of the controller to plant perturbations in a gap metric sense. That is, for any perturbed plant the controller stabilizes the closed-loop with input and measurement disturbances if the gap metric distance between the nominal and perturbed plant is less than a computable constant.

A time delay in feedback control could destroy the stability of a closed-loop system. As an application, we apply the theory we have established to a system with time delay, and prove that if the time delay is suitably small, the controller is able to achieve stability of the closed-loop.

1.4.2 Robust High-gain Designs

For robust high-gain control, we will consider a nonlinear nominal plant in normal form

$$\begin{aligned}\dot{x}_{1i} &= x_{1(i+1)}, \quad 1 \leq i \leq n-1\\ \dot{x}_{1n} &= u_1 + \varphi(y_1), \quad x_{1i}(0) = x_{1i}^0, \quad 1 \leq i \leq n\\ y_1 &= x_{11}\end{aligned}$$

where, for simplicity, we assume φ is Lipschitz continuous.

Since the standard high-gain observer does not guarantee robustness to loop disturbances, we amend the high-gain observer and design a controller. Then we prove the controller is robust to loop disturbances.

By gap metric theory, we show that this controller is able to stabilize the closed-loop for a perturbed plant if the gap metric distance between the nominal plant and the perturbed plant is smaller than a constant, which is independent of the high-gain factor ϵ. That is the plant perturbation margin is independent of ϵ, hence the controller is robust to loop disturbances and plant perturbations.

1.5 Summary of Contents

- In Chapter 2, we outline the standard observer backstepping and high-gain observer design procedures. Then we summarize the major relevant results concerning the two kinds of designs, which we will quote latter.

- In Chapter 3, a non-singular cost functional for non-optimal output feedback designs is introduced to measure the performance of a controller. Then we prove a proposition about backstepping design for a two dimension system to illustrate that a good performance comes from a small initial error.

- In Chapter 4, we show that a *Khalil* design out-performs a *KKK* design when the information on initial state is poor and leads to a large initial observer error.

- In Chapter 5, we establish a result in the reverse direction to that of Chapter 4. We consider an output feedback system with an unknown parameter, and then show that an adaptive *KKK* design out-performs an adaptive *Khalil* design as the information on the size of the parameter becomes conservative.

- In Chapter 6, the required background knowledge on gap metric robustness is given.

- In Chapter 7, a robust state feedback backstepping controller is designed for strict-feedback form nominal plant, and it is proved that this controller is robust to loop disturbances and has a non-zero plant perturbation margin.

- In Chapter 8, for the output-feedback nominal plant, we design output feedback back-stepping controllers, and prove these controllers are robust to loop disturbances and have non-zero plant perturbation margins.
- In Chapter 9, by an amended high-gain observer design, a robust controller is constructed for the nominal plant in normal form. It is proved that this controller is robust to loop disturbances and has a non-zero plant perturbation margin.
- In Chapter 10, overall conclusions and directions for future research are given.

Part I

Performance Comparison

Chapter 2

Preliminaries

In this chapter we introduce the observer backstepping, adaptive observer backstepping and high-gain observer design procedures, and some related standard results which we will use later.

2.1 Observer Backstepping Design Procedure

We simply state the observer backstepping design procedure and some results about the design here, the related material can be found in [55].

The observer backstepping design can be applied to the output-feedback system, in which the nonlinearities only depend on the output

$$
\begin{aligned}
\dot{x}_1 &= x_2 + \varphi_1(y) \\
\dot{x}_2 &= x_3 + \varphi_2(y) \\
&\vdots \\
\dot{x}_{\rho-1} &= x_\rho + \varphi_{\rho-1}(y) \\
\dot{x}_\rho &= x_{\rho+1} + \varphi_\rho(y) + b_m\beta(y)u \\
&\vdots \\
\dot{x}_{n-1} &= x_n + \varphi_{n-1}(y) + b_1\beta(y)u \\
\dot{x}_n &= \varphi_n(y) + b_0\beta(y)u, \quad x_i(0) = x_{0i}, \quad 1 \leq i \leq n \\
y &= x_1
\end{aligned}
\tag{2.1}
$$

It is assumed that the system is minimum phase, that is, $b_m s^m + \cdots + b_1 s + b_0$ is a Hurwitz polynomial, and $\beta(y) \neq 0$ for any $y \in \mathbb{R}$.

To derive an observer for the system, we rewrite the system in the form

$$\begin{aligned} \dot{x} &= Ax + \varphi(y) + b\beta(y)u, \quad x(0) = x_0 \\ y &= Cx \end{aligned}$$

where

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \multicolumn{6}{c}{\dotfill} \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad x_0 = \begin{pmatrix} x_{01} \\ x_{02} \\ \vdots \\ x_{0n} \end{pmatrix}$$

and

$$\varphi(y) = \begin{pmatrix} \varphi_1(y) \\ \varphi_2(y) \\ \vdots \\ \varphi_n(y) \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ b_m \\ \vdots \\ b_0 \end{pmatrix}, \quad C = \begin{pmatrix} 1, 0, \cdots, 0 \end{pmatrix}$$

Let

$$\hat{x} = \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \vdots \\ \hat{x}_n \end{pmatrix}, \quad \hat{x}_0 = \begin{pmatrix} \hat{x}_{01} \\ \hat{x}_{02} \\ \vdots \\ \hat{x}_{0n} \end{pmatrix}$$

then an observer is defined by

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + K(y - \hat{y}) + \varphi(y) + b\beta u, \quad \hat{x}(0) = \hat{x}_0 \\ \hat{y} &= C\hat{x} \end{aligned}$$

where

$$K = \begin{pmatrix} k_1 \\ k_1 \\ \vdots \\ k_n \end{pmatrix}, \quad k_i > 0, \quad 1 \leq i \leq n$$

is chosen such that

$$A_0 = A - KC$$

is Hurwitz, and $\hat{x}_0$ is the initial condition of the observer.

Let

$$\tilde{x} = x - \hat{x}$$

be the observer error. Then $\tilde{x}$ satisfies

$$\dot{\tilde{x}} = A_0\tilde{x}, \quad \tilde{x}(0) = \tilde{x}_0$$

where

$$\tilde{x}_0 = x_0 - \hat{x}_0$$

is the initial error. Hence, the observer error decays exponentially.

Suppose that the tracking reference signal $y_r(t) \in C^\rho[0, \infty)$, and define the following recursive expressions

$$\begin{aligned}
\xi_1(y) &= y - y_r \\
\alpha_1(y) &= -c_1\xi_1 - d_1\xi_1 - \varphi_1(y) \\
\xi_i &= \hat{x}_i - \alpha_{i-1}\left(y, \hat{x}_1, \ldots, \hat{x}_{i-1}, y_r, \cdots, y_r^{(i-2)}\right) - y_r^{(i-1)} \\
\alpha_i &= -c_i\xi_i - \xi_{i-1} - d_i\left(\frac{\partial\alpha_{i-1}}{\partial y}\right)^2 \xi_i - k_i(y - \hat{x}_1) - \varphi_i(y) \\
&\quad + \frac{\partial\alpha_{i-1}}{\partial y}\left(\hat{x}_2 + \varphi_1(y)\right) + \sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial\hat{x}_j}\left(\hat{x}_{j+1} + k_j(y - \hat{x}_1) + \varphi_j(y)\right) \\
&\quad + \sum_{j=1}^{i-2}\frac{\partial\alpha_{i-1}}{\partial y_r^{(j)}} y_r^{(j+1)}, \quad i = 2, 3, \cdots, \rho
\end{aligned}$$

where c_i, d_i, $1 \le i \le n$ are positive constants.

The controller is then defined as

$$u = \frac{1}{b_m\beta(y)}\left(\alpha_\rho - \hat{x}_{\rho+1} - y_r^{(\rho)}\right) \tag{2.2}$$

For this controller, the following theorem (see [55]) holds.

Theorem 2.1. *For the system (2.1), suppose that $b_m s^m + \cdots + b_1 s + b_0$ is a Hurwitz polynomial, and $\beta(y) \neq 0$ for any $y \in \mathbb{R}$, and the reference signal $y_r(t) \in C^\rho[0, \infty)$. Then the controller (2.2) guarantees global boundedness of signal $x(t)$, $\hat{x}(t)$ and u for any initial condition x_0 and initial observer $\hat{x}_0$, furthermore, achieves regulation of the tracking error*

$$\lim_{t\to\infty}(y(t) - y_r(t)) = 0$$

Proof. The proof follows [55]. First, the resulting error system is

$$
\begin{aligned}
\dot{\xi}_1 &= -c_1\xi_1 + \xi_2 - d_1\xi_1 + \tilde{x}_2 \\
\dot{\xi}_i &= -c_i\xi_i + \xi_{i-1} + \xi_{i+1} - d_i\left(\frac{\partial\alpha_{i-1}}{\partial y}\right)^2\xi_i - \frac{\partial\alpha_{i-1}}{\partial y}\tilde{x}_2, \quad 2 \le i \le \rho - 1 \\
\dot{\xi}_\rho &= -c_\rho\xi_\rho + \xi_{\rho-1} - d_\rho\left(\frac{\partial\alpha_{\rho-1}}{\partial y}\right)^2\xi_\rho - \frac{\partial\alpha_{\rho-1}}{\partial y}\tilde{x}_2 \\
\dot{\tilde{x}} &= A_0\tilde{x}
\end{aligned}
$$

Let P_0 be the positive definite symmetric solution of the Lyapunov equation

$$P_0A_0 + A_0^TP_0 = -I$$

and define the Lyapunov function

$$V(\xi, \tilde{x}) = \sum_{j=1}^{\rho}\left(\frac{1}{2}\xi_j^2 + \frac{1}{d_j}\tilde{x}^TP_0\tilde{x}\right)$$

Then, along the solution of the closed-loop, it holds that

$$\dot{V} \le -\sum_{j=1}^{n}\left(c_j\xi_j^2 + \frac{3}{4d_j}|\tilde{x}|^2\right) \le 0$$

Thus, $\xi_1, \cdots, \xi_\rho$ are bounded, and the tracking error ξ_1 tends to zero as t goes to infinity.

The boundedness of other signals is established as follows. Since $\tilde{x}$ and y_r are bounded, y is bounded. Hence, $\hat{x}_1 = y - \tilde{x}_1$ is bounded. Since ξ_2 is bounded, $\hat{x}_2 = \xi_2 + \alpha_1(y, \hat{x}_1)$ is bounded. In the same manner, it can be shown that $\hat{x}_1, \cdots, \hat{x}_\rho$ are bounded. Note that the observer error $\tilde{x}$ satisfies $\dot{\tilde{x}} = A_0\tilde{x}$, and A_0 is Hurwitz, then $\tilde{x}$ exponentially decays to zero. Hence, $x_1, \cdots, x_\rho$ are bounded.

The boundedness of signals $x_{\rho+1}, \cdots, x_n$ and $\hat{x}_{\rho+1}, \cdots, \hat{x}_n$ comes from the fact that the boundedness of y implies the boundedness of ζ.

Finally, the control u is bounded because $b_m\beta(y)$ is bounded away from zero. □

2.2 Adaptive Backstepping Design Procedure

The adaptive observer backstepping design can be applied to the parametric output-feedback system

$$\begin{aligned}
\dot{x}_1 &= x_2 + \varphi_{0,1}(y) + \sum_{j=1}^{p} \theta_j \varphi_{j,1}(y) \\
\dot{x}_2 &= x_3 + \varphi_{0,2}(y) + \sum_{j=1}^{p} \theta_j \varphi_{j,2}(y) \\
&\vdots \\
\dot{x}_{\rho-1} &= x_\rho + \varphi_{0,\rho-1}(y) + \sum_{j=1}^{p} \theta_j \varphi_{j,\rho-1}(y) \\
\dot{x}_\rho &= x_{\rho+1} + \varphi_{0,\rho-1}(y) + \sum_{j=1}^{p} \theta_j \varphi_{j,\rho}(y) + b_m \beta(y) u \\
&\vdots \\
\dot{x}_n &= \varphi_{0,n}(y) + \sum_{j=1}^{p} \theta_j \varphi_{j,n}(y) + b_0 \beta(y) u, \quad x_i(0) = x_{0i}, \quad 1 \leq i \leq n \\
y &= x_1
\end{aligned} \tag{2.3}$$

where $\theta_1, \cdots, \theta_p$ and $b_0, \cdots, b_m$ are unknown constant parameters; the sign of b_m is known; the polynomial

$$b_m s^m + \cdots + b_1 s + b_0$$

is Hurwitz; $\beta(y) \neq 0$ for all $y \in \mathbb{R}$; only the output y is measured.

The control objective is to track a given reference signal $y_r(t)$ with the output y while keeping all signals bounded. Assume that reference signal $y_r(t)$ and its first ρ derivatives are known and bounded, and $y_r^{(\rho)}(t)$ is continuous.

First, rewrite the system in the form

$$\begin{aligned}
\dot{x} &= Ax + \varphi_0(y) + \sum_{j=1}^{p} \theta_j \varphi_j(y) + b\beta(y)u, \quad x(0) = x_0 \\
y &= Cx
\end{aligned}$$

where

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \multicolumn{6}{c}{\dotfill} \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad x_0 = \begin{pmatrix} x_{01} \\ x_{02} \\ \vdots \\ x_{0n} \end{pmatrix}$$

and

$$\varphi_j(y) = \begin{pmatrix} \varphi_{j,1}(y) \\ \varphi_{j,2}(y) \\ \vdots \\ \varphi_{j,n}(y) \end{pmatrix}, \ 0 \le j \le p, \quad b = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ b_m \\ \vdots \\ b_0 \end{pmatrix}, \quad C = \begin{pmatrix} 1, 0, \cdots, 0 \end{pmatrix}$$

Choose the vector

$$K = \begin{pmatrix} k_1 \\ k_1 \\ \vdots \\ k_n \end{pmatrix}, \quad k_i > 0, \quad 1 \le i \le n$$

such that

$$A_0 = A - KC$$

is Hurwitz, and define the filters

$$\begin{aligned} \dot{\xi}_0 &= A_0 \xi_0 + Ky, \quad \xi_0(0) = \xi_0^0 \\ \dot{\xi}_j &= A_0 \xi_j + \varphi_j(y), \quad \xi_j(0) = \xi_j^0, \quad 1 \le j \le p \\ \dot{v}_j &= A_0 v_j + e_{n-j} \beta(y) u, \quad v_j(0) = v_j^0, \quad 1 \le j \le m \end{aligned}$$

where e_i is the ith coordinate vector in $\mathbb{R}^n$.

The controller is defined as

$$
\begin{aligned}
u &= \frac{1}{\beta(y)}\left(\alpha_\rho - v_{m,\rho+1} + \vartheta_{1,1} y_r^{(\rho)}\right)\\
\dot{\vartheta}_1 &= \mathrm{sgn}(b_m)\Gamma\omega_1(y, \bar{\xi}^{(2)}, \bar{v}^{(2)}, y_r^{(1)} - \dot{y}_r e_1)\zeta_1\\
\dot{\vartheta}_2 &= \Gamma\left(\omega_2(y, \bar{\xi}^{(2)}, \bar{v}^{(2)}, \bar{\vartheta}^{(2)}, \bar{y}_r^{(1)}) + \zeta_1 e_{p+m+1}\right)\zeta_2\\
\dot{\vartheta}_i &= \Gamma\omega_i(y, \bar{\xi}^{(i)}, \bar{v}^{(i)}, \bar{\vartheta}^{(i-1)}, \bar{y}_r^{(i-1)}))\zeta_i, \quad i = 2, \cdots \rho\\
\hat{x}(0) &= \hat{x}_0 = \xi_0^0 + \sum_{j=1}^{p}\theta_j\xi_j^0 + \sum_{j=0}^{m} b_j v_j^0\\
\vartheta(0) &= \vartheta_0 = (\vartheta_{01}, \vartheta_{02}, \cdots, \vartheta_{0n})^T
\end{aligned}
\tag{2.4}
$$

where ζ_i, ω_i, α_i, $i = 1, \cdots, n$ and $\bar{y}_r$ are defined by the following recursive expressions

$$
\begin{aligned}
\zeta_1 =& y - y_r\\
\zeta_i =& v_{m,i} - \alpha_{i-1}(y, \bar{\xi}^{(i)}, \bar{v}^{(i)}, \bar{\vartheta}^{(i)}, \bar{y}_r^{(i-1)}) - \vartheta_{1,1} y_r^{(i)}\\
\alpha_1 =& -\vartheta_1^T\omega_1\\
\alpha_2 =& -c_2\zeta_2 - \vartheta_{2,p+m+1}\zeta_1 - d_2\left(\frac{\partial\alpha_1}{\partial y}\right)^2\zeta_2 + \frac{\partial\alpha_1}{\partial y}(\xi_{0,2} + \varphi_{0,1(y)}) - \vartheta_2^T\omega_2 + k_2 v_{m,1}\\
&+ \frac{\partial\alpha_1}{\partial\xi_0}(A_0\xi_0 + Ky + \varphi_0(y)) + \sum_{j=1}^{p}\frac{\partial\alpha_1}{\partial\xi_j}(A_0\xi_j + \varphi_j(y))\\
&+ \sum_{j=1}^{p}\frac{\partial\alpha_1}{\partial v_j}A_0 v_j + \left(\frac{\partial\alpha_1}{\partial\vartheta_j} + \dot{y}_r e_1^T\right)\mathrm{sgn}(b_m)\Gamma(\omega_1 - \dot{y}_r e_1)\zeta_1 + \frac{\partial\alpha_1}{\partial y_r}\dot{y}_r\\
\alpha_i =& -c_i\zeta_i - \zeta_{i-1} - d_i\left(\frac{\partial\alpha_{i-1}}{\partial y}\right)^2\zeta_i + \frac{\partial\alpha_{i-1}}{\partial y}(\xi_{0,2} + \varphi_{0,1(y)}) - \vartheta_i^T\omega_i + k_i v_{m,1}\\
&+ \frac{\partial\alpha_{i-1}}{\partial\xi_0}(A_0\xi_0 + Ky + \varphi_0(y)) + \sum_{j=1}^{p}\frac{\partial\alpha_{i-1}}{\partial\xi_j}(A_0\xi_j + \varphi_j(y))\\
&+ \sum_{j=1}^{p}\frac{\partial\alpha_{i-1}}{\partial v_j}A_0 v_j + \left(\frac{\partial\alpha_1}{\partial\vartheta_j} + \dot{y}_r e_1^T\right)\mathrm{sgn}(b_m)\Gamma(\omega_1 - \dot{y}_r e_1)\zeta_1\\
&+ \frac{\partial\alpha_{i-1}}{\partial\vartheta_2}\Gamma(\omega_2 + \zeta_1 e_{p+m+1})\zeta_2 + \sum_{j=3}^{i-1}\frac{\partial\alpha_{i-1}}{\partial\vartheta_j}\Gamma\omega_j\zeta_j\sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial y_r^{(j)}}y_r^{(j+1)}, \quad i = 3, \cdots, \rho
\end{aligned}
$$

and

$$
\begin{aligned}
\omega_1^T =& (c_1\zeta_1 + d_1\zeta_1 + \xi_{0,2} + \varphi_{0,1} + \zeta_{1,2}, \cdots, \varphi_{p,1} + \xi_{p,2}, \xi_{p,2}, v_{0,2}, \cdots, v_{m-1,2})\\
\omega_i^T =& -\frac{\partial\alpha_{i-1}}{\partial y}(\varphi_{1,1} + \xi_{1,2}, \cdots, \varphi_{p,1} + \xi_{p,2}, \cdots, v_{0,2}, \cdots, v_{m-1,2}, v_{m,2}), \quad i = 2, \cdots, \rho
\end{aligned}
$$

and

$$\begin{aligned}
\bar{\xi}^{(i)} &= (\xi_{0,1}, \cdots, \xi_{0,i}, \cdots, \xi_{p,1}, \cdots, \xi_{p,i}), \quad i = 1, \cdots, \rho - 1 \\
\bar{v}^{(i)} &= (v_{0,1}, \cdots, v_{0,i}, \cdots, v_{m-1,1}, \cdots, v_{m-1,i}, \cdots, v_{m,1} \cdots, v_{m,i}), \quad i = 1, \cdots n \\
\bar{\vartheta}^{(i)} &= (\vartheta_1^T, \cdots, \vartheta_i^T), \quad i = 1, \cdots \rho \\
\bar{y}_r^{(i)} &= (y_r, \dot{y}_r, \cdots, y_r^{(i)}), \quad i = 1, \cdots \rho
\end{aligned}$$

For this controller, the closed-loop has following property.

Theorem 2.2. *Consider the system (2.3) and the reference* $y_r(t) \in C^\rho[0, \infty)$. *Then, for any initial condition of the state* x_0 *and any initial observer state* $\hat{x}_0$, *the controller (2.4) guarantees the boundedness of all signals and regulation of the tracking error*

$$\lim_{t \to \infty} (y(t) - y_r(t)) = 0$$

Proof. The complete proof can be found in [55]. We only give an outline here.

First, define

$$\bar{\theta}_0 = \left(\frac{1}{b_m}, \frac{\theta_1}{b_m}, \cdots, \frac{\theta_n}{b_m} \right)^T$$

also define the estimate of $\bar{\theta}_0$ by ϑ_1, and

$$V_1 = \frac{1}{2}\zeta_1^2 + \frac{|b_m|}{2}(\bar{\theta}_0 - \vartheta_1)^T \Gamma^{-1} (\bar{\theta}_0 - \vartheta_1) + \frac{1}{d_1}\varepsilon^T P_0 \varepsilon$$

where P_0 is the positive definite solution of

$$P_0 A_0 + A_0^T P_0 = -I$$

and

$$\varepsilon =: x - (\xi_0 + \theta\xi_1 + v_0)$$

Then it can be shown that

$$\dot{V}_1 \leq -c_1\zeta_1^2 - \frac{3}{4d_1}\varepsilon^T \varepsilon + b_m \zeta_1 \zeta_2$$

Define

$$\bar{\theta} = (\theta_1, \cdots, \theta_p, b_0, \cdots, b_m)^T$$

and

$$V_2 = V_1 + \frac{1}{2}\zeta_2^2 + \frac{1}{2}(\bar{\theta} - \vartheta_2) T \Gamma^{-1} (\bar{\theta} - \vartheta_2) + \frac{1}{d_2}\varepsilon^T P_0 \varepsilon$$

Then

$$\dot{V}_2 \leq -c_1\zeta_1^2 - c_2\zeta_2^2 - \frac{3}{4}\left(\frac{1}{d_1} + \frac{1}{d_2} \right) \varepsilon^T \varepsilon + \zeta_2 \zeta_3$$

Finally, define

$$V_\rho = \sum_{i=1}^{\rho}\left(\frac{1}{d_2}\zeta_i + \frac{1}{d_i}\varepsilon^T P_0 \varepsilon\right) + \frac{|b_m|}{2}(\bar{\theta}_0 - \vartheta_1)T\Gamma^{-1}(\bar{\theta} - \vartheta_2) + \sum_{i=2}^{\rho}\frac{1}{d_2}(\bar{\theta} - \vartheta_i)T\Gamma^{-1}(\bar{\theta} - \vartheta_i)$$

Then it can be shown that

$$\dot{V}_\rho \leq -\sum_{i=1}^{\rho}\left(c_i\zeta_i^2 + \frac{3}{4d_i}\varepsilon^T\varepsilon\right)$$

Hence, the nonnegative function V_ρ is non-increasing, thus, $\zeta_1, \cdots, \zeta_\rho, \bar{\vartheta} - \vartheta_1, \cdots, \bar{\vartheta} - \vartheta_\rho$ are bounded, and thus $\vartheta_1, \cdots, \vartheta_\rho$ are bounded by constants depending only on the initial conditions of the adaptive system. From this it can be proven that the other signals are bounded.

The convergence of tracking error can be obtained by LaSalle-Yoshizawa theorem (see, e.g., [55]) since $\zeta_1, \cdots, \zeta_\rho$ and ε converge to zero as $t \to \infty$. □

2.3 High-gain Observer Design Procedure

The basic idea of high-gain observer designs is as follows (see, e.g., [45, 3, 30, 31, 32]). First, design a globally bounded state feedback controller, which is usually obtained by saturation. Second, a high-gain observer, which is defined through a high-gain factor ϵ, is designed to estimate the states. Third, replace the states by their observer variables and obtain an output feedback controller.

If ϵ is sufficiently small, the behavior of the closed-loop under the output feedback controller achieves the same properties of the closed-loop under the state feedback controller. Since the state feedback controller and the high-gain observer can be designed separately, this class of designs achieves a separation principle for nonlinear systems.

In this class of designs, the requirement of global boundedness of the state feedback controllers is essential. So, saturation is usually applied to achieve this property. However, the saturation levels must be high enough to guarantee stability of closed-loop under the state feedback control.

Here we give the design procedure of high-gain observer and related assumptions and theorems of the design in [45, 3].

Consider the system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ &\vdots \\ \dot{x}_{n-1} &= x_n \\ \dot{x}_n &= \psi(y,\theta) + u, \quad x_1(0) = x_{0i}, \quad 1 \le i \le n \\ y &= x_1 \end{aligned} \tag{2.5}$$

where u is the control input, y is the measured output, θ is the unknown parameter, the function ψ is sufficiently smooth and locally Lipschitz continuous in its arguments, in addition, $\psi(0,\theta) = 0$.

Introducing

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad x_0 = \begin{pmatrix} x_{01} \\ x_{02} \\ \vdots \\ x_{0n} \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

and

$$C = \begin{pmatrix} 1, 0, \cdots, 0 \end{pmatrix}$$

we rewrite the system into

$$\begin{aligned} \dot{x} &= Ax + B(\psi(x,\theta) + u), \quad x(0) = x_0 \\ y &= Cx \end{aligned}$$

The state estimate is generated by the high-gain observer

$$\dot{\hat{x}} = A\hat{x} + H(y - \hat{x}_1), \quad \hat{x}(0) = \hat{x}_0$$

where $\hat{x}_0$ is the initial condition of the observer, and

$$H = H(\epsilon) = \begin{pmatrix} \frac{\beta_1}{\epsilon} \\ \frac{\beta_2}{\epsilon^2} \\ \vdots \\ \frac{\beta_n}{\epsilon^n} \end{pmatrix}$$

and the positive constants β_i , $1 \le i \le n$, are chosen such that the roots of the equation

$$s^n + \beta_1 s^{n-1} + \cdots + \beta_{n-1}s + \beta_n = 0$$

are in the open left-half plane, and ϵ is a small positive constant to be specified.

Control Design 1

First we follow the separation principle in [3] to design an output feedback controller.

We suppose there is a state feedback control

$$\begin{aligned} u &= \Gamma(x, \hat{\vartheta}) \\ \dot{\hat{\vartheta}} &= \gamma(x, \hat{\vartheta}), \quad \hat{\vartheta}(0) = \hat{\vartheta}_0 \end{aligned}$$

and the assumptions made about the system and controller are as follows.

Assumption 2.1. Let Γ and γ satisfy

1. Γ and γ are local Lipschitz in their arguments over the domain of interest. $\Gamma(0,0) = 0$ and $\gamma(0,0) = 0$.
2. Γ and γ are globally bounded function of x.
3. The origin is an globally asymptotically stable equilibrium point of the closed-loop system.

The output feedback controller is given by

$$u = \Gamma(\hat{x}, \hat{\vartheta}) \tag{2.6a}$$

$$\dot{\hat{\vartheta}} = \gamma(\hat{x}, \hat{\vartheta}), \quad \hat{\vartheta}(0) = \hat{\vartheta}_0 \tag{2.6b}$$

Then for the closed-loop system, the following theorems hold, which are directly quoted from [3].

Theorem 2.3. *Suppose Assumption 2.1 is satisfied, then there exists ϵ_1^* such that, for every $\epsilon : 0 < \epsilon < \epsilon_1^*$, the trajectories of the closed-loop under the output feed back controller, starting in any compact set, are bounded for all $t > 0$.*

Theorem 2.4. *Suppose Assumption 2.1 is satisfied, then there exists ϵ_2^* such that, for every $\epsilon : 0 < \epsilon < \epsilon_2^*$, the origin of the closed-loop system under the output feedback controller is globally asymptotically stable.*

Control Design 2

Now we follow the design in [45] to design an output feedback controller.

For some constant θ_m, write

$$\Omega = \{|\theta| \leq \theta_m\}$$

and

$$\Omega_\delta = \{|\theta| \leq \theta_m + \delta\}$$

where $\hat{\Omega}$ is any compact set which satisfies $\hat{\Omega} \supseteq \Omega_\delta$.

Then for the system (2.5), a state feedback controller is given by

$$u = \mu(x, \hat{\theta}) = kx - \hat{\theta}\psi(y, \theta) \tag{2.7a}$$

$$\dot{\hat{\theta}} = \nu(x, \hat{\theta}) = \text{Proj}(\hat{\theta}, \phi), \quad \hat{\theta}(0) = \hat{\theta}_0 \tag{2.7b}$$

where the vector

$$k = (k_1, k_2, \cdots, k_n)$$

is chosen such that matrix $A + Bk$ is Hurwitz, and the projection Proj$(\hat{\theta}, \phi)$ is defined by

$$\text{Proj}(\hat{\theta}, \phi) = \begin{cases} \phi, & \text{if } |\hat{\theta}| \leq \theta_m \\ \phi - \frac{1}{\delta}(\hat{\theta} - \theta_m)\phi, & \text{otherwise} \end{cases}$$

and

$$\phi(x) = 2x^T P_1 B \psi(y)$$

The signals of the closed-loop under the state feedback controller are bounded. Take

$$U_0 \geq \max |\mu(x, \hat{\theta})|, \quad V_0 \geq \max |\nu(x, \hat{\theta})|$$

and we saturate the function μ and ν as follows

$$\mu_s(x, \hat{\theta}) = U_0 \text{sat}\left(\frac{\mu(x, \hat{\theta})}{U_0}\right)$$

$$\nu_s(x, \hat{\theta}) = V_0 \text{sat}\left(\frac{\nu(x, \hat{\theta})}{V_0}\right)$$

We again use the high-gain observer

$$\dot{\hat{x}} = A\hat{x} + H(y - \hat{x}_1), \quad \hat{x}(0) = \hat{x}_0$$

to estimate the states. Then we define an output feedback controller as

$$u = \mu_s(\hat{x}, \hat{\theta}) \tag{2.8a}$$

$$\dot{\hat{\theta}} = \nu_s(\hat{x}, \hat{\theta}), \quad \hat{\theta}(0) = \hat{\theta}_0 \tag{2.8b}$$

For the closed-loop, we have following theorem.

Theorem 2.5. *For the system (2.5) and any initial condition x_0, suppose that the output feedback controller is defined by (2.8). Then for $\theta_0 \in \Omega$ and $\hat{\theta}_0 \in \hat{\Omega}$, there exists $\epsilon^* > 0$ such that for all $0 < \epsilon < \epsilon^*$, all the signals of the closed-loop system are bounded, and mean square of the* *output is of order* $O(\epsilon)$

$$\lim_{T\to\infty}\frac{1}{T}\int_0^{\mathcal{T}} y(t)^2dt = O(\epsilon)$$

Proof. The system is of the form in [45]. The Assumptions 1 and 2 in [45] are naturally satisfied. The controller is that in [45]. Therefore, this is the result of the first part of Theorem 2 in [45]. □

In the next chapter, we will introduce a performance measurement and prove a result about the choice of initial condition of the observer.

Chapter 3

Performance and Initialization of Observer

In this chapter we consider a non-singular cost functional penalizing both the output transient and the control effort to measure the performance of a controller. Then we prove a result which illustrates that ‘good’ performance results from small initial observer error.

3.1 Performance of Controller

It should be observed that whilst there are many results concerning the transient performance of the output, see, e.g., [55], there is little work in the literature on non-singular costs for non-optimal designs, see however [26, 27, 7] and [29] for related results and techniques.

In particular, for a system Σ with input u and output y, and a controller Ξ mapping $y \mapsto u$, we consider the following cost which penalizes both the control and the output signal.

$$\begin{aligned} P(\Sigma, \Xi) &= \|y\|^2_{L^2(\mathcal{T}_\eta)} + \|u\|_{L^\infty(\mathbb{R}_+)} \\ &= \int_{L^2(\mathcal{T}_\eta)} y^2 dt + \sup_{t \in \mathbb{R}_+} |u(t)| \end{aligned}$$

where the time set $\mathcal{T}_\eta$ is defined by

$$\mathcal{T}_\eta = \left\{ t \geq 0 \;\middle|\; |y(t)| > \eta \right\}$$

and η is a small positive number.

Such a cost penalizes the input and output response of the system whilst $y(t) \notin [-\eta, \eta]$, hence for a closed-loop whose goal is to regulate y to zero, keeping y, u bounded, this cost is finite and is a reasonable penalty on the transient behavior.

Ever for a design which regulates y to zero, it is possible that $y \notin L^2(\mathbb{R}_+)$, for example, $y(t) = 1/\sqrt{t} \in L^2(\mathcal{T}_\eta)$, but $1/\sqrt{t} \notin L^2(\mathbb{R}_+)$. Therefore, in the definition of the performance, the $\|y\|_{L^2(\mathcal{T}_\eta)}$ norm is introduced, instead of the norm $\|y\|_{L^2(\mathbb{R}_+)}$, to guarantee the finiteness of the cost. For the control input, we are concerned with the maximum value of the control input. Hence, the norm $\|u\|_{L^\infty(\mathbb{R}_+)}$ is a proper measurement of this value.

Note that whilst a direct L^2 penalty on the output could be considered for some designs, the relaxation of the output penalty is physically meaningful, and considerably simplifies the technical treatment.

3.2 Initialization of Observer

Let us first consider a generic observer based controller $\Xi(\hat{x}_0)$, where $\hat{x}_0$ is the initial condition for the observer. The performance of the closed-loop $[\Sigma(x_0), \Xi(\hat{x}_0)]$ is dependent on both the initial state x_0 and the initial condition for the observer $\hat{x}_0$. Whilst the initial state x_0 is the property of a system, the control designer has the freedom to choose the initial condition $\hat{x}_0$ for the observer.

It is intuitive that good performance results from initializing the observer state $\hat{x}_0$ close to the actual initial state x_0. Of course, in practice, the initial state is often unknown, so it can be hard to initialize in this manner. Nevertheless standard practice is to try to minimize the initial error

$$\|\tilde{x}_0\| = \|x_0 - \hat{x}_0\|$$

according to the best information available. To establish a rigorous justification for this intuitive idea (or more precisely: to characterize the situations when it is valid) remains an open research problem; here we simply illustrate the validity of this approach on a single example, as discussed next.

Consider the 2-state system defined by

$$\begin{aligned} \Sigma^0(x_0): \quad \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \varphi(y) + u, \quad x(0) = (x_{01}, x_{02})^T \\ y &= x_1 \end{aligned}$$

where $\varphi(y)$ is a Lipschitz continuous function. We consider a *KKK* controller (see Chapter 2) defined as follows.

$$\begin{aligned}\Xi_O^0(\hat{x}_0):\ u &= \alpha_2(y,\hat{x}_1,\hat{x}_2)\\ \dot{\hat{x}}_1 &= \hat{x}_2 + k_1(y-\hat{x}_1)\\ \dot{\hat{x}}_2 &= k_2(y-\hat{x}_1)+\varphi(y)+u, \quad \hat{x}(0)=(\hat{x}_{01},\hat{x}_{02})^T\end{aligned}$$

where

$$\begin{aligned}\xi_1(y) =& y\\ \alpha_1(y) =& -c_1\xi_1 - d_1\xi_1\\ \xi_2(y,\hat{x}_1,\hat{x}_2) =& \hat{x}_2 - \alpha_1(y,\hat{x}_1)\\ \alpha_2(y,\hat{x}_1,\hat{x}_2) =& -c_2\xi_2 - \xi_1 - d_2\left(\frac{\partial\alpha_1}{\partial y}\right)^2\xi_2 - k_2(y-\hat{x}_1) - \varphi(y) + \frac{\partial\alpha_1}{\partial y}\hat{x}_2\end{aligned}$$

and k_1, k_2, c_i, d_i, $1 \le j \le 2$ are positive constants.

Since we can measure x_1, we can always take $\hat{x}_{01} = x_{01}$. However, x_{02} may be unknown, and so it is meaningful to compare the behaviour of the closed-loop s with the alternative choices of

$$\hat{x}_{02} = x_{02}, \quad \hat{x}_{02} = 0$$

We can then establish the following proposition.

Proposition 3.1. *Consider the system* $\Sigma^0(x_0)$ *and the controller* $\Xi^0(\hat{x}_0)$*, then there exist* $c_i, d_i, k_i,\ i = 1,2$ *such that*

$$\lim_{x_{02}\to+\infty}\left(P(\Sigma^0(x_0),\Xi_O^0((x_{01},0)^T)) - P(\Sigma^0(x_0),\Xi_O^0((x_{01},x_{02})^T))\right) = +\infty$$

Proof. Consider the closed-loop $(\Sigma^0(x_0),\Xi_O^0(x_0))$. First, observe that the observation error

$$\tilde{x} = x - \hat{x}$$

satisfies

$$\dot{\tilde{x}}_1 = -k_1\tilde{x}_1 + \tilde{x}_2 \tag{3.1a}$$

$$\dot{\tilde{x}}_2 = -k_2\tilde{x}_1, \quad \tilde{x}(0) = \tilde{x}_0 \tag{3.1b}$$

hence satisfies the equation

$$\ddot{\tilde{x}}_1 + k_1\dot{\tilde{x}}_1 + k_2\tilde{x}_1 = 0 \tag{3.2a}$$

$$\tilde{x}_1(0) = x_{01} - \hat{x}_{01} \tag{3.2b}$$

$$\dot{\tilde{x}}_1(0) = x_{02} - \hat{x}_{02} - k_1(x_{01} - \hat{x}_{01}) \tag{3.2c}$$

where

$$\tilde{x}_0 = x_0 - \hat{x}_0 = \begin{pmatrix} x_{01} - \hat{x}_{01} \\ x_{02} - \hat{x}_{02} \end{pmatrix}$$

Secondly, note that the control signal u can be expressed as

$$\Xi^0_O(\hat{x}_0): \; u = \alpha_2(y, \hat{x}_1, \hat{x}_2) = k_2\hat{x}_1 - h_2\hat{x}_2 - hy - \varphi(y) \tag{3.3}$$

where

$$h = \left(c_2 + d_2(c_1 + d_1)^2\right)(c_1 + d_1) + k_2 + 1$$

$$h_2 = c_2 + d_2(c_1 + d_1)^2 + c_1 + d_1$$

Hence, the closed-loop system can be written as

$$\dot{x}_1 = x_2 \tag{3.4a}$$

$$\dot{x}_2 = -hx_1 + k_2\hat{x}_1 - h_2\hat{x}_2 \tag{3.4b}$$

$$\dot{\hat{x}}_1 = k_1x_1 - k_1\hat{x}_1 + \hat{x}_2 \tag{3.4c}$$

$$\dot{\hat{x}}_2 = -h_1x_1 - h_2\hat{x}_2 \tag{3.4d}$$

where

$$h_1 = h - k_2 = \left(c_2 + d_2(c_1 + d_1)^2\right)(c_1 + d_1) + 1$$

Consider the first situation $\hat{x}_0 = x_0$, namely, $\tilde{x}_0 = 0$. The solution of (3.1) is $\tilde{x} = 0$, so $\hat{x}(t) \equiv x(t)$, and the closed system (3.4) reduces to

$$\dot{x}_1 = x_2 \tag{3.5a}$$

$$\dot{x}_2 = -h_1x_1 - h_2x_2 \tag{3.5b}$$

Thus we have

$$\ddot{x}_1 + h_2\dot{x}_1 + h_1 x_1 = 0 \tag{3.6a}$$

$$x_1(0) = x_{01}, \quad \dot{x}_1(0) = x_{02} \tag{3.6b}$$

Write the solution of above equation as $x_1^0(t)$, and observe that $x_1^0(t)$ can be expressed as

$$x_1^0(t) = x_{01}q_1(t) + x_{02}q_2(t)$$

where q_1, q_2 are functions which are independent of x_{01}, x_{02}. Moreover, we can choose[1] $c_i, d_i,\ i = 1, 2$ such that $q_2(t) > 0$ for $t > 0$, and further [2] $x_1^0(t) > 0$ for $t > 0$ if $x_{02} > 0$.

Now consider the second situation $\hat{x}_{01} = x_{01}$ and $\hat{x}_{02} = 0$, namely, $\tilde{x}_{01} = 0$ and $\tilde{x}_{02} = x_{02}$. Hence, the problem (3.2) becomes

$$\ddot{\tilde{x}}_1 + k_1\dot{\tilde{x}}_1 + k_2\tilde{x}_1 = 0 \tag{3.7a}$$

$$\tilde{x}_1(0) = 0, \quad \dot{\tilde{x}}_1(0) = x_{02} \tag{3.7b}$$

The solution to the above problem can be written as

$$\tilde{x}_1 = x_{02}f_1(t) \tag{3.8}$$

where $f_1(t)$ is a continuous function which is independent of x_{02}. At the same time, $\tilde{x}_2$ can also be written as

$$\tilde{x}_2 = x_{02}f_2(t) \tag{3.9}$$

where $f_2(t)$ is a continuous function which is independent of x_{02}.

[1]For example, we can choose $c_i, d_i,\ i = 1, 2$ such that $h_2^2 > 4h_1$, and let

$$\lambda_1 = -\frac{1}{2}\left(h_2 - \sqrt{h_2^2 - 4h_1}\right), \quad \lambda_2 = -\frac{1}{2}\left(h_2 + \sqrt{h_2^2 - 4h_1}\right)$$

then $q_1(t)$ and $q_2(t)$ can be written as

$$q_1(t) = \frac{1}{\lambda_1 - \lambda_2}\left(\lambda_1 e^{\lambda_2 t} - \lambda_2 e^{\lambda_1 t}\right), \quad q_2(t) = \frac{1}{\lambda_1 - \lambda_2}\left(e^{\lambda_1 t} - e^{\lambda_2 t}\right) > 0, \quad t > 0$$

[2]Here, $x_1^0(t)$ can also be written as

$$x_1^0(t) = \frac{1}{\lambda_1 - \lambda_2}\left(\left(x_{02} - \lambda_1 x_{01}\right)e^{\lambda_1 t} - \left(x_{02} - \lambda_2 x_{01}\right)e^{\lambda_2 t}\right)$$

It can verify that if $x_{02} > 0$ then $\dot{x}_1^0(t) > 0$ for $t > 0$. Note that $x_1^0(0) = 0$, then $x_1^0(t) > 0$ for $t > 0$.

Now substitute $\hat{x} = x - \tilde{x}$ into the closed-loop (3.4), and rewrite the first two equations as

$$\dot{x}_1 = x_2 \tag{3.10a}$$

$$\dot{x}_2 = -h_1 x_1 - h_2 x_2 - k_2 \tilde{x}_1 + h_2 \tilde{x}_2 \tag{3.10b}$$

Alternatively this can be expressed as

$$\ddot{x}_1 + h_2 \dot{x}_1 + h_1 x_1 = x_{02} f(t) \tag{3.11}$$

where

$$f(t) = -k_2 f_1(t) + h_2 f_2(t)$$

is also independent of x_{02}. Again we can choose k_1, k_2 such that[3] $f(t) > 0$.

Solving the following problem

$$\ddot{x}_1 + h_2 \dot{x}_1 + h_1 x_1 = x_{02} f(t) \tag{3.12a}$$

$$x_1(0) = x_{01}, \quad \dot{x}_1(0) = x_{02} \tag{3.12b}$$

we can express the solution of (3.12) as

$$x_1(t) = x_1^0(t) + \int_0^t \phi(t-\tau) x_{02} f(\tau) d\tau$$

where $\phi(t)$ is the solution of (3.6a) which satisfies $\phi(0) = 0$ and $\dot{\phi}(0) = 1$, namely $\phi(t) = q_2(t)$. Write

$$g(t) = \int_0^t q_2(t-\tau) f(\tau) d\tau$$

then

$$x_1(t) = x_1^0(t) + x_{02} g(t)$$

where $g(t) > 0$.

[3]E.g., we can choose $k_2^2 > 4k_2$ and $h_2 k_1 > 2k_2$, and let

$$\mu_1 = -\frac{1}{2}(k_1 - \sqrt{k_1^2 - 4k_2}), \quad \mu_2 = -\frac{1}{2}(k_1 + \sqrt{k_1^2 - 4k_2})$$

then we get

$$f_1(t) = \frac{1}{\mu_1 - \mu_2}\left(e^{\mu_1 t} - e^{\mu_2 t}\right), \quad f_2(t) = \frac{1}{\mu_1 - \mu_2}\left(-\mu_2 e^{\mu_1 t} + \mu_1 e^{\mu_2 t}\right)$$

$$f(t) = \frac{1}{\mu_1 - \mu_2}\left((-h_2\mu_2 - k_2)e^{\mu_1 t} + (h_2\mu_1 + k_2)e^{\mu_2 t}\right) > 0$$

.

Writing

$$\mathcal{T}_\eta = \{t \geq 0 \big| |x_1(t)| > \eta\}$$

$$\mathcal{T}_\eta^0 = \{t \geq 0 \big| |x_1^0(t)| > \eta\}$$

then $\mathcal{T}_\eta^0 \subset \mathcal{T}_\eta$ since $x_1(t) > x_1^0(t)$. Hence,

$$\begin{aligned} \|x_1\|^2_{L^2(\mathcal{T}_\eta)} - \|x_1^0\|^2_{L^2(\mathcal{T}_\eta^0)} \geq & \int_{\mathcal{T}_\eta^0} \left((x_1(t))^2 - (x_1^0(t))^2\right) dt \\ = & x_{02}^2 a + x_{02} b \end{aligned} \tag{3.13}$$

where

$$a = \int_{\mathcal{T}_\eta^0} \left(g(t)^2 + 2q_2(t)g(t)\right) dt$$

is a positive constant since $g(t), q_2(t) > 0$, and

$$b = \int_{\mathcal{T}_\eta^0} 2x_{01}q_1(t)dt$$

is a constant which is independent of x_{02}.

Write the control input of controller $\Xi_O^0(x_{01}, x_{02})$ as u^0, and the control input of controller $\Xi_O^0(x_{01}, 0)$ as u^1. Then by a calculation, we can obtain

$$\|u^0\| \leq x_{02}a_0 + b_0 \tag{3.14a}$$

$$\|u^1\| \leq x_{02}a_1 + b_1 \tag{3.14b}$$

since φ is Lipschitz continuous, where $a_i, b_i,\ i = 1, 2$, are positive constants which are independent of x_{02}. Therefore, from (3.13) and (3.14), we obtain

$$\begin{aligned} & \lim_{x_{02} \to +\infty} \frac{1}{x_{02}^2} \left(P(\Sigma^0(x_0), \Xi_O^0(x_{01}, 0)) - P(\Sigma^0(x_0), \Xi_O^0(x_{01}, x_{02}))\right) \\ = & \lim_{x_{02} \to +\infty} \frac{1}{x_{02}^2} \left(\|x_1\|^2_{L^2(\mathcal{T}_\eta)} - \|x_1^0\|^2_{L^2(\mathcal{T}_\eta^0)}\right) + \frac{1}{x_{02}^2} \left(\|u^1\| - \|u^0\|\right) \geq a > 0 \end{aligned}$$

So, finally, we obtain that

$$\lim_{x_{02} \to +\infty} \left(P(\Sigma^0(x_0), \Xi_O^0(x_{01}, 0)) - P(\Sigma^0(x_0), \Xi_O^0(x_{01}, x_{02})\right) = +\infty$$

This completes the proof. □

This proposition shows that as x_{02} becomes large, the difference of performance can be larger than any positive constant. Therefore, for this system, it is advantageous to initialize the second state of the observer close to the actual state rather than to initialize it at 0, that is, a better performance comes from a small initial error. In the following chapter, we will study the performance behaviour of *KKK* and *Khalil* designs as the initial error becomes large.

Chapter 4

Performance of Output-feedback System

From the discussion in previous chapter, we know that we should minimize the initial error to optimize performance. However, we may well not possess complete information concerning the value of the initial condition of the states, and hence we have to take the initial observer to be the best estimate to initial condition of the states. Then we are interested in studying the situation in which our estimate of initial condition of the states is not accurate and leads to a large initial error, and study how the poor information on initial condition of the states affect the performance of the controllers.

4.1 Problem Formulation

In this chapter we consider a system which can be expressed in the output-feedback form

$$\Sigma(x_0): \quad \dot{x}_i = x_{i+1} + \varphi_i(y), \quad 1 \leq i \leq n-1 \tag{4.1a}$$

$$\dot{x}_n = u + \varphi_n(y), \quad x_i(0) = x_{0i}, \quad 1 \leq i \leq n \tag{4.1b}$$

$$y = x_1 \tag{4.1c}$$

where u is the control input, y is the only measured output, and

$$x_0 = \begin{pmatrix} x_{01} \\ x_{02} \\ \vdots \\ x_{0n} \end{pmatrix}$$

is the initial condition of the states, and the functions $\varphi_i, 1 \le i \le n$ are sufficiently smooth and locally Lipschitz continuous, and $\varphi_i(0) = 0,\ 1 \le i \le n$.

This is an output-feedback system of full relative degree. General output-feedback systems and the observer backstepping designs have been given in Chapter 2. One characteristic of such systems is that the nonlinearities only depend on the output y. For this output-feedback system of full relative degree both adaptive observer backstepping (*KKK*) and high-gain observer designs (*Khalil*) can be used to achieve regulation of output. We consider the situation where we do not exactly know x_0, and hence we have to take $\hat{x}_0$ (the initial observer) to be the best estimate of x_0. Then we study the performance of the *KKK* and *Khalil* designs based on the situation in which our estimate of x_0 is not accurate and

$$\tilde{x}_0 = x_0 - \hat{x}_0$$

is 'large'.

As discussed in Chapter 3 the following cost functional

$$P(\Sigma(x_0), \Xi) = \|y\|_{L^2(\mathcal{T}_\eta)} + \|u\|_{L^\infty(\mathbb{R}_+)}, \quad \mathcal{T}_\eta = \big\{t \ge 0 \ \big|\ |y(t)| > \eta\big\}$$

is employed to measure the performance of a controller Ξ. Through this performance measure we aim to characterize when one design is preferable to another.

4.2 Observer Backstepping Design

Let us first consider a generic *KKK* design observer based controller. Following the observer backstepping design procedure in Chapter 2 or [55] with $\rho = n,\ y_r(t) \equiv 0$, the *KKK* design for system $\Sigma(x_0)$ is as follows.

First, rewrite the system $\Sigma(x_0)$ as

$$\begin{aligned} \Sigma(x_0):\quad & \dot{x} = Ax + \varphi(y) + Bu, \quad x(0) = x_0 \\ & y = Cx \end{aligned}$$

where

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \qquad x_0 = \begin{pmatrix} x_{01} \\ x_{02} \\ \vdots \\ x_{0n} \end{pmatrix}, \qquad \varphi(y) = \begin{pmatrix} \varphi_1(y) \\ \varphi_2(y) \\ \vdots \\ \varphi_n(y) \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \qquad C = \begin{pmatrix} 1, 0, \cdots, 0 \end{pmatrix}$$

An observer is defined by

$$\dot{\hat{x}} = A\hat{x} + K(y - \hat{y}) + \varphi(y) + Bu, \quad \hat{x}(0) = \hat{x}_0 \tag{4.2a}$$

$$\hat{y} = C\hat{x} \tag{4.2b}$$

where

$$K = (k_1, k_2, \cdots, k_n)^T, \quad k_i > 0,\ 1 \leq i \leq n$$

is chosen such that

$$A_0 = A - KC$$

is Hurwitz, and $\hat{x}_0$ is the initial condition of the observer.

Then define

$$\begin{aligned}
\xi_1(y) =& y \\
\alpha_1(y) =& -c_1\xi_1 - d_1\xi_1 - \varphi_1(y) \\
\xi_i(y, \hat{x}_1, \cdots, \hat{x}_i) =& \hat{x}_i - \alpha_{i-1}(y, \hat{x}_1, \ldots, \hat{x}_{i-1}) \\
\alpha_i(y, \hat{x}_1, \cdots, \hat{x}_i) =& -c_i\xi_i - \xi_{i-1} - d_i \left(\frac{\partial \alpha_{i-1}}{\partial y}\right)^2 \xi_i - k_i(y - \hat{x}_1) - \varphi_i(y) \\
& + \frac{\partial \alpha_{i-1}}{\partial y}(\hat{x}_2 + \varphi_1(y)) + \sum_{j=1}^{i-1} \frac{\partial \alpha_{i1}}{\partial \hat{x}_j}(\hat{x}_{j+1} + k_j(y - \hat{x}_1) + \varphi_j(y)) \\
& i = 2, 3, \cdots, n
\end{aligned}$$

where c_i, d_i, $1 \leq i \leq n$ are positive constants. The controller is then defined as

$$\begin{aligned}
\Xi_O(\hat{x}_0): \ & u = \alpha_n(y, \hat{x}_1, \cdots, \hat{x}_n) \\
& \dot{\hat{x}} = A\hat{x} + K(y - \hat{y}) + \varphi(y) + Bu, \quad \hat{x}(0) = \hat{x}_0 \\
& \hat{y} = C\hat{x}
\end{aligned}$$

The following result summarizes the standard properties of the closed-loop.

Proposition 4.1. *Consider the closed-loop system* $(\Sigma(x_0), \Xi_O(\hat{x}_0))$. *For any initial condition* $x_0 \in \mathbb{R}^n$ *and* $\hat{x}_0 \in \mathbb{R}^n$, *the following hold*

1. *The signals x, $\hat{x}$, u and y are bounded;*

2. *The output is regulated to zero*

$$\lim_{t\to\infty} y(t) = 0$$

3. *The performance measure is finite*

$$P\big(\Sigma(x_0), \Xi_O(\hat{x}_0)\big) < \infty$$

Proof. From Theorem 2.1, we directly obtain 1 and 2, and we only prove 3 here.

For any positive number η, since the design guarantees the regulation of the output, then after a finite time, we have $y(t) < \eta$, that is the measure of $\mathcal{T}_\eta : m(\mathcal{T}_\eta)$ is finite. Hence, the boundedness of $\|y\|_{L^2(\mathcal{T}_\eta)}$ is achieved. $\|u\|_{L^\infty}$ is also finite since $u(t)$ is bounded. The boundedness of the performance follows directly. □

Although the *KKK* design achieves global regulation of the output, which has a global region of attraction (in $(x_0, \hat{x}_0)$), we will prove that the performance of the controller can degrade arbitrarily as the initial error $\|\tilde{x}_0\|$ becomes large for any fixed initial state condition x_0.

We now establish the critical performance property for the *KKK* design, which states that the performance gets arbitrarily large as the initial observer error increases.

Theorem 4.2. *For any choice of the controller gains k_i, $1 \le i \le n$, and for any fixed initial state x_0 of the system $\Sigma(x_0)$, let $\hat{x}_0$ be the initial observer state, and*

$$\tilde{x}_0 = x_0 - \hat{x}_0$$

Then the performance of the controller $\Xi_O(\hat{x}_0)$ has the following property

$$\limsup_{\|\tilde{x}_0\|\to\infty} P\big(\Sigma(x_0), \Xi_O(\hat{x}_0)\big) = \infty \tag{4.3}$$

Proof. For convenience of notation, introduce

$$\xi_i(0) = \xi_i(y, \hat{x}_1, \cdots, \hat{x}_i)|_{t=0}$$

$$\alpha_i(0) = \alpha_i(y, \hat{x}_1, \cdots, \hat{x}_i)|_{t=0}$$

$$j = 1, 2, \cdots, n$$

To prove this theorem, it suffices to show that

$$\limsup_{\|\tilde{x}_0\|\to\infty} \|u\|_{L^\infty(\mathbb{R}_+)} = \infty$$

Since $u(t)$ is continuous, to establish the above condition, we only need to show

$$\limsup_{\|\tilde{x}_0\|\to\infty} u(0) = \limsup_{\|\tilde{x}_0\|\to\infty} \alpha_n(0) = \infty \tag{4.4}$$

Let $C \subset \mathbb{R}^{n-1}$ be a compact set, define

$$C_r = \left\{ \hat{x}_0 \in \mathbb{R}^n \middle| (\hat{x}_{01}, \cdots, \hat{x}_{0,n-1}) \in C; \hat{x}_{0n} = r \right\}$$

and consider the initial condition of the observer $\hat{x}_0 \in C_r$. Then since x_0 is fixed, if we can prove that

$$\lim_{r\to\infty} \sup_{\hat{x}_0 \in C_r} \alpha_n(0) = \infty \tag{4.5}$$

then (4.4) will hold.

We now establish (4.5). Since all φ_i and their derivatives are continuous functions it follows that α_i and ξ_i are continuous functions of their variables. Note that

$$\begin{aligned} \xi_i(0) =& \hat{x}_{0i} - \alpha_{i-1}(0) \\ \alpha_i(0) =& -c_i\xi_i(0) - \xi_{i-1}(0) - d_i \left(\left. \frac{\partial \alpha_{i-1}}{\partial y} \right|_{t=0} \right)^2 \xi_i(0) \\ & - k_i\big(x_{01} - \hat{x}_{01}\big) - \varphi_i\big(x_{01}\big) \\ & + \sum_{j=1}^{i-1} \left(\left. \frac{\partial \alpha_{i-1}}{\partial \hat{x}_j} \right|_{t=0} \right) \big(\hat{x}_{0,j+1} + k_j(x_{01} - \hat{x}_{01}) + \varphi_j(x_{01}) \big) \end{aligned}$$

and hence, for $1 \le i \le n-1$, $\xi_i(0)$,$\alpha_i(0)$ are independent of $\hat{x}_{0n}$, i.e. bounded independently of r. Therefore there exists $M > 0$ dependent on C and x_{01} but not on r, for which

$$\sup_{\hat{x}_0 \in C_r} |\xi_i(0)| \le M, \quad \sup_{\hat{x}_0 \in C_r} |\alpha_i(0)| \le M, \qquad 1 \le i \le n-1$$

Now we compute $\alpha_n(0)$. First, we have

$$\xi_n(0) = \hat{x}_{0n} - \alpha_{n-1}(0) = r - \alpha_{n-1}(0)$$

and so

$$\begin{aligned}\alpha_n(0) = &- c_i\xi_n(0) - \xi_{n-1}(0) - d_n \left(\frac{\partial \alpha_{n-1}}{\partial y}\bigg|_{t=0} \right)^2 \xi_n(0) \\ &- k_n\big(x_{01} - \hat{x}_{01}) - \varphi_n(x_{01})\big) \\ &+ \sum_{j=1}^{n-1} \left(\frac{\partial \alpha_{n-1}}{\partial \hat{x}_j}\bigg|_{t=0} \right) \big(\hat{x}_{0,j+1} + k_j(x_{01} - \hat{x}_{01}) + \varphi_j(x_{01})\big) \\ = &\left(-c_n - d_n \left(\frac{\partial \alpha_{n-1}}{\partial y}\bigg|_{t=0} \right)^2 \right) r + \left(\frac{\partial \alpha_{n-1}}{\partial_{n-1}}\bigg|_{t=0} \right) r \\ &+ F(x_{01}, \hat{x}_{01}, \cdots, \hat{x}_{0,n-1})\end{aligned}$$

where

$$\begin{aligned}&F(x_{01}, \hat{x}_{01}, \cdots, \hat{x}_{0,n-1}) \\ = &\left(c_n + d_n \left(\frac{\partial \alpha_{n-1}}{\partial y}\bigg|_{t=0} \right)^2 - \frac{\partial \alpha_{n-1}}{\partial \hat{x}_{n-1}}\bigg|_{t=0} \right) \alpha_{n-1}(0) \\ &+ \xi_{n-1}(0) - k_n\big(x_{01} - \hat{x}_{01}) - \varphi_n(x_{01})\big) \\ &+ \sum_{j=1}^{n-2} \left(\frac{\partial \alpha_{n-1}}{\partial \hat{x}_j}\bigg|_{t=0} \right) \big(\hat{x}_{0,j+1} + k_j(x_{01} - \hat{x}_{01}) + \varphi_j(x_{01})\big)\end{aligned}$$

is independent of $\hat{x}_{0n}$, namely r. Now consider the second term of the expression for $\alpha_n(0)$.

$$\begin{aligned}\frac{\partial \alpha_i}{\partial \hat{x}_i} &= -c_i \frac{\partial \xi_i}{\partial \hat{x}_i} - d_i \left(\frac{\partial \alpha_{i-1}}{\partial y} \right)^2 \frac{\partial \xi_i}{\partial \hat{x}_i} + \frac{\partial \alpha_{i-1}}{\partial \hat{x}_{i-1}} \\ &= -c_i - d_i \left(\frac{\partial \alpha_{i-1}}{\partial y} \right)^2 + \frac{\partial \alpha_{i-1}}{\partial \hat{x}_{i-1}}\end{aligned}$$

Therefore, by recursive substitution we obtain

$$\begin{aligned}\frac{\partial \alpha_{n-1}}{\partial \hat{x}_{n-1}} &= \sum_{j=2}^{n-1} \left(-c_j - d_j \left(\frac{\partial \alpha_{j-1}}{\partial y} \right)^2 \right) + \frac{\partial \alpha_1}{\partial \hat{x}_1} \\ &= \sum_{j=2}^{n-1} \left(-c_j - d_j \left(\frac{\partial \alpha_{j-1}}{\partial y} \right)^2 \right)\end{aligned}$$

since α_1 is independent of $\hat{x}_1$.

Hence,

$$\alpha_n(0) = r\sum_{j=2}^{n}\left(-c_j - d_j\left(\left.\frac{\partial\alpha_{j-1}}{\partial y}\right|_{t=0}\right)^2\right) + F(x_{01}, \hat{x}_{01}, \cdots, \hat{x}_{0,n-1})$$

Since c_j and d_j are all positive numbers, and F is independent of r, this establishes (4.5) as required. □

4.3 High-gain Observer Design

By a suitable coordinate transformation the system $\Sigma(x_0)$ can also be written as integrator chain with a matched nonlinearity[1]. Concretely, we define a coordinate transformation

$$\mathbf{T}\colon \mathbb{R}^n \to \mathbb{R}^n, \qquad z = \mathbf{T}(x)$$

by

$$\mathbf{T}:\; z_1 = x_1,\; z_2 = x_2 + \psi_1(x_1),\; \cdots,\; z_n = x_n + \psi_{n-1}(x_1, x_2, \cdots, x_{n-1})$$

where

$$\psi_i(x_1, \cdots, x_i) = \varphi_i(x_1) + \sum_{j=1}^{i-1}\frac{\partial\psi_{i-1}}{\partial x_j}\left(x_{j+1} + \varphi_j(x_1)\right),\; 1 \le i \le n \tag{4.6}$$

Then in the z coordinates, $\Sigma(x_0)$ is of the form

$$\Sigma(z_0):\quad \dot{z} = Az + B(\psi(z) + u), \quad z(0) = z_0 \tag{4.7a}$$

$$y = Cz \tag{4.7b}$$

where

$$z_0 = \mathbf{T}(x_0)$$

$$\psi(z) = \psi_n\left(\mathbf{T}^{-1}(z)\right) \tag{4.8}$$

$$\psi_n(x) = \psi_n(x_1, \cdots, x_n)$$

Remark 4.3. $\Sigma(z_0)$ and $\Sigma(x_0)$ actually present the same system in different coordinates, but, for convenience, we will use $\Sigma(x_0)$ to denote the original system and $\Sigma(z_0)$ to denote (4.7) respectively.

Remark 4.4. It can be seen from the definition of transform that $\mathbf{T}$ is invertible. Furthermore, both $\mathbf{T}$ and $\mathbf{T}^{-1}$ are smooth since $\varphi_i, 1 \le i \le n$ are smooth. Hence, the mapping $\mathbf{T}$ is a global

[1] A high-gain observer can also be designed for the original systems, see, e.g., [30, 32]. Here, we define the controller via this transformation.

diffeomorphism in $\mathbb{R}^n$.

Remark 4.5. Since the output y is unchanged by the transformation $\mathbf{T}$, and the control input u is independent of the change of variables, the performance P is independent of $\mathbf{T}$.

The *Khalil* designs considered in [15, 82, 48, 82, 3] can be applied to the system $\Sigma(z_0)$. Typical results establish semi-global regulation of the output. The *Khalil* designs utilize a high-gain observer and a nonlinear separation principle which allow the observer and a globally bounded state feedback controller to be designed separately, and then combined using certainty equivalence, to ensure semi-global results and closeness of the output feedback controllers trajectory to the underlying state feedback controller's trajectory. For the system $\Sigma(x_0)$, if φ_i and its higher derivatives are globally bounded, it is straightforward to design a globally bounded state feedback controller achieving bounded performance. Hence through the high-gain observer we can design an output feedback controller, which, for fixed initial condition of the state $z_0 = \mathbf{T}(x_0)$ and any initial condition of the observer $\hat{z}_0$ also has bounded performance. Furthermore, if the initial error

$$\tilde{z}_0 = z_0 - \hat{z}_0$$

becomes 'large', this design still achieves a bounded performance independent of the initial condition of the observer.

To design an output feedback controller, we first give a state feedback controller for $\Sigma(z_0)$. The controller

$$u = -\psi(z) + v \tag{4.9}$$

feedback linearizes the system $\Sigma(z_0)$, yielding

$$\dot{z} = Az + Bv, \quad z(0) = z_0 \tag{4.10a}$$

$$y = Cz \tag{4.10b}$$

We first design a bounded state feedback controller for the linear system (4.10a). For this purpose we introduce the asymptotically null controllability with bounded control (ANCBC), which was studied in [78]. Then the existence of a bounded state feedback controller is equivalent to ANCBC.

Definition 4.6. A linear system is called asymptotically null controllable with bounded control (ANCBC) with bound M if for every state z there exists an open-loop control $v : [0, \infty)$ that steers z to the origin in the limit as $t \to +\infty$ and satisfies $|v(t)| < M$ for all t.

The study of such problems is motivated by the possibility of actuator saturation or constraints on actuators, reflected sometimes also in bounds on available power supply or rate limits.

The theory of controllability of linear systems with bounded control is a well-studied topic. Schmitendorf and Barmish [73] published the fundamental paper, and Sontag [77] discussed the different and more algebraic approach. Sussmann, Sontag and Yang [78] gave a well known property that ANCBC is equivalent to a algebraic condition , which is stated as the following lemma.

Lemma 4.7. *The system (4.10a) is asymptotically null controllable with bounded control if and only if*

1. A *has no eigenvalues with positive real part;*
2. *The pair* (A, B) *is stabilizable in the ordinary sense, i.e., there exists a matrix* F *such that* $A + BF$ *is Hurwitz.*

From this lemma, we have following lemma.

Lemma 4.8. *The system (4.10a) is asymptotically null controllable with bounded control.*

Proof. First, all the eigenvalues of A are zero, namely, without positive real parts. Second, the pair (A, B) is stabilizable. Hence, the system (4.10a) is null controllable with bounded control by Lemma 4.7. □

Furthermore, such a bounded state feedback controller for the system (4.10a) is given in [78], that is we have

Lemma 4.9. *The bounded state feedback controller for the system (4.10a)*

$$v = -\sum_{i=1}^{n} \delta^i \mathrm{sat}(h_i(z)) \tag{4.11}$$

achieves global asymptotic stability for the resulting closed-loop system, where $0 < \delta \le \frac{1}{4}$*, each* $h_i : \mathbb{R}^n \to \mathbb{R},\ 1 \le i \le n$*, is a linear function, and* $\mathrm{sat}(\cdot)$ *is the saturation function defined by*

$$\mathrm{sat}(w) = \begin{cases} -1, & w < -1 \\ w, & -1 \le w \le 1 \\ 1, & w > 1 \end{cases}$$

Proof. The detail of the proof can be found in [78]. We give an outline here.

First, for every $\varepsilon > 0$, there exists a linear change of coordinates $(z_1, \cdots, z_n) \to (\xi_1, \cdots, \xi_n)$ which transform (4.10a) into the form

$$\begin{aligned}
\dot{\xi}_1 &= \epsilon^{n-1}\xi_2 + \varepsilon^{n-2}\xi_3 + \cdots + \varepsilon\xi_n + v \\
\dot{\xi}_2 &= \varepsilon^{n-2}\xi_3 + \cdots + \varepsilon\xi_n + v \\
&\vdots \\
\dot{\xi}_{n-1} &= \varepsilon\xi_n + v \\
\dot{\xi}_n &= v
\end{aligned}$$

We will show that, when $\varepsilon \le \frac{1}{4}$, the feedback controller

$$v = -\varepsilon \mathrm{sat}(\xi_n) - \varepsilon^2 \mathrm{sat}(\xi_{n-1}) - \cdots - \varepsilon^n \mathrm{sat}(\xi_1)$$

stabilizes (4.10a). In fact, for any trajectory $t \to \xi$ of the resulting closed-loop system, the nth coordinate ξ_n will enter and stay in the interval $(-\frac{1}{2}, \frac{1}{2})$ after a finite time. So, $\mathrm{sat}(\xi_n)$ will be equal to ξ_n, and the expression for v gives

$$v = -\varepsilon\xi_n - \varepsilon^2 \mathrm{sat}(\xi_{n-1}) - \cdots - \varepsilon^n \mathrm{sat}(\xi_1)$$

Next, consider the equation $\dot{\xi}_{n-1} = \varepsilon\xi_n + v$. Then it follows that, after a finite time, this equation has the form

$$\dot{\xi}_{n-1} = \varepsilon^{n-2}\xi_3 + \cdots + \varepsilon\xi_n$$

We now conclude that ξ_{n-1} will enter and stay in the interval $(-\frac{1}{2}, \frac{1}{2})$ after a finite time, and v will be given by the expression

$$v = -\varepsilon\xi_n - \varepsilon^2\xi_{n-1} - \cdots - \varepsilon^n \mathrm{sat}(\xi_1)$$

Continuing in this way, we see that after a finite time, v will be given by

$$v = -\varepsilon\xi_n - \varepsilon^2\xi_{n-1} - \cdots - \varepsilon^n\xi_1 \tag{4.12}$$

It is clear that the closed-loop system of (4.10a) under the state feedback (4.12) is asymptotically stable. □

From Lemma 4.9, the state feedback controller

$$\Xi_s : \quad u = -\psi(z) - \sum_{i=1}^{n} \delta^i \mathrm{sat}(h_i(z)) \tag{4.13}$$

globally asymptotically stabilizes the origin of system $\Sigma(z_0)$.

Now we design a output feedback controller for $\Sigma(z_0)$. Following [15, 3], we define a high-gain observer as

$$\dot{\hat{z}} = A\hat{z} + H(y - \hat{z}_1), \qquad \hat{z}(0) = \hat{z}_0 \tag{4.14}$$

where

$$H = H(\epsilon) = \begin{pmatrix} \frac{\beta_1}{\epsilon} \\ \frac{\beta_2}{\epsilon^2} \\ \vdots \\ \frac{\beta_n}{\epsilon^n} \end{pmatrix}$$

and ϵ is a positive constant to be specified. The positive constants $\beta_i\,,\ 1 \le i \le n$, are chosen such that the roots of the equation

$$s^n + \beta_1 s^{n-1} + \cdots + \beta_{n-1} s + \beta_n = 0$$

are in the open left-half plane.

To apply the nonlinear separation principle, the state feedback controller is required to be globally bounded. Generally, this property can be achieved by saturating the controller outside some set. But in our case we are interested in the initial condition of the observer becoming large. Instead, we introduce further assumptions on φ_i to ensure that ψ is globally bounded.

Lemma 4.10. *For system $\Sigma(x_0)$, suppose $\varphi_i \in C^{n-i}(\mathbb{R})$, $\varphi_i^{(k)} \in L^\infty(\mathbb{R}),\ 1 \le i \le n;\ 1 \le k \le n$, then ψ defined by (4.8) lies in $L^\infty(\mathbb{R}^n)$.*

Proof. Since $\varphi_i \in C^{n-i}(\mathbb{R})$, $\varphi_i^{(k)} \in L^\infty(\mathbb{R})$, from (4.6) we have that $\psi_n(x)$ is continuous and in $L^\infty(\mathbb{R}^n)$. Noting that the mapping $\mathbf{T}$ is a global diffeomorphism, we know that $\psi(z)$ also is continuous and in $L^\infty(\mathbb{R}^n)$. $\square$

Suppose that the conditions of Lemma 4.10 are satisfied, then the state feedback controller (4.13) is globally bounded, so an output feedback controller for system $\Sigma(z_0)$ can be taken as

$$\Xi_{H(\epsilon)}(\hat{z}_0) : \ u = -\psi(\hat{z}) - \sum_{i=1}^{n} \delta^i \mathrm{sat}(h_i(\hat{z})) \tag{4.15a}$$

$$\dot{\hat{z}} = A\hat{z} + H(y - \hat{z}_1), \qquad \hat{z}(0) = \hat{z}_0 \tag{4.15b}$$

For the system $\Sigma(z_0)$ and the output feedback controller $\Xi_{H(\epsilon)}(\hat{z}_0)$, the relevant properties of the closed-loop are summarized below.

Proposition 4.11. *For system $\Sigma(z_0)$, suppose that $z_0 = T(x_0)$, x_0 is fixed, and the assumption of Lemma 4.10 is satisfied. Then for any $\tilde{z}_0 = z_0 - \hat{z}_0$ there exists ϵ^* such that for all $\epsilon : 0 < \epsilon < \epsilon^*$ the output feedback controller $\Xi_{H(\epsilon)}(\hat{z}_0)$ guarantees the following:*

1. *The signals $z,\ \hat{z},\ u$ and y are bounded;*
2. *The output is regulated to zero*
$$\lim_{t\to\infty} y(t) = 0$$
3. *The performance is finite*
$$P\big(\Sigma(z_0), \Xi_{H(\epsilon)}(\hat{z}_0)\big) < \infty$$

Proof. First, the function
$$\pi(z) = -\psi(z) - \sum_{i=1}^{n} \delta^i \mathrm{sat}(h_i(z))$$
is locally Lipschitz continuous since $\psi(z)$ is continuous and $\sum_{i=1}^{n} \delta^i \mathrm{sat}(h_i(z))$ is bounded. Second, $\pi(z)$ is bounded from Lemma 4.10. Third, the origin is an asymptotically stable equilibrium of the closed-loop under state feedback control. Hence the conditions of Assumption 2.1 in Chapter 2 are satisfied.

Take any compact set $C \in \mathbb{R}^n$ and $\hat{C} \in \mathbb{R}^n$ such that $z_0 \in C$ and $\hat{z}_0 \in \hat{C}$, then 1, 2 follow directly from Theorem 2.3 and 2.4 in Chapter 2. For 3, the finiteness of $\|y\|_{L^2(T_\eta)}$ is obtained from 2. Note that ψ is continuous and $\hat{z}$ is bounded by 1. Hence, $\|u\|_{L^\infty(\mathbb{R}_+)}$ is also finite. So, $P\big(\Sigma(x_0), \Xi_{H(\epsilon)}(\hat{z}_0)\big)$ is finite. □

Now it is straightforward to uniformly bound the performance of system $\Sigma(z_0)$ for the *Khalil* design.

Theorem 4.12. *Let x_0 be fixed and consider the system $\Sigma(x_0)$, and let $z_0 = T(x_0)$. Let $\varphi_i \in C^{n-i}(\mathbb{R})$, $\varphi_i^{(k)} \in L^\infty(\mathbb{R}),\ 1 \le i \le n;\ \ 1 \le k \le n$. Then there is a positive constant M, such that for any $\tilde{z}_0$ there exists $\epsilon > 0$ for which the controller $\Xi_{H(\epsilon)}(\hat{z}_0)$ achieves a uniformly bounded performance*
$$P\big(\Sigma(z_0), \Xi_{H(\epsilon)}(\hat{z}_0)\big) < M \tag{4.16}$$

Proof. First note that

$$P\big(\Sigma(z_0), \Xi_{H(\epsilon)}(\hat{z}_0)\big) = \int_{\mathcal{T}_\eta} |y|^2 dt + \|u\|_{L^\infty(\mathbb{R}_+)} = \int_{\mathcal{T}_\eta} |z_1(t,\epsilon)|^2 dt + \|u\|_{L^\infty(\mathbb{R}_+)} \tag{4.17}$$

From Lemma 4.10, we know that $\psi(\hat{z})$ is bounded. So, the control input u has a bound which is independent of $\hat{z}_0$. By Proposition 4.11, if ϵ is small enough, then $z_1(t,\epsilon)$ tends uniformly in t to $\bar{z}_1(t)$, which is independent of $\hat{z}_0$ and uniformly bounded. Hence, $\bar{z}_1(t)$ has a bound that is independent of $\hat{z}_0$. Also the measure of the time set $\mathcal{T}_\eta$ is also independent of $\hat{z}_0$ and finite. Hence the integral in (4.17) is finite and the bound is independent of $\hat{z}_0$. Therefore, we can find a constant M such that (4.16) holds. □

4.4 Comparison of Performance

Theorem 4.2 shows that for fixed initial state x_0, when the initial error $\|\tilde{x}_0\|$ becomes large, the performance of the *KKK* design is not uniformly bounded even if φ_i and its higher derivatives are globally bounded. On the other hand, Theorem 4.12 shows that for the *Khalil* design, if φ_i and its higher derivatives are globally bounded, then for any initial error $\tilde{z}_0$, through the high-gain factor, we can design a globally bounded controller, achieving a uniformly bounded performance. Hence we obtain the following comparative result.

Corollary 4.13. *For the system $\Sigma(x_0)$, if $\varphi_i \in C^{n-i}(\mathbb{R}), \varphi_i^{(k)} \in L^\infty(\mathbb{R}), 1 \le i \le n$, then for any initial condition of the observer $\hat{z}_0$ there exist $\epsilon > 0$ and $\hat{x}_0$ such that*

$$P\big(\Sigma(z_0), \Xi_{H(\epsilon)}(\hat{z}_0)\big) < P\big(\Sigma(x_0), \Xi_O(\hat{x}_0)\big)$$

Proof. This follows directly from Theorem 4.2 and 4.12. □

We have now established the following results: For output feedback system, the performance of *KKK* design is sensitive to the initial conditions of the observer. The performance of the *KKK* design is not uniformly bounded in the initial error between the initial conditions of the state and the initial conditions of the observer. When the initial error becomes large, the performance becomes large. Whereas, for the *Khalil* design, for any initial error, by choosing small high-gain factor, we can design a globally bounded controller, achieving an uniformly bounded performance. Therefore, if the initial error is large or in the case that we have poor information on the initial conditions of the state, the *Khalil* design has better performance than the *KKK* design. In the next chapter, we will consider the second problem, that is, when do the *Khalil* designs have superior output transients to the *KKK* designs?

Chapter 5

Performance of Parametric Output-feedback System

In the Chapter 4, we compared *KKK* and *Khalil* designs on the output-feedback system when the initial error is large. In this chapter, we are going to compare the two kinds of designs on the system in output-feedback normal form with an uncertain parameter. We will consider the situation when the a-priori estimate for the unknown parameter becomes conservative and leads to a choice of 'large' bound for the unknown parameter, and also study how the 'bad' choice affects the performance of controllers.

5.1 Problem Formulation

We consider a parametric output-feedback system of the form

$$\Sigma(\theta, x_0): \quad \dot{x}_i = x_{i+1}, \quad 1 \le i \le n-1 \tag{5.1a}$$

$$\dot{x}_n = u + \theta\psi(y), \quad x_i(0) = x_{0i}, \quad 1 \le i \le n \tag{5.1b}$$

$$y = x_1 \tag{5.1c}$$

where u is the control input, y is the measured output, and

$$x_0 = \begin{pmatrix} x_{01} \\ x_{02} \\ \vdots \\ x_{0n} \end{pmatrix}$$

is the initial condition of the state, and the functions ψ are sufficiently smooth and locally Lipschitz continuous, and $\psi(0) = 0$; and $\theta \in \mathbb{R}$ is an unknown constant.

This is a parametric output-feedback system in normal form. General parametric output-feedback systems and the adaptive observer backstepping design was given in Chapter 2. Again, for this parametric output-feedback system in normal form, adaptive versions of both parametric observer backstepping and high-gain observer designs can be used to achieve regulation of the output.

To investigate the performance of the two designs and compare them, the same cost functional

$$P(\Sigma(\theta, x_0), \Xi) = \|y\|^2_{L^2(T_\eta)} + \|u\|_{L^\infty(\mathbb{R}_+)}, \quad \mathcal{T}_\eta = \{t \geq 0 \mid |y(t)| > \eta\}$$

is introduced to measure the performance of a controller Ξ. By this performance we again would like to be able to characterize another situation in which one design is preferable to another.

To design a *Khalil* -type output feedback controller with a high-gain observer, we need first to design a globally bounded state feedback controller. Generally, this is achieved by saturation of the state feedback controller. But we also need that the saturated controller stabilizes the system. For this purpose, we need to determine suitable saturation levels. However, the required saturation levels are typically dependent on θ, the unknown constant. Therefore, we have to first quantify an a-priori estimated for the magnitude of θ. Since θ is assumed to be unknown our knowledge of it is typically poor. Hence we have to estimate θ conservatively. But when our a-priori upper bound for $|\theta|$ is conservative, we will show that the performance of the *Khalil* design becomes poor. For a *KKK* design, however, the performance is independent of any a-priori upper bound for $|\theta|$. Therefore, the performance keeps uniformly bounded as the a-priori upper bound for $|\theta|$ becomes conservative. Hence, for this system we will establish a result with the opposite performance relationship to obtained in Chapter 4.

5.2 Adaptive Observer Backstepping Design

We first consider a *KKK* design for the system. Following the adaptive observer backstepping design procedure in Chapter 2, the construction of a controller is obtained as follows.

Rewrite the system $\Sigma(\theta, x_0)$ in the form

$$\Sigma(\theta, x_0): \quad \dot{x} = Ax + B(\theta\psi(y) + u), \quad x(0) = x_0 \tag{5.2a}$$

$$y = Cx \tag{5.2b}$$

where A, B, C, as in previous chapters, are defined by

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \qquad x_0 = \begin{pmatrix} x_{01} \\ x_{02} \\ \vdots \\ x_{0n} \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \qquad C = \begin{pmatrix} 1, 0, \cdots, 0 \end{pmatrix}$$

The *KKK* design for the parametric output-feedback system $\Sigma(\theta, x_0)$ is given as follows.

Choose a vector K such that

$$A_0 = A - KC$$

is Hurwitz, and define the filters[1]

$$\begin{aligned} \dot{\xi}_0 &= A_0 \dot{\xi}_0 + Ky, \quad \xi_0(0) = \xi_0^0 \\ \dot{\xi}_1 &= A_0 \xi_1 + B\psi(y), \quad \xi_1(0) = \xi_1^0 \\ \dot{v}_0 &= A_0 v_0 + e_n u, \quad v_0(0) = v_0^0 \end{aligned}$$

The controller is defined by

$$\begin{aligned} \Xi_A(\vartheta_0, \hat{x}_0): \qquad u &= \alpha_n \\ \dot{\vartheta}_1 &= \Gamma \omega_1(y, \bar{\xi}^{(2)}, \bar{v}^{(2)})\zeta_1 \\ \dot{\vartheta}_2 &= \Gamma \left(\omega_2(y, \bar{\xi}^{(2)}, \bar{v}^{(2)}, \bar{\vartheta}^{(2)}) + \zeta_1 e_2 \right) \zeta_2 \\ \dot{\vartheta}_i &= \Gamma \omega_i(y, \bar{\xi}^{(i)}, \bar{v}^{(i)}, \bar{\vartheta}^{(i-1)})\zeta_i, \quad i = 3, \cdots n \\ \hat{x}(0) &= \hat{x}_0 = \xi_0^0 + \theta \xi_1^0 + v_0^0 \\ \vartheta(0) &= \vartheta_0 = (\vartheta_{01}, \vartheta_{02}, \cdots, \vartheta_{0n})^T \end{aligned}$$

where e_i denotes the ith coordinate vector in $\mathbb{R}^n$, and ζ_i, ω_i, α_i, $\bar{\xi}^{(i)}$, $\bar{v}^{(i)}$, $\bar{\vartheta}^{(i)}$, $i = 1, \cdots, n$ are

[1] The filters can removed, see [67].

defined by the following recursive expressions

$$
\begin{aligned}
\zeta_1 =& y \\
\zeta_i =& v_{0,i} - \alpha_{i-1}(y, \bar{\vartheta}^i) \\
\alpha_1 =& -\vartheta_1^T \omega_1 \\
\alpha_2 =& -c_2\zeta_2 - \vartheta_{2,2}\zeta_1 - d_2\left(\frac{\partial \alpha_1}{\partial y}\right)^2 \zeta_2 + \frac{\partial \alpha_1}{\partial y}\xi_{0,2} - \vartheta_2^T \omega_2 + k_2 v_{0,1} \\
&+ \frac{\partial \alpha_1}{\partial \xi_0}(A_0\xi_0 + Ky) + \frac{\partial \alpha_1}{\partial \xi_1}(A_0\xi_1 + B\psi(y)) + \frac{\partial \alpha_1}{\partial v_0}A_0 v_0 + \frac{\partial \alpha_1}{\partial \vartheta_1}\Gamma\omega_1\zeta_1 \\
\alpha_i =& -c_i\zeta_i - d_i\left(\frac{\partial \alpha_{i-1}}{\partial y}\right)^2 \zeta_i + \frac{\partial \alpha_{i-1}}{\partial y}\xi_{0,2} - \vartheta_i^T \omega_i + k_i v_{0,1} \\
&+ \frac{\partial \alpha_{i-1}}{\partial \xi_0}(A_0\xi_0 + Ky) + \frac{\partial \alpha_{i-1}}{\partial \xi_1}(A_0\xi_1 + B\psi(y)) \\
&+ \frac{\partial \alpha_{i-1}}{\partial v_0}A_0 v_0 + \frac{\partial \alpha_{i-1}}{\partial \vartheta_1}\Gamma\omega_1\zeta_1 \\
&+ \frac{\partial \alpha_{i-1}}{\partial \vartheta_2}\Gamma(\omega_2 + \zeta_1 e_2)\zeta_2 + \sum_{j=3}^{i-1} \frac{\partial \alpha_{i-1}}{\partial \vartheta_j}\Gamma\omega_j\zeta_j, \quad i = 3, \cdots, n
\end{aligned}
$$

and

$$
\begin{aligned}
\omega_1^T =& (c_1\zeta_1 + d_1\zeta_1 + \xi_{0,2}, v_{0,2}) \\
\omega_i^T =& -\frac{\partial \alpha_{i-1}}{\partial y}(\xi_{1,2}, v_{0,2}), \quad i = 2, \cdots, n-1 \\
\omega_n^T =& -\frac{\partial \alpha_{n-1}}{\partial y}(\psi + \xi_{1,2}, v_{0,2})
\end{aligned}
$$

and

$$
\begin{aligned}
\bar{\xi}^{(i)} =& (\xi_{0,1}, \cdots, \xi_{0,i}, \cdots, \xi_{1,1}, \cdots, \xi_{1,i}), \quad i = 1, \cdots, n \\
\bar{v}^{(i)} =& (v_{0,1}, \cdots, v_{0,i}), \quad i = 1, \cdots n \\
\bar{\vartheta}^{(i)} =& (\vartheta_1^T, \cdots, \vartheta_i^T), \quad i = 1, \cdots n
\end{aligned}
$$

We summarize the relevant properties of this controller in the following proposition.

Proposition 5.1. *Consider the system* $\Sigma(\theta, x_0)$*, then for any* $\hat{x}_0, \vartheta_0$*, the controller* $\Xi_A(\vartheta_0, \hat{x}_0)$ *guarantees global boundedness of all signals* $x(t), \xi_i(t), v_i(t)$*, and regulation of the output, i.e.*

$$
\lim_{t \to \infty} y(t) = 0
$$

Moreover, the controller achieves bounded performance

$$P\big(\Sigma(\theta, x_0), \Xi_A(\vartheta_0, \hat{x}_0)\big) < \infty$$

for fixed x_0 *and* $\hat{x}_0$.

Proof. The boundedness of signals and regulation of y are obtained from Theorem 2.2. We only prove the performance is bounded.

For any positive number η, because the design guarantees the regulation of the output, then after a finite time, we have $y(t) < \eta$, that is to say the measure of $\mathcal{T}_\eta$ is finite. Hence, the boundedness of $\|y\|_{L^2(\mathcal{T}_\eta)}$ is achieved, and $\|u\|_{L^\infty}$ is also finite since $u(t)$ is uniformly bounded. Then boundedness of the performance follows directly. □

5.3 High-gain Observer Design

We design a *Khalil* controller using the nonlinear separation principle [3]. The standard steps in this synthesis procedure is as follows: first design a state feedback controller; then saturate the controller outside some sets based on our a-priori knowledge of the worst case bounds for the closed-loop signals; next replace the unmeasurable state variables by the estimated states from a high-gain observer. This defines an output feedback control.

5.3.1 Control Design

First, we design a state feedback controller based on Lyapunov theory, and obtain an a-priori worst case estimates for the bounds of the closed-loop signals.

We chose a vector

$$k = (k_1, k_2, \cdots, k_n)$$

such that matrix $A + Bk$ is Hurwitz, and let matrix P_1 be the positive definite symmetric matrix solution of the Lyapunov equation

$$(A + Bk)^T P_1 + P_1(A + Bk) = -I$$

Suppose that θ_m is the a-priori estimate of upper bound for the magnitude of the unknown parameter θ, and we define a state feedback controller as in Chapter 2 (also see [45])

$$\Xi_s(\hat{\theta}_0, x_0): \quad u = \mu(x, \hat{\theta}) = kx - \hat{\theta}\psi(y) \tag{5.3a}$$

$$\dot{\hat{\theta}} = \nu(x, \hat{\theta}) = \text{Proj}(\hat{\theta}, \phi), \quad \hat{\theta}(0) = \hat{\theta}_0 \tag{5.3b}$$

where

$$\phi(x) = 2x^T P_1 B \psi(y)$$

Consider the Lyapunov function

$$V(x, \theta - \hat{\theta}) = x^T P_1 x + \frac{1}{2}(\theta - \hat{\theta})^2$$

then along the solutions of the closed-loop , we have

$$\begin{aligned}
\dot{V} =& x^T P_1 \dot{x} + \dot{x}^T P_1 x - (\theta - \hat{\theta})\dot{\hat{\theta}} \\
=& x^T P_1 (Ax + B(\theta\psi(y) + u)) + (Ax + B(\theta\psi(y) + u))^T P_1 x - (\theta - \hat{\theta})\dot{\hat{\theta}} \\
=& x^T P_1 (Ax + B(\theta\psi(y) + kx - \hat{\theta}\psi(y))) + (Ax + B(\theta\psi(y) + kx - \hat{\theta}\psi(y)))^T P_1 x \\
&- (\theta - \hat{\theta})\dot{\hat{\theta}} \\
=& x^T \left((A + Bk)^T P_1 + P_1(A + Bk)\right) x + (\theta - \hat{\theta}) \left(\phi - \text{Proj}(\hat{\theta}, \phi)\right) \\
\le& - x^T x \le 0
\end{aligned}$$

this suffices to show global stability and regulation of the output to zero by LaSalle's theorem.

To design an output feedback controller through a high-gain observer, the functions μ and ν should be globally bounded [3]. So, we saturate μ and ν outside some suitably defined sets which ensure that the modified controller still stabilizes the system. For this purpose, we utilize a-priori estimates of x and $\hat{\theta}$.

Firstly, from $\dot{V} \le 0$, we have

$$\frac{1}{2}(\theta - \hat{\theta})^2 \le V(t) \le V(0) = x_0^T P_1 x_0 + \frac{1}{2}(\theta - \hat{\theta}_0)^2$$

Hence

$$|\hat{\theta}| \le \theta_m + \sqrt{2\overline{\lambda}(P_1)\chi_m^2 + (\theta_m + |\hat{\theta}_0|)^2} =: \Theta_0 \tag{5.4}$$

where θ_m, χ_m are the a-priori estimates of upper bound for the magnitude of the unknown parameter θ and the magnitude of the initial state x_0, and $\overline{\lambda}(P_1)$ is the largest eigenvalue of P_1.

Similarly

$$\|x\| = (x^T x)^{\frac{1}{2}} \le \left(\frac{1}{\underline{\lambda}(P_1)} x^T P_1 x\right)^{\frac{1}{2}} \le \left(\frac{1}{\underline{\lambda}(P_1)} V(0)\right)^{\frac{1}{2}}$$
$$\le \left(\frac{1}{\underline{\lambda}(P_1)} \left(\overline{\lambda}(P_1)\chi_m^2 + \frac{1}{2}(\theta_m + |\hat{\theta}_0|)^2\right)\right)^{\frac{1}{2}} =: X_0 \tag{5.5}$$

where $\underline{\lambda}(P_1)$ is the smallest eigenvalue of P_1, and

$$|y| = |x_1| \le \|x\| \le X_0 \tag{5.6}$$

Finally, from (5.3a)

$$|\mu| \le n\overline{k}X_0 + \Theta_0 \Psi_0 =: U_0 \tag{5.7}$$

where

$$\overline{k} = \max_{1 \le j \le n} \{|k_j|\}$$

$$\Psi_0 = \sup_{|x_1| \le X_0} \{|\psi(x_1)|\}$$

On the other hand, suppose that p is the biggest element in the last row of P_1, then by (5.3b) we obtain

$$|\nu| \le np\|x\|\Psi_0 \le npX_0\Psi_0 =: V_0 \tag{5.8}$$

Now we saturate μ and ν as follows.

$$\mu_s(x, \hat{\theta}) = U_0 \mathrm{sat}\left(\frac{\mu(x, \hat{\theta})}{U_0}\right)$$

$$\nu_s(x, \hat{\theta}) = V_0 \mathrm{sat}\left(\frac{\nu(x, \hat{\theta})}{V_0}\right)$$

to obtain a globally bounded state feedback controller

$$\Xi_s^b(\theta_m, \chi_m, \hat{\theta}_0, x_0): \quad u = \mu_s(x, \hat{\theta}) \tag{5.9a}$$
$$\dot{\hat{\theta}} = \nu_s(x, \hat{\theta}), \quad \hat{\theta}(0) = \hat{\theta}_0 \tag{5.9b}$$

Consequently a *Khalil* controller can be obtained as

$$\Xi_{H(\epsilon)}(\theta_m, \chi_m, \hat{\theta}_0, \hat{x}_0): \quad u = \mu_s(\hat{x}, \hat{\theta}) \tag{5.10a}$$

$$\dot{\hat{\theta}} = \nu_s(\hat{x}, \hat{\theta}), \quad \hat{\theta}(0) = \hat{\theta}_0 \tag{5.10b}$$

$$\dot{\hat{x}} = A\hat{x} + H(y - \hat{x}_1), \quad \hat{x}(0) = \hat{x}_0 \tag{5.10c}$$

The properties of this controller are summarized in the following result.

Proposition 5.2. *For the system $\Sigma(\theta, x_0)$, if $|\theta| \le \theta_m$ and $|\hat{\theta}_0| < \Theta_0$, then when ϵ is small enough, the controller $\Xi_{H(\epsilon)}(\theta_m, \chi_m, \hat{\theta}_0, \hat{x}_0)$ guarantees global boundedness of all signals, and the mean square of the output is of order $O(\epsilon)$*

$$\lim_{T\to\infty} \frac{1}{T}\int_0^{T} y(t)^2 dt = O(\epsilon)$$

Moreover, if $\epsilon < \eta$, then the controller achieves bounded performance

$$P\big(\Sigma(\theta, x_0), \Xi_{H(\epsilon)}(\theta_m, \chi_m, \hat{\theta}_0, \hat{x}_0)\big) < \infty$$

Proof. The system $\Sigma(\theta, x_0)$ is of the form of (2.5) in Chapter 2, and the controller is defined as per (2.8) in Theorem 2.5. Therefore, we obtain the boundedness of all signals and the regulation of the output by Theorem 2.5.

The proof of the boundedness of the performance follows from the boundedness of the closed-loop signals. □

5.3.2 Performance

First we establish the following lemma, which shows that as $\epsilon \to 0$, the control signal necessarily reaches the saturation level U_0.

Lemma 5.3. *Suppose that the system and controller satisfy the condition of Proposition 5.2, and let*

$$e_{0j} = x_{0j} - \hat{x}_{0j}, \quad 1 \le j \le n$$

and suppose that at least one of $e_{0j}, 1 \le j \le n-1$, is not equal to zero. Then for the closed-loop

$(\Sigma(\theta, x_0), \Xi_{H(\epsilon)}(\theta_m, \chi_m, \hat{\theta}_0, \hat{x}_0))$, *we have*[2]

$$\lim_{\epsilon \to 0} \|u\|_{L^\infty(\mathbb{R}^+)} = U_0 \tag{5.11}$$

Proof. From the definition of the controller $\Xi_{H(\epsilon)}(\theta_m, \chi_m, \hat{\theta}_0, \hat{x}_0)$, it suffices to prove that

$$\lim_{\epsilon \to 0} \left(\sup_{t \in \mathbb{R}^+} \|\hat{x}(t)\| \right) = \infty \tag{5.12}$$

Now let

$$e_j = x_j - \hat{x}_j, \quad 1 \le j \le n$$

$$\zeta_j = \frac{1}{\epsilon^{n-j}} e_j, \quad 1 \le j \le n$$

Then the closed-loop $(\Sigma(\theta, x_0), \Xi_{H(\epsilon)}(\theta_m, \chi_m, \hat{\theta}_0, \hat{x}_0))$ is given by

$$\dot{x} = Ax + B(\hat{\theta}\psi(y) + \mu_s(\hat{x}, \hat{\theta})), \quad x(0) = x_0 \tag{5.13a}$$

$$\dot{\hat{\theta}} = \nu_s(\hat{x}, \hat{\theta}), \quad \hat{\theta}(0) = \hat{\theta}_0 \tag{5.13b}$$

$$\epsilon\dot{\zeta} = D\zeta + \epsilon B(\hat{\theta}\psi(y) + \mu^s(\hat{x}, \hat{\theta})), \quad \zeta(0) = \zeta_0 \tag{5.13c}$$

where $L = (\alpha_1, \alpha_2, \cdots, \alpha_n)^T$, the matrix $D = A - LC$ is Hurwitz and

$$\zeta_0 = \left(\frac{e_{01}}{\epsilon^{n-1}}, \cdots, \frac{e_{0,n-1}}{\epsilon}, e_{0n} \right)^T$$

To prove (5.12) it is enough to show

$$\lim_{\epsilon \to 0} \left(\sup_{t \in \mathbb{R}^+} \|e(t)\| \right) = \infty \tag{5.14}$$

Since $\xi_n = e_n$, it is sufficient to show

$$\lim_{\epsilon \to 0} \left(\sup_{t \in \mathbb{R}^+} |\zeta_n(t)| \right) = \infty \tag{5.15}$$

On the other hand, let

$$t = \epsilon\tau$$

[2]In practice, the limit $\epsilon \to 0$ means that ϵ is sufficiently small.

then (5.13c) can be written as

$$\frac{d\zeta}{d\tau} = D\zeta + \epsilon B(\hat{\theta}\psi(y) + \mu^s(\hat{x}, \hat{\theta})), \quad \zeta(0) = \zeta_0 \tag{5.16}$$

When ϵ is small enough, the output $y = x_1$ converges uniformly in t to the solution of the state feedback closed-loop system, and hence is uniformly bounded, therefore $\psi(y)$ is uniformly bounded. So, the term $B(\hat{\theta}\psi(y) + \mu^s(\hat{x}, \hat{\theta}))$ in (5.16) is bounded uniformly in τ. Therefore, when $\epsilon \to 0$, the solution of (5.16) is convergent uniformly in τ to the solution of following equation

$$\frac{d\eta}{d\tau} = D\eta, \quad \eta(0) = \zeta_0 \tag{5.17}$$

Hence, we only need to show

$$\lim_{\epsilon \to 0} \left(\sup_{t \in \mathbb{R}^+} |\eta_n(t)| \right) = \infty \tag{5.18}$$

Note that

$$D = \begin{pmatrix} -d_1 & 1 & 0 & \cdots & 0 \\ -d_2 & 0 & 1 & \cdots & 0 \\ \multicolumn{5}{c}{\dotfill} \\ -d_{n-1} & 0 & 0 & \cdots & 1 \\ -d_n & 0 & 0 & \cdots & 0 \end{pmatrix}$$

and by induction we can show that

$$D^j = \begin{pmatrix} * & \cdots & *_{1j} & 1 & 0 & \cdots & 0 \\ * & \cdots & * & * & 1 & \cdots & 0 \\ \multicolumn{7}{c}{\dotfill} \\ * & \cdots & * & * & * & \cdots & 1 \\ \multicolumn{7}{c}{\dotfill} \\ * & \cdots & -\beta_n & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad 1 \le j \le n \tag{5.19}$$

where the “*”s are elements which do not need to be specified. Let

$$s = \min_{1 \le j \le n-1} \{ j \,| e_{0j} \neq 0 \}$$

and consider the time

$$t_\epsilon = \epsilon \gamma_\epsilon$$

where

$$\gamma_\epsilon = \epsilon^{\frac{n-s-\beta}{s}}$$

and

$$\beta = \frac{1}{2}\min\left\{\frac{n-s}{s+1}, 1\right\}$$

Note the solution of (5.17) is given by

$$\eta(\tau) = e^{D\tau}\zeta_0$$

i.e. equivalently by

$$\eta(t) = e^{D\frac{t}{\epsilon}}\zeta_0 \tag{5.20}$$

Hence

$$\eta(t_\epsilon) = e^{D\gamma_\epsilon}\zeta_0$$

Now

$$\begin{aligned} e^{D\gamma_\epsilon} =& I + \gamma_\epsilon D + \frac{\gamma_\epsilon^2}{2!}D^2 + \cdots + \frac{\gamma_\epsilon^s}{s!}D^s + o\left(\gamma_\epsilon^{s+\frac{1}{2}}\right) \\ =& \begin{pmatrix} \cdots & *_{1s} & * & \cdots & * & * \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & -\beta_n \frac{\gamma_\epsilon^s}{s!} + o\left(\gamma_\epsilon^{s+\frac{1}{2}}\right) & o\left(\gamma_\epsilon^{s+\frac{1}{2}}\right) & \cdots & o\left(\gamma_\epsilon^{s+\frac{1}{2}}\right) & 1 + o\left(\gamma_\epsilon^{s+\frac{1}{2}}\right) \end{pmatrix} \end{aligned}$$

where the "*"s are elements which do not need to be specified. Noting that $e_{0j} = 0$ for $1 \leq j \leq s-1$, yields

$$\begin{aligned} \eta_n(t_\epsilon) &= \left(-\alpha_n \frac{\gamma_\epsilon^s}{s!} + o\left(\gamma_\epsilon^{s+\frac{1}{2}}\right)\right)\frac{e_{0s}}{\epsilon^{n-s}} + o\left(\gamma_\epsilon^{s+\frac{1}{2}}\right)\frac{e_{0,s+1}}{\epsilon^{n-s-1}} + \cdots \\ &\quad + o\left(\gamma_\epsilon^{s+\frac{1}{2}}\right)\frac{e_{0,n-1}}{\epsilon} + \left(1 + o\left(\gamma_\epsilon^{s+\frac{1}{2}}\right)\right)e_{0n} \\ &= -\frac{\beta_n e_{0s}}{s!\epsilon^\beta} + e_{0n} + o(\epsilon^\lambda) \end{aligned} \tag{5.21}$$

where

$$\lambda = \frac{n-s+\beta}{2s} > 0$$

But by assumption, $\beta_n > 0$, and $e_{0s} \neq 0$. Therefore, (5.21) implies (5.18). This completes the proof. □

From this lemma, we obtain the following theorem.

Theorem 5.4. *Let θ, x_0 be fixed, and suppose $|x_0| \leq \chi_m$. Consider the system $\Sigma(\theta, x_0)$ and the controller $\Xi_{H(\epsilon)}(\theta_m, \chi_m, \hat{\theta}_0, \hat{x}_0)$. Suppose the system and controller satisfy the condition of Proposition 5.2, and at least one of the initial errors $e_{0j}, 1 \leq j \leq n-1$, is not equal to zero. Then for the closed-loop system $(\Sigma(\theta, x_0), \Xi_{H(\epsilon)}(\theta_m, \chi_m, \hat{\theta}_0, \hat{x}_0))$, we have*

$$\lim_{\epsilon \to 0} \left(\lim_{\theta_m \to \infty} P\left(\Sigma(\theta, x_0), \Xi_{H(\epsilon)}(\theta_m, \chi_m, \hat{\theta}_0, \hat{x}_0)\right)\right) = \infty$$

Proof. For the closed-loop system $(\Sigma(\theta, x_0), \Xi_{H(\epsilon)}(\hat{\theta}_0, \hat{x}_0))$, the saturation levels Θ_0 and U_0 for the output feedback controller are dependent on θ_m, the a-priori estimate of upper bound for the unknown parameter θ. When θ_m is large, from (5.4)-(5.7), Θ_0 and U_0 are large. By Lemma 5.3, as the high-gain factor ϵ is small, $\|u\|_{L^\infty(\mathbb{R}^+)}$ is also large, that is the performance becomes large. □

5.4 Comparison of Performance

For the system $\Sigma(\theta, x_0)$, as the a-priori estimate of upper bound θ_m for the uncertain parameter θ becomes conservative, Proposition 5.1 shows that the *KKK* design guarantees uniform bounded performance of the controllers; whereas, Theorem 5.4 shows that the performance of the *Khalil* design becomes large. Here we have the following comparative result.

Corollary 5.5. *For the system $\Sigma(\theta, x_0)$, if the estimate of bound for the unknown parameter θ is conservative enough, and the gain factor ϵ is small enough, then*

$$P\big(\Sigma(\theta, x_0), \Xi_A(\vartheta_0, \hat{x}_0)\big) < P\big(\Sigma(\theta, x_0), \Xi_{H(\epsilon)}(\hat{\theta}_0, \hat{x}_0)\big)$$

Proof. The result follows directly from Proposition 5.1 and Theorem 5.4. □

Therefore we have established the following result for parametric output feedback system.

The performance of the *KKK* design is independent of the a-priori estimate bound of the uncertain parameter. When the a-priori estimate becomes conservative the performance remains uniformly bounded.

Whilst, for the *Khalil* design, the performance is dependent on the saturation levels for the controller and the adaptive law, that is dependent on the a-priori estimate bound of the uncertain parameter, and the performance becomes large as the a-priori estimate becomes conservative.

Hence, if we have poor information for the unknown parameter and the a-priori estimate bound is conservative, the *KKK* design has better performance than the *Khalil* design.

In the next chapters, we will study robust *KKK* and *Khalil* designs.

Part II

Gap Metric Robustness

Chapter 6

Preliminaries

In this chapter, we give the required background on robust stability. We introduce the tools for robustness analysis of linear systems, in which the framework of gap metric is of advantage. Naturally, for the robust stability of nonlinear systems gap metric is also a powerful tool. Hence, we will employ the framework of gap metric of nonlinear systems for the study of robust *KKK* and *Khalil* designs. We give some established related results about the gap metric for nonlinear systems and the definition of local stability.

6.1 Feedback Configuration and Stability

Let $\mathcal{U}, \mathcal{Y}$ be appropriate signal spaces such as $L^p(\mathbb{R}^+, \mathbb{R}^n)$. In this thesis we will be mostly concerned with $p = \infty$. Consider a standard feedback configuration with input and measurement

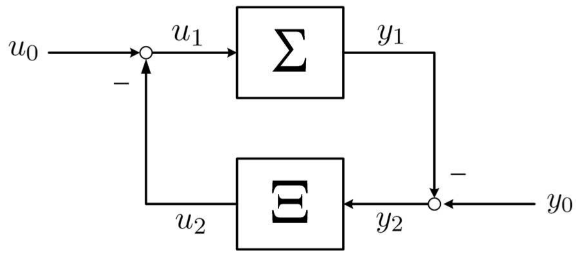

FIGURE 6.1: Standard Feedback Configuration

disturbances shown in FIGURE 6.1, and described by the equations

$$\begin{aligned} y_1 &= \Sigma u_1, & u_2 &= \Xi y_2 \\ y_0 &= y_1 + y_2, & u_0 &= u_1 + u_2 \end{aligned}$$

where Σ is a nominal plant, and Ξ is a controller, $u_0 \in \mathcal{U}, y_0 \in \mathcal{Y}$ are input and measurement disturbances respectively.

Stability of Linear systems

If Σ, Ξ are linear, we use $\Sigma(s), \Xi(s)$, alternatively, Σ, Ξ to denote their respective transfer functions. Then stability can be defined by the transfer functions.

Definition 6.1. Suppose Σ, Ξ are linear, we define the closed-loop $[\Sigma, \Xi]$ to be stable if the transfer function matrix

$$\Pi =: \begin{pmatrix} I \\ \Sigma \end{pmatrix} (I - \Xi\Sigma)^{-1}(I, -\Xi)$$

is stable, that is $\Pi \in \mathcal{H}_\infty$, where $\mathcal{H}_\infty$ is the space of transfer functions of stable linear, time-invariant, continuous time, systems.

In the linear case, the signals satisfy

$$\begin{pmatrix} u_1 \\ y_1 \end{pmatrix} = \Pi \begin{pmatrix} u_0 \\ y_0 \end{pmatrix}$$

Hence, we have that

$$\left\| \begin{pmatrix} u_1 \\ y_1 \end{pmatrix} \right\| \leq \|\Pi\| \left\| \begin{pmatrix} u_0 \\ y_0 \end{pmatrix} \right\|$$

If $\mathcal{U} = \mathcal{Y} = L^p, 1 \leq p \leq \infty$, then $\|\Pi\| < \infty$ if and only if $\Pi \in \mathcal{H}_\infty$.

In particular, if $p = 2$, then

$$\|\Pi\| = \|\Pi(s)\|_{\mathcal{H}_\infty}$$

and if $p = \infty$, then

$$\|\Pi\| = \int_0^\infty \|g(t)\| dt$$

where g is the impulse response of Π, i.e.

$$\begin{pmatrix} u_1 \\ y_1 \end{pmatrix} = g * \begin{pmatrix} u_0 \\ y_0 \end{pmatrix}$$

in the time domain, and '*' denotes convolution.

On the other hand, if there exists a constant Γ such that

$$\left\| \begin{pmatrix} u_1 \\ y_1 \end{pmatrix} \right\| \leq \Gamma \left\| \begin{pmatrix} u_0 \\ y_0 \end{pmatrix} \right\| \tag{6.1}$$

that is, the operator

$$\begin{pmatrix} u_0 \\ y_0 \end{pmatrix} \mapsto \begin{pmatrix} u_1 \\ y_1 \end{pmatrix}$$

is bounded, then the closed-loop system is stable. Therefore, for a linear system, stability is equivalent to (6.1). For nonlinear systems we use boundedness of the operator as a definition of stability.

Stability

We define stability by a closed-loop operator, and generalize the definition of stability to non-linear systems.

Graph of a Plant

Let $\mathcal{U}, \mathcal{Y}$ be appropriate signal spaces, and consider a nominal causal plant Σ and a causal controller Ξ. Write

$$\mathcal{U}_\Sigma = \text{Dom}(\Sigma) = \left\{u \in \mathcal{U} \middle| \Sigma u \in \mathcal{Y}\right\}$$

$$\mathcal{Y}_\Xi = \text{Dom}(\Xi) = \left\{y \in \mathcal{Y} \middle| \Xi y \in \mathcal{U}\right\}$$

then

$$\Sigma : \mathcal{U}_\Sigma \to \mathcal{Y}, \quad \Xi : \mathcal{Y}_\Xi \to \mathcal{U}$$

and let

$$\mathcal{W} = \mathcal{U} \times \mathcal{Y}$$

Then the graph of the plant is defined as

$$\mathcal{G}_\Sigma = \left\{ \begin{pmatrix} u \\ \Sigma u \end{pmatrix} : \ u \in \mathcal{U}_\Sigma, \Sigma u \in \mathcal{Y} \right\} \subset \mathcal{W}$$

Similarly the graph of the control operator is defined as

$$\mathcal{G}_\Xi = \left\{ \begin{pmatrix} \Xi y \\ y \end{pmatrix} : \ y \in \mathcal{Y}_\Xi, \Xi y \in \mathcal{U} \right\} \subset \mathcal{W}$$

Closed-loop Operator and Stability

Write

$$w_0 = \begin{pmatrix} u_0 \\ y_0 \end{pmatrix}, \quad w_1 = \begin{pmatrix} u_1 \\ y_1 \end{pmatrix}, \quad w_2 = \begin{pmatrix} u_2 \\ y_2 \end{pmatrix}$$

Then we define the closed-loop operator by

$$H_{\Sigma,\Xi} : \mathcal{W} \to \mathcal{W} \times \mathcal{W}, \ \ H_{\Sigma,\Xi} : w_0 \mapsto (w_1, w_2)$$

Note that this operator is not always defined, e.g., if the closed-loop is not stable, then $w_1 \notin \mathcal{W}$.

To study the stability of closed-loop systems, another two operators are introduced. Write

$$\mathcal{M} = \mathcal{G}_\Sigma, \quad \mathcal{N} = \mathcal{G}_\Xi$$

and define

$$\Pi_{\mathcal{M}//\mathcal{N}} = \Pi_1 H_{\Sigma,\Xi} : \ \mathcal{W} \to \mathcal{W}$$

$$\Pi_{\mathcal{N}//\mathcal{M}} = \Pi_2 H_{\Sigma,\Xi} : \ \mathcal{W} \to \mathcal{W}$$

where $\Pi_i : \mathcal{W} \times \mathcal{W} \to \mathcal{W}$ denotes the natural projection onto the ith component ($i = 1, 2$) of $\mathcal{W} \times \mathcal{W}$. Hence

$$\Pi_{\mathcal{M}//\mathcal{N}} : \ w_0 \mapsto w_1$$

$$\Pi_{\mathcal{N}//\mathcal{M}} : \ w_0 \mapsto w_2$$

Definition 6.2. The closed-loop $[\Sigma, \Xi]$ is said to be stable if the operator $\Pi_{\mathcal{M}//\mathcal{N}}$ has a finite induced norm, i.e.

$$\|\Pi_{\mathcal{M}//\mathcal{N}}\| = \sup_{w_0 \neq 0} \frac{\|\Pi_{\mathcal{M}//\mathcal{N}} w_0\|}{\|w_0\|} = \sup_{w_0 \neq 0} \frac{\|w_1\|}{\|w_0\|} < \infty$$

Remark 6.3. For linear systems, this definition is equivalent to Definition 6.1 due to inequality (6.1). Further, observe that stability of $\Pi_{\mathcal{M}//\mathcal{N}}$ implies stability of $\Pi_{\mathcal{N}//\mathcal{M}}$, and vice versa. Hence, this is an appropriate generalization applicable to nonlinear systems.

The notion of stability for nonlinear control can be relaxed to the gain-function stability. Here, the gain-function of the operator $\Pi_{\mathcal{M}//\mathcal{N}}$ is defined as

$$g[\Pi_{\mathcal{M}//\mathcal{N}}](\alpha) = \sup_{\|w_0\| \leq \alpha} \|\Pi_{\mathcal{M}//\mathcal{N}} w_0\|, \ \alpha > 0$$

Definition 6.4. The closed-loop $[\Sigma, \Xi]$ is said to be gain-function (gf)-stable if $g[\Pi_{\mathcal{M}//\mathcal{N}}](\alpha)$ remains finite for all $\alpha \geq 0$.

This permits a notion of bounded input−output stability in which large signals can be amplified at different levels to small signals.

It can be seen that if there exists a positive constant Γ such that

$$\|w_1\| \leq \Gamma \|w_0\|, \ \forall w_0 \in \mathcal{W}$$

then $[\Sigma, \Xi]$ is stable; if there exists a continuous function $\gamma(\cdot) > 0$ such that

$$\|w_1\| \leq \gamma(\|w_0\|), \ \forall w_0 \in \mathcal{W}$$

then $[\Sigma, \Xi]$ is gf-stable.

6.2 Plant Uncertainty and Robust Stability

A control design is based on a nominal mathematical model Σ, which approximately describes the physical plant Σ_1. There is always a plant perturbation (or plant uncertainty) Δ.

Uncertainties can be in many forms, and may have complex structures. Generally speaking, the following types of uncertainties are studied in robust stability.

Additive and Multiplicative Uncertainty

The model uncertainties are expressed by additive perturbations

$$\Sigma_1 = \Sigma + \Delta, \quad \Delta \in \mathcal{H}_\infty, \quad \|W_1 \Delta W_2\|_{\mathcal{H}_\infty} < 1 \tag{6.2}$$

where W_1, W_2 are the weights. At frequencies at which the frequency response of the plant is well known, the weights are chosen to be large to force Δ to be small there; at frequencies at which the frequency response of the plant is highly uncertain, the weights are chosen to be small to allow Δ to be large.

Multiplicative Uncertainties are weighted additive uncertainties, where the perturbed plants are of the form

$$\Sigma_1 = (I + \Delta)\Sigma, \quad \Delta \in \mathcal{H}_\infty, \quad \|W_1 \Delta W_2\|_{\mathcal{H}_\infty} < 1 \tag{6.3}$$

and the symbols are the same as those in the additive uncertainty.

Additive and multiplicative uncertainty models are appropriate for describing low frequency (e.g., parametric) uncertainties.

Inverse Multiplicative Uncertainty

Inverse multiplicative uncertainties are those where the perturbed plants are of the form

$$\Sigma_1 = (I - \Delta)^{-1}\Sigma, \quad \Delta \in \mathcal{H}_\infty, \quad \|W_1 \Delta W_2\|_{\mathcal{H}_\infty} < 1 \tag{6.4}$$

and $I - \Delta$ is in invertible.

Inverse multiplicative uncertainty models are appropriate for describing high frequency unmodelled dynamics.

Coprime Factor Uncertainty

Coprime factor uncertainties are a suitable model for combining uncertainties at both low and high frequencies, i.e. they combine features from all three of the simpler models (additive, multiplicative, and inverse multiplicative) outlined above.

Given $M, N \in \mathcal{H}_\infty$, if there exist X, Y such that

$$XM + YN = I, \quad X, Y \in \mathcal{H}_\infty \tag{6.5a}$$

then we say that M and N are *right coprime*.

Let $\Sigma \in \mathcal{RH}_\infty$, where $\mathcal{RH}_\infty$ is the space of rational $\mathcal{H}_\infty$ functions. We say that the ordered pair $\{N, M\}$ is a *right coprime factorization* of Σ if N and M are right coprime, and M is invertible, and

$$\Sigma = NM^{-1}$$

Moreover, if the pair $\{N, M\}$ satisfies

$$M^*M + N^*N = I \tag{6.6}$$

then we say that the ordered pair $\{N, M\}$ is a *normalized right coprime factorization* of the plant Σ, where M^*, N^* are the conjugates of M, N respectively. The condition (6.6) is equivalent to

$$\left\| \begin{pmatrix} N \\ M \end{pmatrix} V \right\| = \|V\|, \quad \forall V \in L^2$$

Left coprime, left coprime factorization, and normalized left coprime factorization can be defined similarly.

Suppose that

$$\Sigma = NM^{-1}, \quad M, N \in \mathcal{RH}_\infty$$

is a normalized right coprime factorization of the plant Σ. Then coprime factor perturbations take the form

$$\Sigma_1 = (N + \Delta_N)(M + \Delta_M)^{-1}, \quad \left\| \begin{pmatrix} \Delta_N \\ \Delta_M \end{pmatrix} \right\|_\infty < \frac{1}{\gamma}$$

where $\gamma > 1$.

An extensive discussion of these different uncertainty descriptions can be found in e.g., [86].

Robust Stability

A stable closed-loop $[\Sigma, \Xi]$ may become unstable because of the plant perturbations. Hence, the robust stability problem is defined as follows.

Definition 6.5. For a set of plants $\mathcal{P}$, a controller Ξ is said to be robust if $[\Sigma, \Xi]$ is stable for all $\Sigma \in \mathcal{P}$.

For our cases, the robustness problem is to design a controller Ξ for the nominal plant Σ such that the closed-loops $[\Sigma_1, \Xi]$ are stable for all Δ or $(\Delta_M, \Delta_N)^T$ in some set.

6.2.1 Gap Metric

The idea of the gap metric robust stability results are as follows. First, for a nominal plant Σ design a controller Ξ to stabilize the closed-loop $[\Sigma, \Xi]$. Second, define a gap metric distance $\delta(\Sigma, \Sigma_1)$ between the nominal Σ and any perturbed plant Σ_1. Third, if the controller has such property that the closed-loop $[\Sigma_1, \Xi]$ is stable if the gap metric $\delta(\Sigma, \Sigma_1)$ is smaller than some computable constant, then we obtain the robust stability.

For linear case, Zames and EI-Sakkary [91] first introduced the gap metric. In L^2 context, some equivalent expressions for the gap metric [33, 35, 86] are as follows.

$$\begin{aligned}
\vec{\delta}_0(\Sigma, \Sigma_1) &= \sup_{m_1 \in \mathcal{M}_1, \|m_1\| \neq 0} \; \inf_{m \in \mathcal{M}, \|m\| \neq 0} \frac{\|m_1 - m\|}{\|m\|} \\
\vec{\delta}_1(\Sigma, \Sigma_1) &= \|(\Pi_{\mathcal{M}_1} - \Pi_{\mathcal{M}})\Pi_{\mathcal{M}}\| \\
\vec{\delta}(\Sigma, \Sigma_1) &= \begin{cases} \inf_{\Phi \in \mathcal{O}} \|(\Phi - I)|_{\mathcal{M}}\|, & \text{if } \mathcal{O} \neq \varnothing \\ \infty, & \text{if } \mathcal{O} = \varnothing \end{cases} \\
\vec{\delta}_g(\Sigma, \Sigma_1) &= \inf_{(\Delta_N, \Delta_M)^T \in \mathcal{H}_\infty} \left\{ \|(\Delta_N, \Delta_M)^T\|_{\mathcal{H}_\infty} \middle| \Sigma_1 = (N + \Delta_N)(M + \Delta_M)^{-1} \right\}
\end{aligned}$$

where

$$\mathcal{O} = \{\Phi : \mathcal{M} \to \mathcal{M}_1 \mid \Phi \text{ is causal, bijective and } \Phi(0) = 0\}$$

and $\mathcal{M}_1 = \mathcal{G}_{\Sigma_1}$, and (M, N) are normalized right coprime factorizations of Σ, and $\Pi_{\mathcal{K}}$ denotes the orthogonal projection onto a closed subspace $\mathcal{K} \subset \mathcal{W}$.

It has been shown that $\vec{\delta}_0(\Sigma, \Sigma_1)$, $\vec{\delta}_1(\Sigma, \Sigma_1)$, $\vec{\delta}(\Sigma, \Sigma_1)$ and $\vec{\delta}_g(\Sigma, \Sigma_1)$ are equal (see, e.g., [35, 86]). The main result for gap metric robustness is given in the following theorem.

Theorem 6.6. *For a linear plant Σ, if there exits a controller Ξ such that the closed-loop $[\Sigma, \Xi]$ is stable, and the gap metric $\vec{\delta}(\Sigma, \Sigma_1)$ is smaller than some positive constant $b_{\Sigma,\Xi}$, the gap robust margin, then the closed-loop $[\Sigma_1, \Xi]$ is also stable.*

If the plant Σ and a controller Ξ have transfer functions $\Sigma(s)$ and $\Xi(s)$, it can be shown that the parallel operator $\Pi_{\mathcal{M}//\mathcal{N}}$ has transfer function

$$\Pi =: \begin{pmatrix} I \\ \Sigma \end{pmatrix} (I - \Xi\Sigma)^{-1}(I, -\Xi)$$

and the gap robust margin is (see [57])

$$b_{\Sigma,\Xi} = \begin{cases} \left\| \begin{pmatrix} I \\ \Sigma \end{pmatrix} (I - \Xi\Sigma)^{-1}(I, -\Xi) \right\|_{\mathcal{H}^\infty}^{-1}, & \text{if } [\Sigma, \Xi] \text{ is stable} \\ 0, & \text{otherwise} \end{cases}$$

A more useful equation for computing the robust margin $b_{\Sigma,\Xi}$ can be obtained by the coprime factorizations. Let Σ have the coprime factorizations

$$\Sigma = NM^{-1}, \quad M, N \in \mathcal{RH}_\infty$$

$$\Sigma = \tilde{M}^{-1}\tilde{N}, \quad \tilde{M}, \tilde{N} \in \mathcal{RH}_\infty$$

and $U, V, \tilde{U}, \tilde{V}$ be matrices over $\mathcal{H}_\infty$ such that

$$VM + UN = I$$

$$\tilde{M}\tilde{V} + \tilde{N}\tilde{U} = I$$

Then for some $Q \in \mathcal{RH}_\infty$, it can be shown [86] that

$$\Pi = \begin{pmatrix} I \\ \Sigma \end{pmatrix} (I - \Xi\Sigma)^{-1}(I, -\Xi) = \begin{pmatrix} M \\ N \end{pmatrix} (\tilde{V} + Q\tilde{N}, -(\tilde{U} + Q\tilde{M}))$$

and $b_{\Sigma,\Xi}$ can be written as [57]

$$b_{\Sigma,\Xi} = \frac{1}{\sqrt{\left\| \tilde{U}\tilde{M}^* + \tilde{V}\tilde{N}^* + Q \right\|_{\mathcal{L}_\infty}^2 + 1}}$$

This is a convenient formula to calculate the robust margin. So far, for linear systems, the robustness analysis is easy to handle using the gap metric.

6.3 Gap Metric of Nonlinear Systems

It can be seen that this framework for studying robustness of linear systems is effective and produces powerful results. So, it is a natural development to generalize this framework to the nonlinear case. In 1997 Georgiou and Smith [35] published a key paper, in which a proper definition of gap metric for nonlinear plants was obtained and a series of results were established. In 2003, Bian and French [9] proved that the gap of Georgiou and Smith was equal to a gap metric which is defined through the coprime factorizations of nonlinear plants.

6.3.1 Gap Metric

The gap metric for nonlinear plants introduced by Georgiou and Smith [35] is defined as follows.

Definition 6.7. For nonlinear plants Σ and Σ_1, we define the gap metric between Σ_1 and Σ as

$$\vec{\delta}(\Sigma, \Sigma_1) = \begin{cases} \inf_{\Phi \in \mathcal{O}} \|(\Phi - I)|_{\mathcal{M}}\|, & \text{if } \mathcal{O} \neq \varnothing \\ \infty, & \text{if } \mathcal{O} = \varnothing \end{cases}$$
$$\delta(\Sigma, \Sigma_1) = \max\{\vec{\delta}(\Sigma, \Sigma_1), \vec{\delta}(\Sigma_1, \Sigma)\}$$

where

$$\mathcal{O} = \{\Phi : \mathcal{M} \to \mathcal{M}_1 \mid \Phi \text{ is causal, bijective and } \Phi(0) = 0\}$$

and $\mathcal{M}_1 = \mathcal{G}_{\Sigma_1}$.

This definition is indeed a generalization of the L^2 linear case. There is no restriction on the underlying signal space norms, and we will be interested in applying the results in the L^∞ setting. Related notion and the results can found in [35, 8, 9].

The significance for the introduction of the gap metric lies in the following theorems.

Theorem 6.8. *Consider the feedback system in Figure 6.1, and let* $[\Sigma, \Xi]$ *be stable. If a plant* Σ_1 *is such that*

$$\vec{\delta}(\Sigma, \Sigma_1) < \frac{1}{\|\Pi_{\mathcal{M}//\mathcal{N}}\|} \tag{6.7}$$

then $[\Sigma_1, \Xi]$ *is also stable, and*

$$\|\Pi_{\mathcal{M}_1//\mathcal{N}}\| \leq \|\Pi_{\mathcal{M}//\mathcal{N}}\| \frac{1 + \vec{\delta}(\Sigma, \Sigma_1)}{1 - \|\Pi_{\mathcal{M}//\mathcal{N}}\| \vec{\delta}(\Sigma, \Sigma_1)} \tag{6.8}$$

The proof of this theorem can be found in [35].

Theorem 6.8 shows that if a robust controller Ξ for the plant Σ is designed, then the controller is able to stabilize another plant Σ_1 provided that the gap metric between Σ and Σ_1 is suitably small. Hence, this theorem provides a framework to design a robust controller in the presence of input and measurement disturbances and plant perturbations.

6.3.2 Local Stability

The above definition of stability is global for the disturbances, which is a very strong requirement. As an alternative to gain-function stability, we relax the notion of stability to stability on a bounded set.

Definition 6.9. Let S be a bounded set in $\mathcal{W}$, the closed-loop $[\Sigma, \Xi]$ is said to be stable on S if the operator $\Pi_{\mathcal{M}//\mathcal{N}}|_S$ has a finite induced norm.

The corresponding relaxations of the notion of global stability are the notions of semi-global and local stability which are defined as follows.

Definition 6.10. Let Σ be a plant, and $S_r \in \mathcal{W}$ be a ball with the radius $r > 0$. If for any positive constant $r > 0$, there exists a controller Ξ_r such that the closed-loop $[\Sigma, \Xi_r]$ is stable on the ball S_r, then we say that the closed-loop $[\Sigma, \Xi]$ is semi-globally stable.

Definition 6.11. Let Σ be a plant, and let Ξ be a controller. If there exists an open bounded set $S : 0 \in S \subset \mathcal{W}$ such that the closed-loop $[\Sigma, \Xi]$ is stable on S, then we say that the closed-loop is locally stable.

For local stability, we have the following theorem.

Theorem 6.12. *Consider the feedback system in Figure.6.1, and let* $[\Sigma, \Xi]$ *be stable on* S_r *with*

$$\|\Phi_{\mathcal{M}//\mathcal{N}}|_{S_r}\| = \alpha$$

For a perturbed plant Σ_1*, suppose there exists a mapping* $\Phi : \mathcal{M} \cap S_{\alpha r} \to \mathcal{M}_1 \cap \mathcal{W}$ *such that*

$$\|(\Phi - I)|_{\mathcal{M} \cap S_{\alpha r}}\| = \pi < \frac{1}{\alpha} \tag{6.9}$$

and $\Psi = (\Phi - I)\Pi_{\mathcal{M}//\mathcal{N}}$ *is continuous and compact with* $\|\Psi|_{S_r}\| < 1$*. Then the closed-loop* $[\Sigma_1, \Xi]$ *is stable on* $S_{(1-\alpha\pi)r}$*, further*

$$\|\Pi_{\mathcal{M}_1//\mathcal{N}}|_{S_{(1-\alpha\pi)r}}\| \leq \frac{(1+\pi)\alpha}{1-\alpha\pi} \tag{6.10}$$

The proof can be found in [35].

Chapter 7

Robust State Feedback Backstepping Designs

In this chapter, we use gap metric robustness framework of Chapter 6 to develop a robust backstepping design procedure for state feedback control.

In 1995, Freeman [17] gave a counterexample to show that for general nonlinear systems, global internal stabilizability does not imply the global external stabilizability for small sensor disturbances. This means that a standard backstepping design does not automatically guarantee robustness to measurement disturbances. On the other hand, Freeman and Kokotović [21] also showed that the plant in strict-feedback form is input/output stabilizable. So, it is possible to design a controller such that the closed-loop is stable in the presence of external disturbances.

We consider the standard feedback configuration in FIGURE 6.1 and a nominal plant in strict-feedback form, and using a backstepping method, we design a robust controller for the nominal plant in the presence of input and measurement disturbances. Then we make use of the robustness results in Chapter 6 to obtain the robustness of the closed-loop to plant perturbations which are small in the sense of the gap metric, that is, we show that the controller stabilizes the closed-loop for any perturbed plant in the presence of input, measurement and system disturbances if the gap metric distance between the nominal and perturbed plant is less than a computable constant.

A related construction of such a gain-function for the stable operator can be found in [23]. However, in that case, only the measurement disturbances in the form $\rho(|x|)B$ (where B denotes the unit ball in a signal space, and ρ is a $\mathcal{K}_\infty$ function) are allowed. So, the measurement disturbances are required to enter the system equations multiplied by a class $\mathcal{K}_\infty$ function of the state magnitude. This means that the effect of measurement disturbances decreases to zero as the states are regulated to zero. However, actual measurement disturbances could be independent of the state size, and have complex structures. In our results this restriction is not required.

The critical step is the construction of a stable operator between the external disturbances and the internal signals of the closed-loop.

7.1 Problem Formulation

We consider a system which is defined by the following nominal plant in strict-feedback form

$$\Sigma(x_1^0): \quad \dot{x}_{1i} = x_{1(i+1)} + \varphi_i(x_{11}, \cdots, x_{1i}), \quad 1 \le i \le n-1 \tag{7.1a}$$

$$\dot{x}_{1n} = u_1 + \varphi_n\left(x_{11}, \cdots, x_{1(n-1)}, x_{1n}\right), \quad x_{1i}(0) = x_{1i}^0, \quad 1 \le i \le n \tag{7.1b}$$

where $u_1 \in \mathbb{R}$ is the input, and

$$x_1^0 = \begin{pmatrix} x_{11}^0 \\ x_{12}^0 \\ \vdots \\ x_{1n}^0 \end{pmatrix}$$

is the initial condition. Throughout this chapter, we always assume that every φ_i satisfies $\varphi_i(0) = 0$. We further assume that every φ_i is globally Lipschitz continuous, that is there exists a constant L_i such that for any $\omega_1^{(i)}, \omega_2^{(i)} \in \mathbb{R}^i$,

$$\left|\varphi_i\left(\omega_1^{(i)}\right) - \varphi_i\left(\omega_2^{(i)}\right)\right| \le L_i \left\|\omega_1^{(i)} - \omega_2^{(i)}\right\|, \quad i = 1, \cdots, n \tag{7.2}$$

Here, we consider state feedback control, hence

$$y_1 = x_1 = \begin{pmatrix} x_{11} \\ x_{12} \\ \vdots \\ x_{1n} \end{pmatrix}$$

We consider the signal spaces

$$\mathcal{U} = L^\infty(\mathbb{R}^+)$$

and

$$\mathcal{Y} = L^\infty(\mathbb{R}^+) \times \cdots \times L^\infty(\mathbb{R}^+) = L^\infty(\mathbb{R}^+, \mathbb{R}^n)$$

Then

$$\Sigma(x_1^0) : \mathcal{U}_\Sigma \to \mathcal{Y} : \Sigma u_1 \mapsto y_1$$

The norm of the space $\mathcal{Y}$ is defined as

$$\|\cdot\|_\infty = \left(\|\cdot\|_\infty^2 + \cdots + \|\cdot\|_\infty^2\right)^{\frac{1}{2}}$$

With the input and measurement disturbances u_0 and y_0, we will use a backstepping procedure to design a controller $\Xi : y_2 \mapsto u_2$ to achieve gain-function stability for the nominal plant Σ, and stability under zero initial condition. Furthermore, by gap metric theory, if the gap $\vec{\delta}(\Sigma, \Sigma_1)$ between a perturbed plant Σ_1 and the nominal plant Σ is small, then the controller Ξ also stabilizes the plant Σ_1.

7.2 Control Design

For the sake of convenience, we introduce following notation

$$z = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix}, \quad x_1^{(i)} = \begin{pmatrix} x_{11} \\ x_{12} \\ \vdots \\ x_{1i} \end{pmatrix}, \quad z^{(i)} = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_i \end{pmatrix}$$

By a similar backstepping design procedure to [55][1] , we define $z_i, \alpha_i, i = 0, 1, \cdots, n$ by

$$\begin{aligned} z_0 &= 0 \\ \alpha_0 &= 0 \\ z_i &= x_{1i} - \alpha_{i-1}\left(x_1^{(i-1)}\right) \\ \alpha_i\left(x_1^{(i)}\right) &= -c_i z_i - \kappa_i z_i - z_{i-1} + \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_{1j}} x_{1(j+1)}, \quad i = 1, \cdots, n-1 \\ \alpha_n\left(x_1\right) &= -c_n z_n - \kappa_n z_n - z_{n-1} + \sum_{j=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_{1j}} x_{1(j+1)} - \kappa z_n \end{aligned}$$

where $c_i, i = 1, \cdots, n$ and κ can be any positive constants, and $\kappa_i, i = 1, \cdots, n$ are to be specified later.

For $z_i, \alpha_i, i = 1, \cdots, n$, we first give three lemmas.

Lemma 7.1. *For $i = 1, \cdots, n$, α_i is linear with respect to its variables. Thus, $\frac{\partial \alpha_{i-1}}{\partial x_{1j}}$, $i = 1 \cdots, n-1; j = 1 \cdots, i-1$ is constant.*

Furthermore, there exists a positive constant a such that for any $\omega \in \mathbb{R}^n$, it holds

$$|\alpha_n(\omega)| \leq a\|\omega\| \tag{7.3}$$

[1]This is not standard backstepping.

Proof. We use mathematical induction to prove the first claim.

Firstly, z_0 and $\alpha_0(x_{11})$ are linear with respect to x_{11}, hence $\frac{\partial \alpha_0}{\partial x_{11}}$ is a constant.

Secondly, suppose that $z_1, \cdots, z_{i-1}$ and $\alpha_1(x_{11}), \cdots, \alpha_{i-1}\left(x_1^{(i-1)}\right)$ are linear with respect to the variables. Then $\frac{\partial \alpha_{i-1}}{\partial x_{1j}},\ j = 1 \cdots, i-1$ are constants, and

$$z_i\left(x_1^{(i)}\right) = x_{1i} - \alpha_{i-1}\left(x_1^{(i-1)}\right)$$

is also linear. Hence, it can be claimed that $\alpha_i\left(x_1^{(i)}\right)$ is also linear from the definition. This completes the proof for this first part of this lemma.

To prove (7.3), first note that $\alpha_n(\omega)$ is linear from above claim, so there exists a vector $\mathbf{a} \in \mathbb{R}^n$ such that

$$\alpha_n(\omega) = \mathbf{a} \cdot \omega$$

By Cauchy-Schwartz Inequality, it follows that

$$|\alpha_n(\omega)| \leq \|\mathbf{a}\| \|\omega\|$$

Hence (7.3) holds with

$$a = \|\mathbf{a}\|$$

□

Lemma 7.2. *Let*

$$\mathbf{T}_i : x_1^{(i)} \mapsto z^{(i)}, \quad i = 1, \cdots, n$$

then the transformations $\mathbf{T}_i, i = 1, \cdots, n$ *are linear and invertible.*

Proof. From above lemma, α_i and z_i, $i = 0, 1, \cdots, n$ are linear with respect to $x_1^{(i)}$. Hence the transformations $\mathbf{T}_i,\ i = 1, \cdots, n$ are also linear.

We use mathematical induction to prove the claim that $\mathbf{T}_i, i = 1, \cdots, n$ are invertible.

First, we have

$$\mathbf{T}_1 : z_1 = x_{11}$$

hence, $\mathbf{T}_1$ is invertible.

Second, we assume that $\mathbf{T}_i$ is invertible, and prove that $\mathbf{T}_{i+1}$ is also invertible. In fact

$$x_1^{(i+1)} = z_1^{(i+1)} + \alpha_i\left(x_1^{(i)}\right)$$

and by the assumption that $\mathbf{T}_i$ is invertible, we have

$$x_1^{(i)} = \mathbf{T}_i^{-1} z_1^{(i)}$$

Therefore, we obtain

$$x_1^{(i+1)} = z_1^{(i+1)} + \alpha_i\left(\mathbf{T}_i^{-1} z_1^{(i)}\right)$$

that is, $\mathbf{T}_{i+1}$ is invertible.

By the principle of induction, we have proved our claim. □

Lemma 7.3. *Write*

$$a_{(i-1)j} = \frac{\partial \alpha_{i-1}}{\partial x_{1j}}, \quad 1 \leq j < i \leq n-1$$

and

$$M_i = L_i\|\mathbf{T}_i^{-1}\| + \sum_{j=1}^{i-1} L_j|a_{(i-1)j}|\|\mathbf{T}_j^{-1}\|,\ i = 1, \cdots, n$$

Then every $M_i,\ i = 1, \cdots, n$ *is constant and independent of* $\kappa_j,\ j = i, \cdots, n$.

Proof. It is easy to obtain that $M_i,\ i = 1, \cdots, n$ are constants since $\alpha_i,\ i = 0, 1, \cdots, n$ are linear with respect to $x_1^{(i)}$.

We use induction to prove the second claim, that is, we prove that every $z_i,\ i = 0, 1, \cdots, n$ only depends on $\kappa_j,\ j = 0, 1, \cdots, i-1$.

First, z_1 is independent of any κ_j.

Second, suppose that z_{i-1} only depends on $\kappa_j,\ j = 0, 1, \cdots, i-2$. Then, by the definition, z_i only depends on α_{i-1}, which only depends on $\kappa_j,\ j = 0, 1, \cdots, i-1$.

By induction, we have proved our claim. □

By above lemma, M_i is depends on $\kappa_1, \cdots, \kappa_{i-1}$, so, we choose $\kappa_i,\ i = 1, \cdots, n$ such that

$$\kappa_i \geq \frac{n}{2c} M_i^2, \quad i = 1, \cdots, n \tag{7.4}$$

where

$$c = \min_{1 \le i \le n} \{c_i\}$$

We assume hereafter that (7.4) holds, and define a controller $\Xi : \mathcal{Y}_\Xi \to \mathcal{U}$ as follows.

$$\Xi : \quad u_2 = -\alpha_n(-y_2) \tag{7.5}$$

We will show that this controller makes the closed-loop gain-function stable, and stable if the initial condition x_1^0 is zero.

7.3 Stability of Closed-loop

As we stated before, let $\|\cdot\|$ denote the Euclidian norm, and $\|\cdot\|_\infty$ denote the L^∞ norm.

Theorem 7.4. *Let the plant $\Sigma(x_1^0)$ and controller Ξ be defined by (7.1) and (7.5). Then there exists a continuous function $\gamma : \mathbb{R}_+^2 \to [0, +\infty)$ such that for all $(u_0, y_0)^T \in L^\infty(\mathbb{R}^+) \times L^\infty(\mathbb{R}^+, \mathbb{R}^n)$*

$$\left\|(u_1, y_1)^T\right\|_\infty \le \gamma\left(\left\|(u_0, y_0)^T\right\|_\infty, \|x_1^0\|\right) \tag{7.6}$$

that is, the closed-loop $[\Sigma(x_1^0), \Xi]$ is gf-stable.

Moreover, if $x_1^0 = 0$, then there exists a positive constant Γ such that for all $(u_0, y_0)^T \in L^\infty(\mathbb{R}^+) \times L^\infty(\mathbb{R}^+, \mathbb{R}^n))$

$$\left\|(u_1, y_1)^T\right\|_\infty \le \Gamma \left\|(u_0, y_0)^T\right\|_\infty \tag{7.7}$$

that is, the closed-loop $[\Sigma(0), \Xi]$ is stable.

Proof. For convenience of notation, we write

$$z_i = z_i\left(x_1^{(i)}\right), \ \alpha_i = \alpha_i\left(x_1^{(i)}\right), \ \varphi_i = \varphi_i\left(x_1^{(i)}\right)$$

in the proof.

Firstly

$$\begin{aligned}
\dot{z}_i &= \dot{x}_{1i} - \dot{\alpha}_{i-1} \\
&= x_{1(i+1)} + \varphi_i - \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_{1j}} \left(x_{1(j+1)} + \varphi_j \right) \\
&= z_{i+1} + \alpha_i + \varphi_i - \sum_{j=1}^{i-1} a_{(i-1)j} \left(x_{1(j+1)} + \varphi_j \right) \\
&= z_{i+1} - c_i z_i - z_{i-1} - \kappa_i z_i + \sum_{j=1}^{i-1} a_{(i-1)j} x_{1(j+1)} + \varphi_i - \sum_{j=1}^{i-1} a_{(i-1)j} \left(x_{1(j+1)} + \varphi_j \right) \\
&= z_{i+1} - c_i z_i - z_{i-1} - \kappa_i z_i + \varphi_i - \sum_{j=1}^{i-1} a_{(i-1)j} \varphi_j, \quad i = 1, 2, \cdots, n-1
\end{aligned}$$

Since z_n and α_n are linear, and

$$y_1 = x_1,\ y_2 = x_2,\ u_1 + u_2 = u_0,\ y_1 + y_2 = y_0$$

we obtain

$$\begin{aligned}
\dot{z}_n &= \dot{x}_{1n} - \dot{\alpha}_{n-1} \\
&= u_1 + \varphi_n - \sum_{j=1}^{n-1} a_{(n-1)j} \left(x_{1(j+1)} + \varphi_j \right) \\
&= u_0 - u_2 + \varphi_n - \sum_{j=1}^{n-1} a_{(n-1)j} \left(x_{1(j+1)} + \varphi_j \right) \\
&= u_0 + \alpha_n(-y_2) + \varphi_n - \sum_{j=1}^{n-1} a_{(n-1)j} \left(x_{1(j+1)} + \varphi_j \right) \\
&= \alpha_n(y_1) + \varphi_n - \sum_{j=1}^{n-1} a_{(n-1)j} \left(x_{1(j+1)} + \varphi_j \right) + u_0 - \alpha_n(y_1 + y_2) \\
&= \alpha_n(x_1) + \varphi_n - \sum_{j=1}^{n-1} a_{(n-1)j} \left(x_{1(j+1)} + \varphi_j \right) + u_0 - \alpha_n(x_0) \\
&= -c_n z_n - z_{n-1} + \varphi_n - \sum_{j=1}^{n-1} a_{(n-1)j} \varphi_j - \kappa z_n + u_0 - \alpha_n(y_0)
\end{aligned}$$

Consider the Lyapunov function

$$V(z_1, \cdots, z_n) = \frac{1}{2} \sum_{i=1}^{n} z_i^2$$

differentiating along the trajectory of the closed-loop, and writing $z_{n+1} = 0$, then we have

$$
\begin{aligned}
\dot{V} &= \sum_{i=1}^{n} z_i \dot{z}_i \\
&= \sum_{i=1}^{n} z_i \left(z_{i+1} - c_i z_i - z_{i-1} - \kappa_i z_i + \varphi_i - \sum_{j=1}^{i-1} a_{(i-1)j} \varphi_j \right) \\
&\quad - \kappa {z_n}^2 + z_n (u_0 - \alpha_n(y_0)) \\
&= - \sum_{i=1}^{n} c_i z_i^2 + \sum_{i=1}^{n} \left(-\kappa_i z_i^2 + \left(\varphi_i + \sum_{j=1}^{i-1} a_{(i-1)j} \varphi_j \right) z_i \right) \\
&\quad - \kappa {z_n}^2 + z_n \big(u_0 - \alpha_n(y_0) \big)
\end{aligned}
$$

By Young's Inequality (see, e.g., [55]), we obtain

$$
\dot{V} \leq - \sum_{i=1}^{n} c_i z_i^2 + \sum_{i=1}^{n} \frac{1}{4\kappa_i} \left(\varphi_i - \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_{1j}} \varphi_j \right)^2 + \frac{1}{4\kappa} \left(u_0 - \alpha_n(y_0) \right)^2
$$

Since $\varphi_i,\ i = 1, 2, \cdots, n$ are globally Lipschitz continuous, and $\varphi_i(0) = 0,\ i = 1, 2, \cdots, n$, then for all $\omega \in \mathbb{R}$ it hold

$$
|\varphi_i(\omega)| \leq L_i |\omega|,\ i = 1, 2, \cdots, n
$$

hence, by (7.4) we have

$$
\begin{aligned}
\left| \varphi_i - \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_{1j}} \varphi_j \right| &\leq |\varphi_i| + \sum_{j=1}^{i-1} \left| \frac{\partial \alpha_{i-1}}{\partial x_{1j}} \right| |\varphi_j| \\
&\leq L_i \| x_1^{(i)} \| + \sum_{j=1}^{i-1} |a_{(i-1)j}| L_j \| x_1^{(j)} \| \\
&\leq L_i \| \mathbf{T}_i^{-1} z^{(i)} \| + \sum_{j=1}^{i-1} L_j |a_{(i-1)j}| \| \mathbf{T}_j^{-1} z^{(j)} \| \\
&\leq L_i \| \mathbf{T}_i^{-1} \| \| z \| + \sum_{j=1}^{i-1} L_j |a_{(i-1)j}| \| \mathbf{T}_j^{-1} \| \| z \| \\
&= M_i \| z \| \\
&\leq \sqrt{\frac{2c\kappa_i}{n}} \| z \|
\end{aligned}
$$

By Lemma 7.1

$$\begin{aligned}|u_0 - \alpha_n(y_0)| &\leq |u_0| + |\alpha_n(y_0)| \\ &\leq \|u_0\|_\infty + a\|y_0\|_\infty\end{aligned}$$

Finally, we obtain

$$\begin{aligned}\dot{V} &\leq -\sum_{i=1}^{n} c_i z_i^2 + \frac{c}{2n}\|z\|^2 + \frac{1}{4\kappa}(\|u_0\|_\infty + a\|y_0\|_\infty)^2 \\ &\leq -\frac{1}{2}\sum_{i=1}^{n} c_i z_i^2 + \frac{1}{4\kappa}(\|u_0\|_\infty + a\|y_0\|_\infty)^2\end{aligned}$$

Hence, $V(t)$ decreases outside the compact set

$$\mathbf{R} = \left\{ z \in \mathbb{R}^n \middle| \sum_{i=1}^{n} c_i z_i^2 \leq \frac{1}{2\kappa}(\|u_0\|_\infty + a\|y_0\|_\infty)^2 \right\}$$

Now define

$$\mathbf{R}_1 = \left\{ z \in \mathbb{R}^n \middle| \sum_{i=1}^{n} z_i^2 \leq \frac{1}{2c\kappa}(\|u_0\|_\infty + a\|y_0\|_\infty)^2 \right\}$$

where

$$c = \min\{c_i : 1 \leq i \leq n\}$$

so, we obtain that if

$$V(0) \leq \frac{1}{2c\kappa}(\|u_0\|_\infty + a\|y_0\|_\infty)^2$$

then $V(t)$ remains in $\mathbf{R}_1$ for all time $t \geq 0$; if

$$V(0) > \frac{1}{2c\kappa}(\|u_0\|_\infty + a\|y_0\|_\infty)^2$$

then $V(t)$ monotonously decrease from $t = 0$ until z reaches $\mathbf{R}_1$. Hence, we obtain

$$V(t) \leq \max\left\{ V(0),\ \frac{1}{2c\kappa}(\|u_0\|_\infty + a\|y_0\|_\infty)^2 \right\}$$

Therefore

$$\begin{aligned}\|z\| = \sqrt{(2V)} &\leq \max\left\{ \sqrt{2V(0)},\ \frac{1}{\sqrt{c\kappa}}(\|u_0\|_\infty + a\|y_0\|_\infty) \right\} \\ &= \max\left\{ \|z^0\|,\ \frac{1}{\sqrt{c\kappa}}(\|u_0\|_\infty + a\|y_0\|_\infty) \right\}\end{aligned}$$

where

$$z^0 = z(0)$$

Let

$$l = \max\{1, a\}$$

and note that

$$\|z^0\| = \|\mathbf{T}_n x_1^0\| \leq \|\mathbf{T}_n\| \|x_1^0\|$$

$$\begin{aligned} \|u_0\|_\infty + a\|y_0\|_\infty &\leq l(\|u_0\|_\infty + \|y_0\|_\infty) \\ &\leq l\sqrt{2}\left(\|u_0\|_\infty^2 + \|y_0\|_\infty^2\right)^{\frac{1}{2}} \\ &= l\sqrt{2}\left\|(u_0, y_0)^T\right\|_\infty \end{aligned}$$

Then we have

$$\|z\|_\infty \leq \max\left\{\|\mathbf{T}_n\|\|x_1^0\|,\ l\sqrt{\frac{2}{c\kappa}}\left\|(u_0, y_0)^T\right\|_\infty\right\}$$

Since

$$\|y_1\| = \|x_1\| = \|\mathbf{T}_n^{-1} z\| \leq \|\mathbf{T}_n^{-1}\|\|z\|$$

we obtain that

$$\begin{aligned} \|y_1\|_\infty &\leq \|\mathbf{T}_n^{-1}\|\|z\|_\infty \\ &\leq \|\mathbf{T}_n^{-1}\| \max\left\{\|\mathbf{T}_n\|\|x_1^0\|,\ l\sqrt{\frac{2}{c\kappa}}\left\|(u_0, y_0)^T\right\|_\infty\right\} \\ &= h\left(\left\|(u_0, y_0)^T\right\|_\infty, \|x_1^0\|\right) \end{aligned}$$

with

$$h\left(\left\|(u_0, y_0)^T\right\|_\infty, \|x_1^0\|\right) = \|\mathbf{T}_n^{-1}\| \max\left\{\|\mathbf{T}_n\|\|x_1^0\|,\ l\sqrt{\frac{2}{c\kappa}}\left\|(u_0, y_0)^T\right\|_\infty\right\}$$

Moreover, we have

$$\begin{aligned}\|u_1\|_\infty &\le \|u_0\|_\infty + \|u_2\|_\infty \\ &= \|u_0\|_\infty + \| - \alpha_n(-x_2)\|_\infty \\ &\le \|u_0\|_\infty + \|\alpha_n(x_1) - \alpha_n(-x_2)\|_\infty + \|\alpha_n(x_1)\|_\infty \\ &\le \|u_0\|_\infty + a\|y_0\|_\infty + a\|x_1\|_\infty \\ &\le l\sqrt{2}\left\|(u_0, y_0)^T\right\|_\infty + ah\left(\left\|(u_0, y_0)^T\right\|_\infty, \|x_1^0\|\right)\end{aligned}$$

Let

$$\begin{aligned}&\gamma\left(\left\|(u_0, y_0)^T\right\|_\infty, \|x_1^0\|\right) \\ &= \left(\left(l\sqrt{2}\left\|(u_0, y_0)^T\right\|_\infty + ah\left(\left\|(u_0, y_0)^T\right\|_\infty, \|x_1^0\|\right)\right)^2 + \left(h\left(\left\|(u_0, y_0)^T\right\|_\infty, \|x_1^0\|\right)\right)^2\right)^{\frac{1}{2}}\end{aligned}$$

then we have

$$\begin{aligned}\left\|(u_1, y_1)^T\right\|_\infty &= (\|u_1\|_\infty^2 + \|y_1\|_\infty^2)^{\frac{1}{2}} \\ &\le \gamma\left(\left\|(u_0, y_0)^T\right\|_\infty, \|x_1^0\|\right)\end{aligned}$$

This completes the proof of (7.6).

As to (7.7), note that if $x_1^0 = 0$, then

$$h\left(\left\|(u_0, y_0)^T\right\|_\infty, \|x_1^0\|\right) = \|\mathbf{T}_n^{-1}\| l \sqrt{\frac{2}{c\kappa}}\left\|(u_0, y_0)^T\right\|_\infty$$

thus

$$\begin{aligned}&\gamma\left(\left\|(u_0, y_0)^T\right\|_\infty, \|x_1^0\|\right) \\ &= \left(\left(l\sqrt{2}\left\|(u_0, y_0)^T\right\|_\infty + a\|\mathbf{T}_n^{-1}\| l\sqrt{\frac{2}{c\kappa}}\left\|(u_0, y_0)^T\right\|_\infty\right)^2 + \left(\|\mathbf{T}_n^{-1}\| l\sqrt{\frac{2}{c\kappa}}\left\|(u_0, y_0)^T\right\|_\infty\right)^2\right)^{\frac{1}{2}} \\ &= l\sqrt{\frac{2}{c\kappa}}\left(\left(\sqrt{c\kappa} + a\|\mathbf{T}_n^{-1}\|\right)^2 + \|\mathbf{T}_n^{-1}\|^2\right)^{\frac{1}{2}}\left\|(u_0, y_0)^T\right\|_\infty \\ &= \Gamma\left\|(u_0, y_0)^T\right\|_\infty\end{aligned}$$

Therefore (7.7) holds with

$$\Gamma = l\sqrt{\frac{2}{c\kappa}}\left(\left(\sqrt{c\kappa} + a\|\mathbf{T}_n^{-1}\|\right)^2 + \|\mathbf{T}_n^{-1}\|^2\right)^{\frac{1}{2}}$$

□

By gap metric theory, we obtain the following result for any perturbed plant.

Theorem 7.5. *Let the nominal plant* $\Sigma(0)$ *and controller* Ξ *be defined by (7.1) and (7.5). Then there exists a positive constant* Γ *such that for any perturbed plant* Σ_1 *which satisfies*

$$\vec{\delta}(\Sigma(0), \Sigma_1) < \frac{1}{\Gamma}$$

the closed-loop $[\Sigma_1, \Xi]$ *is also stable, and*

$$\|\Pi_{\mathcal{M}_1//\mathcal{N}}\| \leq \Gamma\frac{1 + \vec{\delta}(\Sigma(0), \Sigma_1)}{1 - \Gamma\vec{\delta}(\Sigma(0), \Sigma_1)}$$

Proof. By Theorem 7.4, we obtain that there exists a constant $\Gamma > 0$ such that

$$\|\Pi_{\mathcal{M}//\mathcal{N}}\| \leq \Gamma$$

Then, since

$$\vec{\delta}(\Sigma(0), \Sigma_1) < \Gamma^{-1}$$

it holds that

$$\vec{\delta}(\Sigma(0), \Sigma_1) < \frac{1}{\|\Pi_{\mathcal{M}//\mathcal{N}}\|}$$

Lastly, from Theorem 6.8 in Chapter 6, the proof is complete. □

In above work, we have assumed that all the states are measured and used for feedback control. But, in some cases only the first state is measurable and can be used for feedback, this is the problem of output feedback control. Hence, in the next chapter we will consider the case of robust output feedback control.

Chapter 8

Robust Output Feedback Backstepping Designs

In the previous chapter, we studied a robust backstepping design for state feedback control. In this chapter we consider robust backstepping for output feedback control, which is not considered in [23].

We will consider a nominal plant in output-feedback form and the standard feedback configuration in FIGURE 6.1. We design a robust controller for the nominal plant in the presence of input and measurement disturbances. Then we make use of the robustness results in Chapter 6 to obtain the robustness of the closed-loops to plant perturbations which are small in the sense of the gap metric. That is, as in Chapter 7, we show that the controllers stabilize the closed-loops for any perturbed plants in the presence of input, measurement and system disturbances if the gap metric distance between the nominal and a perturbed plant is less than a computable constant.

In this chapter, we will relax the nonlinearities to be locally Lipschitz continuous and get local results. If the nonlinearities are globally Lipschitz continuous, the results are global.

As an application, we use the theory we established to a system with time delay, and prove that if the time delay is suitably small, the controller is able to achieve stability of the closed-loop.

8.1 Problem Formulation

We consider a nominal plant in output-feedback form

$$\Sigma(x_1^0): \quad \dot{x}_{1i} = x_{1(i+1)} + \varphi_i(y_1), \qquad 1 \le i \le n-1 \tag{8.1a}$$

$$\dot{x}_{1n} = u_1 + \varphi_n(y_1), \qquad x_{1i}(0) = x_{1i}^0, \qquad 1 \le i \le n \tag{8.1b}$$

$$y_1 = x_{11} \tag{8.1c}$$

where $y_1 \in \mathbb{R}$ is the measured output, $u_1 \in \mathbb{R}$ is the input, and $\varphi_i : \mathbb{R} \to \mathbb{R}$, $i = 1, 2, \cdots, n$ are assumed to be either locally or globally Lipschitz continuous, and satisfy $\varphi_i(0) = 0$, $i = 1, 2, \cdots, n$, and

$$x_1^0 = \begin{pmatrix} x_{11}^0 \\ x_{12}^0 \\ \vdots \\ x_{1n}^0 \end{pmatrix}$$

is the initial condition.

With respect to the nominal plant Σ, our main purpose is to use backstepping procedure to design a output feedback controller $\Xi : y_2 \mapsto u_2$, achieving gain-function stability for the plant Σ, and stability under zero initial conditions.

We consider the signal spaces

$$\mathcal{U} = \mathcal{Y} = L^\infty(\mathbb{R}^+)$$

then the output-feedback form plant $\Sigma(x_1^0)$ maps $\mathcal{U}_\Sigma \subseteq L^\infty(\mathbb{R}^+)$ into $L^\infty(\mathbb{R}^+)$.

We introduce the following notation

$$x_1 = \begin{pmatrix} x_{11} \\ x_{12} \\ \vdots \\ x_{1n} \end{pmatrix}, \; A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \; B = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \; \varphi(y) = \begin{pmatrix} \varphi_1(y) \\ \varphi_2(y) \\ \vdots \\ \varphi_n(y) \end{pmatrix}$$

and

$$C = (1, 0, \cdots, 0)$$

to rewrite the plant (8.1) as

$$\Sigma(x_1^0): \quad \dot{x}_1 = Ax_1 + \varphi(y_1) + Bu_1, \quad x_1(0) = x_1^0 \tag{8.2a}$$

$$y_1 = Cx_1 \tag{8.2b}$$

For subsequent use, we give two lemmas here.

Lemma 8.1. *If* $\varphi_i,\ i = 1, 2, \cdots, n$ *are locally Lipschitz continuous, and* $\varphi_i(0) = 0,\ i = 1, 2, \cdots, n$, *then for any* $\rho > 0$ *there exist constants* $L_i(\rho) < \infty$, $i = 1, 2, \cdots, n$ *and* $\delta > 0$ *such that for all* $\omega \in [0, \rho]$ *and* $|\omega_0| < \delta$,

$$|\varphi_i(\omega) - \varphi_i(\omega - \omega_0)| \le L_i(\rho)|\omega_0|,\ i = 1, 2, \cdots, n \tag{8.3}$$

and

$$|\varphi_i(\omega)| \le L_i(\rho)|\omega|,\ i = 1, 2, \cdots, n \tag{8.4}$$

Proof. [1] Since every $\varphi_i,\ i = 1, 2, \cdots, n$ is locally Lipschitz continuous, then for any ω there exist constants $L_i(\omega) < \infty$ and $\delta_i(\omega) > 0$ such that for any $|\omega_0| < \delta_i(\omega)$, we have

$$|\varphi_i(\omega) - \varphi_i(\omega - \omega_0)| \le L_i(\omega)|\omega_0|$$

The family of open sets $\{(\omega - \delta_i(\omega), \omega + \delta_i(\omega))\}_{\omega \in [0,\rho]}$ covers the closed set $[0, \rho]$, hence, by the finite cover theorem[2], there exist finite open sets

$$(\omega_j - \delta_i(\omega_j), \omega_j + \delta_i(\omega_j)),\ \omega_j \in [0, \rho],\ j = 1, 2, \cdots, m$$

such that

$$[0, \rho] \subseteq \bigcap_{j=1}^{m} (\omega_j - \delta_i(\omega_j), \omega_j + \delta_i(\omega_j))$$

Then, for any $\omega \in [0, \rho]$, there exists $j:\ 1 \le j \le m$ such that

$$\omega \in (\omega_j - \delta_i(\omega_j), \omega_j + \delta_i(\omega_j))$$

So, for $|\omega_0| < \delta_i(\omega_j)$, we have

$$|\varphi_i(\omega) - \varphi_i(\omega - \omega_0)| \le L_i(\omega)|\omega_0|$$

Now let

$$L_i(\rho) = \max_{1 \le j \le m} \{L_i(\omega_j)\}$$

[1] If we further assume that $\varphi_i,\ i = 1, 2, \cdots, n$ are differentiable, then the proof can be simply obtained by the mean value theorem.

[2] See, e.g., Q. Douglas, Mathematical Analysis, Clarendon Press, Oxford, 1955

$$\delta = \min_{1\le j\le m, 1\le i\le n} \{\delta_i(\omega_j)\}$$

then $L_i(\rho) < \infty$ and $\delta > 0$, furthermore, (8.3) holds for all $\omega \in [0, \rho]$ and $|\omega_0| < \delta$.

As for (8.4), if $\omega = 0$, it holds; if $|\omega| > 0$, take m_ω points $\varpi_j,\ j = 0, 1, \cdots, m_\omega$ such that

$$0 = \varpi_0 < \varpi_1 < \cdots < \varpi_{m_\omega - 1} < \varpi_{m_\omega} = \omega$$

$$\varpi_{m_\omega} - \varpi_{m_\omega - 1} < \delta$$

Then from the result of first part

$$\begin{aligned} |\varphi_i(\omega)| &= \left| \sum_{j=1}^{m_\omega} \varphi_i(\varpi_j) - \varphi_i(\varpi_{j-1}) \right| \\ &\le \sum_{j=1}^{m_\omega} |\varphi_i(\varpi_j) - \varphi_i(\varpi_{j-1})| \\ &\le \sum_{j=1}^{m_\omega} L_i(\rho)|\varpi_j - \varpi_{j-1}| \\ &= L_i(\rho)|\varpi_{m_\omega} - \varpi_0| \\ &= L_i(\rho)|\omega| \end{aligned}$$

This completes the proof. □

Lemma 8.2. *If $\varphi_i,\ i = 1, 2, \cdots, n$ are globally Lipschitz continuous, and $\varphi_i(0) = 0,\ i = 1, 2, \cdots, n$, then there exist constants $L_i,\ i = 1, 2, \cdots, n$ such that for all $\omega \in \mathbb{R}$*

$$|\varphi_i(\omega)| \le L_i|\omega|,\ i = 1, 2, \cdots, n \tag{8.5}$$

Proof. The proof can be obtained from the globally Lipschitz condition and that $\varphi_i(0) = 0,\ i = 1, 2, \cdots, n$. □

8.2 Control Design And Stability Analysis

We first consider the case when the nonlinearities are locally Lipschitz continuous, and design an output feedback control which is valid locally, before considering globally Lipschitz continuous nonlinearities as a special situation and obtaining a global result.

For our purpose, we first use an amended observer backstepping procedure to design a linear transformation, and further define a state feedback linear controller. Next, we introduce an amended observer to obtain our output feedback controller. Then we make use of the robustness results in Chapter 6 to get the robustness of the controller to plant perturbations in a gap metric sense.

8.2.1 Local Lipschitz Condition

For plant $\Sigma(x_1^0)$, suppose that $\varphi_i,\ i=1,2,\cdots,n$ are locally Lipschitz continuous. Since only the local Lipschitz conditions are assumed, our results will be local.

Take any positive constants $c_i, d_i;\ i=1,2,\cdots,n$ and κ. Suppose ρ is a positive constant, and take a positive constant l which satisfies

$$l \geq \frac{1}{4}\sum_{i=1}^{n}\frac{1}{d_i}\sum_{j=1}^{i}L_j(\rho)^2 \tag{8.6}$$

where $L_i(\rho),\ i=1,2,\cdots,n$ are the constants in (8.3) of Lemma 8.1.

Write

$$z=(z_1,z_2,\cdots,z_n)^T$$

and by the backstepping design procedure[3], define a transformation $\mathbf{T}: x_1 \mapsto z$ as follows

$$z_1(x_{11}) = x_{11} \tag{8.7a}$$

$$\alpha_1(x_{11}) = -c_1z_1 - d_1z_1 - lz_1 \tag{8.7b}$$

$$z_i(x_{11},\cdots,x_{1i}) = x_{1i} - \alpha_{i-1}(x_{11},\cdots,x_{1(i-1)}) \tag{8.7c}$$

$$\begin{aligned}\alpha_i(x_{11},\cdots,x_{1i}) =& -c_iz_i - z_{i-1} - d_i\left(1+\sum_{j=1}^{i-1}\left(\frac{\partial\alpha_{i-1}}{\partial x_{1j}}\right)^2\right)z_i \\ &+\sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_{1j}}x_{1(j+1)}, \qquad i=2,3,\cdots,n\end{aligned} \tag{8.7d}$$

$$\begin{aligned}\alpha_n(x_{11},\cdots,x_{1n}) =& \alpha_n(x_1) \\ =& -c_nz_n - z_{n-1} - d_n\left(1+\sum_{j=1}^{n-1}\left(\frac{\partial\alpha_{n-1}}{\partial x_{1j}}\right)^2\right)z_n - \kappa z_n \\ &+\sum_{j=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial x_{1j}}x_{1(j+1)}\end{aligned} \tag{8.7e}$$

For the transformation $\mathbf{T}$ and α_i, we have the following lemmas.

[3] This is different from the standard backstepping in [55]

Lemma 8.3. *For $i = 1, \cdots, n$, z_i and α_i are linear with respect to their variables. The transformation $\mathbf{T}$ is also linear and invertible. Furthermore, there exists a positive constant a such that for any $\omega \in \mathbb{R}^n$*

$$|\alpha_n(\omega)| \le a\|\omega\| \tag{8.8}$$

Proof. We use induction to prove that z_i, α_i are linear.

First, z_1 and $\alpha_1(x_{11})$ are linear with respect to x_{11}, hence $\frac{\partial \alpha_1}{\partial x_{11}}$ is a constant.

Second, suppose that $z_1, \cdots, z_{i-1}$ and $\alpha_1(x_{11}), \cdots, \alpha_{i-1}(x_{11}, \cdots, x_{1(i-1)})$ are linear with respect to the variables. Then $\frac{\partial \alpha_{i-1}}{\partial x_{1j}}$, $j = 1 \cdots, i-1$ are constants, and $z_i(x_{11}, \cdots, x_{1i}) = x_{1i} - \alpha_{i-1}(x_{11}, \cdots, x_{1(i-1)})$ is also linear. Hence, it can be claimed that $\alpha_i(x_{11}, \cdots, x_{1i})$ is also linear from the definition. This completes the proof the claim.

That $\mathbf{T}$ is linear and invertible can be proved the same way as Lemma 7.2.

As to the (8.8), it can be proved the same way as Lemma 7.1. □

As in Chapter 7, a state feedback controller can be defined as

$$\Xi_o : u_1 = \alpha_n(x_1)$$

Consider the Lyapunov function

$$V(z_1, \cdots, z_n) = \frac{1}{2}\sum_{i=1}^{n} z_i^2$$

differentiating along the trajectories of the closed-loop, following the proof of Theorem 7.4, we can prove that the closed-loop $[\Sigma(x_1^0)), \Xi_o]$ is locally gain-function stable, and the closed-loop $[\Sigma(0)), \Xi_o]$ is locally stable.

But in our case, since only the first state x_{11} is measurable, and only $x_{21} = y_0 - x_{11}$ can used for control designs, to implement control, an observer for x_2 is utilized to estimate the other states. First, write

$$\hat{x}_2 = \begin{pmatrix} \hat{x}_{21} \\ \hat{x}_{22} \\ \vdots \\ \hat{x}_{2n} \end{pmatrix}, \quad \hat{x}_2^0 = \begin{pmatrix} \hat{x}_{21}^0 \\ \hat{x}_{22}^0 \\ \vdots \\ \hat{x}_{2n}^0 \end{pmatrix}$$

and define an observer[4] by

$$\dot{\hat{x}}_2 = A\hat{x}_2 - K(y_2 - \hat{y}_2) - \varphi(-y_2) - Bu_2, \quad \hat{x}_2(0) = \hat{x}_2^0 \tag{8.9a}$$

$$\hat{y}_2 = C\hat{x}_2 \tag{8.9b}$$

where

$$K = \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{pmatrix}$$

is chosen such that $A_0 = A - KC$ is Hurwitz. Note that $\hat{x}_2^0$ is the initial observer, and $y_2 = x_{21}$.

To obtain an output feedback controller, write

$$\hat{x}_2^* = \begin{pmatrix} \hat{x}_{22} \\ \vdots \\ \hat{x}_{2n} \end{pmatrix} \in \mathbb{R}^{n-1}$$

and we define the output feedback controller as

$$\begin{aligned} \Xi(\hat{x}_2^0): \ & u_2 = -\alpha_n(-y_2, -\hat{x}_2^*) \\ & \dot{\hat{x}}_2 = A\hat{x}_2 + K(y_2 - \hat{y}_2) - \varphi(-y_2) - Bu_2, \quad \hat{x}_2(0) = \hat{x}_2^0 \\ & \hat{y}_2 = C\hat{x}_2 \end{aligned} \tag{8.10}$$

We first establish a lemma for the estimate error.

Lemma 8.4. *Let x_1 be the state in the plant (8.2), and $\hat{x}_2$ be the observer state in (8.9), and let*

$$\tilde{x} = x_1 + \hat{x}_2 \tag{8.11}$$

be the perturbed observer error, then $\tilde{x}$ satisfies

$$\dot{\tilde{x}} = A_0\tilde{x} + \varphi(y_1) - \varphi(-y_2) - Ky_0 + Bu_0, \quad \tilde{x}(0) = \tilde{x}^0 \tag{8.12}$$

where

$$\tilde{x}^0 = x_1^0 + \hat{x}_2^0$$

[4] This is also different from the observer in [55].

Moreover, if $|y_1| \le \rho$ *and* $|y_0| < \delta$*, then there exist constants* b *and* ν_ρ *such that*

$$\|\tilde{x}\|_\infty \le b\left(\|\tilde{x}^0\| + \nu_\rho\|y_0\|_\infty + \|u_0\|_\infty\right) \tag{8.13}$$

Proof. [5] Note that

$$y_1 + y_2 = y_0, \quad u_1 + u_2 = u_0$$

and then from (8.2) and (8.9), it follows that $\tilde{x}$ satisfies (8.12).

Now we estimate $\tilde{x}$. By (8.12), we obtain that

$$\tilde{x} = \tilde{x}^0 e^{A_0 t} + \int_0^t e^{A_0(t-\tau)}\Big(\varphi(y_1(\tau)) - \varphi(-y_2(\tau)) - Ky_0(\tau) + Bu_0(\tau)\Big)d\tau \tag{8.14}$$

Let $\lambda_i,\ i = 1, \cdots, n$ be the eigenvalues of matrix A_0. Since the matrix A_0 is Hurwitz, the real parts of all its eigenvalues are negative. Let μ be a positive constant such that

$$-\mu > \mathrm{Re}\lambda_i, \quad i = 1, \cdots, n$$

then there exists a positive constant b such that

$$\|e^{A_0 t}\| \le b e^{-\mu t} \tag{8.15}$$

Hence

$$\begin{aligned}
\|\tilde{x}(t)\| \le & \|\tilde{x}^0 e^{A_0 t}\| + \int_0^t \|e^{A_0(t-\tau)}\|\Big(\|\varphi(y_1(\tau)) - \varphi(-y_2(\tau))\| + \|Ky_0(\tau)\| + \|Bu_0(\tau)\|\Big)d\tau \\
\le & \|\tilde{x}^0\|\|e^{A_0 t}\| + \int_0^t \|e^{A_0(t-\tau)}\| \left[\left(\sum_{i=1}^n \Big(\varphi_i(y_1(\tau)) - \varphi_i(-y_2(\tau))\Big)^2\right)^{\frac{1}{2}}\right. \\
& \left. + \left(\sum_{i=1}^n k_i^2\right)^{\frac{1}{2}} |y_0(\tau)| + |u_0(\tau)|\right] d\tau \\
\le & b\|\tilde{x}^0\|e^{-\mu t} + \int_0^t \|e^{A_0(t-\tau)}\| \left[\left(\sum_{i=1}^n \Big(\varphi_i\big(y_1(\tau)\big) - \varphi_i\big(y_1(\tau) - y_0(\tau)\big)\Big)^2\right)^{\frac{1}{2}}\right. \\
& \left. + \left(\sum_{i=1}^n k_i^2\right)^{\frac{1}{2}} |y_0(\tau)| + |u_0(\tau)|\right] d\tau
\end{aligned}$$

[5] By ISS stability, a simple proof can be obtained.

As $|y_1(\tau)| < \rho$ and $|y_0(\tau)| < \delta$, from (8.3) it follows that

$$\begin{aligned}\|\tilde{x}(t)\| \leq & b\|\tilde{x}^0\| + b\int_0^t e^{-\mu(t-\tau)}\left(\left(\sum_{i=1}^n L_i(\rho)^2\big|y_0(\tau)\big|^2\right)^{\frac{1}{2}} + \left(\sum_{i=1}^n k_i^2\right)^{\frac{1}{2}}|y_0(\tau)| + |u_0(\tau)|\right)d\tau \\ \leq & b\|\tilde{x}^0\| + b\left(\left(\left(\sum_{i=1}^n L_i(\rho)^2\right)^{\frac{1}{2}} + \left(\sum_{i=1}^n k_i^2\right)^{\frac{1}{2}}\right)\|y_0\|_\infty + \|u_0\|_\infty\right)\int_0^t e^{-\mu(t-\tau)}d\tau \\ \leq & b\left(\|\tilde{x}^0\| + \frac{1}{\mu}\left(\left(\sum_{i=1}^n L_i(\rho)^2\right)^{\frac{1}{2}} + \left(\sum_{i=1}^n k_i^2\right)^{\frac{1}{2}}\right)\|y_0\|_\infty + \|u_0\|_\infty\right) \\ = & b\left(\|\tilde{x}^0\| + \nu_\rho\|y_0\|_\infty + \|u_0\|_\infty\right)\end{aligned}$$

with

$$\nu_\rho = \frac{1}{\mu}\left(\left(\sum_{i=1}^n L_i(\rho)^2\right)^{\frac{1}{2}} + \left(\sum_{i=1}^n k_i^2\right)^{\frac{1}{2}}\right)$$

Therefore (8.13) holds. □

Since we only assume that the nonlinear terms of the plant are locally Lipschitz continuous, we can only hope for local stability results. For convenience we introduce the following notations:

$$c = \min_{1\leq i\leq n}\{c_i\}, \quad c_0 = \max_{1\leq i\leq n}\{c_i\}, \quad M = \max\left\{1 + 3a^2b^2, a\left(1 + 3b^2\nu_\rho^2\right)\right\}$$

$$\pi_1 = \frac{\rho\sqrt{2c\kappa}}{3\sqrt{M}}, \quad \pi_2 = \frac{\rho\sqrt{2c\kappa}}{3\sqrt{3}ab}, \quad \pi_3 = \frac{2\rho\sqrt{c}}{3\|\mathbf{T}\|\sqrt{c_0}}, \quad \pi_\delta = \min\{\pi_1, \delta\}$$

Theorem 8.5. *Consider the plant $\Sigma(x_1^0)$ defined by (8.1), and let φ_i, $i = 1, 2, \cdots, n$ be locally Lipschitz continuous. Consider the controller $\Xi(\hat{x}_2^0)$ defined by (8.7) and (8.10). Then*

1. *For any disturbance $(u_0, y_0)^T \in L^\infty(\mathbb{R}^+) \times L^\infty(\mathbb{R}^+)$, $\|(u_0, y_0)^T\|_\infty \leq \pi_\delta$, initial state $x_1^0 \in \mathbb{R}^n$, $\|x_1^0\| \leq \pi_3$, and initial error $\tilde{x}^0 \in \mathbb{R}^n$, $\|\tilde{x}^0\| \leq \pi_2$, there exists a positive constant γ_ρ such that*

$$\left\|(u_1, y_1)^T\right\|_\infty \leq \gamma_\rho \tag{8.16}$$

   that is, the closed-loop system $[\Sigma(x_1^0), \Xi(\hat{x}_2^0)]$ is locally gain-function stable.

2. *If $x_1^0 = \hat{x}_2^0 = 0$, then for any disturbance $(u_0, y_0)^T \in L^\infty(\mathbb{R}^+) \times L^\infty(\mathbb{R}^+)$, $\|(u_0, y_0)^T\|_\infty \leq \pi_\delta$, there exists a positive constant Γ_ρ such that*

$$\left\|(u_1, y_1)^T\right\|_\infty \leq \Gamma_\rho\left\|(u_0, y_0)^T\right\|_\infty \tag{8.17}$$

that is, the closed-loop system $[\Sigma(0), \Xi(0)]$ *is locally stable.*

Proof. Let us first establish 1.

Throughout the proof, for simplicity of notation, we will use α_i to denote $\alpha_i(x_{11}, \cdots, x_{1i})$, z_i to denote $z_i(x_{11}, \cdots, x_{1i})$, φ_i to denote $\varphi_i(x_{11})$, and

$$a_{(i-1)j} = \frac{\partial}{\partial x_{1j}} \alpha_{i-1} \left(x_{11}, \cdots, x_{1(i-1)}\right), \quad i = 1, \cdots, n, \; j = 1, \cdots, i-1$$

Consider the Lyapunov function

$$V(z_1, \cdots, z_n) = \frac{1}{2} \sum_{i=1}^{n} z_i^2 \tag{8.18}$$

then we can establish that

$$\dot{V} \leq -\sum_{i=1}^{n} c_i z_i^2 + c \left(\frac{2\rho}{3}\right)^2 \tag{8.19}$$

In fact, along the solution of the closed-loop, we have

$$\begin{aligned} \dot{z}_1 = \dot{y}_1 = \dot{x}_{11} =& x_{12} + \varphi_1 \\ =& z_2 + \alpha_1 + \varphi_1 \\ =& z_2 - c_1 z_1 - d_1 z_1 + \varphi_1 - l z_1 \end{aligned}$$

and

$$\begin{aligned} \dot{z}_i =& \dot{x}_{1i} - \dot{\alpha}_{i-1} \\ =& x_{1(i+1)} + \varphi_i - \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_{1j}} \left(x_{1(j+1)} + \varphi_j\right) \\ =& z_{i+1} + \alpha_i + \varphi_i - \sum_{j=1}^{i-1} a_{(i-1)j} \left(x_{1(j+1)} + \varphi_j\right) \\ =& z_{i+1} - c_i z_i - z_{i-1} - d_i \left(1 + \sum_{j=1}^{i-1} a_{(i-1)j}^2\right) z_i + \sum_{j=1}^{i-1} a_{(i-1)j} x_{1(j+1)} \\ & + \varphi_i - \sum_{j=1}^{i-1} a_{(i-1)j} \left(x_{1(j+1)} + \varphi_j\right) \\ =& z_{i+1} - c_i z_i - z_{i-1} - d_i \left(1 + \sum_{j=1}^{i-1} a_{(i-1)j}^2\right) z_i + \varphi_i - \sum_{j=1}^{i-1} a_{(i-1)j} \varphi_j, \quad i = 2, 3, \cdots, n-1 \end{aligned}$$

and

$$\begin{aligned}\dot{z}_n =&\dot{x}_{1n} - \dot{\alpha}_{n-1}\\ =&u_1 + \varphi_n - \sum_{j=1}^{n-1} a_{(n-1)j}\left(x_{1(j+1)} + \varphi_j\right)\\ =&u_0 - u_2 + \varphi_n - \sum_{j=1}^{n-1} a_{(n-1)j}\left(x_{1(j+1)} + \varphi_j\right)\\ =&u_0 + \alpha_n(-y_2, -\hat{x}_2^*) + \varphi_n - \sum_{j=1}^{n-1} a_{(n-1)j}\left(x_{1(j+1)} + \varphi_j\right)\end{aligned}$$

Noting that α_n is linear with respect to its variables, we obtain

$$\begin{aligned}\dot{z}_n =&u_0 + \alpha_n(y_1, x_1^*) - \alpha_n(y_1 + y_2, x_1^* + \hat{x}_2^*) + \varphi_n - \sum_{j=1}^{n-1} a_{(n-1)j}\left(x_{1(j+1)} + \varphi_j\right)\\ =&u_0 + \alpha_n(x_1) - \alpha_n(y_0, \tilde{x}^*) + \varphi_n - \sum_{j=1}^{n-1} a_{(n-1)j}\left(x_{1(j+1)} + \varphi_j\right)\\ =&- c_n z_n - z_{n-1} - d_n\left(1 + \sum_{j=1}^{n-1} a_{(n-1)j}^2\right) z_n + \varphi_n - \sum_{j=1}^{n-1} a_{(n-1)j}\varphi_j + u_0 - \alpha_n(y_0, \tilde{x}^*)\end{aligned}$$

Write $z_0 = 0,\ z_{n+1} = 0$, then along the solution of the closed-loop, we have

$$\begin{aligned}\dot{V} =&\sum_{i=1}^{n} z_i \dot{z}_i\\ =&\sum_{i=1}^{n} z_i\left(z_{i+1} - c_i z_i - z_{i-1} - d_i\left(1 + \sum_{j=1}^{i-1} a_{(i-1)j}^2\right) z_i + \varphi_i - \sum_{j=1}^{i-1} a_{(i-1)j}\varphi_j\right)\\ &- l z_1^2 - \kappa {z_n}^2 + \big(u_0 - \alpha_n(y_0, \tilde{x}^*)\big) z_n\\ =&- \sum_{i=1}^{n} c_i z_i^2 - l z_1^2 + \sum_{i=1}^{n}\left(-d_i z_i^2 + z_i\varphi_i + \sum_{j=1}^{i-1}\Big(-d_i a_{(i-1)j}^2 z_i^2 - a_{(i-1)j} z_i \varphi_j\Big)\right)\\ &- \kappa {z_n}^2 + z_n\big(u_0 - \alpha_n(y_0, \tilde{x}^*)\big)\end{aligned}$$

By Young's Inequality, we obtain

$$\begin{aligned}\dot{V} \leq&- \sum_{i=1}^{n} c_i z_i^2 - l z_1^2 + \sum_{i=1}^{n} \frac{1}{4d_i}\left(\varphi_i^2 + \sum_{j=1}^{i-1} \varphi_j^2\right) + \frac{1}{4\kappa}\big(u_0 - \alpha_n(y_0, \tilde{x}^*)\big)^2\\ =&- \sum_{i=1}^{n} c_i z_i^2 - l z_1^2 + \frac{1}{4}\sum_{i=1}^{n} \frac{1}{d_i}\sum_{j=1}^{i} \varphi_j^2 + \frac{1}{4\kappa}\big(u_0 - \alpha_n(y_0, \tilde{x}^*)\big)^2\end{aligned}$$

and we now claim that $y_1(t) < \rho$ for all $t \geq 0$.

For a contradiction, assume the claim does not hold, i.e., there at lest exists a finite time $t^* > 0$ such that $y_1(t^*) \geq \rho$. Let t_s be the smallest time at which $|y_1(t_s)| = \rho$. Then we get the following claims: First, $t_s > 0$ since $|y_1(0)| = |x_{11}^0| \leq \|x_1^0\| < \rho$. Second, for $t \in [0, t_s)$, we have $|y_1(t)| < \rho$.

For $t \in [0, t_s)$, it holds that $|y_1(t)| < \rho$. Hence, for $t \in [0, t_s)$, by Lemma 8.1, we obtain that

$$\begin{aligned}
\dot{V} &\leq -\sum_{i=1}^{n} c_i z_i^2 - l z_1^2 + \frac{1}{4}\sum_{i=1}^{n} \frac{1}{d_i} \sum_{j=1}^{i} L_j(\rho)^2 y_1^2 + \frac{1}{4\kappa}\big(u_0 - \alpha_n(y_0, \tilde{x}^*)\big)^2 \\
&= -\sum_{i=1}^{n} c_i z_i^2 - l z_1^2 + \frac{1}{4}\sum_{i=1}^{n} \frac{1}{d_i} \sum_{j=1}^{i} L_j(\rho)^2 z_1^2 + \frac{1}{4\kappa}\big(u_0 - \alpha_n(y_0, \tilde{x}^*)\big)^2 \\
&\leq -\sum_{i=1}^{n} c_i z_i^2 + \frac{1}{4\kappa}\big(u_0 - \alpha_n(y_0, \tilde{x}^*)\big)^2
\end{aligned} \tag{8.20}$$

We now estimate the last term for $\tau \in [0, t_s)$ and $t \in [0, \tau]$. By Lemma 8.4, we have

$$\begin{aligned}
\frac{1}{4\kappa}\big(u_0 - \alpha_n(y_0, \tilde{x}^*)\big)^2 \leq& \frac{1}{2\kappa}\big(u_0^2 + \alpha_n(y_0, \tilde{x}^*)^2\big) \\
\leq& \frac{1}{2\kappa}\left(\|u_0\|_\infty^2 + a^2\|(y_0, \tilde{x}^*)^T\|^2_{L^\infty[0,\tau)}\right) \\
\leq& \frac{1}{2\kappa}\left(\|u_0\|_\infty^2 + a^2\left(\|y_0\|_\infty^2 + \|\tilde{x}^*\|^2_{L^\infty[0,\tau)}\right)\right) \\
\leq& \frac{1}{2\kappa}\Big(\|u_0\|_\infty^2 + a^2\Big(\|y_0\|_\infty^2 + b^2\big(\|\tilde{x}^0\| + \nu\|y_0\|_\infty + \|u_0\|_\infty\big)^2\Big)\Big) \\
\leq& \frac{1}{2\kappa}\Big(\|u_0\|_\infty^2 + a^2\Big(\|y_0\|_\infty^2 + 3b^2\big(\|\tilde{x}^0\|^2 + \nu^2\|y_0\|_\infty^2 + \|u_0\|_\infty^2\big)\Big)\Big) \\
=& \frac{1}{2\kappa}\Big((1 + 3a^2b^2)\|u_0\|_\infty^2 + (a^2 + 3b^2\nu^2)\|y_0\|_\infty^2 + 3a^2b^2\|\tilde{x}^0\|^2\Big) \\
\leq& \frac{1}{2\kappa}M\|(u_0, y_0)^T\|_\infty^2 + \frac{3}{2\kappa}a^2b^2\|\tilde{x}^0\|^2
\end{aligned} \tag{8.21}$$

From $\|(u_0, y_0)^T\|_\infty \leq \pi_\delta \leq \pi_1$ and $\|\tilde{x}^0\|_\infty \leq \pi_2$, we obtain that

$$\begin{aligned}
\frac{1}{4\kappa}\big(u_0 - \alpha_n(y_0, \tilde{x}^*)\big)^2 &\leq \frac{1}{2\kappa}M\pi_1^2 + \frac{3}{2\kappa}a^2b^2\pi_2^2 \\
&\leq \frac{2c\rho^2}{9} + \frac{2c\rho^2}{9} \\
&= c\left(\frac{2\rho}{3}\right)^2
\end{aligned}$$

So, for $t \in [0, \tau)$ we have

$$\dot{V} \leq -\sum_{i=1}^{n} c_i z_i^2 + c\left(\frac{2\rho}{3}\right)^2 \tag{8.22}$$

Therefore $V(t)$ decreases outside the compact set

$$\mathbf{R} = \left\{ z \in \mathbb{R}^n \,\middle|\, \sum_{i=1}^{n} c_i z_i^2 \leq c\left(\frac{2\rho}{3}\right)^2 \right\}$$

Since

$$\sum_{i=1}^{n} c_i (z_i^0)^2 \leq c_0 \|z^0\|^2 \leq c_0 \|\mathbf{T}\|^2 \|x_1^0\|^2 \leq c_0 \|\mathbf{T}\|^2 \pi_3^2 < c\left(\frac{2\rho}{3}\right)^2$$

we have $z^0 \in \mathbf{R}$. So, we obtain that for all $t \in [0, \tau)$, $z(t) \in \mathbf{R}_1$, which is defined by

$$\mathbf{R}_1 = \left\{ z \in \mathbb{R}^n \,\middle|\, = \sum_{i=1}^{n} z_i^2 \leq \left(\frac{2\rho}{3}\right)^2 \right\}$$

Hence

$$y_1^2 = x_{11}^2 = z_1^2 \leq \|z\|^2 < \left(\frac{2\rho}{3}\right)^2, \;\; t \in [0, \tau)$$

or

$$|y_1(t)| < \frac{2\rho}{3}, \;\; t \in [0, \tau)$$

From this we obtain that for any $\tau < t_s$

$$\|y_1\|_{L^\infty[0,\tau]} < \frac{2\rho}{3}$$

This is contrary to the fact that

$$\|y_1\|_{L^\infty[0,t_s]} = \rho$$

and $\|y_1\|_{L^\infty[0,\tau]}$ is continuous with respect to τ since $y_1(t)$ is continuous.

This completes the proof of the claim, and shows that

$$\|y_1\|_\infty = \|y_1\|_{L^\infty[0,+\infty)} < \rho$$

Now we prove that u_1 is also bounded. In fact

$$\begin{aligned} u_1 &= u_0 - u_2 \\ &= u_0 + \alpha_n(-y_2, -\hat{x}_2^*) \\ &= u_0 + \alpha_n(y_1 - y_0, x_1^* - \tilde{x}^*) \\ &= u_0 + \alpha_n(y_1, x_1^*) - \alpha_n(y_0, \tilde{x}^*) \\ &= u_0 + \alpha_n(x_1) - \alpha_n(y_0, \tilde{x}^*) \end{aligned}$$

since α_n is linear. Hence

$$\begin{aligned} \|u_1\|_\infty &\le \|u_0\|_\infty + \|\alpha_n(x_1)\|_\infty + \|\alpha_n(y_0, \tilde{x}^*)\|_\infty \\ &\le \|u_0\|_\infty + a\|x_1\|_\infty + a\|(y_0, \tilde{x}^*)\|_\infty \\ &= \|u_0\|_\infty + a\left(\|x_1\|_\infty + (\|y_0\|_\infty^2 + \|\tilde{x}^*\|_\infty^2)^{\frac{1}{2}}\right) \end{aligned} \tag{8.23}$$

Since $\|u_0\|_\infty, \|y_0\|_\infty$ are bounded by the assumptions of the theorem, we need only show that $\|\tilde{x}^*\|_\infty$ and $\|x_1\|_\infty$ are bounded.

From the first part of the proof, we have obtained that $|y_1(t)| < \rho$ for all $t \in [0, +\infty)$, therefore, (8.13) holds for all $t \in [0, +\infty)$. So

$$\|\tilde{x}^*\|_\infty \le \|\tilde{x}\|_\infty \le b\left(\|\tilde{x}^0\| + \nu_\rho\|y_0\|_\infty + \|u_0\|_\infty\right) \tag{8.24}$$

is bounded. From $z(t) \in \mathbf{R}_1$, and

$$\|x_1\|_\infty = \|\mathbf{T}^{-1}z\|_\infty \le \|\mathbf{T}^{-1}\|\|z\|_\infty \tag{8.25}$$

we know that $\|x_1\|_\infty$ is also bounded. Hence $\|u_1\|_\infty$ is bounded.

Therefore we have established 1. Now we establish 2.

Since $x_1^0 = \hat{x}_2^0 = 0$, we have $\tilde{x}^0 = 0$. From (8.21), we obtain

$$\frac{1}{4\kappa}\left(u_0 - \alpha_n(y_0, \tilde{x}^*)\right)^2 \le \frac{M}{2\kappa}\|(u_0, y_0)^T\|_\infty^2$$

Hence

$$\dot{V} \leq -\sum_{i=1}^{n} c_i z_i^2 + \frac{M}{2\kappa}\|(u_0, y_0)^T\|_\infty^2 \tag{8.26}$$

Similarly, we can obtain

$$\|z\|_\infty \leq \sqrt{\frac{M}{2c\kappa}}\|(u_0, y_0)^T\|_\infty \tag{8.27}$$

So

$$\|y_1\|_\infty \leq \|z\|_\infty \leq \sqrt{\frac{M}{2c\kappa}}\|(u_0, y_0)^T\|_\infty \tag{8.28}$$

By (8.23), (8.24), (8.25) and (8.27), we have

$$\begin{aligned}
\|u_1\|_\infty &\leq \|u_0\|_\infty + a\left(\|x_1\|_\infty + (\|y_0\|_\infty^2 + \|\tilde{x}^*\|_\infty^2)^{\frac{1}{2}}\right) \\
&\leq \|u_0\|_\infty + a\left(\|\mathbf{T}^{-1}\|\|z\|_\infty + \left(\|y_0\|_\infty^2 + (\nu_\rho\|y_0\|_\infty + \|u_0\|_\infty)^2\right)^{\frac{1}{2}}\right) \\
&\leq \|u_0\|_\infty + a\left(\|\mathbf{T}^{-1}\|\sqrt{\frac{M}{2c\kappa}}\|(u_0, y_0)^T\|_\infty + \left(\|y_0\|_\infty^2 + (\nu_\rho\|y_0\|_\infty + \|u_0\|_\infty)^2\right)^{\frac{1}{2}}\right) \\
&\leq \|u_0\|_\infty + a\left(\|\mathbf{T}^{-1}\|\sqrt{\frac{M}{2c\kappa}}\|(u_0, y_0)^T\|_\infty + \left(\|y_0\|_\infty^2 + \left(2\nu_\rho^2\|y_0\|_\infty^2 + 2\|u_0\|_\infty^2\right)\right)^{\frac{1}{2}}\right)
\end{aligned}$$

Let

$$\lambda_\rho = \max\{2, 1 + 2\nu_\rho^2\}$$

then

$$\begin{aligned}
\|u_1\|_\infty &\leq \|(u_0, y_0)^T\|_\infty + a\left(\|\mathbf{T}^{-1}\|\sqrt{\frac{M}{2c\kappa}}\|(u_0, y_0)^T\|_\infty + \sqrt{\lambda_\rho}\|(u_0, y_0)^T\|_\infty\right) \\
&\leq \left(1 + a\|\mathbf{T}^{-1}\|\sqrt{\frac{M}{2c\kappa}} + \sqrt{\lambda_\rho}\right)\|(u_0, y_0)^T\|_\infty
\end{aligned} \tag{8.29}$$

Write

$$\Gamma_\rho = \frac{M}{2c\kappa} + \left(1 + a\|\mathbf{T}^{-1}\|\sqrt{\frac{M}{2c\kappa}} + \sqrt{\lambda_\rho}\right)^2 \tag{8.30}$$

then by (8.28) and (8.29), we obtain that

$$\|(u_1, y_1)^T\|_\infty \leq \Gamma_\rho\|(u_0, y_0)^T\|_\infty$$

Thus, we have established 2. □

Since we only assume that the nonlinearities of the plant are locally Lipschitz continuous, the results are only local, which are weaker than semi-global. It remains an open question as to whether semi-global results can be obtained.

The purpose of the framework of gap metric robustness is to allow plant perturbations. If the plant Σ and controller Ξ satisfy the conditions of Theorem 8.5, and let

$$\mathcal{W} = L^\infty(\mathbb{R}^+) \times L^\infty(\mathbb{R}^+),\ S_r = \{s \in \mathcal{W} \big| \|s\| \leq r\}$$

then $\|\Pi_{\mathcal{M}//\mathcal{N}}|_{S_{\pi_\delta}}\|$ is finite by Theorem 8.5, and we can obtain the following result.

Theorem 8.6. *Let plant $\Sigma(0)$ and controller $\Xi(0)$ satisfy the conditions of Theorem 8.5, and let*

$$\|\Pi_{\mathcal{M}//\mathcal{N}}|_{S_{\pi_\delta}}\| = \alpha$$

Let Σ_1 denote a perturbed plant, and suppose there exists a mapping $\Phi : \mathcal{M} \cap S_{\alpha\pi_\delta} \to \mathcal{M}_1 \cap \mathcal{W}$ such that

$$\|(\Phi - I)|_{\mathcal{M}\cap S_{\alpha\pi_\delta}}\| = \pi < \frac{1}{\alpha} \tag{8.31}$$

and

$$\Psi = (\Phi - I)\Pi_{\mathcal{M}//\mathcal{N}}$$

is continuous and compact with

$$\|\Psi|_{S_{\pi_\delta}}\| < 1$$

then the closed-loop $[\Sigma_1, \Xi(0)]$ is stable on $S_{(1-\alpha\pi)\pi_\delta}$ with

$$\|\Pi_{\mathcal{M}_1//\mathcal{N}}|_{S_{(1-\alpha\pi)\pi_\delta}}\| \leq \frac{(1+\pi)\alpha}{1-\alpha\pi} \tag{8.32}$$

that is, the closed-loop is locally stable.

Proof. Since $\|\Pi_{\mathcal{M}//\mathcal{N}}|_{S_{\pi_\delta}}\|$ is finite by Theorem 8.5, the result follows from Theorem 6.12. □

We will give an application of the global version of this result in Section 8.3.

8.2.2 Global Lipschitz Condition

For the nominal plant $\Sigma(x_1^0)$, if we suppose that the nonlinearities $\varphi_i,\ i = 1, 2, \cdots, n$ are globally Lipschitz continuous, then Lemma 8.2 holds, and we can obtain a global result for $(u_0, y_0)^T$, $\tilde{x}^0$ and x_1^0.

Take $c_i, d_i;\ i = 1, 2, \cdots, n$ and κ as any positive constants. Let $L_i,\ i = 1, 2, \cdots, n$ be the constants defined in Lemma 8.2. We take l such that

$$l \geq \frac{1}{4} \sum_{i=1}^{n} \frac{1}{d_i} \sum_{j=1}^{i} L_j^2 \tag{8.33}$$

Define α_n, and the observer, and the controller $\Xi(\hat{x}_2^0)$ the same way as (8.7), (8.9) and (8.10). Then we have the following lemma.

Lemma 8.7. *Let $\tilde{x}$ be the perturbed observer error defined as in Lemma 8.4. Then*

$$\|\tilde{x}\|_\infty \leq b \left(\|\tilde{x}^0\| + \nu \|y_0\|_\infty + \|u_0\|_\infty \right) \tag{8.34}$$

where b and μ are constants.

Proof. The proof is almost the same as that of Lemma 8.4, hence, it is omitted. □

From this lemma we can prove the following theorem.

Theorem 8.8. *Consider the plant $\Sigma(x_1^0)$ defined by (8.1), and let $\varphi_i,\ i = 1, 2, \cdots, n$ be globally Lipschitz continuous. Let the controller $\Xi(\hat{x}_2^0)$ be defined by (8.7), (8.9) and (8.10). Then*

1. *There exists a continuous function $\gamma : \mathbb{R}_+^3 \to (0, +\infty)$ such that for any $(u_0, y_0)^T \in L^\infty(\mathbb{R}^+) \times L^\infty(\mathbb{R}^+)$, $\tilde{x}^0 \in \mathbb{R}^n$ and $x_1^0 \in \mathbb{R}^n$, we have*

$$\left\| (u_1, y_1)^T \right\|_\infty \leq \gamma \left(\left\| (u_0, y_0)^T \right\|_\infty, \|\tilde{x}^0\|_\infty, \|x_1^0\|_\infty \right) \tag{8.35}$$

   that is, the closed-loop system $[\Sigma(x_1^0), \Xi(\hat{x}_2^0)]$ is globally gf-stable.

2. *If $x_1^0 = \hat{x}_2^0 = 0$, then there exists a positive constant Γ such that for any $(u_0, y_0)^T \in L^\infty(\mathbb{R}^+) \times L^\infty(\mathbb{R}^+)$, we have*

$$\left\| (u_1, y_1)^T \right\|_\infty \leq \Gamma \left\| (u_0, y_0)^T \right\|_\infty \tag{8.36}$$

   that is, the closed-loop system $[\Sigma(0), \Xi(0)]$ is globally stable.

Proof. Again consider the Lyapunov function

$$V(z_1, \cdots, z_n) = \frac{1}{2} \sum_{i=1}^{n} z_i^2$$

Following the proof of Theorem 8.5, we obtain that for $t \in [0, \infty)$

$$\dot{V} \leq -\sum_{i=1}^{n} c_i z_i^2 + \frac{1}{4\kappa}\left(\|u_0\|_\infty + \|\alpha_n(y_0, \tilde{x}^*\|_\infty\right)^2$$

By Lemma 8.3, Lemma 8.7, and by noting that $\|\tilde{x}^*\| \leq \|\tilde{x}\|$, we have

$$\begin{aligned}\|\alpha_n(y_0, \tilde{x}^*)\|_\infty &\leq a(\|y_0\|_\infty^2 + \|\tilde{x}^*\|_\infty^2)^{\frac{1}{2}} \\ &\leq a\left(\|y_0\|_\infty^2 + b^2\left(\|\tilde{x}^0\| + \nu\|y_0\|_\infty + \|u_0\|_\infty\right)^2\right)^{\frac{1}{2}}\end{aligned}$$

Let

$$a^* = \max\{1, a^2\},\ \iota = \max\{1, \nu\}$$

then

$$\begin{aligned}(\|u_0\|_\infty + \|\alpha_n(y_0, \tilde{x}^*)\|_\infty)^2 \leq & \left(\|u_0\|_\infty + a\left(\|y_0\|_\infty^2 + b^2\left(\|\tilde{x}^0\| + \nu\|y_0\|_\infty + \|u_0\|_\infty\right)^2\right)^{\frac{1}{2}}\right)^2 \\ \leq & 2\left(\|u_0\|_\infty^2 + a^2\left(\|y_0\|_\infty^2 + b^2\left(\|\tilde{x}^0\| + \nu\|y_0\|_\infty + \|u_0\|_\infty\right)^2\right)\right) \\ \leq & 2\left(a^*\|(u_0, y_0)^T\|_\infty^2 + a^2 b^2\left(\|\tilde{x}^0\| + \iota\|(u_0, y_0)^T\|_\infty\right)^2\right) \\ = & 2g\left(\|(u_0, y_0)^T\|_\infty, \|\tilde{x}^0\|\right)^2 \end{aligned} \tag{8.37}$$

where

$$g = g(p, q) = \left(a^* p^2 + a^2 b^2 (q + \iota p)^2\right)^{\frac{1}{2}}$$

Thus

$$\dot{V} \leq -\sum_{i=1}^{n} c_i z_i^2 + \frac{1}{2\kappa} g^2\left(\|(u_0, y_0)^T\|_\infty, \|\tilde{x}^0\|\right)$$

Following the same argument in Theorem 8.5, we obtain

$$\|z\|_\infty \leq \max\left\{\|z^0\|,\ \frac{1}{\sqrt{2c\kappa}} g\left(\|(u_0, y_0)^T\|_\infty, \|\tilde{x}^0\|\right)\right\}$$

Note that we have

$$\|y_1\| = \|x_{11}\| = \|z_1\| \leq \|z\|$$

On the other hand,

$$\|y_1\| = \|x_{11}\| \leq \|x_1\| = \|\mathbf{T}^{-1}z_1\| \leq \|\mathbf{T}^{-1}\|\|z\|$$

Hence

$$\|y_1\| \leq \min\{1, \|\mathbf{T}^{-1}\|\}\|z\|$$

Note that

$$\|z^0\| = \|\mathbf{T}x_1^0\| \leq \|\mathbf{T}\|\|x_1^0\|$$

and write

$$\beta = \min\{1, \|\mathbf{T}^{-1}\|\}$$

therefore

$$\begin{aligned} \|y_1\|_\infty &\leq \beta\|z\|_\infty \\ &\leq \beta \max\left\{\|\mathbf{T}\|\|x_1^0\|,\ \frac{1}{\sqrt{2c\kappa}} g\left(\|(u_0, y_0)^T\|_\infty, \|\tilde{x}^0\|\right)\right\} \\ &\leq \beta \max\left\{\|\mathbf{T}\|\|x_1^0\|,\ \frac{1}{\sqrt{2c\kappa}} g\left(\|(u_0, y_0)^T\|_\infty, \|\tilde{x}^0\|\right)\right\} \\ &= h\left(\left\|(u_0, y_0)^T\right\|, \|\tilde{x}^0\|, \|x_1^0\|\right) \end{aligned} \tag{8.38}$$

where

$$h(p, q, s) = \beta \max\left\{\|\mathbf{T}\|s,\ \frac{1}{\sqrt{2c\kappa}} g(p, q)\right\}$$

Moreover, we have

$$\begin{aligned} |\alpha_n(x_1)| &\leq a\|x_1\| = a\|\mathbf{T}^{-1}z\| \leq a\|\mathbf{T}^{-1}\|\|z\| \leq a\|\mathbf{T}^{-1}\|\|z\|_\infty \\ &\leq a\|\mathbf{T}^{-1}\| \max\left\{\|\mathbf{T}\|\|x_1^0\|,\ \frac{1}{\sqrt{2c\kappa}} g\left(\|(u_0, y_0)^T\|_\infty, \|\tilde{x}^0\|\right)\right\} \\ &= h^*\left(\left\|(u_0, y_0)^T\right\|, \|\tilde{x}^0\|, \|x_1^0\|\right) \end{aligned} \tag{8.39}$$

where

$$h^*(p, q, s) = a\|\mathbf{T}^{-1}\| \max\left\{s\|\mathbf{T}\|,\ \frac{1}{\sqrt{2c\kappa}} g(p, q)\right\}$$

Hence

$$\begin{aligned}
\|u_1\|_\infty &\le \|u_0\|_\infty + \|u_2\|_\infty \\
&= \|u_0\|_\infty + \| - \alpha_n(-y_2, -\hat{x}_2^*)\|_\infty \\
&\le \|u_0\|_\infty + \|\alpha_n(y_1, x_1^*) - \alpha_n(-y_2, -\hat{x}_2^*)\|_\infty + \|\alpha_n(y_1, x_1^*)\|_\infty \\
&= \|u_0\|_\infty + \|\alpha_n(y_0, \tilde{x}^*)\|_\infty + \|\alpha_n(x_1)\|_\infty \\
&\le \sqrt{2} g\left(\|(u_0, y_0)^T\|_\infty, \|\tilde{x}^0\|\right) + h^*\left(\left\|(u_0, y_0)^T\right\|, \|\tilde{x}^0\|, \|x_1^0\|\right) \qquad (8.40)
\end{aligned}$$

Therefore from (8.38) and (8.40), we obtain

$$\begin{aligned}
&\|(u_1, y_1)^T\|_\infty \\
&= \left(\|u_1\|_\infty^2 + \|y_1\|_\infty^2\right)^{\frac{1}{2}} \\
&\le \Bigg[\left(\sqrt{2} g\left(\|(u_0, y_0)^T\|_\infty, \|\tilde{x}^0\|\right) + h^*\left(\left\|(u_0, y_0)^T\right\|, \|x_1^0\|, \|\tilde{x}^0\|\right)\right)^2 \\
&\quad + h\left(\left\|(u_0, y_0)^T\right\|, \|\tilde{x}^0\|, \|x_1^0\|\right)^2 \Bigg]^{\frac{1}{2}} \qquad (8.41)
\end{aligned}$$

Let

$$\gamma(p, q, s) = \left(\left(\sqrt{2} g(p, q) + h^*(p, q, s)\right)^2 + h(p, q, s)^2\right)^{\frac{1}{2}} \qquad (8.42)$$

then we have

$$\left\|(u_1, y_1)^T\right\| \le \gamma\left(\left\|(u_0, y_0)^T\right\|, \|\tilde{x}^0\|, \|x_1^0\|\right) \qquad (8.43)$$

This completes the proof of (8.35).

To prove (8.36), note that

$$g(p, 0) = \left(a^* + a^2 b^2 \iota^2\right)^{\frac{1}{2}} |p|$$

$$h^*(p, 0, 0) = \frac{\|\mathbf{T}^{-1}\|}{\sqrt{2c\kappa}} g(p, 0)$$

and

$$h(p, 0, 0) = \frac{\beta}{\sqrt{2c\kappa}} g(p, 0)$$

therefore

$$\begin{aligned}\gamma(p,0,0) &= \left(\left(\sqrt{2}g(p,0)+h^*(p,0,0)\right)^2+h(p,0,0)^2\right)^{\frac{1}{2}}\\ &= \left(\left(\sqrt{2}+\frac{\|\mathbf{T}^{-1}\|}{\sqrt{2c\kappa}}\right)^2+\frac{\beta^2}{2c\kappa}\right)^{\frac{1}{2}}\left(a^*+a^2b^2\iota^2\right)|p|\end{aligned}$$

Let

$$\Gamma = \left(\left(\sqrt{2}+\frac{\|\mathbf{T}^{-1}\|}{\sqrt{2c\kappa}}\right)^2+\frac{\beta^2}{2c\kappa}\right)^{\frac{1}{2}}\left(a^*+a^2b^2\iota^2\right) \tag{8.44}$$

then

$$\left\|(u_1,y_1)^T\right\|_\infty \le \gamma\left(\left\|(u_0,y_0)^T\right\|_\infty,0,0\right) = \Gamma\left\|(u_0,y_0)^T\right\|_\infty \tag{8.45}$$

Therefore (8.36) holds. □

For plant perturbations, we obtain the following robustness result.

Theorem 8.9. *Let the plant* $\Sigma(x^0)$ *and the controller* $\Xi(\hat{x}^0)$ *satisfy the conditions of Theorem 8.8. Then there exists* $\Gamma > 0$ *such that if a plant* Σ_1 *satisfies*

$$\vec{\delta}(\Sigma(0),\Sigma_1) < \frac{1}{\Gamma} \tag{8.46}$$

the closed-loop $[\Sigma_1,\Xi(0)]$ *is also stable, and*

$$\|\Pi_{\mathcal{M}_1//\mathcal{N}}\| \le \Gamma\frac{1+\vec{\delta}(\Sigma(0),\Sigma_1)}{1-\Gamma\vec{\delta}(\Sigma(0),\Sigma_1)} \tag{8.47}$$

Proof. By Theorem 8.8, there exists a constant $\Gamma > 0$ such that

$$\|\Pi_{\mathcal{M}//\mathcal{N}}\| \le \Gamma$$

or

$$\frac{1}{\Gamma} \le \frac{1}{\|\Pi_{\mathcal{M}//\mathcal{N}}\|}$$

Hence, if $\vec{\delta}(\Sigma(0),\Sigma) < \Gamma^{-1}$, we obtain that

$$\vec{\delta}(\Sigma(0),\Sigma_1) < \frac{1}{\|\Pi_{\mathcal{M}//\mathcal{N}}\|}$$

From Theorem 6.8 in Chapter 6, the proof is completed. □

8.3 Application to a System with Time Delay

In this section we analyse the robustness of stability for a system perturbed by a time delay. We first consider a nominal plant without time delay, and design a robust backstepping controller to stabilize the closed-loop. Then we consider the system with time delay as a perturbed plant, by above robustness results we have built up, we show that the controller is able to stabilize the system with time delay if the time delay is less than a computable constant.

Suppose the nominal plant Σ is defined by

$$\begin{aligned} \Sigma: \quad \dot{x}_{11} &= x_{12} - 2y_1 + \sin y_1 \\ \dot{x}_{12} &= u_1 - y_1, \qquad\qquad x_{11}(0) = 0,\ x_{12} = 0 \\ y_1 &= x_{11} \end{aligned}$$

where

$$\varphi_1(y_1) = -2y_1 + \sin y_1, \quad \varphi_2(y_1) = -y_1$$

are globally Lipschitz continuous.

By (8.10), the backstepping robust controller Ξ is designed as

$$\begin{aligned} \Xi: \quad u_2 &= -\alpha_2(-y_2, -\hat{x}_{22}) = -b_1 y_2 - b_2 \hat{x}_{22} \\ \dot{\hat{x}}_{21} &= x_{21} + k_1(y_2 - \hat{y}_2) - 2y_2 + \sin y_2 \\ \dot{\hat{x}}_{22} &= k_2(y_2 - \hat{y}_2) - y_2, \qquad \hat{x}_2(0) = 0 \\ \hat{y}_2 &= \hat{x}_{21} \end{aligned}$$

where

$$\begin{aligned} b_1 &= a_1 c_2 + 1 + d_2(1 + a_1^2)a_1 \\ b_2 &= c_2 + d_2(1 + a_1^2) + a_1 \\ a_1 &= c_1 + d_1 + l \end{aligned}$$

Then, from Theorem 8.8, the closed-loop $[\Sigma, \Xi]$ is stable.

Now we consider the effect of time delay on the closed-loop. Suppose a plant Σ_1 is defined by

$$\begin{aligned} \Sigma_1: \quad \dot{x}_{11}(t) &= x_{12}(t) - 2y_1(t) + \sin y_1(t) \\ \dot{x}_{12}(t) &= u_1(t) - y_1(t), \qquad x_{11}(0) = 0,\ x_{12} = 0 \\ y_1(t) &= x_{11}(t - \varsigma) \end{aligned}$$

where ς is the time delay.

We define a mapping $\Phi : \mathcal{M} \to \mathcal{M}_1$ by

$$\Phi \begin{pmatrix} u_1(t) \\ x_{11}(t) \end{pmatrix} = \begin{pmatrix} u_1(t) \\ x_{11}(t-\varsigma) \end{pmatrix}$$

then we have

$$\begin{aligned} \|\Phi - I\| &= \sup_{\|(u_1(t),x_{11}(t))^T\|_\infty \neq 0} \frac{\|(u_1(t), x_{11}(t-\varsigma))^T - (u_1(t), x_{11}(t))^T\|_\infty}{\|(u_1(t), x_{11}(t))^T\|_\infty} \\ &= \sup_{\|(u_1(t),x_{11}(t))^T\|_\infty \neq 0} \frac{\|(0, x_{11}(t-\varsigma) - x_{11}(t))^T\|_\infty}{\|(u_1(t), x_{11}(t))^T\|_\infty} \\ &\leq \sup_{\|(u_1(t),x_{11}(t))^T\|_\infty \neq 0} \frac{\|\dot{x}_{11}\|_\infty \varsigma}{\|(u_1(t), x_{11}(t))^T\|_\infty} \end{aligned}$$

by the mean value theorem.

To estimate $\|\dot{x}_{11}\|_\infty$, rewrite the plant Σ as

$$\dot{x}_1 = Dx_1 + J_1 \sin x_{11} + J_2 u_1, \qquad x_1(0) = 0 \tag{8.48}$$

where

$$x_1 = \begin{pmatrix} x_{11} \\ x_{12} \end{pmatrix},\ D = \begin{pmatrix} -2 & 1 \\ -1 & 0 \end{pmatrix},\ J_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ J_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

It can be verified that D is Hurwitz, and the two eigenvalues are -1. Hence, there exists a constant b^* such that

$$\|e^{Dt}\| \leq b^* e^{-\frac{1}{2}t}$$

Further, we rewrite (8.48) as the integral equation

$$x_1(t) = \int_0^t e^{D(t-\tau)} \big(J_1 \sin x_{11}(\tau) + J_2 u_1(\tau)\big) d\tau \tag{8.49}$$

So,

$$\begin{aligned} \|x_1\|_\infty &\leq \int_0^t \|e^{Dt(-\tau)}\| \|J_1 \sin x_{11} + J_2 u_1\|_\infty d\tau \\ &\leq \int_0^t b^* e^{-\frac{1}{2}(t-\tau)} \|(\sin x_{11}, u_1)^T\|_\infty d\tau \\ &\leq 2b^* \|(u_1, x_{11})^T\|_\infty \end{aligned} \tag{8.50}$$

since $|\sin x_{11}| \leq |x_{11}|$. Therefore, from the plant Σ, we obtain

$$\begin{aligned}\|\dot{x}_{11}\|_\infty &\leq \|\dot{x}_1\|_\infty \\ &\leq \|D\|\|x_1\|_\infty + \|J_1 \sin x_{11} + J_2 u_1\|_\infty \\ &\leq (2b^*\|D\| + 1)\|(u_1, x_{11})^T\|_\infty \\ &= \sigma\|(u_1, x_{11})^T\|_\infty\end{aligned}$$

where

$$\sigma = 2b^*\|D\| + 1$$

is a positive constant.

Hence

$$\|\Phi - I\| \leq \sigma\zeta$$

By the definition of directed gap, we obtain that

$$\vec{\delta}(\Sigma, \Sigma_1) \leq \sigma\varsigma$$

On the other hand, we know that if

$$\vec{\delta}(\Sigma, \Sigma_1) \leq \frac{1}{\|\Pi_{\mathcal{M}//\mathcal{N}}\|}$$

then the closed-loop $[\Sigma_1, \Xi]$ is stable. Hence, we obtain that if

$$\varsigma \leq \frac{1}{\sigma\|\Pi_{\mathcal{M}//\mathcal{N}}\|}$$

then the closed-loop $[\Sigma, \Xi_1]$ is stable, that is, if the time delay is less than some computable quantity[6], the controller designed for the nominal plant is able to stabilize the closed-loop with the presence of time delay.

So far, we have studied robust backstepping for state feedback and output feed back control. In the next chapter, we will consider the robustness of high-gain observer designs.

[6]The norm $\|\Pi_{\mathcal{M}//\mathcal{N}}\|$ can estimated by following the proof of Theorem 8.5.

Chapter 9

Robust High-gain Observer Designs

When the high-gain observer is applied to output feedback, it is required that the high-gain factor ϵ is small enough. This results in the concern that the robustness to loop disturbances and plant perturbations for this control design may be sensitive to ϵ, and may degrade as ϵ becomes small.

Indeed it is believed that the high-gain observer design is sensitive to loop disturbances and plant perturbations. But it is surprising that the simulation results in [47] show that the high-gain observer design exhibits almost the same level of degradation as other designs in the presence of disturbances. To date, there are no results about the robustness of high-gain designs.

In this chapter, we consider the standard feedback configuration in FIGURE 6.1, and employ an amended high-gain observer design to design a controller, and prove the controller is robust to disturbances and small plant perturbations, and not sensitive to ϵ, *provided* the initial error is zero. For these results, the plant is restricted to have a matched, globally Lipschitz nonlinearity depending on the output only, hence the results in this chapter only represent a preliminary investigation into the robustness of high-gain observer designs.

9.1 Problem Formulation

To investigate the robustness of high-gain designs to loop disturbances and plant perturbations, we consider a nonlinear nominal plant in normal form

$$\Xi(x_1^0): \quad \dot{x}_{1i} = x_{1(i+1)}, \quad 1 \leq i \leq n-1 \tag{9.1a}$$

$$\dot{x}_{1n} = u_1 + \varphi(y_1), \quad x_{1i}(0) = x_{1i}^0, \quad 1 \leq i \leq n \tag{9.1b}$$

$$y_1 = x_{11} \tag{9.1c}$$

where, for simplicity, we assume $\varphi : \mathbb{R} \to \mathbb{R}$ is globally Lipschitz continuous, and $\varphi(0) = 0$.

We first rewrite the system as

$$\Sigma(x_1^0): \quad \dot{x}_1 = Ax_1 + B(\varphi(y_1) + u_1), \quad x_1(0) = x_1^0 \tag{9.2a}$$

$$y_1 = Cx_1 \tag{9.2b}$$

where

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \; B = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \; x_1 = \begin{pmatrix} x_{11} \\ x_{12} \\ \vdots \\ x_{1n} \end{pmatrix}, \; x_1^0 = \begin{pmatrix} x_{11}^0 \\ x_{12}^0 \\ \vdots \\ x_{1n}^0 \end{pmatrix}$$

and

$$C = (1, 0, \cdots, 0)$$

We consider the standard feedback configuration in FIGURE 6.1. We first design an output feedback controller $\Xi : y_2 \mapsto u_2$, which is robust to loop disturbances, then we prove this controller has a non-zero gap metric margin to any plant perturbations.

We will consider the signal spaces

$$\mathcal{U} = \mathcal{Y} = L^\infty(\mathbb{R}^+)$$

then the output-feedback form plant $\Sigma(x_1^0)$ maps $\mathcal{U}_\Sigma \subseteq L^\infty(\mathbb{R}^+)$ into $L^\infty(\mathbb{R}^+)$.

9.2 Control Design

We first amend the standard high-gain observer in [48, 45, 3] so that it can be used for our design purpose. Here, we define a high-gain observer as

$$\dot{\hat{x}}_2 = A\hat{x}_2 - H(y_2 - \hat{y}_2) + Bk\hat{x}_2, \qquad \hat{x}_2(0) = \hat{x}_2^0 \tag{9.3a}$$

$$\hat{y}_2 = C\hat{x}_2 \tag{9.3b}$$

where

$$H = \begin{pmatrix} \frac{\beta_1}{\epsilon} \\ \frac{\beta_2}{\epsilon^2} \\ \cdots \\ \frac{\beta_n}{\epsilon^n} \end{pmatrix} \tag{9.4}$$

and

$$k = (k_1, \cdots, k_n)$$

is chosen such that $A + Bk$ is Hurwitz.

Then we define a controller as

$$\Xi_{H(\epsilon)}(\hat{x}_2^0): \quad u_2 = \varphi(-y_2) + k\hat{x}_2 \tag{9.5a}$$

$$\dot{\hat{x}}_2 = A\hat{x}_2 + H(y_2 - \hat{y}_2) + Bk\hat{x}_2, \quad \hat{x}_2(0) = \hat{x}_2^0 \tag{9.5b}$$

$$\hat{y}_2 = C\hat{x}_2 \tag{9.5c}$$

9.3 Robustness Analysis

First we prove a lemma about the estimate of the observer error.

Lemma 9.1. *Let x_1 be the state of the plant in (9.2), and $\hat{x}_2$ be observer state in (9.3), and let*

$$\tilde{x} = x_1 + \hat{x}_2$$

be the perturbed observer error. Then there exist positive constants b and β such that

$$\|\tilde{x}\|_\infty \leq \frac{b}{\epsilon^{n-1}}\|\tilde{x}^0\| + \epsilon\beta\|(u_0, y_0)^T\|_\infty$$

Proof. The closed-loop $[\Sigma(x_1^0), \Xi_{H(\epsilon)}(\hat{x}_2^0)]$ can be written as

$$\dot{x}_1 = Ax_1 + B\big(\varphi(y_1) - \varphi(-y_2) - k\hat{x}_2 + u_0\big)$$

$$\dot{\hat{x}}_2 = A\hat{x}_2 + H\big(y_0 - (x_{11} + \hat{x}_{21})\big) + Bk\hat{x}_2$$

Write

$$\tilde{x} = x_1 + \hat{x}_2, \quad \tilde{x}_i = x_{1i} + \hat{x}_{2i}, \ i = 1, 2, \cdots, n$$

then, from above two equations, we obtain

$$\dot{\tilde{x}} = A\tilde{x} - H\tilde{x}_1 + Hy_0 + B(\varphi(y_1) - \varphi(-y_2) + u_0) \tag{9.6}$$

Let

$$\xi_i = \frac{\tilde{x}_i}{\epsilon^{n-i}}, \quad i = 1, 2, \cdots, n$$

and we write (9.6) as

$$\epsilon\dot{\xi} = D\xi + \epsilon\big(Ey_0 + B(\varphi(y_1) - \varphi(-y_2) + u_0)\big) \tag{9.7}$$

where

$$D = A - EC$$

and

$$E = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \cdots \\ \beta_n \end{pmatrix}$$

It can be verified that the matrix D is Hurwitz (see Chapter 2).

By a time transformation $t = \epsilon\tau$, (9.7) can be written as

$$\frac{d\xi}{d\tau} = D\xi + \epsilon\big(Ey_0 + B(\varphi(y_1) - \varphi(-y_2) + u_0)\big) \tag{9.8}$$

Solving (9.8), we obtain

$$\xi(\tau) = e^{D\tau}\xi^0 + \epsilon\int_0^\tau e^{D(\tau-s)}\left(Ey_0(s) + B\Big(\varphi\big(y_1(s)\big) - \varphi\big(-y_2(s)\big) + u_0(s)\Big)\right)ds \tag{9.9}$$

where

$$\xi^0 = \begin{pmatrix} \tilde{x}_1^0 \\ \vdots \\ \frac{\tilde{x}_i^0}{\epsilon^{i-1}} \\ \vdots \\ \frac{\tilde{x}_n^0}{\epsilon^{n-1}} \end{pmatrix}$$

Since D is Hurwitz, all the real parts of the eigenvalues of D are negative. We take a positive constant μ such that $-\mu$ is greater than all the real parts of the eigenvalues of D, then there exists a positive constant b such that

$$\|e^{D\tau}\| \leq be^{-\mu\tau}$$

On the other hand, by Lipschitz condition there exists a positive constant L such that

$$|\varphi(y_1) - \varphi(-y_2)| \leq L|y_1 + y_2| = L|y_0|$$

Since ϵ is a small constant, without loss of generality, we assume that $\epsilon < 1$. Therefore, from (9.9), we obtain

$$\begin{aligned}
\|\xi(\tau)\| &\leq \|\xi^0\|\|e^{D\tau}\| + \epsilon \int_0^\tau \|e^{D(\tau-s)}\| \left(\|Ey_0\| + \left(\|\varphi(y_1) - \varphi(-y_2)\|\right) + \|u_0\|\right) ds \\
&\leq \|\xi^0\| b e^{-\mu\tau} + \epsilon \int_0^\tau b e^{-\mu(\tau-s)} \left((\|E\| + L)\|y_0\|_\infty + \|u_0\|_\infty\right) ds \\
&\leq \frac{b}{\epsilon^{n-1}}\|\tilde{x}^0\| + \frac{b\epsilon}{\mu}\left((\|E\| + L)\|y_0\|_\infty + \|u_0\|_\infty\right) \\
&\leq \frac{b}{\epsilon^{n-1}}\|\tilde{x}^0\| + \epsilon\beta\|(u_0, y_0)^T\|_\infty
\end{aligned}$$

where

$$\beta = \frac{b\sqrt{2}}{\mu} \max\{\|E\| + L, 1\}$$

Therefore

$$\|\xi\|_\infty \leq \frac{b}{\epsilon^{n-1}}\|\tilde{x}^0\| + \epsilon\beta\|(u_0, y_0)^T\|_\infty$$

Again from $\epsilon < 1$, and

$$\tilde{x}_i = \epsilon^{n-i}\xi_i, \quad i = 1, 2, \cdots, n$$

we obtain $\|\tilde{x}\| \leq \|\xi\|$, further

$$\|\tilde{x}\|_\infty \leq \|\xi\|_\infty \leq \frac{b}{\epsilon^{n-1}}\|\tilde{x}^0\| + \epsilon\beta\|(u_0, y_0)^T\|_\infty$$

and the proof is complete. □

Now we state and prove the main result of this chapter.

Theorem 9.2. *Let the plant $\Sigma(x_1^0)$ and controller $\Xi_{H(\epsilon)}(\hat{x}_2^0)$ be defined by (9.1) and (9.5). Then*

1. *For any $\epsilon < 1$, there exists a continuous function $\gamma_\epsilon : \mathbb{R}_+^3 \to [0, +\infty)$ such that for all $(u_0, y_0)^T \in L^\infty(\mathbb{R}^+) \times L^\infty(\mathbb{R}^+)$*

$$\left\|(u_1, y_1)^T\right\|_\infty \leq \gamma_\epsilon\left(\left\|(u_0, y_0)^T\right\|_\infty, \|\tilde{x}_0\|, \|x_1^0\|_\infty\right) \tag{9.10}$$

that is, the closed-loop $[\Sigma(x_1^0), \Xi_{H(\epsilon)}(\hat{x}_2^0)]$ is gf-stable.

2. *If $x_1^0 = \hat{x}_2^0 = 0$, then for any $\epsilon < 1$, there exists a positive constant Γ, which is independent of ϵ, such that for all $(u_0, y_0)^T \in L^\infty(\mathbb{R}^+) \times L^\infty(\mathbb{R}^+)$*

$$\left\|(u_1, y_1)^T\right\|_\infty \leq \Gamma \left\|(u_0, y_0)^T\right\|_\infty \tag{9.11}$$

that is, the closed-loop $[\Sigma(0), \Xi(0)]$ is stable.

Proof. Let Q be the solution of the equation

$$(A + Bk)^T Q + Q(A + Bk) = -2I$$

and consider the Lyapunov function

$$V(x_{11}, \cdots, x_{1n}) = x_1^T Q x_1 \tag{9.12}$$

then along the trajectories of the closed-loop, we have

$$\begin{aligned}
\dot{V} &= \dot{x}_1^T Q x_1 + x_1^T Q \dot{x}_1 \\
&= \big(Ax_1 + B(\varphi(y_1) + u_1)\big)^T Q x_1 + x_1^T Q \big(Ax_1 + B(\varphi(y_1) + u_1)\big) \\
&= \Big(Ax_1 + B\big(\varphi(y_1) + u_0 - u_2\big)\Big)^T Q x_1 + x_1^T Q \Big(Ax_1 + B\big(\varphi(y_1) + u_0 - u_2\big)\Big) \\
&= \Big(Ax_1 + B\big(\varphi(y_1) + u_0 - k\hat{x}_2 - \varphi(-y_2)\big)\Big)^T Q x_1 \\
&\quad + x_1^T Q \Big(Ax_1 + B\big(\varphi(y_1) + u_0 - k\hat{x}_2 - \varphi(-y_2)\big)\Big) \\
&= x_1^T \big((A + Bk)^T Q + Q(A + Bk)\big) x_1 + 2B^T Q x_1 \big(\varphi(y_1) + \varphi(-y_2) - k\tilde{x} + u_0\big) \\
&= -2x_1^T x_1 + 2B^T Q x_1 \big(\varphi(y_1) - \varphi(-y_2) - k\tilde{x} + u_0\big) \\
&= -2\|x_1\|^2 + 2B^T Q x_1 \big(\varphi(y_1) - \varphi(-y_2) - k\tilde{x} + u_0\big)
\end{aligned}$$

Let

$$Q = \{q_{ij}\}_{n \times n}$$

and

$$q_1 = \max_{1 \leq j \leq n} \{|q_{1j}|\}$$

then

$$B^T Q x_1 \leq q_1 \|x_1\|$$

On the other hand, from the Lipschitz condition and Lemma 9.1, we obtain

$$
\begin{aligned}
&\varphi(y_1) - \varphi(-y_2) - k\tilde{x} + u_0 \\
\leq &L\|y_0\|_\infty + \|k\|\|\tilde{x}\|_\infty + \|u_0\|_\infty \\
\leq &l\sqrt{2}\|(u_0, y_0)^T\|_\infty + \|k\| \left(\frac{b}{\epsilon^{n-1}}\|\tilde{x}^0\| + \epsilon\beta\|(u_0, y_0)^T\|_\infty \right) \\
\leq &\frac{b^*}{\epsilon^{n-1}}\|\tilde{x}^0\| + \beta^*\|(u_0, y_0)^T\|_\infty
\end{aligned}
$$

where

$$l = \max\{L, 1\}$$

$$b^* = \|k\|b$$

$$\beta^* = l\sqrt{2} + \|k\|\beta$$

and ϵ is assumed to be smaller than 1. Hence

$$2B^TQx_1\big(\varphi(y_1) - \varphi(-y_2) - k\tilde{x} + u_0\big) \leq 2q_1 \left(\frac{b^*}{\epsilon^{n-1}}\|\tilde{x}^0\| + \beta^*\|(u_0, y_0)^T\|_\infty \right) \|x_1\|$$

Therefore

$$
\begin{aligned}
\dot{V} &= -2\|x_1\|^2 + 2B^TQx_1\big(\varphi(y_1) - \varphi(-y_2) - k\tilde{x} + u_0\big) \\
&\leq -\|x_1\|^2 - \|x_1\|^2 + 2q_1 \left(\frac{b^*}{\epsilon^{n-1}}\|\tilde{x}^0\| + \beta^*\|(u_0, y_0)^T\|_\infty \right) \|x_1\|
\end{aligned}
$$

By Young's Inequality, we obtain that

$$\dot{V} \leq -\|x_1\|^2 + q_1^2 \left(\frac{b^*}{\epsilon^{n-1}}\|\tilde{x}^0\| + \beta^*\|(u_0, y_0)^T\|_\infty \right)^2$$

Define a compact set as follows

$$\mathbf{R} = \left\{ x_1 \in \mathbb{R}^n \middle| \|x_1\| \leq q_1 \left(\frac{b^*}{\epsilon^{n-1}}\|\tilde{x}^0\| + \beta^*\|(u_0, y_0)^T\|_\infty \right) \right\}$$

then V decreases monotonically outside $\mathbf{R}$. Hence

$$V(x_1(t)) \leq \max \left\{ V(0),\ \sup \left\{ V(x_1) \middle| \|x_1\| = q_1 \left(\frac{b^*}{\epsilon^{n-1}}\|\tilde{x}^0\| + \beta^*\|(u_0, y_0)^T\|_\infty \right) \right\} \right\}$$

On the other hand,

$$\underline{\lambda}(Q)\|x_1\|^2 \leq V(x_1) \leq \bar{\lambda}(Q)\|x_1\|^2$$

and

$$V(0) \leq \bar{\lambda}(Q)\|x_1^0\|^2$$

Therefore

$$\|x_1\|_\infty \leq \max\left\{\sqrt{\frac{\bar{\lambda}(Q)}{\underline{\lambda}(Q)}}\|x_1^0\|,\ q_1\sqrt{\bar{\lambda}(Q)}\left(\frac{b^*}{\epsilon^{n-1}}\|\tilde{x}^0\| + \beta^*\|(u_0, y_0)^T\|_\infty\right)\right\}$$

Write

$$g(p, q, r) = \max\left\{\sqrt{\frac{\bar{\lambda}(Q)}{\underline{\lambda}(Q)}}r, q_1\sqrt{\bar{\lambda}(Q)}\left(\frac{b^*}{\epsilon^{n-1}}q + \beta^* p\right)\right\}$$

then the above inequality can be rewritten as

$$\|x_1\|_\infty \leq g\left(\|(u_0, y_0)^T\|_\infty, \|\tilde{x}^0\|, \|x_1^0\|\right)$$

Hence

$$\begin{aligned}\|y_1\|_\infty &= \|x_{11}\|_\infty \\ &\leq \|x_1\|_\infty \\ &\leq g\left(\|(u_0, y_0)^T\|_\infty, \|\tilde{x}^0\|, \|x_1^0\|\right)\end{aligned}$$

Next we estimate u_1. First

$$\begin{aligned}u_1 &= u_0 - u_2 \\ &= u_0 - \varphi(-y_2) - k\hat{x}_2 \\ &= u_0 + \varphi(y_1) - \varphi(-y_2) - k(x_1 + \hat{x}_2) - \varphi(y_1) + kx_1 \\ &= u_0 + \varphi(y_1) - \varphi(-y_2) - k\tilde{x} - \varphi(y_1) + kx_1\end{aligned}$$

Note that φ is Lipschitz, and $\varphi(0)$ is zero, hence

$$\begin{aligned}
\|u_1\|_\infty &\leq \|u_0\|_\infty + \|\varphi(y_1) - \varphi(-y_2)\|_\infty + \|k\|\|\tilde{x}\|_\infty + \|\varphi(y_1)\|_\infty + \|k\|\|x_1\|_\infty \\
&\leq \|u_0\|_\infty + L\|y_1 + y_2\|_\infty + \|k\|\|\tilde{x}\|_\infty + L\|y_1\|_\infty + \|k\|\|x_1\|_\infty \\
&\leq \|u_0\|_\infty + L\|y_0\|_\infty + \|k\| \left(\frac{b}{\epsilon^{n-1}} \|\tilde{x}^0\| + \epsilon\beta\|(u_0, y_0)^T\|_\infty \right) \\
&\quad + Lg\left(\|(u_0, y_0)^T\|_\infty, \|\tilde{x}^0\|, \|x_1^0\|\right) + \|k\|g\left(\|(u_0, y_0)^T\|_\infty, \|\tilde{x}^0\|, \|x_1^0\|\right) \\
&\leq l\sqrt{2}\|(u_0, y_0)^T\|_\infty + \|k\| \left(\frac{b}{\epsilon^{n-1}} \|\tilde{x}^0\| + \epsilon\beta\|(u_0, y_0)^T\|_\infty \right) \\
&\quad + (L + \|k\|)g\left(\|(u_0, y_0)^T\|_\infty, \|\tilde{x}^0\|, \|x_1^0\|\right)
\end{aligned}$$

Write

$$h(p, q, r) = l\sqrt{2}p + \|k\| \left(\frac{b}{\epsilon^{n-1}} q + \beta p \right) + (L + \|k\|)g\left(p, q, r\right)$$

then we obtain

$$\|u_1\|_\infty \leq h\left(\|(u_0, y_0)^T\|_\infty, \|\tilde{x}^0\|, \|x_1^0\|\right)$$

where we again have used $\epsilon \leq 1$.

Therefore, write

$$\gamma_\epsilon(p, q, r) = \left(g(p, q, r)^2 + h(p, q, r)^2\right)^{\frac{1}{2}}$$

then we have built up the following inequality

$$\begin{aligned}
\|(u_1, y_1)^T\|_\infty &= \left(\|u_1\|_\infty^2 + \|u_1\|_\infty^2\right)^{\frac{1}{2}} \\
&\leq \left(g\left(\|(u_0, y_0)^T\|_\infty, \|\tilde{x}^0\|, \|x_1^0\|\right)^2 + h\left(\|(u_0, y_0)^T\|_\infty, \|\tilde{x}^0\|, \|x_1^0\|\right)^2\right)^{\frac{1}{2}} \\
&= \gamma_\epsilon\left(\|(u_0, y_0)^T\|_\infty, \|\tilde{x}^0\|, \|x_1^0\|\right)
\end{aligned}$$

that is, the closed-loop is gf-stable.

If $x_1^0 = 0$ and $\hat{x}_2^0 = 0$, then $\tilde{x}^0 = 0$. From the definitions of functions g and h

$$g(p, 0, 0) = q_1\beta^*\sqrt{\bar{\lambda}(Q)}p$$

hence

$$\begin{aligned} h(p,0,0) &= (l\sqrt{2} + \|k\|\beta)p + (L + \|k\|)g(p,0,0) \\ &= (l\sqrt{2} + \|k\|\beta)p + (L + \|k\|)q_1\beta^* p \\ &= \left(l\sqrt{2} + \|k\|\beta + (L + \|k\|)q_1\beta^* \sqrt{\bar{\lambda}(Q)} \right) p \end{aligned}$$

and

$$\begin{aligned} \gamma_\epsilon(p,0,0) &= \big(g(p,0,0)^2 + h(p,0,0)^2\big)^{\frac{1}{2}} \\ &= \left(\left(q_1\beta^* \sqrt{\bar{\lambda}(Q)} p \right)^2 + \left(\left(l\sqrt{2} + \|k\|\beta + (L + \|k\|)q_1\beta^* \sqrt{\bar{\lambda}(Q)} \right) p \right)^2 \right)^{\frac{1}{2}} \\ &= \left((q_1^2(\beta^*)^2\bar{\lambda}(Q) + \left(l\sqrt{2} + \|k\|\beta + (L + \|k\|)q_1\beta^* \sqrt{\bar{\lambda}(Q)} \right)^2 \right)^{\frac{1}{2}} p \end{aligned}$$

Let

$$\Gamma = \left((q_1^2(\beta^*)^2\bar{\lambda}(Q) + \left(l\sqrt{2} + \|k\|\beta + (L + \|k\|)q_1\beta^* \sqrt{\bar{\lambda}(Q)} \right)^2 \right)^{\frac{1}{2}}$$

then, it follows that (9.11) holds, and Γ is independent of ϵ. □

A robust stability result can be given as follows.

Theorem 9.3. *Let the plant $\Sigma(x_1^0)$ and controller $\Xi_{H(\epsilon)}(\hat{x}_2^0)$ be defined by (9.1) and (9.5). Then there exists $\Gamma > 0$ such that if a plant Σ_1 satisfies*

$$\vec{\delta}(\Sigma(0), \Sigma_1) < \frac{1}{\Gamma} \tag{9.13}$$

the closed-loop $[\Sigma_1, \Xi(0)]$ is also stable, and

$$\|\Pi_{\mathcal{M}_1//\mathcal{N}}\| \leq \Gamma \frac{1 + \vec{\delta}(\Sigma(0), \Sigma_1)}{1 - \Gamma\vec{\delta}(\Sigma(0), \Sigma_1)} \tag{9.14}$$

Proof. By Theorem 9.2, we have shown that there exists $\Gamma > 0$ such that

$$\|\Pi_{\mathcal{M}//\mathcal{N}}\| \leq \Gamma$$

Then, if

$$\vec{\delta}(P, P_1) < \frac{1}{\Gamma}$$

it holds that

$$\vec{\delta}(P, P_1) < \frac{1}{\|\Pi_{\mathcal{M}//\mathcal{N}}\|}$$

Hence, by Theorem 6.8 in Chapter 6, the closed-loop $[\Sigma_1, \Xi(0)]$ is stable, and (9.14) holds. □

Since Γ is independent of $\epsilon < 1$, the allowed plant margin is not sensitive to ϵ as $\epsilon \to 0$. However, it is very important to observe that these results depends heavily on the assumption that there is no initial observer error.

The bounds obtained in (9.10) are sensitive to small ϵ, and so one would expect that any robust stability result for non-zero initial conditions will indicate a sensitivity to $\epsilon > 0$.

Chapter 10

Conclusions and Future Work

We summarize the results obtained in this thesis and give some possible areas for future work.

Part I Through the comparison of performances for *KKK* and *Khalil* designs, we have established the following results.

- For output feedback system, the performance of *KKK* design is sensitive to the initial condition of the observer. The performance of the *KKK* design is not uniformly bounded in the initial error between the initial condition of the state and the initial condition of the observer. When the initial error gets large, the performance gets large. Whereas, for the *Khalil* design, for any initial error, by choosing small high-gain factor, we can design a globally bounded controller, achieving uniformly bounded performance. Therefore, if the initial error is large or in the case that we have poor information for the initial condition of the state, the *Khalil* design has better performance than the *KKK* design.

- For parametric output feedback system, the performance of the *KKK* design is independent of the a-priori estimate bound of the uncertain parameter. When the a-priori estimate becomes conservative the performance remains uniformly bounded. Whilst, for the *Khalil* design, the performance is dependent on the saturation levels for the controller and the adaptive law, that is dependent on the a-priori estimate bound of the uncertain parameter, and the performance becomes large as the a-priori estimate becomes conservative. Hence, if we have poor information for the unknown parameter and the a-priori estimate bound is conservative, the *KKK* design has better performance than the *Khalil* design.

The primary contribution of this part is to provide rigorous statements and proofs of the intuitively reasonable trade-offs in performance between the differing classes of designs. The results have been expressed in qualitative terms only, the purpose of the thesis is to illustrate the asymptotic differences between the designs. It should also be noted that the results are asymptotic in nature, that is they require some parameter (either an initial condition or an uncertainty level) to be large in order to make the required comparison. Of course, in practice these parameters cannot be arbitrarily large without causing the control to run into physical limits. A more quantitative approach is challenging, as achieving tight bounds on non-singular performance is difficult. This is an interesting avenue for future research.

Part II Within the framework of nonlinear gap metric, we have established the following results.

- Following the backstepping design approach, we have built up a design procedure to design a controller for plant in strict-feedback form. This controller is robust to input and measurement disturbances and plant perturbation. The controller achieves gain-function stability for the plant with input and measurement disturbances. If the initial states are zero, the controller achieves stability for the plant with input and measurement disturbances, and achieves stability for any perturbed plant with input and measurement disturbances if the gap metric between the plants and the strict-feedback plant is less than some constant.

- We have established a robust backstepping design procedure for a nominal plant in output feedback form. This output-feedback controller is robust to input and measurement disturbances and plant perturbations within the framework of nonlinear gap metric.

  If the nominal plant nonlinearities are locally Lipschitz continuous, the controller achieves local gain-function stability for the plant with input and measurement disturbances; further, if the initial states are zero, the controller achieves local stability for the plant with input and measurement disturbances, and achieves stability for any perturbed plant with input and measurement disturbances if the gap metric between the plant and the output-feedback plant is less than some constant.

  If the nominal plant nonlinearities are globally Lipschitz continuous, the controller achieves global gain-function stability for the plant with input and measurement disturbances; further, if the initial states are zero, the controller achieves global stability for the plant with input and measurement disturbances, and achieves stability for any perturbed plant with input and measurement disturbances if the gap metric between the plant and the output-feedback plant is less than some constant.

- We have developed a robust high-gain observer design procedure for the nominal plant in output feedback normal form. The controller achieves gain-function stability for the plant with input and measurement disturbances. If the initial states are zero, the controller achieves stability for the plant with input and measurement disturbances, achieves stability for any perturbed plant with input and measurement disturbances if the gap metric between the plants and the strict-feedback plant is less than some constant. The allowed plant perturbation margin is bounded independently of the high-gain factor.

The contributions of this part is to show that by proper amendments of designs, we achieve the robustness of backstepping and high-gain designs. For the amended high-gain designs, the robust stability margin for plant perturbations is independent of the high-gain factor.

This thesis therefore represents the start of an approach to apply recent operator based techniques to address long-stability robustness questions in constructive nonlinear control. The study of performance of control designs for nonlinear systems is largely an open field in control theory, especially for output feedback designs. There are still many problems which need to be studied. Next we list some of the possible topics for future work.

Topics related to Part I:

- For high dimensional output feedback systems, to show that observer backstepping design has better performance in the situation when $\tilde{x}_0$ is small to that when $\tilde{x}_0$ is large.
- To compare the performance of adaptive observer backstepping design with high-gain observer design for a system with uncertain parameters and nonlinearities dependent on x, rather than only dependent on the output y. When the bound for the uncertain parameters becomes large it is anticipated that the adaptive observer backstepping design is superior to the high-gain observer design.
- To study the performance of other output feedback designs and compare them. For example, Khalil [47] used simulation to compare the performance of a variety of different output feedback nonlinear adaptive controllers, we may compare those techniques analytically.

Topics related to Part II:

- To construct semi-global results under the locally Lipschitz assumption on the nonlinearities of the systems, possibly by designing nonlinear controllers.
- To calculate gap metric distances for a variety of plant perturbations other than time delay, to widen applications, see, e.g., [34, 35].
- To study how to choose the gains in the controllers to optimize the robustness margins.
- To compare the *KKK* and *Khalil* designs in the framework of gap metric, e.g., compare the two designs by comparing their robustness margins.
- For the plant in normal form, which the nonlinearity depends on all the states other than the output, design a high-gain observer controller in the framework of gap metric.
- To investigate the sensitivity of the robustness margin in high-gain observer designs to the high-gain factor ϵ, in the presence of initial observer errors.

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