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Automorphism Groups of Primitive Distance-Bitransitive Graphs are Almost Simple

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1. Introduction, Definitions and Initial Results

We apply a result of Praeger, Saxl and Yokoyama [4] concerning the automorphism groups of primitive distance-transitive graphs to primitive distance-bitransitive graphs. Imprimitive distance-bitransitive graphs are discussed in Section 2. One of the cases of the Praeger, Saxl and Yokayama Theorem is considered in Section 3 and the following main result is proved in Section 4.

THEOREM 1.1. *If (Γ, G) is a primitive distance-bitransitive pair, then Γ is almost simple.*

We begin with the definition of a distance-bitransitive graph. Let G be a graph. By VG we denote the vertex set of G and by EG the edge set. For u, v in VG we write $u \sim v$ if (u, v) in EG. With $d_G(u, v) = d(u, v)$ we denote the usual distance in G between vertices u and v. The complement of a graph G is a graph G^c with $VG^c = VG$ and $u \sim v$ in G^c if $u \nsim v$ in G. The subdivision graph $S(G)$ of a graph G has vertex set $VS(G) = VG \cup EG$, and adjacency between elements of VG and elements of EG incident in G. Let $A(G)$ denote the usual adjacency matrix of a graph G. The set of eigenvalues of a square matrix $\mathbf{M}$ is denoted by $\lambda(\mathbf{M})$. We also write $\lambda(G) = \lambda(A(G))$ for the set of eigenvalues of a graph G.

A pair (Γ, G) where G is a connected graph and Γ a subgroup of aut(G), is *distance-bitransitive* if G is a nonregular, not complete bipartite graph with bipartition $A \cup B = VG$ satisfying that for any four vertices u, v, u', v' with u and u' both in the same part and $d(u, v) = d(u', v')$, there exists an automorphism g in Γ such that $(u)g = u'$ and $(v)g = v'$. A graph G is also called *distance-bitransitive* if the pair (aut(G), G) is distance-bitransitive.

A distance-bitransitive pair (Γ, G) is *imprimitive* if either the permutation group (Γ, A) or the permutation group (Γ, B) is imprimitive. Note that Γ fixes A setwise as vertices in B have different degree to those in A. For a graph G we denote by $G^{(k)}$ the graph with vertex set VG and adjacency defined by $u \sim v$ iff $d_G(u, v) = k$, for u, v in VG.

We now define a distance-regularised and distance-biregular graph as introduced in [2]. For v in VG and i in $\mathbb{N}$, $G_i(v)$ denotes the set of vertices at distance i from v. For u in VG and v in $G_i(u)$ we write $c(u, v) = |G_{i-1}(u) \cap G_1(v)|$, $b(u, v) = |G_{i+1}(u) \cap G_1(v)|$, $a(u, v) = |G_i(u) \cap G_1(v)|$ and $k_i(u) = |G_i(u)|$.

We are interested in vertices u in VG for which the numbers $b(u, v)$, $a(u, v)$ and $c(u, v)$ are independent of the choice of v in $G_i(u)$. In this case we say u is *distance-regularised* and we denote $b(u, v)$, $a(u, v)$ and $c(u, v)$ by $b_i(u)$, $a_i(u)$ and $c_i(u)$. Let d be the diameter of G. Then the array:

$$i(u) = \begin{bmatrix} * & c_1(u) & \dots & c_{d-1}(u) & c_d(u) \\ 0 & a_1(u) & \dots & a_{d-1}(u) & a_d(u) \\ b_0(u) & b_1(u) & \dots & b_{d-1}(u) & * \end{bmatrix}$$

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is called the intersection array for u, and the matrix:

$$\mathbf{I}(u) = \begin{bmatrix} 0 & c_1(u) & 0 & \cdot & \cdot & \cdot \\ b_0(u) & a_1(u) & c_2(u) & \cdot & \cdot & \cdot \\ 0 & b_1(u) & a_2(u) & \cdot & \cdot & \cdot \\ 0 & 0 & b_2(u) & \cdot & \cdot & \cdot \\ & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & b_{d-2}(u) & a_{d-1}(u) & c_d(u) \\ & & & 0 & b_{d-1}(u) & a_d(u) \end{bmatrix}$$

is called the intersection matrix for u. In the matrix we omit last rows and columns if they are identically zero.

We will call a connected graph in which every vertex is distance-regularised a *distance-regularised* graph. A special case of distance-regularised graphs are bipartite distance-regularised graphs in which vertices in the same partition or colour class have the same intersection array. These graphs are called *distance-biregular*. It is shown in [2] that distance-regularised graphs are distance-regular or distance-biregular.

Unless explicitly stated, we use the following standardised notation for a distance-biregular graph. Sets A and B denote the colour partition of VG, d is the diameter of G, u is a vertex in A and has intersection array:

$$i(A) = \begin{bmatrix} * & 1 & c_2 & \ldots & c_d \\ 0 & 0 & 0 & \ldots & 0 \\ r & b_1 & b_2 & \ldots & * \end{bmatrix} \quad \text{or just} \quad \begin{bmatrix} * & 1 & c_2 & \ldots & c_d \\ r & b_1 & b_2 & \ldots & * \end{bmatrix}$$

v is a vertex in B and has intersection array:

$$i(B) = \begin{bmatrix} * & 1 & f_2 & \ldots & f_d \\ s & e_1 & e_2 & \ldots & * \end{bmatrix}.$$

The corresponding intersection matrices are denoted by $\mathbf{I}(A)$ and $\mathbf{I}(B)$ respectively. Note that $\deg(u) = r$ and $\deg(v) = s$. We denote with k_i the numbers $|G_i(u)|$ and with l_i the numbers $|G_i(v)|$, $i = 0, \ldots, d$. Note that $l_{d-1} \neq 0$ and $k_{d-1} \neq 0$ though one of l_d and k_d may be zero.

Distance-bitransitive graphs are clearly distance-biregular, as distance-transitive graphs are distance-regular.

A special class of distance-regular graphs which we will refer to is that of (k, g)-graphs. These are distance-regular graphs with valency k, girth g and diameter $\lfloor g/2 \rfloor$, which are also bipartite when g is even. The subdivision graphs of such graphs are distance-biregular (see [3]) and we meet an example of such a graph in Proposition 3.2.

We will first give some examples of distance-biregular and distance-bitransitive graphs in order to show that they form an important and natural generalisation of distance-regular graphs. Two classes of distance-biregular graphs that we will only mention in passing are generalised polygons and the incidence graphs of partial geometries. The following two examples are of distance-bitransitive graphs.

EXAMPLE 1. Consider a vector space V of dimension m over the Galois field GF(q), where q is a prime power. The vertices of the graph G are the k-dimensional and $(k + 1)$-dimensional subspaces of V with (X, Y) an edge in G if $X \subseteq Y$. To ensure G is not a regular graph we require $m \neq 2k+1$ (in the case $m = 2k+1$ the graph obtained is the q-analogue of the double cover of the odd graph O_k). The group GL(m, q) acts as a group of automorphisms on G and it is not hard to check that (GL(m, q), G) is a distance-bitransitive pair. The group GL(m, q) has simple socle PSL(m, q).

EXAMPLE 1.3. It is well known that the 2-(21, 5, 1) design consisting of the points and lines of PG(2, 4) can be extended to a 3-(22, 6, 1) design by adding an additional vertex to each line and a class of 56 ovals, determined by an equivalence relation on the set of all ovals in PG(2, 4) (an oval is a maximal set of points no three of which are collinear and the relation is given by $O \sim O'$ if $|O \cap O'| = 0, 2$ or 6). The graph G has vertex set the points of PG(2, 4) and the 56 ovals of a chosen class. The pair (x, O) is an edge of G if x is a point of the oval O. The group PSL(3, 4) is a group of automorphisms of G as it is the vertex stabiliser of M_{22}, the automorphism group of the 3-(22, 6, 1) design. Using the fact that PSL(3, 4) acts transitively on quadruples of points, exactly three of which are collinear and that three non collinear points uniquely determine an oval vertex adjacent to them in G, we can check that the simple group PSL(3, 4) acts distance-bitransitively.

The intersection arrays of G are:

$$\begin{bmatrix} * & 1 & 2 & 12 & 6 \\ 6 & 15 & 4 & 4 & * \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} * & 1 & 4 & 6 \\ 16 & 5 & 12 & * \end{bmatrix}.$$

We being with a useful lemma on the relation between the two arrays of a distance-biregular graph.

LEMMA 1.4. *Let G be a distance-biregular graph with the standard notation. Then* $c_{2i+1}c_{2i} = f_{2i+1}f_{2i}$ *and* $b_{2i}b_{2i-1} = e_{2i}e_{2i-1}$, *for* $i = 1, \ldots, \lceil d/2 \rceil - 1$.

Proof. Let $1 \leqslant i \leqslant \lceil d/2 \rceil - 1$ and consider two vertices u, v with $d(u, v) = 2i + 1$ ($\leqslant d$), with u in A and v in B. We wish to evaluate the size of the set $G_j(u) \cap G_{2i+1-j}(v)$. We claim that

$$m_j = |G_j(u) \cap G_{2i+1-j}(v)| = (f_{2i+1} \ldots f_{2i+2-j})/(c_1 \ldots c_j).$$

We prove the claim by induction on j. For $j = 1$, $m_1 = f_{2i+1}$ by the definition of the intersection numbers. Suppose the equation holds for smaller numbers than j. Each vertex in $G_{j-1}(u) \cap G_{2i+2-j}(v)$ is adjacent to f_{2i+2-j} vertices in $G_{2i+1-j}(v)$ each of which lies in $G_j(u)$, while each vertex in $G_j(u) \cap G_{2i+1-j}(v)$ is adjacent to c_j vertices in $G_{j-1}(u)$, each of which lies in $G_{2i+2-j}(v)$. Hence $m_{j-1}f_{2i+2-j} = m_jc_j$. Using the induction hypothesis the claim follows. But then $m_{2i+1} = |G_{2i+1}(u) \cap G_0(v)| = 1$ and so $(f_{2i+1} \ldots f_1)/(c_1 \ldots c_{2i+1}) = 1$, and $f_{2i+1} \ldots f_1 = c_{2i+1} \ldots c_1$. As $f_{2i-1} \ldots f_1 = c_{2i-1} \ldots c_1 \neq 0$, we have $f_{2i+1}f_{2i} = c_{2i+1}c_{2i}$.

To prove the second equation of the lemma we partition $G_{2i}(u)$, for u in A and $1 \leqslant i \leqslant \lceil d/2 \rceil - 1$, into two subsets, $G_{2i}(u) \cap G_{2i-1}(v)$ and $G_{2i}(u) \cap G_{2i+1}(v)$, where v is a vertex adjacent to u. We now estimate $k_{2i} = |G_{2i}(u)|$ in two ways. Firstly in the obvious fashion

$$k_{2i} = (b_0b_1 \ldots b_{2i-1})/(c_1c_2 \ldots c_{2i}) \neq 0, \quad \text{as} \quad 2i < d.$$

To get the second estimate we first prove a claim that

$$n_j = |G_{j+1}(u) \cap G_j(v)| = (b_1b_2 \ldots b_j)/(f_1f_2 \ldots f_j).$$

We again proceed by induction on j. For $j = 1$ it is true by the definition of b_1. Now assume it holds for integers less than j. Each vertex in $G_j(u) \cap G_{j-1}(v)$ is adjacent to b_j vertices in $G_{j+1}(u)$ all of which are distance j from v. Each vertex in $G_{j+1}(u) \cap G_j(v)$ is adjacent to f_j vertices in $G_{j-1}(v)$ all of which are distance j from u. Hence $n_{j-1}b_j = n_jf_j$. Using the induction hypothesis

$$n_j = (b_1 b_2 \dots b_j)/(f_1 f_2 \dots f_j).$$

By the symmetry of the definition of a distance-biregular graph

$$|G_{j+1}(v) \cap G_j(u)| = (e_1 e_2 \dots e_j)/(c_1 c_2 \dots c_j).$$

Hence $k_{2i} = |G_{2i}(u) \cap G_{2i-1}(v)| + |G_{2i}(u) \cap G_{2i+1}(v)|$

$$= \frac{b_1 b_2 \dots b_{2i-1}}{f_1 f_2 \dots f_{2i-1}} + \frac{e_1 e_2 \dots e_{2i}}{c_1 c_2 \dots c_{2i}}$$

By the first part $c_1 c_2 \dots c_{2i-1} = f_1 f_2 \dots f_{2i-1}$, and so $b_0 b_1 \dots b_{2i-1} = b_1 b_2 \dots b_{2i-1} c_{2i} + e_1 e_2 \dots e_{2i}$, and $b_1 b_2 \dots b_{2i-1}(b_0 - c_{2i}) = e_1 e_2 \dots e_{2i}$, or $b_1 b_2 \dots b_{2i} = e_1 e_2 \dots e_{2i}$. For $i > 1$, $b_1 b_2 \dots b_{2i-2} = e_1 e_2 \dots e_{2i-2} \neq 0$ and so $b_{2i} b_{2i-1} = e_{2i} e_{2i-1}$ as required.

Lemma 1.4 shows that one array of a distance-biregular graph determines the other. This result was proved in [2], but the formulas of Lemma 1.4 give a much simpler connection. The next lemma will not be used in the paper but is presented to give some justification for the exclusion of regular graphs from our definition of a distance-bitransitive pair.

LEMMA 1.5. *A regular distance-biregular graph is distance-regular.*

PROOF. We prove by induction that $i(A) = i(B)$. As G is regular of degree $r = s$, the first two columns in each array are identical. Suppose now that the arrays are identical up to and including the $(2i - 1)$-st column. Then by Lemma 1.4 $b_{2i} b_{2i-1} = e_{2i} e_{2i-1}$, and so $b_{2i} = e_{2i}$. As $r = s$ this gives $c_{2i} = f_{2i}$. But again by Lemma 1.4 $c_{2i+1} c_{2i} = f_{2i+1} f_{2i}$ and so $c_{2i+1} = f_{2i+1}$, yielding $b_{2i+1} = e_{2i+1}$ and agreement of the next two columns of the intersection arrays.

LEMMA 1.6. *The diameter d of a non regular distance-biregular graph is even.*

PROOF. Suppose w.l.o.g. that $G_d(u) \neq \varnothing$ for u in A. By arranging the rows and columns of the adjacency matrix $\mathbf{A}(G)$ of G so that the vertices of A precede those of B we obtain a block pattern:

$$\mathbf{A}(G) = \left[\begin{array}{c|c} \mathbf{0} & \mathbf{M} \\ \hline \mathbf{M}^{\mathrm{T}} & \mathbf{0} \end{array}\right]$$

with $\mathbf{M}$ an $n \times m$ matrix. Then rank $(\mathbf{A}(G)) \leqslant 2\min(n, m)$ as at most this many rows may be linearly independent. As $n \neq m$, $\mathbf{A}(G)$ is not full rank and so 0 is an eigenvalue of G. In [2] it is shown that $\lambda(G) = \lambda(\mathbf{I}(A)) \cup \lambda(\mathbf{I}(B))$. But then 0 is an eigenvalue of $\mathbf{I}(A)$ or $\mathbf{I}(B)$. But $\mathbf{I}(A)$ and $\mathbf{I}(B)$ both have zero diagonal and so 0 is one of their eigenvalues iff they have odd order. Suppose d is odd. Let $v \in G_d(u) \neq \varnothing$. But then u in $G_d(v)$ and as v in B both $i(A)$ and $i(B)$ have an even number of non zero columns. Hence $\mathbf{I}(A)$ and $\mathbf{I}(B)$ both have even order, a contradiction.

It is shown in [3] that if G is a distance-biregular graph then $G^{(2)}$ is the disjoint union of two distance-regular graphs called the derived graphs of G. The following lemma presents an analogous result for distance-bitransitive graphs.

LEMMA 1.7. Let (Γ, G) be a distance-bitransitive pair. Then $G^{(2)}$ is the disjoint union of two connected graphs D and E on each of which Γ acts faithfully and distance-transitively.

PROOF. Let $A \cup B = VG$ be the bipartition of G. In $G^{(2)}$ no vertex of A is adjacent to a vertex of B. Hence $G^{(2)}$ is the disjoint union of two graphs D and E with $VD = A$ and $VE = B$. For u, u' vertices in A, $d_{G^{(2)}}(u, u') = d_G(u, u')/2$. Similarly for v, v' vertices in B. So D and E are connected graphs and Γ acts transitively on pairs at a given distance apart in both D and E. It remains to show that the action of Γ is faithful. Suppose g in Γ is the identity on D. Let v in B and $u_1, \ldots, u_s$ be the neighbours of v in G. Since g fixes $u_1, \ldots, u_s$, $(v)g$ is also adjacent to precisely $u_1, \ldots, u_s$. Suppose $(v)g \neq v$. Considering the intersection array for v we must have:

$$i(B) = \begin{bmatrix} * & 1 & s \\ s & r-1 & * \end{bmatrix}.$$

So $G = K_{r,s}$ the complete bipartite graph excluded in our definition of a distance-bitransitive pair. We conclude that g fixes every vertex of G. Hence g is the identity and Γ acts faithfully on D. Similarly Γ acts faithfully on E.

Note that if (Γ, G) is an imprimitive distance-bitransitive graph then one of the distance-transitive derived graphs D or E of the lemma will also be imprimitive. It is a well known result of Smith [5] that an imprimitive distance-transitive graph is either bipartite or antipodal. This prompts the following generalisation of the definition of primitivity to distance-regular and distance-biregular graphs. Imprimitive distance-biregular graphs are the subject of Section 2.

A distance-regular graph G of diameter d is *antipodal* if $G^{(d)}$ is disconnected. A distance-regular graph is *primitive* if it is neither bipartite nor antipodal, otherwise it is *imprimitive*. A non regular distance-biregular graph is *primitive* if both of its derived graphs are primitive, otherwise it is *imprimitive*. A non regular distance-biregular graph is *antipodal* if at least one derived graph is antipodal.

The exclusion of regularity in these definitions is justified by Lemma 1.5.

2. Imprimitive Distance-biregular Graphs

It is known that the intersection array of an antipodal distance-regular graph is 'palindromic'. To be precise if a distance-regular graph G has intersection array

$$\begin{bmatrix} * & c_1 & & c_{d-1} & c_d \\ 0 & a_1 & \cdots & a_{d-1} & a_d \\ b_0 & b_1 & & b_{d-1} & * \end{bmatrix}$$

then G is antipodal if and only if $b_i = c_{d-i}$, $i = 0, 1, \ldots, d$, $i \neq \lfloor g/2 \rfloor$. The proof of this is in [1] though it is not explicitly stated there. This result means that one of the intersection arrays of an antipodal distance-biregular graph must be 'palindromic', as the next proposition makes explicit.

PROPOSITION 2.1. *Let G be a non regular distance-biregular graph with derived graph D on vertex set $VD = A$. Then D is antipodal if and only if $G_d(u) \neq \varnothing$ for u in A, and $i(A)$ satisfies $b_i = c_{d-i}$, $i = 0, 1, \ldots, d$, $i \neq d/2$.*

PROOF. ($\Rightarrow$) Suppose $G_d(u) = \varnothing$. Then $c_{d-1} = s$ and the derived graph D has diameter $d' = d/2 - 1$, as d is even by Lemma 1.6. Let D have intersection array:

$$i(D) = \begin{bmatrix} * & c'_1 & & c'_{d'} \\ 0 & a'_1 & \cdots & a'_{d'} \\ b'_0 & b'_1 & & * \end{bmatrix}.$$

Then $a'_{d'} = (c_{d-2}(b_{d-3} - 1) + b_{d-2}(c_{d-1} - 1))/c_2$ (see [3]). But $c_{d-1} = s > 1$. Hence $a'_{d'} > 0$ and so $b'_0 \neq c'_{d'}$, and D is not antipodal. We conclude that $G_d(u) \neq \varnothing$. Suppose now that $b_j = c_{d-j}$ for $j < i$, for some i, $1 \leqslant i < d/2$. This is true for $i = 1$, as $b_0 = c_d = r$. We consider the possible parities of i separately.

Case (*a*): i odd. Here $b_{i-1}b_i/c_2 = b'_{(i-1)/2} = c'_{(d/2)-[(i-1)/2]} = c'_{(d-i+1)/2} = c_{d-i+1}c_{d-i}/c_2$, as $(i-1)/2 \neq \lfloor d'/2 \rfloor$. But $b_{i-1} = c_{d-i+1}$ and so $b_i = c_{d-i}$ as required.

Case (*b*): i even. Here $c_i c_{i-1}/c_2 = c'_{i/2} = b'_{(d/2)-(i/2)} = b'_{(d-i)/2} = b_{d-i}b_{d-i+1}/c_2$, as $d/2 - i/2 \neq \lfloor d'/2 \rfloor$. But $b_{i-1} = c_{d-i+1}$ so $c_{i-1} = b_{d-i+1}$, as $b_{i-1} + c_{i-1} = s = c_{d-i+1} + b_{d-i+1}$. We conclude that $c_i = b_{d-i}$ and so $b_i = c_{d-i}$. The result follows by induction.

($\Leftarrow$) Let $d' = d/2$, the diameter of D as $G_d(u) \neq \varnothing$. Then

$$\begin{aligned} b'_j &= b_{2j}b_{2j+1}/c_2 = c_{d-2j}c_{d-2j+1}/c_2 \\ &= c'_{d'-j}, \qquad j = 0, 1, \ldots, d' \quad j \neq \lfloor d'/2 \rfloor. \end{aligned}$$

We conclude this section by showing that both derived graphs of a non regular distance-biregular graph cannot be imprimitive.

PROPOSITION 2.2. *Let G be a non-regular distance-biregular graph. Then at least one of the derived graphs is primitive. Suppose the derived graph E is imprimitive. Then one of the following holds*:
(*a*) *G is the subdivision graph of E, which is a bipartite (k, g)-graph,*
(*b*) *E is an antipodal, non-bipartite graph with* $\mathrm{diam}(E) \geqslant \mathrm{diam}(D)$.

PROOF. We consider first the case when G has vertices of valency two.

Case (*a*): G has vertices of valency 2. The main theorem of [3] states that in this case G is the subdivision graph of one of its derived graphs, which is a (k, g)-graph. Let E be this derived graph. Then in the standard notation $r = 2$ and $s = k$ the degree of E. The intersection array of the second derived graph D may be computed as:

$$\begin{bmatrix} * & 1 & & 1 & 1 & 4 \\ 0 & k-2 & \ldots & k-2 & k-1 & 2(k-3) \\ 2(k-1) & k-1 & & k-1 & k-2 & * \end{bmatrix},$$

if g is odd,

$$\begin{bmatrix} * & 1 & & 1 & 2 \\ 0 & k-2 & \ldots & k-2 & 2(k-2) \\ 2(k-1) & k-1 & & k-1 & * \end{bmatrix},$$

if g is even. In no case is D bipartite, as we must have $k = s > 2 = r$ for non-regularity. The only case when the array is antipodal is when $k = 3$, $g = 3$. This means that E is K_4 and $G = S(K_4)$. E is primitive while D is antipodal and non-bipartite with $\mathrm{diam}(D) > \mathrm{diam}(E)$. This is example (*b*) of the proposition, with D and E interchanged. For other values of k and g, D is primitive, while the (k, g)-graph E is imprimitive only if bipartite (g even). This is example (*a*) of the proposition.

Case (*b*): G has no vertices of valency 2. It is immediate that both derived graphs contain triangles and so neither is bipartite. If derived graph E is antipodal, then $\mathrm{diam}(E) = d/2$ by Proposition 2.1. But for the second derived graph D, $\mathrm{diam}(D) \leqslant d/2$ and so $\mathrm{diam}(E) \geqslant \mathrm{diam}(D)$. Hence it remains to prove that both derived graphs cannot be antipodal. Suppose this to be the case. By Proposition 2.1 both intersection arrays for G are 'palindromic' with

$G_d(u)$ and $G_d(v)$ non-empty for u in A, v in B.

Let

$$i(A) = \begin{bmatrix} * & c_1 & \dots & c_{l-1} & c_l & b_{l-1} & \dots & b_1 & r \\ r & b_1 & & b_{l-1} & b_l & c_{l-1} & & c_1 & * \end{bmatrix}$$

and

$$i(B) = \begin{bmatrix} * & f_1 & \dots & f_{l-1} & f_l & e_{l-1} & \dots & e_1 & s \\ s & e_1 & & e_{l-1} & e_l & f_{l-1} & & f_1 & * \end{bmatrix},$$

where $l = d/2$. Consider first l odd. Here by Proposition 2.1 and Lemma 1.4 $b_l c_{l-1} = e_l f_{l-1}$ and $c_l c_{l-1} = f_l f_{l-1}$. Adding we obtain $c_{l-1}(b_l + c_l) = f_{l-1}(e_l + f_l)$ and so $f_{l-1}/c_{l-1} = s/r$. But then $b_l/e_l = f_{l-1}/c_{l-1} = s/r$. For l even $b_{l-1}b_l = e_{l-1}e_l$ and $b_{l-1}c_l = e_{l-1}f_l$, by Lemma 1.4. Adding we have $b_{l-1}(c_l + b_l) = e_{l-1}(e_l + f_l)$ and so $b_{l-1}/e_{l-1} = s/r$. Now suppose that for some $2i + 1 \leqslant l$, $b_{2i+1}/e_{2i+1} = s/r$. As $c_{2i+1} + b_{2i+1} = s$ and $e_{2i+1} + f_{2i+1} = r$, we have $c_{2i+1}/f_{2i+1} = (s - b_{2i+1})/(r - e_{2i+1}) = s/r$. Then as $c_{2i+1}c_{2i} = f_{2i+1}f_{2i}$, $f_{2i}/c_{2i} = c_{2i+1}/f_{2i+1} = s/r$, and as $e_{2i} + f_{2i} = s$, while $b_{2i} + c_{2i} = r$, we have $e_{2i}/b_{2i} = (s - f_{2i})/(r - e_{2i}) = s/r$. Further as $b_{2i-1}b_{2i} = e_{2i-1}e_{2i}$, $b_{2i-1}/e_{2i-1} = e_{2i}/b_{2i} = s/r$. Hence by induction $b_1/e_1 = (s - 1)/(r - 1) = s/r$ and so $r = s$, a contradiction.

3. Distance-biregular Graphs with Hamming Derived Graph

In this section we consider which non-regular distance-biregular graphs have the Hamming graph $H(d, q)$ or its complement when $d = 2$ as one of their derived graphs. These results will be used in Section 4.

First we define the Hamming graph $H = H(d, q)$. H has vertex set the d-vectors over a q-element set X, d, $q > 1$. Two vectors are adjacent in H if they differ in precisely one component.

The following lemma relating the two derived graphs of a distance-biregular graph will prove useful in this section.

Lemma 3.1. *Let D and E be the derived graphs of a distance-biregular graph G with $VD = A$. Suppose $G_4(u) \neq \varnothing$, for u in A. Then the vertices of E correspond to maximal cliques in D.*

Proof. Consider a vertex $v \in B = VE$ as a vertex of G. Its neighbours $u_1, \dots, u_s$ will form a clique in the derived graph D. We must show that this clique is maximal. Suppose a further vertex u is adjacent to each of $u_1, \dots, u_s$ in D. Now v is distance 3 from u in G, but every neighbour of v is distance 2 from u. Hence $c_3 = s$ and $G_4(u) = \varnothing$, a contradiction.

Proposition 3.2. *The only distance-biregular graph with Hamming derived graph is $S(K_{q,q})$, the subdivision graph of $K_{q,q}$. This graph is imprimitive and has derived graph $H(2, q)$.*

Proof. Suppose G is a distance-biregular graph with derived graph D on vertex set A isomorphic to $H(d, q)$. By Lemma 3.1 the vertices of the other derived graph E correspond to maximal cliques of $H(d, q)$ as $G_4(u) \neq \varnothing$, for u in A. The maximal cliques of $H(d, q)$ are indexed by d-vectors over $X' = X \cup \{*\}$ in which precisely one component is $*$, a symbol not in the set X used to define $H(d, q)$. The clique indexed by $c = (i_1, \dots, i_d)$, with

$i_k = *$, consists of all the vertices of $H(d, q)$ which agree with c in every component except the kth. We claim that every such clique must correspond to a vertex of E. We prove this for the general clique c. The two vertices $(i_1, \ldots, i'_k, \ldots, i_d)$ and $(i_1, \ldots, i''_k, \ldots, i_d)$, where i'_k and i''_k are two distinct elements of X, are adjacent in $H(d, q)$, so there must be a vertex v of B adjacent to both of them in G. The only maximal clique containing both of them is c and so c must correspond to v. Hence the claim holds and G has vertex set the vertices of $H(d, q)$ together with its maximal cliques, with adjacency given by inclusion of a vertex in a clique.

Now suppose $d > 2$. The clique $v = (*, i_2, \ldots, i_d)$ is distance 4 from $v' = (i_1, i'_2, *, i_4, \ldots, i_d)$ and $v'' = (*, i'_2, i_3, \ldots, i_d)$. But every neighbour of v'' is distance 3 from v, while just one neighbour $(i_1, i'_2, i_3, \ldots, i_d)$ of v' is distance 3 from v. This contradicts G being distance-biregular.

If $d = 2$ the maximal cliques are indexed by $\{(i, *), (*, i) \mid i \in X\}$. Each vertex (i_1, i_2) of $H(2, q)$ can be viewed as the edge joining $(i_1, *)$ to $(*, i_2)$ in the complete bipartite graph with parts $X_1 = X \times \{*\}$ and $X_2 = \{*\} \times X$. Hence $G \cong S(K_{q,q})$. The derived graphs of G are $K_{q,q}$ and $L(K_{q,q}) \cong H(2, q)$. As $K_{q,q}$ is bipartite G is imprimitive.

PROPOSITION 3.3. *Let $q > 2$. The existence of a distance-biregular graph G with derived graph $H(2, q)^c$ is equivalent to the existence of a projective plane P of order q. The graph G is the incidence graph of the structure P' obtained from P by choosing two distinct points x and y and deleting all the lines through either of them and all the points on the line xy. The graph G is antipodal.*

PROOF. ($\Rightarrow$) Suppose G is a distance-biregular graph with derived graph $D \cong H(2, q)^c$ on vertex set A in the standard notation. Let X denote the set used to define $H(2, q)$ so that the set A can be regarded as $A = \{(i, j) \mid i, j \in X\}$, with $d((i, j), (i', j')) = 2$ iff $i \neq i'$ and $j \neq j'$. Then $G_4(u) \neq \varnothing$, for u in A, so by Lemma 3.1 the vertices of B correspond to maximal cliques of D.

We claim that any maximal clique of D has q elements, for suppose $C = \{(i_1, j_1), \ldots, (i_l, j_l)\}$ is a maximal clique of $H(2, q)^c$. Then each pair differ in both coordinates and so $i_1, \ldots, i_l$ are all distinct and likewise $j_1, \ldots, j_l$. Hence $l \leqslant q = |X|$. If $l < q$ we can choose $i_{l+1} \in X - \{i_1, \ldots, i_l\}$ and $j_{l+1} \in X - \{j_1, \ldots, j_l\}$. Then $(i_{l+1}, j_{l+1}) \sim (i_t, j_t)$, for $t = 1, \ldots, l$, contradicting the maximality of C.

We conclude that $s = q$ and as $H(2, q)^c$ has intersection array:

$$\begin{bmatrix} * & 1 & (q-1)(q-2) \\ 0 & (q-2)^2 & q-1 \\ (q-1)^2 & 2(q-2) & * \end{bmatrix},$$

we can compute:

$$i(A) = \begin{bmatrix} * & 1 & r/(q-1) & q-2 & r \\ r & q-1 & r(q-2)/(q-1) & 2 & * \end{bmatrix}.$$

By Lemma 1.4 $e_1 e_2 = b_1 b_2$ and so $e_2 = r(q-2)/(r-1)$. Hence $r - 1 \mid q - 2$ and $q - 1 \mid r$. This forces $r = q - 1$ and so

$$i(A) = \begin{bmatrix} * & 1 & 1 & q-2 & q-1 \\ q-1 & q-1 & q-2 & 2 & * \end{bmatrix}$$

and

$$i(B) = \begin{bmatrix} * & 1 & 1 & q-2 & q \\ q & q-2 & q-1 & 1 & * \end{bmatrix}.$$

By Proposition 2.1 G is antipodal. The derived graph E on the vertex set B has intersection array:

$$\begin{bmatrix} * & 1 & q(q-2) \\ 0 & q(q-3) & 0 \\ q(q-2) & q-1 & * \end{bmatrix}.$$

This is an antipodal graph of diameter 2 with $|\{u\} \cup G_2(u)| = q$. Hence $E \cong K_{(q-1)(q)}$, the complete $(q-1)$-partite graph with each part having q vertices. We label the parts of E from 1 to $q-1$. To complete the first half of the proof it remains to construct a projective plane P of order q from G. The points of the plane P will be the vertices of $A = VH$ together with $q+1$ points labelled $x, y, p_1, \ldots, p_{q-1}$. The lines of P will be labelled by the vertices of B together with $2q+1$ additional lines $l_i, m_i, i \in X$ and l_∞. Vertex v of B in block k of E labels a line composed of the points $\{u \in A \mid u \sim v\} \cup \{p_k\}$. The line l_i is the set of points $\{(i,j) \mid j \in X\} \cup \{x\}$ while m_i is the set $\{(j,i) \mid j \in X\} \cup \{y\}$. Finally l_∞ is the set of points $\{x, y, p_1, \ldots, p_{q-1}\}$. It is fairly straightforward to check that each pair of points lie on exactly one line and that each pair of lines intersect in exactly one point. Finally the four points $x, y, (i,i), (j,j)$ $(i, j \in X, i \neq j)$ form a four-point. So P is a projective plane of order q.

($\Leftarrow$) Suppose P is a projective plane of order q. Let x, y, P' and G be as in the proposition statement. Let u be any point of P'. The point u lies on $q+1$ lines in P, but the line through x and the line through y (distinct because u is not on xy) have been deleted, so u lies on $q-1$ lines in P'. Let v be a line of P'. The line v intersects xy in P in a point $p \neq x$ or y. Hence v is incident with q points in P' and G is a semi-regular graph. Two points lie on one line in P so the incidence graph of P has girth greater than 4. Hence girth(G) $\geqslant 6$. Now consider a point u of P' and a line v of P' not incident with u. Let u' be a point on v. The line uu' is in P' iff x and y are not on uu'. Now ux and uy intersect v in two distinct points of v as u is not on xy. Hence precisely $q-2$ points of v are collinear with u in P'. We thus see that a point vertex of G has the first seven intersection numbers well defined as follows:

$$\begin{bmatrix} * & 1 & 1 & q-2 & \\ & & & & \cdots \\ q-1 & q-1 & q-2 & 2 & \end{bmatrix}$$

But in the argument above we took any line not incident with u and found it was distance 3 from u. So $G_5(u) = \varnothing$ and the point vertices of G are distance-regularised with array:

$$\begin{bmatrix} * & 1 & 1 & q-2 & q-1 \\ q-1 & q-1 & q-2 & 2 & * \end{bmatrix}$$

Finally consider a line v of P' and a point u not incident with it. The only line through u which does not intersect v in P' is the line through the point $v \cap xy$. Again we choose any point u not incident with v, so $G_5(v) = \varnothing$ and the line vertices of G are distance-regularised with array:

$$\begin{bmatrix} * & 1 & 1 & q-2 & q \\ q & q-2 & q-1 & 1 & * \end{bmatrix}.$$

So G is a distance-biregular graph. We now investigate its derived graph on the point vertices. A point vertex u of G can be labelled by an ordered pair of lines (ux, uy), which clearly determine u as their intersection. Conversely a pair of lines (l, m) with x on l and y on m, but neither the line xy, determine a point vertex of G. We now use this labelling, so that the point vertices of G are

$$A = \{(l, m) | l \in L(x) - \{xy\} \text{ and } m \in L(y) - \{xy\}\}.$$
$$\cong X \times X, \text{ with } |X| = q.$$

The distinct vertices (l, m) and (l', m') are adjacent in the derived graph of G iff they are collinear in P'. This will be true iff the line through them in P was not deleted, i.e. did not go through x or y. But the line $(l, m)(l', m')$ of P is incident with x iff it is $l = l'$, while it is incident with y iff it is $m = m'$. We conclude that (l, m) is adjacent to (l', m') in the derived graph iff $l \neq l'$ and $m \neq m'$, and so the derived graph is $H(2, q)^c$.

4. Primitive Distance-bitransitive Graphs

We begin this section by introducing the group theoretical definitions we will need.

The *socle* of a group Γ is the product of its minimal normal subgroups. Note that the socle is a characteristic subgroup and so certainly normal. It is a standard group theoretical result that the socle is the direct product of mutually isomorphic simple groups. A group Γ is *almost simple* if there is a finite non abelian simple group T, such that $T \leqslant \Gamma \leqslant \operatorname{aut}(T)$. A permutation group (Γ, X) is *affine* if it is primitive and the socle N of Γ is elementary abelian.

The next well known result tells us more about affine permutation groups. We include a proof for completeness.

Proposition 4.1. *If (Γ, X) is an affine permutation group with socle $N \cong Z_p$, then N acts regularly on X and $\Gamma \leqslant \mathrm{AGL}(m, p)$.*

Proof. First note that N acts transitively on X as the orbits of N would otherwise be non-trivial blocks of imprimivity:

Let g in Γ and O an orbit of N. Then Og is an orbit of N, as for $n \in N$, $x \in O$,

$$xgn = xn'g \in Og,$$

as $n' = gng^{-1} \in N$, and $xng = xgn''$, where $n'' = g^{-1}ng \in N$.

Now suppose n in N fixes x in X. Let x' be any element of X and n' in N such that $xn' = x'$. Then

$$x'n = x'n'^{-1}nn'$$
$$= xnn' = xn' = x',$$

so n is the identity and N acts regularly.

Finally an element g in Γ acts on $N \cong Z_p$ by conjugation. As $g^{-1}nn'g = g^{-1}ngg^{-1}n'g$, this gives us a map $\alpha: \Gamma \to \mathrm{GL}(m, p)$. The kernel of α is $C_\Gamma(N)$. Let g in $C_\Gamma(N)$, so that g fixes an element x. Then as above for any x' in X, choose n' in N so that $xn' = x'$ and we have $x'g = xng = xgn = xn = x'$. So g is the identity, $C_\Gamma(N)$ acts regularly and as $C_\Gamma(N) \geqslant N$, $C_\Gamma(N) = N$. In conclusion we can write $\Gamma = \Gamma_x \ltimes N$ for some fixed x in X and α embeds Γ_x into $\mathrm{GL}(m, p)$. Hence $G \leqslant \mathrm{AGL}(m, p)$.

We now state Praeger, Saxl and Yokoyama's result mentioned in the introduction.

Theorem 4.2. [4]. *Let G be a finite primitive distance-transitive graph of diameter d with Γ a group acting distance-transitively on G. Then one of the following holds:*

(a) G is the Hamming graph or $d = 2$ and G is the complement of the Hamming graph,

(b) Γ is almost simple,

(c) (Γ, VG) is affine.

We now present the proof of our main result.

PROOF OF THEOREM 1.1. By Lemma 1.7 Γ acts distance-transitively (and faithfully) on each of the derived graphs D and E of G. As G is primitive, we can apply Theorem 4.2 to each of graphs D and E. We consider the three possible cases for graph D:

(a) $D \cong H(d, q)$ or $D \cong H(2, q)^c$. By Propositions 3.2 and 3.3 this cannot occur if G is primitive.

(b) Γ is almost simple.

(c) (Γ, VD) is affine. In this case the socle N of Γ acts regularly on VD and so $|VD| = |N|$. But consider the action of Γ on E. As Γ is not almost simple and we can exclude the case when E is of Hamming type, (Γ, VE) is also affine and so $|VE| = |N|$. But then $|VD| = |VE|$ and so G is regular contradicting (Γ, G) being a distance-bitransitive pair.

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