AUTOMORPHISMS AND COVERINGS OF KLEIN SURFACES

by

Wendy Hall

A thesis submitted for the degree of

Doctor of Philosophy

Faculty of Mathematical Studies

University of Southampton

August 1977

To my parents

CONTENTS

UNIVERSITY OF SOUTHAMPTON

ABSTRACT

FACULTY OF MATHEMATICAL STUDIES

Doctor of Philosophy

AUTOMORPHISMS AND COVERINGS OF KLEIN SURFACES

by Wendy Hall

In this thesis the theory of automorphisms and coverings of compact Klein surfaces is discussed by considering a Klein surface as the orbit space of a non-Euclidean crystallographic group. In chapter 1 we set out some of the well-established theory concerning these ideas.

In chapter 2 maximal automorphism groups of compact Klein surfaces without boundary are considered. We solve the problem of which groups PSL (2,q) act as maximal automorphism groups of non-orientable Klein surface without boundary.

In chapter 3 we discuss cyclic groups acting as automorphism groups of compact Klein surfaces without boundary. It is shown that the maximum order for a cyclic group to be an automorphism group of a compact non-orientable Klein surface without boundary of genus $g \geq 3$ is $2g$, if g is odd and $2(g-1)$ if g is even.

Chapter 4 is the largest section of the thesis. It is concerned with coverings (possibly folded and ramified) of compact Klein surfaces, mainly Klein surfaces with boundary. All possible two-sheeted connected unramified covering surfaces of a Klein surface are classified and the orientability of a normal n-sheeted cover, for odd n, is determined

ACKNOWLEDGEMENTS

I would like to express my sincere gratitude to my supervisor Dr. D. Singerman for his immense help and encouragement during the preparation of this thesis.

I would also like to thank the Science Research Council for financing me over the last three years.

INTRODUCTION

Historically, Riemann surfaces were introduced as devices which render certain mappings as one-one mappings and were originally defined to be without boundary and orientable. The notion of a Klein surface is attributable to Klein because of his remarks in 1882 on the closing pages of [9]. Riemann surfaces have been studied extensively during the last century. Klein surfaces which are not Riemann surfaces were occasionally mentioned but work on them did not really begin until the appearance of [23]. In this work Schiffer and Spencer refer to Riemann surfaces as surfaces which can be orientable or non-orientable, with or without boundary. In [2] the term Riemann surface will infer an orientable surface without boundary.

Pioncaré introduced Fuchsian groups in order to generalize elliptic functions and subsequently realized that they were identical with groups of orientation preserving isometries of the non-Euclidean plane geometry of Lobatschewsky.

The orbit space of Fuchsian group is Riemann surface and recently Fuchsian groups have become very significant in the study of Riemann surfaces (e.g. [3], [14], [11]).

Non-Euclidean crystallographic (NEC) groups are discontinuous groups of isometries of the non-Euclidean plane which contain orientation reversing elements. The orbit space of a NEC group is a Klein surface. Thus, Klein surfaces can be studied by way of NEC groups. In chapter 1, we give the preliminary definition and results (obtained from the large volume of work already published on the subject) which we require to develop these ideas.

In chapter 2, we consider maximal automorphism groups of compact Klein surfaces without boundary. Hurwitz [8] showed that the order of a group orientation preserving automorphisms of a compact Riemann surface, of genus $g \geq 2$, cannot exceed 84 (g – 1). He also showed that this bound is attained when g = 3. Macbeath [13], [16] has shown that this bound is attained for infinitely many values of g. Maximal groups of orientation preserving automorphisms of compact Riemann surfaces are called Hurwitz group. Macbeath [16] gives the condition for PSL (2,q) to be a Hurwitz group. The orders of the automorphism groups of compact non-orientable Klein surfaces without boundary, of genus $g \geq 3$, are bounded above y 84 (g – 2) and a group of this order acting on a Klein surface of genus g is called an H* -group. Every H* -group is a Hurwitz group. Singerman [24] showed that the Huwitz group PSL (2,7) is not an H* -group while the Hurwitz group PSL (2,8) is. We establish general conditions which determine when PSL (2,q) is an H* -group given that it is a Hurwitz group and show that infinitely many such groups appear.

It is know (e.g. [7]) that the maximum order for a cyclic group to be a group of orientation preserving automorphisms of a compact Riemann surface of genus $g \geq 2$ is 2(2g +2) and May [22] has considered the problem for Klein surfaces without boundary. We show that the maximum order for a cyclic group to be a group of automorphisms of such a surface of genus $g \geq 3$ is 2g, if g is odd and 2(g-1) if g is even.

In chapter 4 we discuss coverings of Klein surfaces. Including ramified and folded covers. These have been studied in some detail by Alling and Greenleaf [2]. Initially, we consider 2-sheeted connected unramified covering of compact Klein surfaces with boundary. By determining all subgroups of index two in certain NEC groups with compact orbit space Γ we classify all possible connected unramified 2-sheeted coverings of the orbit space of Γ.

We then extend the problem to connected n-sheeted coverings of compact Klein surfaces. We determine the number of boundary components of a normal subgroup of prime index p, in a NEC group and the orientability of a normal subgroup of odd index n, in a NEC group. We give an example to show that in general these results cannot be extended to non-normal subgroups.

CHAPTER 1

Preliminary definitions and results.

1. Non Euclidean crystallographic groups.

1). Let U denote the upper –half complex plane, {z ε £ : Im z > 0}. U can be made into a model of the non-Euclidean (written N.E.) plane as follows.

Define the N.E. length of a piecewise differentiable arc C by

$$y1(C) = \int c \; \frac{\sqrt{dx^2 + dy^2}}{y}$$

and the N.E. area of a measurable set E by

$$\mu\,(E) = \iint_E \frac{dxdy}{y^2}$$

The geodesics of this metric are circles and lines orthogonal to the real axis ¡ (see [14]) and are called N.E. lines.

The N.E. distance between two points in U is the length of the unique N.E. line joining them.

The metric induces a topology on U which is the same as the topology induced from the usual topology on £ .

Let g denote the group of transformations of the extended complex plane, $\mathbb{C} \cup \{\infty\}$ of the form

$$\text{A) } z \to \frac{az+b}{cz+d}, \text{ a,b,c,d real, ad - bc = 1}$$

$$\text{B) } z \to \frac{a\bar{z}+b}{c\bar{z}+d}, \text{ a,b,c,d, real, ad - bc = -1}$$

The set of transformations of type A forms a subgroup of index two in g, denoted by g+ (although in other contexts it is more usually denoted by PSL $(2, \mathbb{R})$).

Each element of g is a conformal (type A) or anti-conformal (type B) homeomorphism of U onto itself. Every conformal homeomorphism of U lies in g+ (see e.g. Springer [27]) so that g is the group of conformal and anti-conformal homeomorphisms of U onto itself. Every element of g maps N.E. lines to N.E. lines and preserves N.E. distance.

We topologise g as the subset of $\mathbb{R}^4$

$$\{a, b, c, d: ad - bc = \pm 1\}$$

by identifying (a, b, c, d) and (-a,-b, -c, -d) and taking the identification topology. The topological group g has two components, namely g+ and g\g+. A discrete subgroup of g is called a <u>non-Euclidean crystallographic group</u> which we shall always abbreviate to NEC group. An NEC group contained in g+ is called a <u>Fuchsian group</u>. If an NEC group contains elements of type B, i.e. orientation reversing elements, we shall call it a proper NEC group.

2) We can classify the elements of g by their orientation and their fixed point set.

Elements of type A are orientation preserving. Their fixed point set is found by solving the quadratic equation

$$z = \frac{az + b}{cz + d} \quad ad - bc = 1$$

There are three types.

(i) <u>Hyperbolic</u> if $|a + d| > 2$, with two fixed points on $\mathbb{R} \cup \{\infty\}$

(ii) <u>Elliptic</u> if $|a + d| < 2$, with two complex conjugate fixed points, one of which is in U.

(iii) <u>Parabolic</u> if $|a + d| = 2$, with one fixed point on $\mathbb{R} \cup \{\infty\}$.

Elements of type B are orientation reversing. Their fixed point set is found by solving the equation

$$z = \frac{a\bar{z} + b}{c\bar{z} + d} \quad ad - bc = -1$$

They are two types.

(i) <u>Glide reflections</u> if $a + d \neq 0$, with two fixed points on $\mathbb{R} \cup \{\infty\}$.

(ii) <u>Reflections</u> if $a + d = 0$, with N.E. line of fixed points.

As the elements are classified by their trace (a + d) and their determinant, conjugate elements of g are of the same type. Each of the five types of transformations has a canonical form are listed below.

Type of element	Canonical form
Hyperbolic	$z \to \lambda z\ (\lambda > 1)$
Elliptic	$z \to w$, where $\frac{w - i}{w + i} = e^{i\theta}\left(\frac{z - i}{z + i}\right), \theta \neq 2n\pi$
Parabolic	$z \to z + 1$
Glide reflection	$z \to \lambda\bar{z}\ (\lambda - 1)$
Reflection	$z \to -\bar{z}$

If g ε g is a hyperbolic elemkent then g is conjugate to the transformation w (z) = $\lambda z, > 1$ and λ is known as the multiplier of the transformation. λ is an ivariant of the conjugacy class. Now $\lim_{n\to\infty} g^n$ (z) must exist and is a fixed point of g, called the attracting fixed point. Simarlarly $\lim_{n\to\infty} g^{-n}$ (z) is called the repelling fixed point. A hyperbolic element is uniquely determined by its multiplier and its fixed points. The same remarks apply to glide reflections.

Reflections are of order two. The only other elements which can have finite order are elliptic elements and conversely every elliptic element in an NEC group is of finite order.

3) A NEC group Γ acts properly discontinuously on U in the sense that every point z ε U has a neighbourhood V such that if $\gamma \varepsilon \Gamma$ and γV n $V \neq \theta$, then $\gamma z = z$.

The Γ-orbit of z ε U is { γz: γ ε Γ} and we for the orbit (or quotient) space, U/ Γ, by giving the set of all orbits the identificaiton topology.

Definition 1.1. A surface is a connected Hausdorff space on which there is an open covering by sets homeomorphic to open sets in ¡ ^{2} .

Definition 1.2. A connected Hausdorff space is called a surface with boundary if it is not a surface and if it possesses an open covering by sets which can be mapped homeomorphically onto relatively open sets of a closed half-plane.

Definition 1.3. A Γ – fundamental region is closed set F with the properties

1) F contains at least one element of every orbit,
2) Int F contains at most one element of every orbit,
3) The N.E. area μ (I\int F) = 0.

It has been shown by Wilkie [28] that for every NEC group Γ with compact quotient space there exists a canonical surface symbol of a fundamental region for Γ from which a canonical presentation for Γ can be derived. U/ Γ is a surface, with or without boundary, orientable or non-orientable, depending on the structure of Γ.

It is easy to see that U/ Γ is a surface with boundary if and only if Γ contains reflections.

Throughout this thesis we shall only be concerned with NEC groups with compact quotient space. By a well known result, such groups contain no parabolic elements (see Bers [3]). Also the classification of compact surfaces is well-known (see e.g. Massey [19], Lefschetws [10], Griffiths [6]).

Every compact oreintable surface is homeomorphic to a sphere with g handles attached.

Every compact non-orientable surface is homeomorphic to a sphere with g cross-caps attached.

Every compact orientable surface with boundary is homeomophic to a sphere with g handles attached and k discs removed.

Every compact non-orientable surface with boundary is homeomorphic to a sphere with g cross-caps attached and k discs removed.

We now give a brief description of how a presentation of a NEC group Γ may be obtained from a given fundamental region. The method is found in detail in [14] and [28].

Let p ε U be a point not fixed by any element of Γ. Let F be the set of points satisfying

$$D\ (z,p) \leq d\ (gz,p) \qquad \text{for all } g \in \Gamma,$$

where d 9z,p) dentotes the N.E. distance of a point z ε U to p. F is a fundamental region for Γ and is called the Dirichlet region. It is a convex set bounded by N.E. lines, with all its vertices in Ū (the closure of U). As Γ has compact quotient space F will be a bounded convex polygon with afinite numer of sides.

Two vertices are called congruent if they lie in the same Γ – orbit . Two edges are congruent if there is an element of Γ which maps one edge to the other.

If F meets one of its images gF (g ε Γ) in an edge then g^{-1} F meets F in an edge. These edges are distinct uless $g^2 = 1$, i.e. unless g is an elliptic transformation of order 2 or a reflection. If g is an elliptic transformation of order 2, Fn gF is an edge of F, say AB, which is mapped onto itself by g. The mid-point C of AB is fixed point of g and AC is mapped on CB by g. We add C to the set of vertices of F and regard AC and CB as two separate but congruent edges of F. If, however, g is a reflection every point of Fn gF is fixed under g. Such and edge of F is congruent to no other edge of F under Γ.

F has the following properties (see [28]).

1. F is homeomorphic to a closed disc.
2. F\int F is a polygonal Jordan curve, i.e. a curve which is a finite union of N.E. line segments.
3. There are a finite number of points on F\int F (the vertices) dviding F\int F into Jordan arcs (the edges).
4. The edges of F are divided into three categories as follows:
   a) Congruent pairs s, s' , where s, s' are the edges Fn gF, Fn g^{-1}F respectively and g ε Γ but $g^2 \neq 1$. Here $s = gs'$.
   b) Congruent pairs s, s' where $s = gs'$ and g is an elliptic transformation of order 2. In this case $sus' =$ Fn gF.

c) Edges s" where s" is Fn gF and g is a refletion. Such and edge is congruent to no other edge of F and is an N.E. line segment.

5. If Fn gF $\neq \theta$ where g ε Γ and F, gF do not have an edge in common then Fn gF is at most a finite number of vertices.

(A fundamental region with the above properties is called a <u>regular fundamental region</u>.)

The set [gF: g ε Γ) forms a tesselation which fits together to cover U. Any face of the tesselaion with the g' F for some g' g ε Γ and so will determine a unique face of the tesselation. Faces with an edge in common are called neighbours.

Let F be a face and F' another face meeting F in an edge α. Denote the group element which maps F to F' by a so that F' = aF. If $\hat{\alpha}$ is the edge congruent to α the a (α) = α.

To associate a surface symbol with a regular fundamenal region, e.g. the Dirichlet region, F for Γ we first label the edges of type c). The remaining edges occur in congruent pairs and we now label one edge from each congruent pair. If α is the label of such an edge, the edge congruent to α is labelled α' or α * according as the transformation which maps it onto the edge α preserves or reerses orientation. If we now write dow the labels of the edges of F in order anti-clockwise we obtain the surface symbol for F which will determine the topological strcture of U/ Γ.

Starting from the Dirichlet region for Γ (or any regular fundamental region for Γ) F we can otbain a new fundamental region as follows. Let α and $\hat{\alpha}$ be two congruent edges of F and split F into two regions F_1, F_2 by a polygon arc joining two vertices of F such that

^ ^

$\alpha \varepsilon F_1$, $\alpha \varepsilon F_2$. Then if a $(\alpha) = \alpha$, F_1 U aF_2 will be a new fundamental region for Γ which will have a different surface symbol. The side of this fundamental region may not now be N.E. lines. However, edges which are axes of reflection will still be N.E. lines. In this way a canonical form of the surface symbol is obtained (see [28]).

There are two types of canonical forms of surface symbols. One is for groups with orientable quotient space and one is for groups with non-orientabel quotient space. The wsurface symbol for a group with orientable quotient space is

$$(1.4) \quad \xi_1 \xi_1' \xi_2 \xi_2' \cdots \xi_k \xi_k' \varepsilon_1 \gamma_{10} \gamma_{11} \cdots \gamma_{1s} \varepsilon_1' \varepsilon_2 \gamma_{20} \gamma_{21} \cdots \gamma_{2s_2} \varepsilon_2' \cdots$$
$$\cdots \varepsilon_r \gamma_{r0} \cdots \gamma_{rs} \quad \varepsilon_r' \alpha_1 \beta_1 \alpha_1' \beta_1' \cdots \alpha_g \beta_g \alpha_g' \beta_g'$$

and the surface symbol for groups with non-orientable quotient space is

$$(1.5) \ \xi_1 \xi_1' \cdots \xi_k \xi_k' \varepsilon_1 \gamma_{10} \cdots \gamma_{rs_r} \varepsilon_r \alpha_1 \alpha_1^* \alpha_2 \alpha_2^* \cdots \alpha_g \alpha_g^{*\prime}$$

which differs from (1.4) only in the last part of the symbol. N_{is_i}

If we identify corresponding points on the related edges of the fundamental region with surface symbol (1.4) we obtain an orientablN_{is_i} rfacN_{i1} ith boundary which is a sphere with r discs removed and g handles added.

Similarly with surface symbol (1.5) we obtain a non-orientable surface with boundary which is a sphere with r discs removed and g cross-caps added.

On these surfaces, the edges α (in the non-orientable case) and α, β (in the orientable case) determin a canonical system of cross-cuts meeting at a base-point Q, say. There are k distinguished points M_i in the interior of the surface and s_i distinguished points

Nil, · · ·   on the ith boundary component. The lines ε_i joins Q to the points M and the line ε_i joins Q to a point on the ith boundary component between   and .

It can be shown that the set of group elements which map F on a neighbour generate Γ. We obtain the relations in Γ in the following way. There are a infinite number of faces meeting at each vertex, each face being a neighbour of the preceding face. If F is one of the faces going round the vertex we shall meet in order the faces a_1F, a_1a_2F, $a_1a_2a_3F$, · · · etc. (a_i ε Γ). After a finite number of steps we come back to F so that for some n, $a_1a_2 \cdots a_nF = F$ and we obtain the relation $a_1a_2 \cdots a_n = 1$ for the vertex, known as the canonical relation for that vertex. Congruent vertices give rise to the same canonical relation and it is shown that every relation in the group is a consequence of the canonical relations.

Denote by a, b, c, e, x the transformation which map F across the sides α, β, γ, ε, ξ. Then a group with surface symbol (1.4) will have presentation.

(1.6) generators

$$a_i, b_i \quad i = 1,2, K \; g$$
$$x_i \quad i = 1,2, K \; k$$
$$e_i \quad i = 1,2, K \; r$$
$$c_{ij} \quad i = 1,2, K \; r, j = 0,1,2, K \; s_i$$

and relations

$$x_i^{m}i = 1$$
$$c_{isi} = e_i \; {-1}_{ci0} e_i$$
$$c^2_i, j\text{-}1 = c^2_{ij} = (c_i, j\text{-}c_{ij})^n ij = 1$$
$$x_1x_2 \; K \; x_k e_1 e_2 \; K \; e_r a_1 b_1 a_1^{-1} b_1^{-1} \; K \; a_g b_g^{-1b}{}_g^{-1} = 1.$$

The presentation for groups with surface symbol (1.5) (i.e. with non-orientable quotient space) will have generators a_i, $i = 1,2, K \; g \; x_i, e_i, c_{ij}$ as in (1.6). The relations will be as in (1.6) except for the final relation which becomes

(1.7) $$x_1x_2 \ldots x_ke_1e_2 \ldots e_ra_1^2a_2^2 \ldots a_g^2 = 1$$

In (1.6) the elements a_i, b_i will be hyperbolic, x_i elliptic, c_{ij} reflections and the e_i will usually be hyperbolic although in exceptional cases they may be elliptic. In (1.7), similar remarks apply, except now the a_i are glide reflections.

The number m_i, n_{ij} are the orders of the orientation preserving elements of Γ and are called the periods of Γ. We call the m_i proper periods. We can associate with each group of NEC signature.

The NEC signature of the group Γ with presentation (1.6), i.e. with orientable quotient space is

(1.8) $$(g, +, [m_1, \ldots m_k], \{(n_{11}, \ldots n_{1s1}), \ldots (n_{r1} \ldots n_{rsr})\})$$

and the NEC signature of the group with presentation (1.7), i.e. with non-orientable quotient space is

(1.9) $$g, -, [m_1, \ldots m_k], \{n_{11}, \ldots n_{1s1}), \ldots (n_{r1}, \ldots n_{rsr})\}).$$

Brackets such as $(n_{11}, \ldots n_{1s1})$ are called period cycles. Note that once we are given a signature of a group the surface symbol and presentation are uniquely determined. So given a signature for an NEC group Γ we can immediately determine the topological structure of $U\Gamma$.

The integer g is known as the genus of the surface and called the orbit-genus of the group. The genus is an invariant of the surface as is the number of discs removed. A removed disc will be called a hole or a boundary component.

An NEC group may have empty period cycles and signature of the form

$$(g, \pm, [m_1, \ldots m_k], \{(\,), (\,), \ldots (\,)\})$$

which, if the number of empty period cycles if Γ, we shall write as

$$(g, \pm, [m_1, \ldots m_k], \{(\,)^r\}).$$

A Fuchsian group will have an orientable quotient space with no holes. All its periods are proper periods and it is determined by its orbit-genus and its periods. Its NEC signature is

$$(g, +, [m_1, K \;\; m_k], \{\ \})$$

and is usually written

$$g; m_1, K \;\; m_k).$$

Groups with no periods are no reflections are known as <u>surface groups</u>. If the orbit space is orientable it is called an <u>orientable surface group</u> (sometimes known as a Fuchsian surface group) and will have signature

$$(g, +, [\ ], \{\ \}) \text{ or } \qquad (g; \text{-------}).$$

If the orbit space is non-orientable it is known as a <u>non-orientable surface group</u> and will have signature

$$g, -, [\ ], \{\ \}).$$

Groups with no periods but with reflections are known as <u>bordered surface groups</u>. A group with signature

$$(g, +, [\ ], \{(\ )^r\})$$

will be called an orientable bordered surface group (with r boundary components) and a group with signature

$(g, -, [\ ], \{(\ )^r\})$

will be called a non-orientable bordered surface group (with r boundary components).

4)

Lemma 1.10 ([17]) Let Γ have signature (1.8) or (1.9). Then an element of finite order in Γ is conjugate to one of the following:

(i) A power of some x_i $(1 \le i \le k)$

(ii) A power of some $c_{i}, j\text{-}1\ c_{ij}$ $(1 \le i \le r, 1 \le j \le s_i)$

(iii) Some c_{ij} $(1 \le i \le r, 0 \le j \le s_i)$.

Proof.

An element of finite order in Γ is either an elliptic element or a reflection and thus has a fixed point $p \varepsilon U$. If F is a fundamental region for Γ, F contains an element in the orbit of p, say gp ε F for some g ε Γ. Thus gp, being a fixed point of gtg^{-1}, lies on the boundary of F. The stabilizers of fixed points on the boundary of F are those listed in (i) (ii) or (iii) above and as gtg^{-1} belongs to the stabilizer of one of these points our assertion is proved.

In [28] Wilkie gave some sufficient conditions for two NEC groups to be isomorphic, his work was purely algebraic, Macbeath [17] found necessary and sufficient conditions for two NEC groups to be isomorphic but these results were not obtained algebraically.

Definition 1.11. Let Γ and Γ' be two isomorphic NEC groups and let $\Phi: \Gamma \rightarrow \Gamma'$ be the isomorphism. Γ and Γ' are called geometrically isomorphic if there exists a homeorphism w of U onto itself such that

$$\Phi(g) = wgw^{-1} \text{ for all } g \in \Gamma.$$

We say that the isomorphism Φ can be realized geometrically.

If the isomorphism Φ can be realized geometrically then the groups Γ and Γ' are conjugate in the group of all homeomorphisms of U. If $z \in U$, the geometrical isomorphism w maps the Γ-orbit of z on the Γ'-orbit of wz, thus it induces a homeomorphism between the quotient spaces.

Theorem 1.12. (1[1.7]) Let $\Phi: \Gamma \rightarrow \Gamma'$ be an isomorphism (of the group structure only) between two NEC groups. Then Φ can be realized geometrically.

Macbeath proved this result using Teichmuller's theorem on external quasiconformal mappings and used it to determine the necessary and sufficient conditions for Γ and Γ' to be isomorphic.

The genus and orientability of a surface are geometric invariants of that surface so clearly if two NEC groups are isomorphic then the orientability and genera of their orbit spaces are the same.

4) From lemma 1.10 we see that every reflection in an NEC group is conjugate to one of the (canonical) generating reflections. When trying to determine the number of boundary components of an NEC group, as we shall be in chapter 4, we are in fact counting conjugacy classes of reflection.

Lemma 1.15. Let Γ_1 be a bordered surface group and let $\Gamma_2 < \Gamma_1$ with index n. Let $g_1, \ldots g_n \in \Gamma_1 \setminus \Gamma_2$ such that

$$\Gamma_1 = \Gamma_2 g_1 + \Gamma_2 g_2 + \ldots + \Gamma_2 g_n,$$

and let $c \in \Gamma_2 \subset \Gamma_1$ be a reflection. Then any conjugate of c in Γ_1 will be conjugate to $g_i c g_i^{-1}$ in Γ_2 for some $i = 1, \ldots n$.

<u>Proof</u>.

Clearly any conjugate of c in Γ_1 will be an element of Γ_2. Consider hch^{-1}, where $h \varepsilon \Gamma_1$. Then $h \varepsilon \Gamma_2 g_i$, for some i=1, K n. So we can express h in the form h = xg_i, $x \varepsilon \Gamma_2$ and then

$$hch^{-1}\ xg_i\ c\ (xg_i)^{-1} = x(g_i\ cg_i\ ^{-1})\ x\ ^{-1}$$

which is a conjugate of $g_i\ cg_i^{-1}$ in Γ_2.

Singerman [25] has investigated some of the algebraic properties of NEC groups in particular their reflections.

Theorem 1.16. ([25]) Let $c \varepsilon \Gamma$ be a reflection, Γ a NEC group. Then Z (c) , the centralizer of c in Γ, is infinite. In particular if c is the generating reflection associated with an empty period cycle and e is the generator in the canonical presentation for Γ commuting with c, then Z (c) = (c, e), the group generated by c and e.

Clearly, from this result, an 'e generator' associated with an empty period cycle must have infinite order, i.e. is hyperbolic. In the case of Γ being a bordered surface group this is obvious since the only elements of finite order are the generating reflections and their conjugates.

In [25] the N.E. area of a fundamental region of an NEC group was determined. This is independent of a fundamental region chosen for the group and thus will depend only on the signature of the group. Thus we can denote the N.E. area of a fundamental region for Γ by $\mu(\Gamma)$.

Theorem 1.17. ([25])

(a) Let Γ be a NEC group with signature (1.8). Then the N.E. area of a fundamental region for Γ is given by

$$\mu(\Gamma) = 2\pi\left(2g - 2 + \sum_{i=1}^{k}\left(1-\frac{1}{m_i}\right) + r + \sum_{i=1}^{r}\sum_{j=1}^{s_i}\frac{1}{2}\left(1-\frac{1}{n_{ij}}\right)\right).$$

(b) If Γ has signature (1.9) then

$$\mu(\Gamma) = 2\pi\left(g - 2 + \sum_{i=1}^{k}\left(1 - \frac{1}{m_i}\right) + r + \sum_{i=1}^{r}\sum_{j=1}^{s_i}\frac{1}{2}\left(1-\frac{1}{n_{ij}}\right)\right).$$

Let Γ be a NEC group and $\wedge g_1 + \wedge g_2 + K \wedge g_n$.

If F is a fundamental region for Γ then it is easily verified that $g_1 F \upsilon g_2 F \upsilon \ldots \upsilon g_n F$

$g_1 F \upsilon g_2 F \upsilon K \upsilon g_n F$

is a fundamental region Λ. But the N.E. area is invariant and we deduce the formula for the index of Λ in Γ, known as the Riemann-Hurwitz formula,

$$[\Gamma : \Lambda] = \mu(\Lambda)/\mu(\Gamma),$$

(6) Let Γ be a proper NEC group. Then the Fuchsian subgroup of Γ consisting of all elements which preserve orientation has index two in Γ and will be denoted by Γ^+.Γ^+ is called the canonical Fuchsian group of Γ.

If Γ is a NEC group with signature (1.8) or (1.9) then the elements of finite order in Γ are given in (i) , (ii) and (iii) of lemma 1.10. But the elements $x_i, c_{i,j-1}c_{ij}$ lie in Γ^+ and so the periods of Γ^+ contain the periods of Γ.

In [25] it is shown that each proper period, m_i, is repeated twice and only twice in Γ^+ but each period of the form n_{ij} occurs only once among the periods of Γ^+. Using this and the Riemann-Hurwitz formula which says that $\mu(\Gamma^+) = 2\mu(\Gamma)$ we deduce the following theorem.

Theorem 1.18.

(a) Let Γ be a proper NEC group with signature (1.8). Then Γ^+ has signature

$$(2g + r - 1; m_1, m_1, \ldots m_k, m_k, n_{11}, n_{12}, \ldots n_{rs_r}).$$

(b) If Γ has signature (1.9) then Γ^+ has signature

$$(g + r - 1; m_1, m_1, \ldots m_k, m_k, n_{11}, n_{12}, \ldots n_{rs_r}).$$

7). If Γ is a NEC group we denote by $N G + (\Gamma)$ the normaliser of G+ of Γ and by NG (Γ) the normaliser of G of Γ. Let Γ^+ be the canonical Fuchsian group of Γ and let $t \in \Gamma \setminus \Gamma^+$ so that $\Gamma = \Gamma^+ + t\Gamma^+$. If $g \in Ng(\Gamma)$ then $g\Gamma g^{-1} = \Gamma$ and

$$g(\Gamma^+ + t\Gamma^+)g^{-1} = \Gamma^+ + t\Gamma^+,$$

i.e. $$g\Gamma^+g^{-1} + gt\Gamma^+g^{-1} = \Gamma^+ + t\Gamma^+$$

By equating together the set of orientation preserving elements on both sides of the equation we see that $g\Gamma^+g^{-1} = \Gamma^+$ so that $g \in N g (\Gamma^+)$ which implies that

$$Ng(\Gamma) \subset Ng(\Gamma^+).$$

Now Ng + (Γ^+) is a subgroup of index one or two in Ng (Γ^+) and it is well-known that Ng + (Γ^+) is a Fuchsian group (see e.g. [14]), and thus discrete. Therefore Ng (Γ^+) and hence Ng(Γ) is discrete. At $\Gamma \subset$ Ng (Γ), Ng (Γ) has a compact fundamental region (for NEC groups Γ with compact orbit space) and hence has compact orbit space.

We have thus proved the following.

Lemma 1.19. Let Γ be a NEC group.

(a) If Γ is a proper NEC group and Γ^+ its canonical Fuchsian group then Ng (Γ) $\subset$ Ng (Γ^+).

(b) If Γ has compact orbit space the Ng (Γ) is a NEC group with compact orbit space.

II Klein surfaces and their automorphisms

1).

Definition 1.20. A complex chart on a surface S consists of a pair (U,z) where U is an open set and z is a homeomorphism and U onto an open set in the complex plane. £ . If S is a surface with boundary then z is a homeomorphism of U onto either an open set in £ or a relatively open set in a closed upper-half plane.

Definition 1.21. A family of charts $\Omega = \{U_i, z_i)\}_{i\varepsilon I}$ where I is an index set, is called a dianalytic (or complex) atlas for S if

(i) $U_{i\varepsilon I} U_i = S$,

(ii) if $((U_i, z_i), (U_j, z_j)$ ε $U_i \cap U_j \neq \emptyset$ then $z_i z_j^{-1}$ conformal or anti-conformal homeomorphism defined on z_j $(U_i \cap U_j)$ intersects the boundary of $\overline{U}$, the closed upper-half plane, then we require $z_i z_j^{-1}$ to have an analytic or anti-analytic extension to an open subset of the plane.

The maps $z_i z_j^{-1}$ are called <u>co-ordinate transformations</u> (or <u>transition functions</u>).

If S is a surface with boundary then the boundary δS of S consists of the points s ε S such that s ε U_i with z_i (U_i) open in $\overline{U}$. But not open in £ and z_i (s) ε IR. We denote the interior of S, i.e. s \δS, by S^o.

<u>Definition 1.22</u>. A <u>Klein surface</u> is a surface, or a surface with boundary, S with a dianalytic atlas Ω. It will be denoted by (S, Ω) or just by S.

The dianalytic atlas Ω is said to define a <u>dianalytic structure</u> on S. Another atlas $\Omega' = \{(v_j, w_j)\}_{j \quad J}$ defines the same structure provided $\{U_i, z_i) \cup (V_j, w_j)\}_{ieI, jeJ}$ is a dianalytic atlas for S. We say Ω and Ω' are <u>dianalytically equivalent.</u>

A dianalytic atlas in which all the co-ordinate transformations are conformal (sense-preserving) maps will be called <u>analytic atlas</u>. We say an analytic atlas on a surface S defines an <u>analytic structure</u> on S.

By a <u>Riemann surface</u> we shall mean a surface without boundary with an analytic atlas. The meaning of the term <u>Riemann surface with boundary</u> should be clear. Clearly a Riemann surface is an orientable Klein surface without boundary.

If (S, Ω) is an orientable Klein surface then there are tow analytic structures on S each of which is dianalytically equivalent to Ω (see Alling and Greenleaf [2], theorem 1.2.4.). (The proof basically involves choosing a maximal analytic atlas for S.)

Choosing between the two analytic structures is equivalent to choosing an orientation for S. (The two resulting Riemann surfaces are anti-conformally equivalent) So without real ambiguity we may consider an orientable Klein surface to be a Riemann surface or a Riemann surface with boundary.

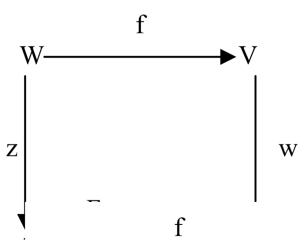
We now wish to define a morphism f: S $\to$ T of Klein surfaces. This differs from the corresponding concept for Riemann surfaces principally in that S may "fold" along δT. For this reason we need to define the folding map. This is the map $\varnothing$:C $\to$ $\overline{\text{U}}$ given by

$$\Phi\ (x + iy) = (x + i/y/).$$

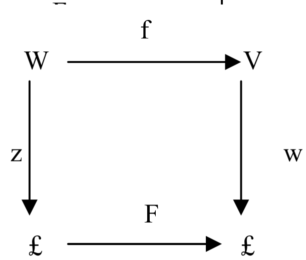
We define a <u>positive c</u>hart (V,w) to be a chart such that w(V) C $\overline{\text{U}}$.

Definition 1.23. ([2]) A morphism f: S $\to$ T of Klein surfaces is a continuous map f of S into T, with f(δS)C δT, such that for all s ε S there exist dianalytic charts (W,Z) and (V,w) about s and f(s) respectively, and an anlytic function F on z (W) such that the following diagram commutes.

(1.24)



(1.25)



In this case if F is anti-analytic then we can replace z by $\overline{z}$, which will make F analytic so that f is still a morphism.

Let f: S $\to$ T be a non-constant morphism of Klein surfaces.

Let s ε S. Alling and Greenleaf [2] have shwon that we can find dianalytic charts (U,z) and (V,w) at s and f (s) respectively, such that z (s) = 0 = w(f(s)), f (U)CV and such that g/U has the form

$$g/U = \begin{cases} w^{-1} \circ \varphi \circ (\pm z^{e}) & \text{if } f(s) \; \varepsilon \; \delta T \\ w^{-1} \circ (\pm z^{e}) & \text{if } f(s) \; \varepsilon \; T^{o} \end{cases}$$

where e is an iteger, $e \geq 1$. The integer e is called the ramification index of f at s and will be denoted by e_f (s). We say that f is ramified at s if e_f (s) > l; otherwise we say thaqt f is unramified at s.

We define the relative degree of s ε S over f (s), d_f (s), to be

$$d_f(s) = \begin{cases} 2 & \text{if } s \; \varepsilon \; S^{o} \text{and } f(s) \; \varepsilon \; \delta T \\ 1 & \text{otherwise.} \end{cases}$$

Definition 1.26. A non-constant morphism f: $S \rightarrow T$ between two Klein surfaces will be called an n-sheeted covering of T if for every point t ε T

$$\sum_{s \varepsilon f^{-1}(t)} e_f(s) d_f(s) = n.$$

If $e_f(s) = 1$ for all s ε S, f: $S \rightarrow T$ is an unramified n-sheeted covering, otherwise it is a ramified n-sheeted covering.

In [2] Alling and Greenleaf give detailed proof to show that every non-constant morphism between two compack Klein surfaces is an n-sheeted covering for some n. Also if S, T and X are Klein surfaces and f: $S \rightarrow T$, g: $T \rightarrow X$ non-constant morphisms then gf: $S \rightarrow X$ is a non-constant morphism. If f is an n-sheeted covering of T and g is an m-sheeted covering of X then gf is an mn-sheeted covering of X.

Let S_1, S_2 be two homeomorphic orientable Klein surfaces. An orientation preserving (reversing) homeomorphism f: $S_1 \to S_2$ is called a conformal (anti-conformal) homeomorphism if f is a morphism with respect to the dianalytic structures on S_1 and S_2.

A conformal (anti-conformal) homeomorphism from S_1 to S_1 will be called a +automorphism (-automorphism). An automorphism is either a + or a – automorphism. For any orientable Klein surface S the set of all automorphisms form a group Aut S, which contains as a subgroup of indes 1 or 2 the group of all + automorphisms of S, denoted by + Aut S.

If S_1, S_2 are two homeomorphic non-orientable Klein surfaces, then a homeomorphism f: $S_1 \to S_2$ is called a conformal homeomorphism if f is a morphism with respect to the dianalytic structures of S_1 and S_2. A conformal hemeomorphism of a non-orientable surface onto itself will be called a automorphism. The set of automorphism of non-orientable surface forms a group, Aut S.

If S_1 and S_2 are two Klein surfaces and F: $S_1 \to S_2$ is a conformal homeomorphism then S_1, S_2 are called conformally equivalent or isomorphic.

2) We shall only be concerned with compact Klein surfaces (either with or without boundary). We now discuss how the surface U/Γ, where Γ is an NEC group (with compact quotient space) may be given a dianalytic structure.

Theorem 1.27. Let Γ be a NEC group. Then the quotient space U/Γ has a unique dianalytic structure such that the quotient map $\pi: U \to U/\Gamma$ is a morphism of Klein surfaces.

Proof.

Since Γ acts properly discontinuously of U, this follows immediately from a result of Alling and Greenleaf ([2]) theorem 1.8.4).

The map π is folded over the boundary U/Γ and ramified over the distinguished points of the surface. If Γ is a surface group or a bordered surface group the π is unramified. Also, it is easy to see that for $z \varepsilon U$, $\pi(z) \; \varepsilon \; \delta(U/\Gamma)$ if and only if there exists a reflection $c \, \varepsilon \, \Gamma$ such that $c(z) = z$. If Γ is a Fuchsian group then U/Γ has an induced analytic structure with which it is a Reimann surface.

Let Γ be a non-orientable surface group or a bordered surface group, then the quotient space U/Γ^+, where Γ^+ is a canonical Fuchsian group of Γ, is a Riemann surface. If $z \; \varepsilon U$ let

$$\pi_{\Gamma^+}(z) = [z]_\Gamma$$

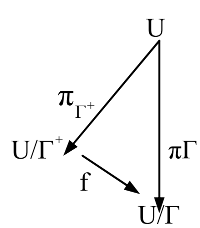
and

$$\pi_{\Gamma^+}(z) = [z]_\Gamma.$$

If f: $U/\Gamma^+ \rightarrow U/\Gamma$ is the natural projection defined by

$$f([z]_{\Gamma^+}) = [z]_\Gamma$$

then the following diagram commutes



and f is an ur ring of U/Γ.

If U/ J/Γ^+ is a uniquely defined two-sheeted orientable covering sur boundary then U/Γ^+ is a uniquely defined two-sheeted orientable covering surface without boundary of U/Γ.

It is a well-known result that any compact Riemann surface of genus $g \geq 2$ can be represented in the form U/Γ, where Γ is a Fuchsian surface group and $\mu(\Gamma) = 2\pi(2g - 2)$

(see Springer [27]). This is because U is the universal covering space of all compact Riemann surfaces except the sphere and the torus.

Schiffer and Spencer [23] describe the double of a compact Klein surface S, which if S is a surface with boundary or a non-orientable surface with or without boundary, is a connected compact Riemann surface. If S has genus g and r boundary coponents then the genus of the double of S is $2g{+}r{-}1$ if S is orientable and g + r –1 if S is non-orientable. The same double is described by Alling and Greenleaf [2]. They call it the complex double and denote it by S_c. We shall describe in detail in chapter 4 the construction of S_c but for the moment we shall assume its existence and use it to prove the following theorem.

<u>Theorem 1.28</u>. Let S be a compact Klein surface with genus g and r boundary components such that $g \geq 2$ if S is orientable without boundary, $2g + r \geq 3$ if S is orientable with boundary and $g + r \geq 3$ if S is non-orientable. Then $S = U/\Gamma$, where Γ is either a surface group or a bordered surface group.

<u>Proof</u>.

If S is orientable without boundary and genus $g \geq 2$ then $S = U/\Gamma$, where Γ is a Fuchsian surface group.

If S is orientable with boundary and $2g + r \geq 3$ then the genus of S_c is

$$Y_1 = 2g + r = 1 \geq 2.$$

If S is non-orientable (with or without boundary) and $g + r - 1 \geq 3$ then the genus of S_c, the double of S is

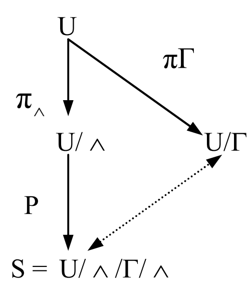
$$Y_2 = g + r - 1 \geq 2.$$

Therefore in both cases S_c = U/Λ, where Λ is an orientable surface group. S_c is symetric and so admits and anti-conformal involution which we may represent by $[z]_\wedge \to [gz]_\wedge$ where g ε g is an orientation reversing transformation with the property that $g\Lambda = \Lambda g$ and $g^2 \varepsilon \Lambda$. Let $\Gamma = \Lambda + g\Lambda$. Γ/Λ has a natural action of U/Λ sending $[z]_\Lambda$ to$[gz]_\Lambda$ and the orbit U/Λ/Γ/Λ

When given the induced dianalytic structure is conformally equivalent to S. (We note here that if S has boundary then we can take g to be a reflection.)

Let p be the natural projection of U/Λ onto U/Λ/Γ/Λ and let $p[z]_\Lambda = \{[z]_\Lambda\} = \{[gz]_\Lambda\}$.

The correspondence $\{[z]_\Lambda = \{[z]_\Lambda\} \to [z]_\Gamma$ is one-one and is a conformal homeomorphism from the following diagram



As the maps π_Γ, π_Λ and p are all open, continuous and analytic. Therefore U/Γ is conformally equivalent to S.

From theorem 1.28 we see that the only compact Klein surfaces not representable as U/Γ, where Γ is a surface group or a bordered surface group, are in the orientable case the sphere (g = 0, r = 0), the torus (g = 1, r = 0), the closed disc) g = 0, r = 1) and the closed annulus (g = 1, r = 1) and in the non-orientable case the projective plane (g = 1, r = 0), the Mobius band (g = 1, r = 1) and the Klein bottle (g = 2, r = 0).

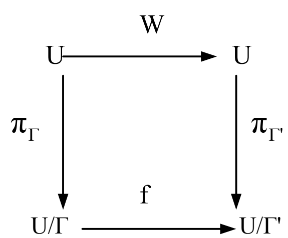
3) We shall now develop the theory of automorphisms of Klein surfaces without boundary.

The representation of homeomorphisms between compact Riemann surfaces by homeomorphisms of U is well –known (see e.g. Macbeath [13], Bers [3]) and the results extend to Klein surfaces without boundary.

Let Γ, Γ' be two surface groups. Put

$$\pi_\Gamma(z) = [z]_\Gamma$$

We say a homeomorphism w: $U \to U$ induces a homeomorphism f: $U/\Gamma \to U/\Gamma'$ if the following diagram commutes.



Clearly if w: $U \to U$ is a homeomorphism such that $w\Gamma w^{-1} = \Gamma'$ then the mapping

$$f)[z]\Gamma) = [wz]_{\Gamma'}$$

is well defined and f is a homeomorphism.

If f: $U/\Gamma \to U/\Gamma'$ is a homeomorphism then using results in the theory of covering spaces we can deduce that there exists a homeomorphism w: $U \to U$ which induces f. This mapping w is not uniquely defined for f is also induced by $w\gamma, \gamma \, \varepsilon \, \Gamma$, as $f\pi_\Gamma = f\pi_\Gamma = f\pi_\Gamma \gamma$. It follows that w also induces an isomorphism i: $\Gamma \to \Gamma'$ defined by $i\ (\gamma) = w\gamma w^{-1}$ and so $w\Gamma w^{-} = \Gamma'$. F is conformal or anti-conformal if and only if w ε g. Also if U/Γ and U/Γ' are orientable surfaces then f is orientation preserving if and only if w is orientation preserving so that f is conformal if and only if w ε g. $w\Gamma w^{-1} = \Gamma'$ we deduce the following well-known result. If Γ, Γ' are orientable (non-orientable) surface groups then U/Γ, U/Γ' are conformally equivalent if and only if there exists $w \in g+$ ($w \in g$) such that $w\Gamma w^{-1} = \Gamma'$.

If we put $\Gamma = \Gamma'$ we see that f is an automorphism of U/Γ if and only if $w \in Ng$ (Γ). The group of automorphisms of U/Γ is isomorphic to $Ng(\Gamma)/\Gamma$. If Γ is an orientable surface group then the goup of + automorphisms of U/Γ is isomorphic to $Ng+(\Gamma)/\Gamma$.

May [21] has infact extended this result and has shown that if Γ is a bordered surface group then the group of automorphisms of U/Γ is $Ng(\Gamma)/\Gamma$. We discuss these ideas as related to Klein surfaces with boundary in chapter 4.

Groups of + automorphisms of compact Riemann surfaces have been well-studied by Hurwitz [8], Macbeath [13], [14], [15], Harvey [7], Maclachlan [18] and Singerman[24], [26].

If S is a compact Riemann surface of genus $g \geq 2$ then = U/Λ where Λ is an orientable surface group and

$$+\text{Aut } S = Ng+(\Lambda)/\Lambda$$

which is a quotient of two Fuchsian groups. Any subgroup G of + Aut S is therefore of the form

$$G = \Gamma/\Lambda$$

Where Γ is a Fuchsian group. Conversely, any element of $Ng+(\Lambda)$ induces a + automorphism of U/Λ so that Γ/Λ acts as a group of + automorphisms of U/Λ.

Therefore a necessary and sufficient condition for a group G to be a group of + automorphism of a compact Riemann surface $S = U/\Lambda$ is that there is a homeomorphism from a Fuchsian group Γ onto G such that the kernel is the orientable surface group Λ.

Using this we can compact orientable Klein surface without boundary since

$$|G| = \frac{\mu(\Lambda)}{\mu(\Gamma)} = \frac{2\pi(2g - 2)}{\mu(\Gamma)}.$$

If Γ has signature $(h;\ m_1, m_2, \ldots\ m_k)$ then

$$\mu(\Gamma) = 2\pi(2h - 2 + \sum_{i=1}^{k}(1_i - \frac{1}{m_i}))$$

and by considering all possibilities (see Macbeath [14]) we can show that $\mu(\Gamma) \geq \pi/21$ with equality holding only when Γ is the Fuchsian traing (0; 2,3,7). We have thus shown that $|G| \leq 84\,(g - 1)$. This bound was first obtained by Hurwitz [8], who showed that it was attained when g = 3. Since then it has been shown to be attained for infinitely many g (e.g. Macbeath [13]). We shall look more closely at this problem in chapter 2.

Let S be a non-orientable compact Klein surface without boundary of genus $g \geq 3$ so that $S = U/\Lambda$, where Λ is a non-orientable surface group, then

$$\text{Aut } S \cong N_{\varsigma}(\Lambda)/\Lambda$$

And since $N_{\varsigma}(\Lambda)$ is a NEC group with compact quotient space any group of automorphism of U/Λ will be isomorphic to Γ/Λ where Γ is a NEC group. Conversely Γ/Λ acts as a group of automorphisms of U/Λ.

Thus a group of automorphisms of a non-orientable Klein surface without boundary of genus $g \geq 3$ is finite.

Theorem 1.30. ([24]) A necessary and sufficient condition for a finite group G to be a group of automorphisms of a compact non-orientable Klein surface without boundary S of genus $g \geq 3$ is that there exists a proper NEC group Γ and a homomorphism $\theta : \Gamma \rightarrow G$ such that the kernel of θ is a surface group and $\theta(\Gamma^+) = G$.

Proof.

If G is a group of automorphisms of S then $G = \Gamma / \Lambda$ where Λ is a non-orientable surface group such that S = U/Λ and Γ is a proper NEC group. Hence there exists a homomorphism $\theta : \Gamma \to G$ whose kernel is a non-orientable surface group. Thus there exists

$$t\, \varepsilon \ker \theta \cap (\Gamma \setminus \Gamma^{+}).$$

Then $\Gamma = \Gamma^{+} + t\Gamma^{+}$. Let $\theta(\Gamma^{+}) = G^{+}$. So

$$G = \theta(\Gamma) = \theta(\Gamma^{+} + t\Gamma^{+}) = \theta\ (\Gamma^{+}) + \theta(t)\ \theta(\Gamma^{+}) = G^{+} + G^{+} = G^{+}$$

Which implies that $\theta(\Gamma^{+}) = G$

Conversely suppose there exists a homomorphism $\theta\ \Gamma \to G$ such that $\theta(\Gamma^{+}) = G$ and ker θ is a surface group Λ. Now if Λ were an orientable surface group $\Lambda < \Gamma^{+}$, so that Λ is the kernel of the restriction of θ to Γ^{+}. Thus

$$G = \Gamma^{+} / \Lambda \ ; \quad \Gamma / \Lambda,$$

Which is impossible as $\mu(\Gamma^{+}) = 2\mu(\Gamma)$. Therefore Λ is a non-orientable surface group and G is a group of automorphisms of a non-orientable Klein surface without boundary of genus $g \geq 3$.

If we let $\tilde{S} = U/\Lambda^{+}$ then $\tilde{S}$ is the uniquely defined two-sheeted orientable covering surface of $S = U/\Lambda$.

Corollary 1.31. If G is a group of automorphisms of S then G is a group of + automorphisms of $\tilde{S}$, its orientable two-sheeted covering surface.

Proof.

$S = U/\Lambda$, so by theorem 1.30 $G = \Gamma/\Lambda$ where Γ is a proper NEC group. The group Γ/Λ has a natural action on S. If $\gamma \in \Gamma$ then

$$\gamma\Lambda([z]_{\Lambda}) = [\gamma z]_{\Lambda}.$$

As Λ is a non-orientable we can without loss of generality assume that γ by $\gamma\bar{\lambda}$ where $\bar{\lambda} \varepsilon \Lambda \setminus \Lambda^{+}$. Clearly, as $\Lambda^{+} \subset \Lambda$,

$$\gamma\Lambda([z]_{\Lambda^{+}}) = [\gamma z]_{\Lambda^{+}}$$

so that G has a well-defined action on U/Λ^{+}. It follows that G is a group of + automorphisms of $\tilde{S} = U/\Lambda^{+}$.

<u>Lemma 1.32</u>. If σ is an anti-conformal involution of $\tilde{S}$ such that $S = \tilde{S}/\langle\sigma\rangle$ (where $\langle\sigma\rangle$ denotes the group generated by σ) and G is a group of automorphisms of S (which by corollary 1.31 is a group of + automorphisms of S (which by corollary 1.31 is a group of + automorphisms of $\tilde{S}$) then σ commutes with every element of G.

<u>Proof</u>.

$S = U/\Lambda$, $\tilde{S} = U/\Lambda^{+}$ and by anti-conformal involution σ of $\tilde{S}$ such that $S = \tilde{S}/\langle\sigma\rangle$ is of the form.

$$\sigma : [z]\ \sigma : [z]_{\Lambda^{+}} \to [\lambda z]_{\Lambda^{+}},\ \lambda \varepsilon \Lambda \setminus \Lambda b^{+}$$

By theorem 1.30 $G = \Gamma/\Lambda$, where Γ is a proper NEC group. Let $g = \gamma\Lambda\ \varepsilon\ \Gamma/\Lambda$, which from the proof of corollary 1.31 is a + automorphism of $\tilde{S}$, Then

$$g\sigma([z]_{\Lambda^{+}}) = g[\lambda z]_{\Lambda^{+}} = [\gamma\lambda z]_{\Lambda^{+}}$$

and

$$\sigma g([z]_{\Lambda^{+}}) = \sigma[\gamma z]_{\Lambda^{+}} = [\lambda\gamma z]_{\Lambda^{+}}$$

But $\lambda^{-1}\gamma^{-1}\lambda\gamma \in \Lambda$ because $\Lambda < \Gamma$ so $\lambda^{-1}\gamma^{-1}\lambda\gamma$ is orientation preserving . Therefore $\lambda^{-1}\gamma^{-1}\lambda\gamma \in \Lambda^{+}$ and hence $g\sigma = \sigma g$.

We note here that if S is a non-orientable Klein surface without boundary of genus g then S has genus g – 1 from the Riemann-Hurwitz formula.

Definition 1.33. A homomorphism from a NEC group onto a finite group whose kernel is a surface group is called a surface-kernel homomorphism.

Lemma 1.34. A homomorphism θ from a NEC group Γ onto a finite group G is surface-kernel if and only if for every element x of finite order in Γ, $\theta(x)$ has the same finite order.

Proof.

A NEC group is a surface group if and only if it contains no elements of finite order. It is then clear that if θ preserves the orders of the elements of finite order in Γ, ker θ must be a surface group. Converseley, if θ is a surface-kernel homomorphism and $x \in \Gamma$ is an element with finite order m then $\theta(x)$ has order d dividing m. This implies $x^d \in \ker \theta$ and as ker θ is a surface group, d = m.

Note: As every element of finite order in an NEC group Γ is conjugate to a generator of Γ, θ is surface –kernel if and only if θ preserves the orders of the elliptic and refelection generators.

CHAPTER 2

Maximal automorphism groups of compact Klein surfaces without boundary.

1). In chapter 1 we deduced Hurwitz's result that the order of a group of + automorphisms of an orientable Klein surface without boundary (i.e . a Riemann surface) of genus g cannot be bigger than 84(g –1). Because this bound was first obtained by Hurwitz we have the following definition.

Definition 2.1. A group of 84(g-1) + automorphisms of an orientable Klein surface without boundary of genus g is called a Hurwitz group.

The problem of finding Hurwitz groups has been considered by Macbeath [13], [16], by Lehner and Newman [12] and by Singerman [24].

Let S be an orientable Klein surface without boundary of genus $g \geq 2$, which admits a group of 84 (g – 1) + automorphisms. Le Λ be an orientable surface group such that $S = U/\Lambda$, then as shown in chapter 1, the group + automorphisms of S is isomorphic of $N\varsigma^{+}(\Lambda)/\Lambda$, $\mu(N\varsigma^{+}(\Lambda)) = \pi/21$ and $N\varsigma^{+}(\Lambda)$ is the Fuchsian group with signature

$$(0; 2,3,7)$$

i.e. the group with presentation

$$\{x,y;\ x^2 = y^3 = (xy)^7 = 1\}.$$

This group may be obtained as follows. Let Δ be the proper NEC group generated by the reflections c_1, c_2, c_3, in the three sides of a non-Euclidean triangle with angles $\pi/2, \pi/3, \pi7$. Δ has the presentation $\{c_1,c_2,c_3;\ c_1^2 = c_2^2 = c_3^2 = (c_1c_2)^2 = (c_2c_3)^3 = (c_1c_3)^7 = 1\}$.

Δ^{+}, the canonical Fuchsian group of Δ, is the (0; 2,3,7) group with presentation

$\{x,y; x^2 = y^3 = (xy)^7 = 1\}$, $x = c_1c_2$, $y=c_2c_3$.

Lemma 2.2. Every normal subgroup of Δ^+ is a surface group.

Proof.

Every element of finite order in Δ^+ is conjugate to either x, y or xy. If Λ is a normal subgroup of finite index Δ^+ which contains an element of finite order it must contain one of these elements . Suppose $x \varepsilon \Lambda$. Then under the canonical homomorphism from Δ^+ to Δ^+/Λ, x must mapt to 1, the identity. Suppose that y maps to $\bar{y}$. Then from the presentation of Δ^+, we must have

$$\bar{y}^3 = \bar{y}^7 = 1$$

which implies that $\bar{y} = 1$ and $\Lambda = \Delta^+$. Similarly if $\wedge$ contains y or xy, $\Lambda = \Delta^+$. Hence every normal subgroup of Δ^+ is a surface group.

We have thus shown that a finite group G is a Hurwitz group if and only if it is a homomorphic image of Δ^+ , i.e. it has two generators X,Y such that $X^2 = Y^3 = (XY)^7 = 1$.

2). Maximal groups of automorphisms of non-orientable Klein surfaces without boundary have been studied by Singerman [24] and of Klein surfaces with boundary by May [20], [21], [22]. In [20] May has shown that a compact Klein surface of algebraic genus $\gamma \geq 2$ with non-empty boundary cannot have more than 12 (γ-1) automorphisms, the algebraic genus of a surface being the non-negative integer that makes the algebraic version of the Riemann – Roch theorem wor [4], the field of meromorphic functions of Klein surface being an algebraic function field in one variable over ¡ . (It can be shown that as long as the boundary is non-empty the algebraic genus of a Klein surface $S = U/\Gamma$, where Γ is a bordered surface group, is equal to the topological genus of the surface U/Γ^+. This will be discussed more fully in chpter 4). In [21] May shows that the bound 12(γ-1) is attained for infinitely many values of the algebraic genus γ and exhibits some infinite families of surfaces with boundary which admit groups of 12(γ-1) automorphisms. In [22] it is shown that there are an infinite number of values of the algebraic genus γ for which there is no Klein surface with boundary with $12(\gamma - 1)$ automorphisms.

In this work we shall be concerned with maximal groups of automorphisms of non-orientable Klein surfaces without boundary and it may be assumed throughout the rest of this chapter that the Klein surfaces considered are without boundary.

Lemma 2.3. If G is a group of automorphisms of a non-orientable Klein surface, S of genus g then $|G| \leq 84(g-2)$. If $|G| = 84(g-2)$ then G is a Hurwitz group.

Proof.

By Collary (1.31) every group of automorphisms of S is isomorphic to a group of + automorphisms of $\tilde{S}$, the orientable two sheeted covering surface of S. If S has genus g, then $\tilde{S}$ has genus $\gamma = g - 1$. Therefore $|G| \leq 84(g-2)$. If $|G| = 84(g-2)$ if and only if $|G| = 84(\gamma-1)$ i.e. if and only if G is a Hurwitz group.

Definition 2.4. A group of 84(g-2) automorphisms of a non-orientable Klein surface of genus g will be called an H* - group.

From lemma 2.3 we can see that if G acts as an H* - group on a non-orientable surface S, then G acts as a Hurwitz group on $\tilde{S}$. In particular every H* - group is a Hurwitz group.

Now suppose S is a non-orientable Klein surface of genus g $\geq$ 3, which admits a group of 84(g-2) automorphisms. Let Λ be a non-orientable surface group such that S = U/Γ, then

$$|N\varsigma(\Lambda)/\Lambda| = 84(g\text{-}2).$$

But

$$\left|N\varsigma(\Lambda)/\Lambda\right| = \frac{\mu(\Lambda)}{\mu(N\varsigma(\Lambda))} = \frac{2\pi(g-2)}{\mu(N\varsigma(\Lambda))}$$

and thus $\mu(N\varsigma(\Lambda)) = \pi/42$. Thus $N\varsigma(\Lambda)$ is the NEC group with signature

$$(0, +, [\ ],\ \{2,3,7)\}).$$

This is the NEC group (unique upto isomorphism) with the smallest possible area of fundamental region and is the group Δ with presentation

$$\{c_1,c_2,c_3;\ c_1^2 = c_2^2 = c_3^2 = (c_1c_2)^2 = (c_2c_3)^3 = (c_1c_3)^7 = 1\},$$

By exactly the same methods used in the proof of lemma 2.2 we can show that every normal subgroup of Δ of finite index greater than two is a surface group.

Theorem 2.5 ([25]). A finite group G is an H* - group if and only if it contains three generators C_1, C_2, C_3 which obey the relations

$$C_1^2 = C_2^2 = C_3^2 = (C_1C_2)^2 = (C_2C_3)^3 = C_1C_3)^7 = 1,$$

And G is generated by C_1C_2 and C_2C_3.

Proof.

If G is an H* - group, there exists a homomorphism $\theta : \Delta \rightarrow G$ wuch that $\theta(\Delta^+)$ = G, and so has gnerators as described in the theorem. Conversely, if G has these genertors, there exists a homomorphism $\theta : \Delta \rightarrow G$ such that $\theta(\Delta^+)$ = G and the kernel of θ must be a surface group as all normal subgroups of Δ of index greater than two are surface groups. By applying theorem 1.30 we deduce that G is an H* - group.

Corollary 2.6. A Hurwitz group G generated by X,Y which obey the relations

$$X^2 = Y^3 = (XY)^7 = 1$$

Is an H* - group if and only if there exists Z ε G such that

$$Z^2 = (ZX)^2 = (ZY)^2 = 1$$

Proof.

If such a Z exists, then G is generated by $C_1 = ZX$, $C_2 = Z$, $C_3 = ZY$ obeying the relations

$$C_1^2 = C_2^2 = C_3^2 = (C_1C_2)^2 = (C_2C_3)^3 = (C_1C_3)^7 = 1.$$

Thus, by theorem 2.5, G is an H* - group.

Conversely, if G is an H* - group generated by C_1, C_2, C_3 obeying the same relations as above and also generated by $X – C_1C_2$, $Y = C_2C_3$ then $Z = C_2$ obeys the relations

$$Z^2 = (ZX)^2 = (ZY)^2 = 1.$$

3). In searching for Hurwitz groups an obvious first step is to look amongst simple groups. This is because no non-trivial Hurwitz group is cyclic and any factor group of a Hurwitz group is a Hurwtiz group. So if we factor out a Hurwitz group by a maximal normal subgroup we obtain a simple Hurwtiz group (see [14]). Since the projective unimodular groups, PSL(2,q) (Dickson's LF(2,q) [5]), are simple for $q \geq 3$, it is natural to look amongst these for Hurwitz groups. Macbeath [16] has determined for which values of q PSL (2,q) is a Hurwitz group. His results will be discussed later. Here we give the definition of PSL (2,q) and some of its properties.

For each prime power, $q = p^n$, there is a field of order q. Moreover for every prime power, q, there is, upto isomorphism, precisely one field of order q, namely GF(q), and there are no fields of order q if q is not a prime power.

e.g. if n = 1, q = p, prime, then GF(p) $\equiv$ residues mod p.

Let $q = p^n$. Then the general linear group, GL(2,q) is defined as

$$GL(2,q) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,b,c,d, \varepsilon\ GF(q),\ ad - bc \neq 0 \right\}.$$

The centre of GL(2,q), denoted by Z(GL(2,q)), is

$$\left\{ \begin{pmatrix} a & o \\ o & a \end{pmatrix} : a \neq o \right\}$$

and we define the projective general linear group, PGL (2,q), as

$$PGL(2,q) = GL(2,q) / Z(GL(2,q)).$$

We define the special linear group, SL(2,q), as

$$SL(2,q) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \varepsilon\ GL(2,q)\ :\ ad - bc = 1 \right\}$$

And the projective special linear group, PSL(2,q), is then

$$SL(2,q)/Z(SL(2,q)).$$

Z(SL(2,q)) is $\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}$ if $p \neq 2$ and $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ if $p = 2$. The order of PSL(2,q) is q(q-1) (q+1)/2 if $p \neq 2$ and q(q-1) (q+1) if p = 2.

Under the natural homomorphism from SL(2,q) onto PSL(2,q). a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in SL(2,q) induces a unique element in PSL(2,q), namely the coset $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}$.
Without ambiguity we can represent an element of PSL(2,q) by either of the two matrices in SL(2,q) which induce it.

The trace of a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in GL (2,q) and (a+d) and it is easy to show the following:-

(i) matrices in SL(2,q) which induce an element of order 2 in PSL(2,q) have trace 0

(ii) matrices in SL(2,q) which induce an element of roder 3 in PSL(2,q) have trace 0

(iii) matrices in SL(2,q) which induce an element of order 7 in PSL(2,q) have trace ξ, where $\xi^3 + \xi^2 - 2\xi - 1 = 0$.

2) We now state Macbeath's result.

Theorem 2.7. PSL(2,q) is a Hurwitz group if and only if

(i) $q = p$ p prime $\equiv \pm 1 (\text{mod } 7)$

(ii) $q = p^3$ p prime $\not\equiv 0, \pm 1$ (mod 7)

(i) $q = 7$

In case (i) there are three distinct orientable Klein surfaces upon which the group acts as a Hurwitz group. In cases (ii) and (iii) there is only one such Klein surface.

The two smallest Hurwitz groups are PSL(2,7) and PSL(2,8) which act on orientable surfaces of genus g = 3, g = 7 respectively. Singerman [24] has shown that PSL(2,7) is not an H* - group but PSL(2,8) is an H* - group and this will follow from our results aswell. Thus the smallest value of the genus for which a non-orientable Klein surface admits 84(g –2) automorphisms is g = 8. Our problem is to establish a general result to determine when PSL
(2,q) is an H* - group given that it is a Hurwitz group. (Since every H* - group is a Hurwitz group when looking for H* - groups we need only look amongst Hurwitz groups).

Macbeath has shown that all the groups listed above are Hurwitz groups i.e. that there always exist generators X,Y such that

$$X^2 = Y^2 = (XY)^7 = 1.$$

If X ε PSL(2,q) with order 2 then trace X = 0, so by con-jugation we can always assume that

$$X = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Let $Y = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$ ε PSL(2,q). If X and Y generate PSL(2,q) then the quadratic form $Q = Q_{\alpha\beta\gamma}$, defined by

$$Q(\xi,\eta,\zeta) = \xi^2 + \eta^2 + \zeta^2 + \alpha\eta\zeta + \beta\zeta\xi + \gamma\xi\eta$$

is non-singular, where α = trace X, β = trace Y, γ = trace XY (see [16]).

Now $Q(\xi,\eta,\zeta)$ is non-singular if and only if

$$\det\begin{pmatrix} 1 & \gamma/2 & \beta/2 \\ \gamma/2 & 1 & \alpha/2 \\ \beta/2 & \alpha/2 & 1 \end{pmatrix} \neq 0$$

But α = trace X = 0

β = trace Y = x+w

γ = trace XY = z-y.

So we can deduce that if X and Y generate PSL(2,q) then

$$X^2 + y^2 + z^2 + w^2 \neq 2.$$

5). We know from corollary 2.6 that if PSL(2,q) is a Hurwitz group generated by X and Y such that

$$X^2 = Y^3 = (XY)^7 = 1$$

Then it is an H* - group if and only if there exists Z ε PSL(2,q) such that

$$Z^2 = (ZX)^2 = (ZY)^2 = 1.$$

It is left oly for us to determine when such an element Z exists.

<u>Lemma 2.8</u>. Suppose that PSL(2,q) is generated by X,Y where

$$X = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} x & y \\ z & w \end{pmatrix}.$$

Then there always Z ε PGL(2,q) such that

$$Z^2 = (ZX)^2 = (ZY)^2 = 1$$

<u>Proof</u>.

Let $Z = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ ε PGL(2,q).

We want $Z^2 = 1$, so we require trace Z = 0 i.e.

$$\delta = -\alpha$$

(Note: It is easy to shwo that a matrix in GL(2,q) which induces an element of order two in PGL(2,q) must have trace 0).

We also wan $(ZX)^2 = 1$, so we require trace ZX = 0 i.e.

$$\gamma = \beta$$

Now

$$ZY = \begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} \alpha x + \beta z & \alpha y + \beta w \\ \beta x - \alpha z & \beta y - \alpha w \end{pmatrix}$$

so that

$$\text{trace } ZY = \alpha(x - w) + \beta(y+z)$$

and since $(ZY)^2 = 1$ we must have trace ZY = 0 i.e.

$$\alpha(x - w) = -\beta(y+z).$$

Hence given any value for β we can always find a value for α such that Z satisfies the required relations (if x = w, put $\beta = 0$ and then α can take any value). But we must ensure that Z ε PGL(2,q) i.e. that det Z $\neq$ 0.

Now det Z = - ($\alpha^2 + \beta^2$) and we know that

$$\alpha(x\text{-}w) = -\beta(y+z).$$

If we square both sides of this equation and add $\beta^2 (x\text{-}w)^2$ to both sides we get

$$\alpha^2(x\text{-}w)^2 + \beta^2(x\text{-}w)^2 = \beta^2(y+z)^2 + \beta^2(x\text{-}w)^2.$$

From which, since xw-yz = 1 (because Y ε PSL(2,q)), we can deduce that

$$\frac{\beta^2}{\alpha^2+\beta^2} = \frac{(x\text{-}w)^2}{(x^2+y^2+z^2+w^2\text{-}2}$$

Put

$$\frac{(x-w)^2}{(x^2+y^2+z^2+w^2-2)}.$$

For X and Y to generate PSL(2,q), $x^2+y^2+z^2+w^2 \neq 2,$ so T always exists.

If x = w, T=0 which implies that $\beta = 0$ and α can take any value except zero, so

$$\det Z = (\alpha^2+\beta^2) \neq 0.$$

If X $\neq$ w, since T $\neq$ 0, we can write

$$\alpha 2 = \beta 2\ \frac{(1-T)}{T} = \beta 2 \left\{\frac{1}{T} - 1\right\}.$$

Now $\frac{1}{T} \neq 0$ and so $\frac{1}{T} - 1 \neq -1$ which implies that $\alpha^2 \neq -\beta^2$ and hence again det $Z \neq 0$.

Therefore there always exists an element $Z \in PGL(2,q)$ such that

$$Z^2 = (ZX)^2 = (ZY)^2 = 1.$$

<u>Theorem 2.9</u>. If PSL(2,q) is a Hurwitz group generated by X,Y such that $X^2 = Y^3 = (XY)^7 = 1$ then it is an H* - group if and only if $(3 - \xi^2)$ is a square in GF(q), where ξ = trace XY.

<u>Proof</u>.

As X,Y generate PSL(2,q) we can assume that

$$X = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and

$$Y = \begin{pmatrix} x & y \\ z & w \end{pmatrix},$$

$xw - yz = 1,\ x^2 + y^2 + z^2 + w^2 \neq 2$. Since the order of Y is three and the order of XY is seven we know that

$$x + w = \text{trace } Y = \pm 1$$

and if

$$z - y = \text{trace } XY = \xi$$

then ξ is a solution of the equation $\xi^3 + \xi^2 - 2\xi - 1 = 0$ in GF(q). By the proof of lemma 2.8 there exists an element Z ε PGL(2,q) of the form

$$Z = \begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix}, \ \alpha^2+\beta^2 \neq 0$$

satisfying the relations

$$Z^2 = (ZX)^2 = (ZY)^2 = 1$$

and

$$\beta^2 \ (x^2+xy^2+z^2+w^2-2) = (\alpha^2+\beta^2)\ (x-w)^2 K \ (*)$$

By corollary 2.6 PSL (2,q) is an H* - group if and only if Z ε PSL(2,q) i.e. if and only if det Z = $-(\alpha^2+\beta^2)$ = 1 (in GF(q)).

We know that x+w = 1 and z-y = ξ, so squaring both expressions and using the fact that xw – yz = 1 we can deduce that

$$x^2 + y^2 + z^2 + w^2 = \xi 2 - 1$$

substituting this expression in(*) we obtain

$$\beta^2(\xi^2 - 3) = (\alpha^2 + \beta^2)\ (x - w)^2$$

so $-(\alpha 2 + \beta 2) = 1$ if and only if

$$\beta^2 = \frac{(x\text{-}w)^2}{(3\text{-}\xi^2)}$$

(Note: $3-\xi2 \neq 0$ since $x^2+y^2+z^2+w^2 \neq 2$)

This equation has a solution for β if and only if ($3 - \xi^2$) is a square in GF(q). Given a value for β we can find α as in the proof of lemma 2.8.

Hence PSL(2,q) is an H* - group if and only if ($3 - \xi^2$) is a square in GF(q).

We now now the conditions under which PSL(2,q) is an H* - group. We can, in fact, directly relate the condition imposed on ξ to properties of GF(q).

If the equation

$$\xi3+\xi2-2\xi-1=0$$

has three roots $\tau_1+\tau_2+\tau_3=-1$

$$\tau_1\tau_2+\tau_1\tau_3+\tau_2\tau_3=-2$$

$$\tau_1\tau_2\tau_3=1$$

so if -1 is a square in GF(q) then either ($3 - \tau_i^2$) is a square for all i (i = 1,2,3) or it is a square for one value of i only. If -1 is not a square in GF(q) then either ($3 - \tau_i^2$) is never a square or it is a square for two values of i.

In GF(q) ($q = p^n$, p prime), if q = 1(mod 4) or p=2 then -1 is a square, otherwise (i.e. if q = (i.e if $q \equiv 3 \pmod 4$)) -1 is not a square. Thus we can tell immediately for how many values of i (=1,2,3) it is possible for ($3 - \tau_i^2$) to be a square in GF(q).

Let us consider the three cases in theorem 2.7 seperately.

(i) q = p, prime $p \equiv \pm 1 \pmod 7$. In this case there are three distinct traces τ_1, τ_2, τ_3 yielding elements of PSL(2,q) of order seven. As Macbeath [16] has shown that in this cse there are three distinct orientable Klein surfaces on which PSL(2,q) acts as a Hurwitz group it is clear that the number of values of I(=1,2,3) for which $(3 - \tau_i^2)$ is a square in GF(q) corresponds to the number od distinct non-orientable Klein surfaces on which PSL(2,p) acts and an H* - group, for if two Klein surfaces have non-conformally equivalent orientable two-sheeted covers then they themselves must be distinct.

(ii) $q = p^3$, p prime $p \equiv \pm 2$ or $\pm 3 (\text{mod } 7)$. In this case the three traces τ_1, τ_2, τ_3 are conjugate under the automorphism group of $GF(p^3)$ which induces automorphisms of $PSL(2,p^3)$ (see [16]). Since automorphisms preserve squares $(3 - \tau_i^2)$ is either a square for all values of I or $(3 - \tau_i^2)$ is never a square. In the latter case which occurs when $p \equiv 3(\text{mod } 4)$ $PSL(2,p^3)$ is clearly not and H* - group. In the former case which occurs when either p = 2 or $p \equiv 1(\text{mod } 4)$ PSL $(2,p^3)$ is an H* - group acting on one non-orientable Klein surface only because it acts only on one orientable Klein surface, S say as a Hurwitz group (as shown by Macbeath [16]). So if it acted as an H* - group on more than one non-orientable Klein surface, say on K_1 and K_2, they would have the same orientable two-sheeted covering surface S and there would exist anti-conformal involutions c_i, I = 1,2, of S such that

$$K_i = \frac{S}{\langle \sigma_i \rangle}$$

where $\langle \sigma_i \rangle$ denotes the group generated by σ_i.

From lemma 1.32 we see that σ_1, σ_2 must oth commute with every element of $PSL(2,p^3)$ and since $\sigma_1 \sigma_2^{-1}$ ε PSL $(2,p^3)$ which has a trivial centre, $\sigma_1 = \sigma_2$ and hence $K_1 \equiv K_2$.

(Note: PSL (2.8) is now a special case and clearly from the above is and H* - group acting on one non-orientable Klein surface because $8 = 2^3$.)

(iii) q = 7. The fact that PSL(2,7) is not an H* - group follows directly now since the only solution of $\xi^3 + \xi^2 - 2\xi - 1 = 0$ in GF(q) is $\xi = 2$, and $(3 - 2^2) = -1$ is not a square in GF(7) because $7 \equiv 3 (\mathrm{mod}\ 4)$.

Now $p \equiv \pm 1 (\mathrm{mod}\ 7)$ and $p \equiv 1 (\mathrm{mod}\ 4)$ if and only if $p \equiv 1$ or $13 (\mathrm{mod}\ 28)$,
$p \equiv \pm 1 (\mathrm{mod}\ 7)$ and $p \equiv 3 (\mathrm{mod}\ 4)$ if and only if $p \equiv 1$ or $-13 (\mathrm{mod}\ 28)$,
$p \equiv \pm 2 (\mathrm{mod}\ 7)$ and $p \equiv 1 (\mathrm{mod}\ 4)$ if and only if $p \equiv 5$ or $9 (\mathrm{mod}\ 28)$,
$p \equiv \pm 3 (\mathrm{mod}\ 7)$ and $p \equiv 1 (\mathrm{mod}\ 4)$ if and only if $p \equiv -3$ or $-11 (\mathrm{mod}\ 28)$

and we have proved the following.

Theorem 2.10.
(i) If q = p prime and $p \equiv 1$ or $13 (\mathrm{mod}\ 28)$ then PSL(2,q) is an H* - group acting on one or three distinct non-orientable Klein surfaces. If q = p prime and $p \equiv -1$ or -13(mod 28) then PSL(2,q) is an H* - group acting on two distinct non-orientale Klein surfaces if $(3 - \tau_i^2)$ is a square for two values of i and is not and H* - group if $(3 - \tau_i^2)$ is never a square, where τ_1, τ_2, τ_3 are three roots of the equation $\xi^3 + \xi^2 - 2\xi - 1 = 0$ in GF(q). Otherwise PSL(2,p) is not an H* - group.

(ii) If $q = p^3$, prime then PSL(2,q) is an H* - group if and only if p = 2 or $p \equiv 5, 9, -3$ or -11(mod28). In this case ther eis only one non-orientable surface on which the group acts as an H* - group.

By Dirichlet's theorem on primes in an arithmetic progression there are an infintie number of compact non-orientable Klein surfaces for which the upper bound for the order of the automorphism group is attained. The result shows also that there exist and infinite number os simpe H* - groups.

Example 2.11. q = p = 13

By theorem 2.10, PSL(2,13) is an H* - group. Does it act on one or three non-orientable surfaces?

GF(13) ≡ residues mod 13. Let τ_1, τ_2, τ_3 be the roots of $\xi^3 + \xi^2 - 2\xi - 1 = 0$ in GF(13). So

$$\tau_1 = 7,\ \tau_2 = 8,\ \tau_3 = 10$$

Now (3 - τ_i^2) ≡ 6(mod 13), which is not a square in GF(13),

(3 - τ_2^2) ≡ 4(mod 13), which is a square in GF(13),

(3 - τ_3^2) ≡ 7(mod 13), which is not a square in GF(13).

(Note: 6.4.7.) ≡ -1(mod 13).)

So PSL (2,13) is an H* - group acting on only one non-orientable Klein surface of genus 15.

6). For each prime p ≡ ± 1(mod 7) a computer program was run which solved the equation $\xi^3 + \xi^2 - 2\xi - 1 = 0$ in GF(p) giving the roots τ_1, τ_2, τ_3 and then determined when (3 τ_i^2) was a square in GF(p). The following results were obtained.

<u>p ≡ 1(mod 4)</u>

(a) <u>(3 - τ_i^2) square for one value of I</u>	(b) <u>(3 - τ_i^2) square for all three values of I</u>
p = 13	p = 181
29	293
41	
97	
113	
197	

281

337

349

421

433

449

461

$p \equiv 3 (\mathrm{mod}\ 4)$

(c) $(3 - \tau_i^2)$ square for two values of I	(d) $(3 - \tau_i^2)$ never square
P = 43	P = 167
71	239
83	251
127	379
139	491
211	
223	
307	
419	
463	
503	

In (a) PSL(2,p) is an H* - group acting on one non-orientable Klein surface. In (b) PSL(2,p) is an H* - group acting on three distinct-non-orientable Klein surfaces. In (c) PSL (2,p) is an H* - group acting on two distinct non-orientable Klein surfaces. In (d) PSL(2,p) is not an H* - group.

CHAPTER 3

Cyclic groups of automorphisms of compact Non-orientable Klein surfaces without boundary

1). In this chapter we are going toc consider the problem of when cyclic group acts as a group of automorphisms of a compact non-orientable Klein surface. The problem for + automorphisms of compact Riemann surfaces has been solved by Harvey [7]. His results are stated below.

In [22] May has shown that the order of a cyclic group of automorphisms of a compact Klein surface S with boundary of algebraic genus (as defined in chapter 2) $\gamma \geq 2$ cannot be larger than $2\gamma + 2$ if S is orientable and γ is even; otherwise the order cannot be larger than 2γ. It is shown that for all values of the algebraic genus $\gamma \geq 2$ there are both orientable and non-orientable surface with a cyclic automorphism group of maximum possible order.

In this chapter, as in chapter 2, we shall be considering non-orientable Klein surfaces without boundary and it is interesting to note that in this case the maximum order for a cyclic group of automorphisms of such a surface again depends on whether the genus of the surface is even or odd.

2). We now state Harvey's results. All surfaces from now on are assumed to be without boundary.

Theorem 3.1. ([7]) Let Γ be a Fuchsian group with signature $(g;\ m_1, K\ , m_k)$ and let m be the 1.c.m. of $\{m_1, K\ , m_k\}$. There is a surface-kernel homomorphism $\theta: \Gamma \to Z_n$ (cyclic group of order n) if and only if the following conditions are satisfied.

(i) 1.c.m. $\{m_1, K\ , \hat{m}_i, K\ , m_k\} = m$, for all i, where $\hat{m}_i$ denotes the omission of m_i

(ii) m divides n and if $g = 0$, $m = n$,

(iii) $k \neq 1$, and if $g = 0$, $k \geq 3$,

(iv) if $2 | m$, the number of periods divisible by the maximum power of 2 dividing m is even.

(Note: If $\wedge = \ker\theta$ and the above conditions are satisifed then Z_n acts as a group of orientation preserving automorphisms of U/Λ).

Theorem 3.2 ([7]). The maximum order for a + automorphism of an orientable Klein surface of genus g is 2(2g + 1). This maximum order is attained for each g and hence Z_{4g+2} is a + automorphism group for some surface of genus g, for every value of $g \geq 2$.

Our problem is to find an attainable upper bound for the order of an automorphism of a non-orientable Klein surface.

<u>Lemma 3.3</u>. An upper bound for the order of an automorphism of a non-orientable Klein surface, S, of genus g is 2(2g-1).

<u>Proof</u>.

By corollary 1.31 every group of automorphisms of S is isomorphic to a group of + automorphisms of $\tilde{S}$, the orientable two-sheeted covering surface of S. If S has genus g then $\tilde{S}$ has genus $\gamma = g - 1$. So if Z_n is an automorphism group of S then it is an automoprhism group of $\tilde{S}$ and by theorem 3.2 $n \leq 2(2\gamma + 1) = 2(2g - 1)$.

Thus we have an upper bound for the order of an automorphism of a non-orientable Klein surface, but is this bound actually attained? The answer to this question is in the negative as we see in the following theorem.

<u>Theorem 3.4</u>. The maximum order for an automorphism of a non-orientable Klein surface of genus $g \geq 3$ is

2g, if g is odd,

2(g - 1), if g is even.

The maximum order is attained for every g, hence Z_{2g} is an automorphism group of some non-orientable Klein surface of odd genus $g \geq 3$.

<u>Proof</u>.

By theorem 1.30 if Z_n is an automorphism of a non-orientable Klein surface, S, of genus $g \geq 3$ then there exists a proper NEC group Γ and a homomorphism $\theta :\rightarrow Z_n$ such that ker θ is a surface group and $\theta(\Gamma^+) = Z_n$. Γ^+ must satisfy the conditions of theorem 3.1.

Let ker $\theta = \Lambda$, then Λ will be a non-orientable surface group (with orbit-genus g), S = U/Λ and

$$Z_n \; ; \; \Gamma/\Lambda.$$

Hence

$$n = |\Gamma/\Lambda| = \frac{\mu(\Lambda)}{\mu(\Gamma)} \; \frac{2\pi(g-2)}{\mu(\Gamma)}$$

For g odd, if $n \geq 2g$ then

$$\mu(\Gamma) \leq \frac{2\pi(g-2)}{2g} < \pi$$

and for g even, if $n \geq 2(g-1)$ then again $\mu(\Gamma) < \pi$, so in both cases

$$0 < \mu(\Gamma^+) = 2\mu(\Gamma) < 2\pi.$$

Since Γ^+ is a Fuchsian group it will have signature of the form

$$(\gamma;\ m_1, K\ , m_k)$$

in which case

$$\mu(\Gamma^+) = 2\pi(2\gamma - 2 + \sum_{i=1}^{k}(1 - \frac{1}{m_i}))$$

and so we wish to consider only those signatures which satisfy the condition

$$0 < 2\gamma - 2 + \sum_{i=1}^{k}(1 - \frac{1}{m_i}) < 1.$$

This implies that $\gamma \leq 1$. If $\gamma = 1$ then $k = 1$ and if $\gamma = 0$ then $k \leq 5$. However for Γ^+ to satisfy the condition fo theorem 3.1, $k \neq 1$. Hence $\gamma = 0$ and $k \geq 3$ and it is easy to see from condition (iv) that $k < 5$. Also we note if $k = 3$ then $\sum \frac{1}{m_i} < 1$ and $k = 4$ $\sum \frac{1}{m_i} < 2$.

Let us therefore consider NEC groups Γ such that Γ^+ has signature of the form $(0; m_1,m_2,m_3)$ or $(0; m_1,m_2,m_3,m_4)$. If Γ^+ has signature $(0; m_1,m_2,m_3)$ then by theorem 1.18 there are two possibilities for the signature of Γ namely

(1) $(0, +, [\ ], \{(m_1,m_2,m_3)\}) = \Gamma_1$ say,

(2) if $m_1 = m_2$, $(0, +, [m_1], \{m_3)\}) = \Gamma_2$ say.

If Γ^+ has signature $(0; m_1,m_2,m_3,m_4)$ then again by theorem 1.18 there are four possibilities for the signature of Γ namely

(3) $(0, +, [\ ], \{m_1, m_2, m_3, m_4)\}) = \Gamma_3$ say,

(4) if $m_1, = m_2$, $(0, +, [m_1], \{(m_3, m_4)\}) = \Gamma_4$ say,

(5) if $m_1 = m_2$, $m_3 = m_4$, $(0, +, [m_1, m_3], \{(\ )\}) = \Gamma_5$ say,

(6) if $m_1 = m_2$, $m_3 = m_4$, $(1, -, [m_1, m_3], \{\ \}) = \Gamma_6$ say

We wish to consider surface-kernel homomorphisms θ from Γ onto Z_n such that $\theta(\Gamma^+) = Z_n$ such that $\theta(\Gamma^+) = Z_n$, so to satisfy theorem 3.2 since γ (the orbit-genus of Γ^+) = 0 in all cases, we must have n = m = 1.c.m. $\{m_1, K, m_4\}$. The following lemma shows that a surface-kernel homomorphism onto Z_n for n> 2 does not exist in the first four of the above six cases.

Lemma 3.5. There does not exists a surface-kernel homomorphism $\theta: \Gamma_i \to Z_n$ for n > 2 and I = 1,2,3, or 4.

Proof.

Γ_1 and Γ_3 have presentations

$$\{c_1, c_2, c_3; c_1^2 = c_2^2 = c_3^2 = (c_1c_2)^{m1} = (c_2c_3)^{m2} = (c_1c_3)^{m3} = 1\}$$

and

$$\{c_1, c_2, c_3, c_4;\ c_1^2 = c_2^2 = c_3^2 = c_4^2 = (c_1c_2)^{m1} = (c_2c^3)^{m2} = (c_3c_4)^{m3} = (c_1c_4)^{m4} = 1\}$$

respectively, and are thus generated by elements of order two. So no homomorphism $\theta: \Gamma_i \to Z_n$ exists for n > 2, for I = 1 or 3.

Γ_2

$$\{c, x;\ c^2 = x^{m1} = (xcx^{-1}c)^{m3} = 1$$

and since Z_n is abelian any homomorphism $\theta: \Gamma_2 \rightarrow Z_n$ must have

$$\theta(xcx^{-1}c) = 1$$

and hence θ cannot be surface-kernel.

Γ_4 has presentation

$$\{x, c_1, c_2;\ x^{m1} = c_1^{\ 2} = c_2^{\ 2} = (c_1c_2)^{m3} = (c_2xc_1x^{-1})^{m4} = 1\}.$$

If Z_n has an element of order two then it is unique and so for any homomorphism $\theta: \Gamma_4 \rightarrow Z_n$ we must have

$$\theta(c_1c_2) = (\theta(c_1))^2 = 1$$

and again θ cannot be surface-kernel, which completes the proof of the lemma.

The following lemma shows that there does exist a surface-kernel homomorphism $\theta: \Gamma_5 \rightarrow Z_m$, where m = 1.c.m.$\{m_1, m_3\}$, under certain conditions.

Lema 3.6. Let Γ be a proper NEC group with signature

$$(0,\ +,\ [k,l\,],\ \{(\ )\}).$$

If either K and l are both even or have opposite parity then there exists a surface-kernel hoomorphism $\theta: \Gamma \rightarrow Z_m$, where m = 1.c.m.(k,l), such that $(\theta(\Gamma^+) = Z_m$.

(Note: Γ^{+} has signature (0; k,k, l ,l) and satisfies all the conditions of theorem 3.1)

Proof.

Γ has presentation

$$\{c, x_1, x_2;\ c^2 = x_1^{\,k} = x_2 l = x_1 x_2 c(x_1 x_2)^{-1} c = 1\}.$$

$$Zm = \left\langle 1, z, z^2, \ldots, z^{m-1}, z^m = 1 \right\rangle$$

The condition that either k and l are both even or have opposite parity implies that m is even. Clearly without this condition we could not define a homomorphism $\theta: \Gamma \to Z_m$, since Γ contains an element of order two an dif m was odd Z_m would not contain an element of order two.

If we let t = g.c.d.(k,l), so m = kl /t, then we can define a homomorphism $\theta: \Gamma \to Z_m$, by

$\theta(x1) = z^{l/t}$, which has order exactly k

$\theta(x2) = z^{k/t}$, which has order exactly l,

$\theta(c) = z^{m/2}$, which has order exactly 2.

θ is onto because l /t and k/t are relatively prime, so there exists p,q, ε Z (the set of integers) such that

$$p\frac{l}{t} + q\frac{k}{t} = 1.$$

Therefore

$$\theta(x_1^p x_2^q) = z$$

and z generates Z_m.

Every element of finite order in Γ is mapped to an element of the same finite order by θ and so θ is a surface-kernel homomorphism (by lemma 1.34) onto Z_m. We also have $\theta(\Gamma^+) = Z_m$ because $x_1, x_2 \ \varepsilon\ \Gamma^+$, hence the lemma is proved.

Applying lemma 3.6 to Γ_5 we see that, provided m(=1.c.m.$\{m_1,m_3\}$) is even, we can define a surface-kernel homomorphism $\theta: \Gamma_5 \to Z_m$ such that $\theta(\Gamma^+) = Z_m$. So Z_m acts as a group of automorphisms of the Klein surface $U/\Lambda = \ker\ \theta$ and we know that

$$m = \frac{2\pi(g - 2)}{\mu(\Gamma_5)} = \frac{g - 2}{(1 - 1/m_1 - 1/m_3)} \ ,$$

where g is the orbit-genus of Λ.

The following lemma shows us how to maximise m in terms of g.

<u>Lemma 3.7</u>. Given any two integers r,s such that

$$[r,s]\ (1 - \frac{1}{r} - \frac{1}{s}) = b$$

where b is a fixed integer and [r,s] = 1.c.m.(r,s) then

$$[r,s] \ \le \begin{cases} 2b + 4, & \text{if b is odd,} \\ 2b + 2, & \text{if b is even,} \end{cases}$$

Proof.

The equation

$$[r,s](1 - \frac{1}{r} - \frac{1}{s}) = b$$

is always satisfied because if b is odd put r = 2, s = b + 2 and if b is even put r = 2, s = 2b + 2.

Now suppose [r,s] > 2b + 4. Then, since $\frac{b}{2b+4} < \frac{1}{2}$, the equation

$$[r,s]\ (1 - \frac{1}{r} - \frac{1}{s}) = b$$

implies that

$$\frac{1}{r} + \frac{1}{s} > \frac{1}{2}\ .$$

This inequality is satisfied by only a few integer values of r and s namely (assuming without loss of generality that $r \leq s$)

r = 2, s arbitrary,
r = 3, s = 3,4 or 5

and in each case we can obtain a contradiction.

(1) $r = 3, s = 3$ implies $[r,s] = 3, b = 1$, so $[r,s] < 2b + 4$.

(2) $r = 3, s = 4$ implies $[r,s] = 12, b = 5$, so $[r,s] < 2b + 4$.

(3) $r = 3, s = 5$ implies $[r,s] = 15, b = 7$, so $[r,s] < 2b + 4$.

(4) $r = 2$, s odd implies $[r,s] = 2s, b = 2 – 2$, so $[r,s] = 2b + 4$.

(5) $r = 2$, s even implies $[r,s] = s, b = \frac{s}{2} - 1$, so $[r,s] < 2b + 4$.

Therefore for any value of b, $[r,s] \leq 2b+ 4$. Clearly $2b + 4$ is the least upper bound for b odd since $[r,s] = 2b + 4$ when $r = 2$ and $s = b + 2$. We now wish to show that if b is even then $[r,s] \leq 2b + 2$.

Suppose b is even and $[r,s] > 2b + 2$, then again since $\frac{b}{2b+2} < \frac{1}{2},$ the equation

$$[r,s]\ (1 - \frac{1}{r} - \frac{1}{s}) = b$$

implies that

$$\frac{1}{r} + \frac{1}{s} < \frac{1}{2}\ ,$$

so we have the same cases as before for integer values of r and s. Now only one of these cases, namely case (5), gives us a value of b which could be even, i.e. when $r = 2$, s is even, $[r,s] = s$ and $b = \frac{s}{2} - 1$. But then $[r,s] = 2b + 2$ and so if b is even we must always have $[r,s] \leq 2b + 2$, the upper bound being attained when $r = 2$ and $s = 2b + 2$. This completes the proof of the lemma.

If we put $r = m_1$, $s = m_3$ and $b = g – 2$ in lemma 3.7 then $[r,s] = m$ and we have

$$m \leq \begin{cases} 2g, & \text{if g is odd,} \\ 2(g - 1), & \text{if g is even.} \end{cases}$$

since

$$\mu(\Gamma_6) = 2\pi(1 - \frac{1}{m_1} - \frac{1}{m_3}) = \mu(\Gamma_5)$$

we would obtain no larger values for m using Γ_6.

If $\mu(\Gamma) \geq \pi$, then

$$n = \frac{2\pi(g - 2)}{\mu(\Gamma)} \leq 2(g - 2).$$

Since we have considered all cases with $\mu(\Gamma) < \pi$ we have proved that the maximum order for an automorphism of a non-orientable Klein surface of genus $g \geq 3$ is

$2g$, if g is odd,
$2(g – 1)$, if g is even.

The maximum order is attained for each g since the NEC group with signature (0, +, [2,g], {( )}) admits a surface-kernel homomorphism onto Z_{2g} when g is odd by lemma 3.6 and by the same lemma the NEC group with signature (0, +, [2,2(g – 1)], {( )} admits a surface-kernel homomorphism onto $Z_{2(g-1)}$ when g is even.

CHAPTER 4

Covering of Klein surfaces.

1). In chapter 1 we saw that every nonconstant morphism between two compact Klein surfaces is an n-sheeted covering for some n, possibly ramified. If a morphism f: T $\to$ S of Klein surfaces is a ramified n-sheeted covering then s ε S has a neighbourhood V such that $f^{-1}(V)$ has n components each of which is mapped homeomorphically onto V by f except where the covering is ramified or folded.

If T is ramified over s ε S (we say a point t ε T is over a point s ε S if f(t) = s) then at each point in the set $f^{-1}(s)$ severla sheets of the covering surface T hang together, the number of sheets at one point being the ramification index e of f at the point. Over such points, locally, the covering map f looks lie $z \to z^e$. If S has non-empty boundary and T has no boundary component over one (or more) boundary components of S then the covering is folded over that boundary component of S. for all points t ε T over δ S at which folding occurs $d_f(t) = 2$, where $d_f(t)$ is the relative degree of f(t) as described in chapter 1. The following is an example of a folded covering.

Example 4.1 Let S be an orientable Klein surface with $r \geq 1$ boundary components and genus g. Let S* be a surface homeomorphic to S and let h: S $\to$ S* be the homeomorphism.

If $\Omega = (U_i, z_i)_{i\varepsilon I}$ is an analytic atlas of S we can define an analytic atlas Ω^* and S* by putting Ω^* equal to the set of charts $(h(U_i), \overline{z}_i)_{i\varepsilon I}$ where

$$\overline{z}(h(p)) = \overline{z(p)}, \text{ for all p } \varepsilon \text{ S}.$$

It is easily seen that Ω^* is an analytic atlas.

Now form a new Klein surface T as follows. Consider the space SUS* and 'glue' the borders together by identifying, for p ε δS, p and h(p). An analytic atlas Ω_T is defined on T by $\Omega_T = \Omega_1 U \Omega_2$, where Ω_1 consists of all charts (U, z) on S such that $U \cap \delta S = \varnothing$, together all charts (h(U), $\overline{z}$) on S* and Ω_2 is the set of all charts (Uuh(U),w), for all U such that $U \cap \delta S \neq \varnothing$ and

$$\left.\begin{array}{l} w(p) = z(p) \\ w(h(p)) = \overline{z(p)} \end{array}\right\} \text{for all } p \; \varepsilon \; S$$

This definition is consisent on δS is mapped to the real line. It is trivial to show that this defines an analytic atlas for all charts in Ω_1. We use the reflection principle, which says that if V is an open set in £ symmetric ab out the real line and if g is an analytic funciton defined on V and g(¡ ∩ V) ⊂ ¡) then g(z) = $\overline{g(z)}$, to shwo that the co-ordination transformations associated with Ω_2 are analytic.

The Klein surface (T, Ω_T) obtained in this way is orientable with genus 2g + r – 1 and no boundary. This surface is the 'classical' double of an orientable Klein surface with boundary as described by Schiffer and Spencer [23].

Note that the homeomorphism h now acts as an anti-conformal involution on T.

The covering map f: T $\rightarrow$ S is the identity on all points of T except those which are on the union of the boundary of S and S* which is closed curve in T. So f maps points T in pairs onto the interior of S except the points on the union of the two boundaries, a neighbourhood of such a point being mapped onto a neighbourhood of a point on the boundary of S, i.e. onto a half-closed disc. Hence over the boundary of S the covering map locally has the form of the folding map $\varnothing$ ($\varnothing(x + iy) = x + i|y|$).

2). We shall begin by considering 2-sheeted coverings of Klein surfaces. These are otherwise known as double covers.

In[2] Alling and Greenleaf define a double cover as follows.

Definition 4.2. A morphism f: $T \to S$ of Klein surfaces is a double cover if each $s \varepsilon S$ has a neighbourhood V such that $f^{1}(V)$ has two components, each of which is mapped homeomorphically onto V by f; of $f^{1}(s) = \{t\}$ and there exist dianalytic charts (U_t,z_t) and (U_s,z_s) of t and s respectively such that $z_t(t) = 0 = z_s(s)$, $f(U_t) \subset U_s$ and

$$z_s f|Ut = \begin{cases} \phi z_t & \text{if } s \;\varepsilon\; \delta S \text{ and } t \not\subset \delta T, & \text{(i)} \\ \phi z^2_t & \text{if } s \;\varepsilon\; \delta S \text{ and } t \not\subset \delta T, & \text{(ii)} \\ z^2_t & \text{if } s \not\subset \delta S & \text{(iii)} \end{cases}$$

ϕ being the folding map; f is unramified if (ii) and (iii) never occur.

This is clearly compatible with definition 1.26 with n = 2. Alling and Greenleaf proceed to show the existence and uniqueness of three special double covers, the first of which is described in the following theorem.

Theorem 4.3 ([2]). Let S be a Klein surface. There exists a double cover f: $S_c \to S$ of S by an orientable Klein surface without boundary S_c (here we allow S_c to be disconnected) such that S_c has an anti-conformal involution σ with $f\sigma = f$. If ($S'_c f'$, σ) is any other such triple, then there is a unique conformal homeomorphism $\rho : S'_c \to S_c$ such that $f'\rho = f$.

Further, f is unramified, σ is the only anti-conformal sutomorphism of S_c such that $f\sigma$ = f and S_c is disconnected if and only if S is orientable and $\delta S = \phi$.

The triple (S_c, f, σ) is called the complex double of S and is usually just denoted by S_c . It corresponds to the 'classical' double of a Klein surface described in [23], where it is hwon that if S is orientable with genus g and r $\geq$ 1 boundary components then S_c has genus 2g + r – 1 and if S is non-orientable with genus g and r boundary components then S_c has genus g + r – 1. If S is orientable then the complex double is the same as the double described in example 4.1

Sc can be constructed as follows (see[2]). Let $(U_j,z_j)_{j \varepsilon J}$ be a dianalytic atlas of S. For each j ε J, let $U'_j \equiv U_j \equiv U''_j$, and $z'_j = z_j$, z_j, $z''_j = \overline{z}_j$. Let Ω be the disjoint union of the U'_js and make identifications of the following two types.

(1) If W is a component of $U_j \cap U_k$ and if $z_j z_k$-1 is conformal (respectively anti-conformal) on $z_k(W)$, then identify its image in U'_j with its image in U'_k (respectively its image in U'_j with its image in U''_k) and its image in U''_j with its image in U''_k (respectively its image in U''_j with its image in U'_k).

(2) Let $B_j = \delta S \cap U_j$ and identify its image in U'_j with its image in U''_j .

Let Sc be the quotient space of Ω, with all the above identifications. Let $\hat{U}_j$ be the image of $U'_j \cup U''_j$.

Let S_c be the quotient space of Ω, with all the above identificaitons. Let $\hat{U}_j$ be the image of $U'_j \cup U''_j$ in S_c and let $\hat{z}_j$ map $\hat{U}_j$ into £ as follows:

$\hat{z}_j \big| u'_j = z'_j$ and $\hat{z}_j \big|\ U''_j = z''_j$. It is easily seen that $\hat{z}_j$ is a homeomorphism . Using the reflection principle, we can see that $\hat{z}_k\ \hat{z}_j^{\ -1}$ is analytic on $\hat{z}_j(\hat{U}_j)$: thus $(\hat{U}_j, \hat{z}_j)_{j\varepsilon J}$ is an anlytic atlas of S_c. Let f: $S_c \to$ S be induced by the identity maps $U'_j \to U_j$ and $U''_j \to U_j$.

The two points p and $\hat{p}$ of S_c which lie over the same point $s \in S$ are called conjugate points . If p corresponds to a boundary point of S the $p = \hat{p}$. The correspondence between conjugate points of S_c defines a one-one anti-conformal mapping, σ, of S_c onto itself. Clearly $\sigma^2 = 1$ and $f\sigma = f$.

If $S = U/\Gamma$, where Γ is either a non-orientable surface group or a bordered surface group, then U/Γ^+ is the uniquely defined two-sheeted orientable covering surface without boundary of S. So, as S_c is unique.

$$S_c = U/\Gamma^+$$

This will be discussed in more detail in section 5 of this chapter.

We describe the other two special doubles in a less formal way as they will not be used in any formal proofs.

If we construct Ω in exactly the same way as above but employ only identifications of the first type we obtain an orientable Klein surface with boundary which is an unramified double cover of S. This double cover, called the orienting double by Alling and Greenleaf and denoted by S_o, is disconnected if and only if S is orientable. If S has r boundary components then S_o has 2r boundary components. If $\delta S = \phi$ then $S_o = S_c$.

Examples 4.4. If S is a Mobius strip then S_o is an annulus and S_c is a torus. If S is a Klein bottle with a hole then S_o is a torus with two holes and S_c is a sphere with two handles attached (see example 4.14).

The third double cover is also unramified ad is called the Schottky double . It is obtained by modifying the procedure to construct the complex double so that identifications always occur between U'_j and U'_k (and U''_j and U''_k) or more directly we can take two copies of S with opposite orientations and glue them together on the boundary. We denote the Schottky double of S by S_s. If S is orientable then $S_s = S_c$. If S is non-orientable then so is S_s and S_s is disconnected if and only if $\delta S = \phi$.

Example 4.5. If S as a Klein bottle with one hole then S_s is a sphere with four cross-caps attached. If S is the projective plane with two holes then S_s is again a sphere with four cross-caps.

3)

Definition 4.6 Let F: T $\rightarrow$ S be a covering of Klein surfaces. The fibre of a point s ε S is the set of points f-1(s) in T. A homeomorphism g: T $\rightarrow$ T is called a covering transformation if g takes each fibre to itself, i.e. fg = f. Elarly the set of covering transformations forms a group under composition of maps.

Let Γ_1 be an NEC group so U/Γ_1 is a Klein surface and let Γ_2 be a subgroup of Γ_1 of index n. Then Γ_2 is a NEC group and U/Γ_2 is a Klein surface which is an n-sheeted covering surface of U/Γ_1, possibly ramified.

If $\pi_{\Gamma i} : U \rightarrow U/\Gamma$ is the natural projection and we put $\pi_{\Gamma i}(z) = [z]_{\Gamma i}$ then the covering map is the natural map f: $U/\Gamma_2 \rightarrow U/\Gamma_1$ defined by

$$F([z]_{\Gamma 2}) = [z]_{\Gamma 1}.$$

Since $f\pi_{\Gamma 2} = \pi_{\Gamma 1}$ and $\pi_{\Gamma i}$, I = 1,2, is a morphism f is itself a morphism of Klein surfaces.

Definition 4.7. The covering f: $U/\Gamma_2 \rightarrow \Gamma_1$ is called a normal covering if $\Gamma_2 < \Gamma_1$.

This definition is just an extension of the idea of a normal covering of a Riemann surfac as defined in [1].

If f: $U/\Gamma_2 \to U/\Gamma_1$ is a normal covering then the group G Γ_1/Γ_2 acts as a group of covering transformations and it is easy to show that G acts transitively on each fibre i.e. if $x_1, x_2 \in U/\Gamma_2$ and are in the same fibre then there exits g ε G such that $g(x_1) = x_2$..

If θ is a homorphism from Γ_1 onto a group G of order n, then ker θ is a normal subgroup of Γ_1 of index n. Every subgroup of Γ_1 can be found in this way. So we can find every n-sheeted normal covering of U/Γ_1 by looking at all possile homomorphisms from Γ_1 onto all possible groups of order n.

If Γ_1 is a surface group or a bordered surface group then any subgroup Γ_2 or Γ_1 will be a surface group, possibly bordered and U/Γ_2 will be an unramified covering surface of U/Γ_1.

4). For n = 2 the problem of finding all subgroups of index n in Γ_1 is greatly simplified since there is (up to isomorphism) only one group of order two, namel Z_2, the cyclic group of order two with presentaiton $\{z: z^2 = 1\}$ and any subgroup of index two must be a normal subgroup. Therefore by looking at all homeomorphism from Γ_1 onto Z_2 and considering the kernel of each one we can find all 2-sheeted coverings of U/Γ_1 of the form U/Γ_2 where $\Gamma_2 \subset \Gamma_1$ with index two, every one of which will be normal.

If S is a Klein surface such that $S = U/\Gamma_1$ where Γ_1 is a bordered surface group, and we consider a homomorphism $\theta: \Gamma_1 \to Z_2$ then U/ker θ will be a connected unramified normal double cover of S. The question we ask is: can every connected unramified double cover of S be represented as the orbit sapce of a subgroup of index two in Γ_1 with the natural covering map?

To answer this questionw e need first to establish a more general fact, that is to show that an automorphism of a Klein surface is induced by an automorphism of the upper hal-plane even if the Klein surface has boundary. This result has also been obtained by May [21] but by a different though analogous method.

Proposition 4.8. Let S be a Klein surface such that $S = U/\Gamma$ where Γ is a surface group , possibly bordered, and let g: $S \rightarrow S$ be an automorphism of S. Then there exists a homeomorphism $\omega \ \varepsilon$ g, such that $\omega\Gamma\omega^{-1} = \Gamma$, which induces g.

Proof.

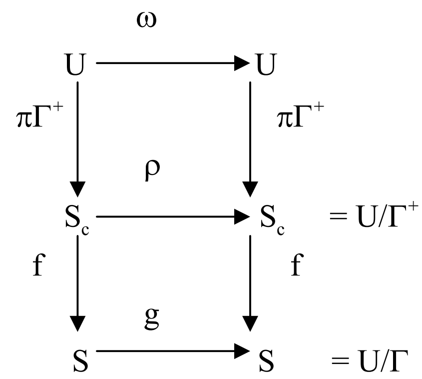
Case (i): If $\delta S = \phi$, then we use ordinary covering space theory as in chapter 1 to show ω exists.

Case (ii) If $\delta S = \phi$, let (S_c, f, σ) be the complex double of S . As δ S is non-empty, S_c is connected. Cosnider the map gf: $S_c \rightarrow$ S.

$$gf(\sigma) = g(f\sigma) = gf \quad \text{(by theorem 4.3).}$$

So (S_c, gf, σ) is another triple representing the complex double of S. Therefore by theorem 4.3 there exists a unique conformal homeomorphism
$\rho: S_c \rightarrow S_c$ such that $gf = f\rho$.

Sc is an orientable Klein surface without bounday so there exists a homeomorphism $\omega \ \varepsilon$ g (infact $\omega \ \varepsilon \ g^+$) which induces ρ as described in chapter 1. If $\pi_{\Gamma^+}: U \rightarrow U/\Gamma^+ = S_c$ is the natural projection, we have the following commutative diagram.



Let $f\pi_{\Gamma^+} = q$, then, as $S = U/\Gamma$,

$$q(z) = \pi_{\Gamma}(z) = [z]_{\Gamma}.$$

Now $gq = q\omega$ and so for $z \ \varepsilon \ U$

$$g[z]_{\Gamma} = [\omega(z)]_{\Gamma} .$$

Let $\gamma \ \varepsilon \ \Gamma$, then

$$[\omega\gamma(z)]_{\Gamma} = [\omega(\gamma z)_{\Gamma} = g[z]_{\Gamma} = [\omega(z)]_{\Gamma}.$$

Therefore there exists $\lambda \ \varepsilon \ \Gamma$ such that $\lambda\,\omega\,\gamma(z) = \omega(z)$ and so (because of the discreteness of Γ and the continuity of ω)

$$\omega\gamma\omega^{-1} = \lambda \ \varepsilon \ \Gamma$$

which implies that

$$\omega\Gamma\omega^{-1} = \Gamma.$$

Proposition 4.9. Let f: $T \to S$ be a connected unramified double cover of Klein surfaces such that $T = U/\Gamma$, where Γ is a surface group, possibly bordered. Then there exists a surface group Δ, possibly bordered, such that $S = U/\Gamma$ and $\Gamma < \Delta$ with index two.

Proof.

If f: $T \to S$ is a double cover then there exists an automorphism $\tau : T \to T$ such that $\tau^2 = 1$ (τ is the correspondence between point(s) in the same fibre). If $T = U/\Gamma$ then by proposition 4.8 there exixts a homeomorphism $\omega \varepsilon g$ which induces τ such that $\omega\Gamma\omega^{-1} = \Gamma$. and since $\tau^2 = 1$, $\omega^2 \varepsilon \Gamma$.

Let Δ be the group generated by Γ and ω, i.e.

$$\Delta = \Gamma = \omega\Gamma.$$

Then $\Gamma < \Delta$ with index two and

$$S = \frac{U/\Gamma}{\Delta\Gamma} = U/\Delta$$

Clearly f can now be defined by $f([z]_\Gamma) = [z]_\Delta$.

From this proposition we can deduce that all double covers of $S = U/\Gamma_1$, where Γ_1 is a bordered surface group, are of the form U/Γ_2, where $\Gamma_2 < \Gamma_1$ with index two, and that by looking at U/ker θ for all homomorphisms θ from Γ_1 onto Z_2 we can determine all possible connected unramified double covers of S.

Lemma 4.10. Let Γ_1 be a bordered surface group with orbit-genus g and r boundary components. Then there are $2^a - 1$ homomorphisms θ from Γ_1 onto Z_2 where

$$a = \begin{cases} 2g + 2r - 1, \text{ if } \Gamma_1 \text{ is orientable} \\ g + 2r - 1, \text{ if } \Gamma_1 \text{ is non-orientable.} \end{cases}$$

Proof. We define θ on the canonical generators of Γ_1 and then the proof is a simple process of counting all such homomorphisms. We divide the proof into two cases.

(i) When Γ_1 is an orientable bordered surface group. Then Γ_1 has signature

$$(g, +, [\,], \{(\,)^r\})$$

and generators

$$\begin{array}{ll} a_i, b_i & i = 1, \ldots, g \\ e_i & i = 1, \ldots, r \\ c_i & i = 1, \ldots, r \end{array}$$

with relations

$$c_i^2 = 1$$
$$e_i c_i e_i^{-1} = c_i$$
$$e_1 e_2 \ldots e_r a_1 b_1 a_1^{-1} b_1^{-1} \ldots a_g b_g a_g^{-1} b_g^{-1} = 1.$$

Since Z_2 is abelian of order two, all the relations in Γ_1 will be preserved automatically by θ except we must ensure that

$$\theta(e_1)\ \theta(e_2) \ldots \theta(e_r) = 1.$$

So we can choose $\theta(a_i)$ and $\theta(b_i)$ in each of two ways for each i = 1, … g, $\theta(c_i)$ in each of two ways for each i = 1, … r, $\theta(e_i)$ in each of two ways for each i = 1, … r-1 and then

$\theta(e_r)$ is uniquely determined by the relation $\theta(e1)\ \theta(2) \ldots \theta(er) = 1$. For θ to be onto it cannot map everything to the identity of Z_2. Thus the number of homomorphisms from Γ_1 onto Z_2 is

$$2^g 2^g 2^r 2^{r-1} - 1 = 2^{2g+2r-1} - 1$$

(ii) When Γ_1 is a non-orientable bordered surface group. Then Γ_1 has signature

$$(g, -, [\ ], \{(\ )^r\})$$

and generators

$$\begin{array}{ll} a_i & i = 1, \ldots, g \\ e_i & i = 1, \ldots, r \\ c_i & i = 1, \ldots, r \end{array}$$

with relations

$$c_i^2 = 1$$
$$e_i c_i e_i^{-1} = c_i$$
$$e_1 e_2 \ldots e_r a_1^2 \ldots a_g^2 = 1.$$

We thus have the same situation as in the orientable case except that there are no b_i's giving us that the number of homomorphisms in this case is

$$2^{g+2r-1} - 1$$

Since there are $2^a - 1$ homomorphisms $\theta : \Gamma_1 \to Z_2$, Γ_1 has $2^a - 1$ subgroups, ker θ, of index two. Some of these subgroups will be isomorphic but as they come from different homomorphisms each subgroup corresponds to a different 2-sheeted covering of

U/Γ_1. By proposition 4.9 every connected double cover of U/Γ_1 can be found in this way. We have thus proved the following.

<u>Theorem 4.11</u>. Let S be a Klein surface of genus g with r boundary components such that $S = U/\Gamma$ where Γ is a bordered surface group. Then there are $2^a - 1$ connected unramified double covers of S, where

$$a = \begin{cases} 2g + 2r - 1, & \text{if S is orientable} \\ g + 2r - 1, & \text{if S is non-orientable} \end{cases}$$

(Note we specify that S has non-empty boundary because this theorem does not hold for r = 0 as, from the proof of lemma 4.10, it is easy to see that if r = 0 the number of homomorphisms is $2^{2g} - 1$ or $2^g - 1$ and not $2^{2g-1} - 1$ or $2^{g-1} - 1$. We are more interested in the case when r > 0 because then the complex double and the Schottky double are always connected).

The result in theorem 4.11 agrees with the number of connected unramified double covers of a Klein surface with boundary found by Alling and Greenleaf from their topological approach ([2]).

If we can in some way determine $\ker\theta$ from the construction of the homomorphism $\theta: \Gamma_1 \rightarrow Z_2$, defined on the canonical generators of Γ_1, we can identify the surface $U/\ker\theta$ and hence classify all connected double covers of U/Γ_1.

If Γ_1 has orbit-genus g and r boundary components then it has signature

$$(g, \pm, [\ ], \{(\ )^r\})$$

and generators

(A) if Γ_1 is orientable

$$a_i, b_i \quad i = 1, \ldots g$$
$$c_i \quad i = 1, \ldots r$$
$$e_i \quad i = 1, \ldots r$$

(B) if Γ_1 is non-orientable

$$a_i \quad i = 1, \ldots g$$
$$c_i \quad i = 1, \ldots r$$
$$e_i \quad i = 1, \ldots r$$

with relations

(A)

$$c_i^2 = 1$$
$$e_i c_i e_i^{-1} = c_i$$
$$e_1 e_2 \ldots e_r a_1 b_1 a_1^{-1} b_1^{-1} \ldots$$
$$..a_g b_g a_g^{-1} b_g^{-1} = 1.$$

(B)

$$c_i^2 = 1$$
$$e_i c_i e_i^{-1} = c_i$$
$$e_1 e_2 \ldots e_r a_1^2 a_2^2 \ldots a_g^2 = 1$$

In presentation (A) the orientation preserving generators are the a_i's, b_i's and e_i's (all hyperbolic), the only orientation reversing generators are the reflections, c_i. The only difference in presentation (B) is that the a_i's are orientation reversing (glide reflections).

A subgroup, Γ_2, of index two in Γ_1 will have signature of the form.

$$(h, \pm, [\;], \{(\;)^s\}).$$

The Riemann-Hurwitz formula gives us that

$$\mu(\Gamma_2) = 2\mu(\Gamma_1)$$

from which we can determine h if we know s and the orientability of Γ_2.

<u>Theorem 4.12</u>. Let Γ_1 be a bordered surface group with orbit-genus g and r boundary components. Let $\theta : \Gamma_1 \to Z_2$ be a homomorphism defined on the canonical generators of Γ_1 (described above) and let ker $\theta = \Gamma_2$. Define a map τ_θ from $\{c_1, c_2, \ldots c_r\}$ (the set of generating reflections) to {0,1,2) such that

$$\tau_\theta(ci) = \begin{cases} 2 \ \text{if } c_i, e_i \ \varepsilon \ \Gamma_2 \\ 1 \ \text{if } c_i \ \varepsilon \ \Gamma_2, ei \not\varepsilon \ \Gamma_2, & \text{for all } i = 1 \ldots r \\ 0 \ \text{if } c_i, \ \varepsilon \ \Gamma_2 \end{cases}$$

Then

(i) the number of boundary components of Γ_2 is

$$s = \sum_{i=1}^{r} \tau_\theta(c_i),$$

(ii) if Γ_1 is orientable then Γ_2 is non-orientable if and only if $\Gamma_1 \backslash \Gamma_2$ contains both orientation reversing and orientation preserving generators of Γ_1.

(iii) if Γ_1 is non-orientable then Γ_2 is non-orientable if and only if $\Gamma_1 \backslash \Gamma_2$ contains both orientation reversing and orientation preserving generators of Γ_1 or Γ_2 contains any of the glide reflection generators of Γ_1.

<u>Proof</u>.

(i) If two reflections are conjugate in an NEC group Γ, then they represent the same boundary component in U/Γ. We are considering a bordered surface group Γ_1 with r boundary components, so all reflections in Γ_1 will be conjugate to one of the generating reflections c_1, ... c_r. Our aim is to count the number of conjugacy classes of reflections in Γ_2.

We define a reflection c' ε Γ_2 to be induced by the reflection c_i in Γ_1 if c' is conjugate to c_i in Γ_1. Let g ε $\Gamma_1 \backslash \Gamma_2$ so that $\Gamma_1 = \Gamma_2 + g\Gamma_2$ then, by lemma 1.15, if c_i ε Γ_1 is also in Γ_2, c' is either conjugate to c_i or gc_ig^{-1} in Γ_2. If c_i $\not\varepsilon$ Γ_2 then it induces no reflections in Γ_2. If c_i ε Γ_2 then it induces 1 or 2 conjugacy classes of reflections in Γ_2 depending on whether c_i is conjugate to gc_ig^{-1} in Γ_2 or not.

If gc_ig^{-1} is conjugate to c_i in Γ_2 then there exists h ε Γ_2 such that

$$h(gc_ig^{-1})h^{-1} = c_i.$$

This implies that hg is in the centralizer of ci in Γ_1 which, from theorem 1.16 is $\langle c_i, e_i \rangle$. So h = xg^{-1}, for some x ε $\langle c_i, e_i \rangle$.

If $e_i \not\subset \Gamma_2$, put x = e_i ε $\langle c_i, e_i \rangle$, so h = e_ig^{-1}. Since $e_i, g^{-1} \not\subset \Gamma_2$, e_ig^{-1} ε Γ_2 and

$$h(gc_ig^{-1})h^{-1} = e_ig^{-1}gc_ig^{-1}ge_i^{-1} = e_ic_ie_i^{-1} = c_i.$$

so gc_ig^{-1} is conjugate to c_i in Γ_2.

If e_i ε Γ_2, then $\langle c_i, e_i \rangle \subset \Gamma_2$. So if x ε $\langle c_i, e_i \rangle$ then x ε Γ_2 and, since $g^{-1} \not\subset \Gamma_2$, h = $xg^{-1} \not\subset \Gamma_2$. Therefore there does not exist h ε Γ_2 such that gc_ig^{-1} is conjugate to c_i in Γ_2.

Hence if c_i ε Γ_2, ci is conjugate to gc_ig^{-1} in Γ_2 if and only if $e_i \not\subset \Gamma_2$. So we can define a map $\tau_\theta : \{c_i, \ldots c_r\} \rightarrow \{0,1,2\}$ such that

$$\tau_\theta(c_i) = \begin{cases} 2 & \text{if } c_i, e_i \ \varepsilon\ \Gamma_2 \\ 1 & \text{if } c_i \ \varepsilon\ \Gamma_2,\ e_i \not\subset \Gamma_2, \\ 0 & \text{if } c_i \ \varepsilon\ \Gamma_2 \end{cases} \qquad \text{for all } i = 1 \ldots r$$

and the number of boundary components of Γ_2 is

$$s = \sum_{i=1}^{r} \tau_\theta(c_i)$$

(iii) Γ_1 is orientable and will have generators (A) as described above.

Let $\Gamma_1 \backslash \Gamma_2$ contain both orientation reversing and orientation preserving generators of Γ_1. Let us assume that $\Gamma_1 \backslash \Gamma_2$ contains c_i (orientation reversing) for some i = 1, … 4 and a_j (orientation preserving) for some j = 1, …. g. Without loss of generality we can choose $\{1,c_i\}$ as coset representative so that

$$\Gamma_1 = \Gamma_2 + c_1 \Gamma_2 .$$

Throughout the proof (ii) and (iii) let F be the canonical fundamental region for Γ_1 associated with the canonical generators of Γ_1 and using the notation developed in chapter 1 denote by $\alpha, \beta, \gamma, \varepsilon$ the sides across which F is mapped by the transformations a,b,c,e. As, in this case, Γ_1 is orientable the surface symbol for Γ_1 is

$$\varepsilon_1\gamma_1\varepsilon_1{}'\varepsilon_2\gamma_2\varepsilon_2{}'\mathrm{K}\ \ \varepsilon_r\gamma_r\varepsilon_r{}'\alpha_1\beta_1\alpha_1{}'\beta_1{}'\ \mathrm{K}\ \ \alpha_g\beta_g\alpha_g{}'\beta_g{}'.$$

Also, as $\Gamma_1 = \Gamma_2 + c_i \Gamma_2$, F U$c_i$F is a fundamental region for Γ_2. We need to look at how the sides of F Uc_iF are identified under Γ_2, remembering that c_i is orientation reversing. The following is a diagram showing the structure of F Uc_iF.

If F, under Γ_1,

$$a_j(a_j') = \alpha j.$$

So

$$C_i a_j(\alpha_j') = c_i \alpha_j.$$

Now aj $\not\varepsilon$ Γ_2 but c_i a_j ε Γ_2 because c_i, a_j $\not\varepsilon$ Γ_2 and $c_i a_j$ is orientation reversing. Therefore in FUc_iF the side α_j' is a congruent to the side $c_i \alpha$ j by an orientation reversing transformation in Γ_2. Hence when the fundamental region FUC_iF is folded up to form the surface U/Γ_2, the sides α_j' and $c_i \alpha_j$ will be identified as shown by the arrows in the following diagram.

If we draw a strip from α_j' to $c_i \alpha_j$' then when the fundamental region is folded up this strip will become a Mobius band. Therefore we can embed a Mobius band in the surface U/Γ_2 and so U/Γ_2 must be a non-orientable surface, i.e. Γ_2 must be non-orientable. From this proof it is clear that whatever mixture of orientation reversing and orientation preserving transformation $\Gamma_1 \backslash \Gamma_2$ contains, Γ_2 must be non-orientable.

Conversely, let Γ_2 be non-orientable. Let g ε $\Gamma_1 \backslash \Gamma_2$ be one of the generators of Γ_1. Then FUgF is a fundamental region for Γ_2. As Γ_2 is non-orientable there must be at least one pair of sides of FUgF which are congruent by an orientation reversing transformation, x say.

The sides of F are congruent in pairs by the generators of Γ_1 unless they are sides fixed by one of the generating reflections in which case they are fixed pointwise and are congruent to no other side of F. So if η is a side of F then there exists a side of F, η, congruent to F by $h \ \varepsilon \ \Gamma_1$ such that $h(\hat{\eta}) = \eta$, where h is one of the generators of Γ_1 or the inverse of one of the generators. (If η is a side fixed by a reflection then $\hat{\eta} = \eta$). So in gF, $g\hat{\eta}$ and $g\hat{\eta}$ are congruent by $ghg^{-1} \ \varepsilon \ \Gamma_1$.

Now η, $\hat{\eta}$, $g\eta$ and $g\hat{\eta}$ are sides of FUgF and all sides of FUgF can be found if this way. If $h \ \varepsilon \ \Gamma_2$ then $\hat{\eta}$ is congruent to η under Γ_2 by h and $g\hat{\eta}$ is congruent to $g\eta$ under Γ_2 by ghg^{-1}. If $h \not\varepsilon \ \Gamma_2$ then $gh, hg^{-1} \ \varepsilon \ \Gamma_2$ and $g\hat{\eta}$ is congruent to η under Γ_2 by hg^{-1} and $\hat{\eta}$ is congruent to $g\eta$ under Γ_2 by gh. Therefore $x \ \varepsilon \ \Gamma_2$ and is of the form h, ghg^{-1}, gh or hg^{-1} where h is one of the generators of Γ_1 or the inverse of one of the generators.

However x is orientation reversing. So if x = h or $x = ghg^{-1}$, for some h, then since Γ_1 contains no glide reflection generators x must be a reflection. But then the sides congruent by h, η and $\hat{\eta}$, must coincide $\eta = \hat{\eta} = \gamma$ say, so that $g\eta = g\hat{\eta} = g\gamma$ and x fixes either γ or $g\gamma$ pointwise. Therefore, as we require the sides congruent by x to be distinct, $x = hg^{-1}$ or x = gh, for some $h \not\varepsilon \ \Gamma_2$, and since x is orientation reversing one of g and h must be orientation reversing and the other orientation preserving. Since $h \not\varepsilon \ \Gamma_2$ and $g \not\varepsilon \ \Gamma_2$, $\Gamma_1 \backslash \Gamma_2$ contains a mixture of orientation reversing and orientation preserving generators of Γ_1.

(iii) Γ_1 is non-orientable and will have generators and relations (B) as described above. In this case the surface symbol associated with the canonical generators of Γ_1 is

$$\varepsilon_1\gamma_1\varepsilon_1'\varepsilon_2\gamma_2\varepsilon_2' K \ \varepsilon_r\gamma_r\varepsilon_r'\alpha_1\alpha_1^* K \ \alpha_g\alpha_g^*.$$

If $\Gamma_1 \backslash \Gamma_2$ contains both orientation preserving and orientation reversing generators of Γ_1 then the proof that Γ_2 is non-orientable is exactly the same as in the case when Γ_1 is orientable (except that in this case the only orientation preserving generators are e_1, … e_r).

Clearly if Γ_2 contains any of the glide reflection generators of Γ_1 then Γ_2 is non-orientable. For if two sides of F are congruent by a glide reflection generator of Γ_1 which is also in Γ_2 then these two sides will also be congruent in FUgF, where g ε $\Gamma_1 \backslash \Gamma_2$, by the same (orientation reversing) generator. Since FUgF is a fundamental region for Γ_2 this means that we can again embed a Mobius band in the surface U/Γ_2 and hence Γ_2 is non-orientable.

Conversely let Γ_2 be non-orientable and suppose Γ_2 does not contain any of the glide reflection generators of Γ_1. Then, as in the case when Γ_1 is orientable, if g ε $\Gamma_1 \backslash \Gamma_2$ is a generator of Γ_1, FUgF is a fundamental region for Γ_2 and there exists an x ε Γ_2 which is orientation reversing and maps one side of FUgF to another. As before x must be of the form hg^{-1} or gh, where h ε $\Gamma_1 \backslash \Gamma_2$ is one of the generators of Γ_1 or the inverse of one of the generators, because Γ_2 contains none of the glide reflection generators of Γ_1 so if x is to be orientation reversing but not a reflection it cannot be of the form h or ghg^{-1}.

Again for x to be orientation reversing one of g and h must be orientation reversing and the other orientation preserving and hence $\Gamma_1 \backslash \Gamma_2$ contains a mixture of orientation reversing and orientation preserving generators of Γ_1.

If however Γ_2 is non-orientable but $\Gamma_1 \backslash \Gamma_2$ does not contain a mixture of orientation preserving and orientation reversing generators of Γ_1 then either

(a) $\Gamma_1 \backslash \Gamma_2$ contains only orientation preserving generators of Γ_1

or

(b) $\Gamma_1 \backslash \Gamma_2$ contains only orientation reversing generators of Γ_1

We must show that in both cases Γ_2 must contain at least one of the glide reflection generators of Γ_1.

In case (a) as $\Gamma_1 \backslash \Gamma_2$ contains only orientation preserving generators, Γ_2 must contain the glide reflection generators of Γ_1. (Here we note that as Γ_1 is non-orientable it cannot have zero genus and hence must have glide reflection generators.)

In case (b) let us assume $\Gamma_1 \backslash \Gamma_2$ contains all the glide reflection generators of Γ_1 so Γ_2 contains none but will contain all the orientation preserving generators and possibly some of the reflection generators. Let where $g \ \varepsilon \ \Gamma_1 \backslash \Gamma_2$ be one of the generators of Γ_1, then g must be orientation reversing and FUgF is a fundamental region for Γ_2.

As described in the proof of (ii) above all sides of FUgF are of the form η or $g\eta$ where η is a side of F and the sides to which they are congruent under Γ_1, $\hat{\eta}$ or $g\hat{\eta}$. Let h be the generator of Γ_1 (or the inverse of a generator of Γ_1) such that $h(\hat{\eta}) = \eta$, so $ghg^{-1}(g\hat{\eta}) = g\eta$.

If h is a reflection then $\eta = \hat{\eta} = \gamma$ say and $g\eta = g\hat{\eta} = \gamma$ and $g\eta = g\hat{\eta} = g\gamma$. If $h \ \varepsilon \ \Gamma_2$ then γ and $g\gamma$ are fixed pointwise by h and ghg^{-1} and are congruent to no other sides of FUgF. If $h \not\varepsilon \ \Gamma_2$ then $ghg^{-1} \not\varepsilon \ \Gamma_2$ but hg^{-1}, $gh \not\varepsilon \ \Gamma_2$ and $g\hat{\eta}$ is congruent to η under Γ_2 by an orientation preserving transformation and $\hat{\eta}$ is congruent to $g\eta$ under Γ_2 similarly by an orientation preserving transformation.

If h is a glide reflection then h, $ghg^{-1} \not\in \Gamma_2$ because we are assuming that Γ_2 contains none of the glide reflection generators of Γ_1. But hg^{-1}, $gh \not\in \Gamma_2$, both of which are orientation preserving and we have the same situation as when h is a reflection no in Γ_2.

If h is hyperbolic then $h, ghg^{-1} \not\in \Gamma_2$ because $\Gamma_1 \backslash \Gamma_2$ contains only orientation reversing generators. So $\hat{\eta}$ is congruent to η under Γ_2 by an orientation preserving transformation and similarly $g\hat{\eta}$ is congruent to $g\eta$ under Γ_2 by an orientation preserving transformation.

Thus the sides of FUgF, except for the sides fixed by reflections, are congruent in pairs under Γ_2 by orientation preserving transformations. So Γ_2 must be orientable, which is a contradiction.

Therefore at least one of the glide reflection generators of Γ_1 must be in Γ_2 for Γ_2 to be non-orientable when $\Gamma_1 \backslash \Gamma_2$ contains only orientation reversing generators.

We have thus shown that if Γ_2 is non-orientable then either $\Gamma_1 \backslash \Gamma_2$ must contain a mixture of orientation preserving and orientation reversing generators of Γ_1 or Γ_2 must contain at least one of the glide reflection generators of Γ_1 (or both) and this completes the proof of the theorem.

(It is easy to verify that if Γ_1 contains elliptic generators x_i, i = 1, … k, with the relations $x_i^{m_i} = 1$, then theorem 4.12 is still true and if the signature of Γ_1 has non-empty period cycles then the conditions determining the orientability of the subgroup Γ_2 remain unaltered. In these cases the double cover f: $U/\Gamma_2 \rightarrow U/\Gamma_1$ may be ramified.)

We now give some examples to show how theorem 4.12 can be used.

Example 4.13. Let S be an orientable Klein surface with genus g = 1 and r = 1 boundary components (i.e. a torus with one hole), then S = U/Γ_1, where Γ_1 is and orientable bordered surface group with signature

$$(1, +, [\ ], \{(\ )\}).$$

Γ_1 will have generators a, b, c, e with relations

$$c^2 = 1$$
$$ece^{-1} = c$$
$$eaba^{-1}b^{-1} = 1$$

If Γ_2 is a subgroup of index two in Γ_1 then it will have signature of the form

$$(h, \pm, [\ ], \{(\ )^s\})$$

and all possible numerical pairs of values for h and s can be found from the Riemann-Hurwitz formula

$$\mu(\Gamma_2) = 2\mu(\Gamma_1)$$

But using theorem 4.12 we can determine which pairs of values actually occur and the orientability of Γ_2 in each case.

Let θ be a homomorphism from Γ_1 onto $Z_2 = \langle 1, z\rangle$ defined on the canonical generators of Γ_1. By lemma 4.10 (with g = 1, r = 1 Γ_1 orientable) there seven such homomorphisms which will give us seven double covers of S of the form U/Γ_2 where Γ_2

$= \ker \theta$. By proposition 4.9 any connected unramified double cover of S will be isomorphic to one of these.

We now list the homomorphisms. From the proof of lemma 4.10 we see that we must have in all cases $\theta(e) = 1$. For brevity we shall use the abbreviation o.p.g. for orientation preserving generator and o.r.g. for orientation reversing generator.

(1) $\theta(c) = z$, $\theta(a) = \theta(b) = \theta(e) = 1$, here $c \not\in \Gamma_2$, $\Gamma_1 \setminus \Gamma_2$ contains only o.r.g.'s of Γ_1

(2) $\theta(a) = \theta(b) = z, \theta(c) = \theta(e) = 1$
(3) $\theta(a) = z, \theta(b) = \theta(c) = \theta(e) = 1$
(4) $\theta(b) = z, \theta(a) = \theta(c) = \theta(e) = 1$

here $c,e \in \Gamma_2$, $\Gamma_1 \setminus \Gamma_2$ Only o.p.g. 's of Γ_1

(5) $\theta(c) = \theta(a) = z, \theta(b) = \theta(e) = 1$
(6) $\theta(c) = \theta(b) = z, \theta(a) = \theta(e) = 1$
(7) $\theta(c) = \theta(a) = \theta(b) = z, \theta(e) = 1$

Here $c \not\in \Gamma_2$, $\Gamma_1 \setminus \Gamma_2$ contains a Mixture of o.p.g. 's and o.r.g's of Γ_1

By applying theorem 4.12 in each case we obtain the number of boundary components of Γ_2 and its orientability. Then we use the Riemann-Hurwitz formula to determine the orbit-genus, h.

e.g. in (1), theorem 4.12 implies that Γ_2 is orientable and has no boundary components (s = 0) so

$$\mu(\Gamma_1) = 2\pi(2g - 2 + r) = 2\pi$$

So from the Riemann-Hurwitz formula we deduce that h – 2.

We can now list the signatures of Γ_2 in each of the above cases.

(1) $(2, +, [\ ], \{\ \})$

$$\left.\begin{matrix}(2)\\(3)\\(4)\end{matrix}\right\}(1,+,[],\{()^2\})$$

$$\left.\begin{matrix}(5)\\(6)\\(7)\end{matrix}\right\}(4,-,[],\{\}).$$

In (1) the Klein surface U/Γ_2 is orientable with genus 2 and no boundary. (2), (3) and (4) represent different double covers of S because they come from different homomorphisms but in each case U/Γ_2 is orientable with genus 1 and 2 boundary components (i.e. a torus with 2 holes). In (5), (6) and (7) U/Γ_2 is non-orientable with genus 4 and no boundary, each case representing a different double cover of S.

<u>Example 4.14</u> Let S be a non-orientable Klein surface with genus g = 2 and r = 1 boundary components (i.e. a Klein bottle with one hole). Then $S = U/\Gamma_1$, where Γ_1 is non-orientable bordered surface group with signature

$$(2, -, [\ ], \{(\ )\}).$$

Γ_1 will have generators a1, a2, c, e with relations

$$c^2 = 1$$
$$ece^{-1} = c$$
$$ea_1{}^2a_2{}^2 = 1$$

As in example 4.13 if Γ_2 is a subgroup of index two in Γ_1 then it will have signature of the form

$$(h, \pm, [\ ], \{(\ )^s\})$$

and

$$\mu(\Gamma_2) = 2\mu(\Gamma_1).$$

By lemma 4.10 the number of homomorphism $\theta: \Gamma_1 \to Z_2$ is seven, again we must have $\theta(e) = 1$ in each case. Let ker $\theta = \Gamma_2$. For brevity we abbreviate glide reflection generator to g.f.g.. The homomorphisms are

(1) $\theta(c) = \theta(a_i) = \theta(a_2) = z, \theta(e) = 1,$ here c $\not\in \Gamma_2$, $\Gamma_1 \backslash \Gamma_2$ contains only o.r.g.'s of Γ_1

(2) $\theta(a_1) = \theta(a_2) = z,\ \theta(c) = \theta(e) = 1,$ here c,e ε Γ_2, $\Gamma_1 \backslash \Gamma_2$ contains only o.r.g. 's of Γ_1

(3) $\theta(a_1) = z,\ \theta(a_2) = \theta(c) = \theta(e) = 1$
(4) $\theta(a_2) = z,\ \theta(a_1) = \theta(c) = \theta(e) = 1$
here c,e ε Γ_2, $\Gamma_1 \backslash \Gamma_2$ contains only o.r.g.'s of Γ_1 but Γ_2 contains a g.f.g. of Γ_1

(5) $\theta(c) = z, \theta(a_1) = \theta(a_2) = \theta(e) = 1$
(6) $\theta(c) = \theta(a_1) = z,\ \theta(a_2) = \theta(e) = 1$
(7) $\theta(c) = \theta(a_2) = z, \theta(a_1) = \theta(e) = 1$
here c $\not\in \Gamma_2$, $\Gamma_1 \backslash \Gamma_2$ contains only o.r.g. 's of Γ_1 but Γ_2 contains g.f.g'(s) of Γ_1.

Applying theorem 4.12 and the Riemann-Hurwitz formula in each case we obtain the signature of Γ_2 to be

(1) $(2, +, [\,], \{\,\})$

(2) $(1, +, [\,], \{(\,)^2\})$

$\left.\begin{matrix}(3)\\(4)\end{matrix}\right\}$ $(2, -, [\,], \{(\,)^2\})$

$\left.\begin{matrix}(5)\\(6)\\(7)\end{matrix}\right\}$ $(4, -, [\,], \{(\,)^2\})$.

As before different homomorphisms giving the same signature for Γ_2 represent different double covers of S. Any connected unramified double cover of S will be isomorphic to one of the seven Klein surfaces, U/Γ_2, represented above.

Example 4.15. Let S be an orientable Klein surface with genus g = 1 and r = 2 boundary components (i.e. a torus with two holes). Then $S = U/\Gamma_1$ where Γ_1 is an orientable bordered surface group with signature.

$$(1, +, [\,], \{(\,)^2\}).$$

Γ_1 will have generators a, b, c_1, c_2, e_1, e_2 with relations

$$c_1^{\,2} = c_2^{\,2} = 1$$
$$e_1c_1e_1^{-1} = c_1, \quad e_2c_2e_2^{-1} = c_2$$
$$e_1e_2aba^{-1}b^{-1} = 1.$$

If Γ_2 is a subgroup of index two in Γ_1 then it will have signature of the form

$$(h, \pm, [\,], \{(\,)^8\})$$

and

$$\mu(\Gamma_2) = 2\mu(\Gamma_1).$$

By lemma 4.10 the number of homomorphisms $\theta: \Gamma_1 \to Z_2$ is 31, we must have $\theta(e_1)\theta(e_2) = 1$ in each case. Let ker $\theta = \Gamma_2$. The homomorphisms are

(1) $\theta(c_1) = \theta(c_2) = z, \theta(a) = \theta(b) = \theta(e_1) = \theta(e_2) = 1,$

(2) $\theta(c_1) = z, \theta(c_2) = \theta(a) = \theta(b) = \theta(e_1) = \theta(e_2) = 1,$
(3) $\theta(c_2) = z, \theta(c_1) = \theta(a) = \theta(b) = \theta(e_1) = \theta(e_2) = 1,$
(4) $\theta(e_1) = \theta(e_2) = z, \theta(c_1) = \theta(c_2) = \theta(a) = \theta(b) = 1,$
(5) $\theta(e_1) = \theta(e_2) = \theta(a) = z, \theta(c_1) = \theta(c_2) = \theta(b) = 1,$
(6) $\theta(e_1) = \theta(e_2) = \theta(b) = z, \theta(c_1) = \theta(c_2) = \theta(a) = 1,$
(7) $\theta(e_1) = \theta(e_2) = \theta(a) = \theta(b) = z, \theta(c_1) = \theta(c_2) = 1,$
(8) $\theta(a) = \theta(b) = z, \theta(c_1) = \theta(c_2) = \theta(e_1) = \theta(e_2) = 1,$
(9) $\theta(a) = z, \theta(b) = \theta(c_1) = \theta(c_2) = \theta(e_1) = \theta(e_2) = 1,$
(10) $\theta(b) = z, \theta(a) = \theta(c_1) = \theta(c_2) = \theta(e_1) = \theta(e_2) = 1,$

(11) $\theta(c_1) = \theta(c_2) = \theta(a) = \theta(b) = z, \theta(e_1) = \theta(e_2) = 1,$
(12) $\theta(c_1) = \theta(c_2) = \theta(a) = z, \theta(b) = \theta(e_1) = \theta(e_2) = 1,$
(13) $\theta(c_1) = \theta(c_2) = \theta(b) = z, \theta(a) = \theta(e_1) = \theta(e_2) = 1,$
(14) $\theta(c_1) = \theta(c_2) = \theta(e_1) = \theta(e_2) = z, \theta(a) = \theta(b) = 1,$
(15) $\theta(c_1) = \theta(c_2) = \theta(e_1) = \theta(e_2) = \theta(a) = z, \theta(b) = 1,$
(16) $\theta(c_1) = \theta(c_2) = \theta(e_1) = \theta(e_2) = \theta(b) = z, \theta(a) = 1,$
(17) $\theta(c_1) = \theta(c_2) = \theta(e_1) = \theta(e_2) = \theta(a) = \theta(b) = z,$

(18) $\theta(c_1) = \theta(e_1) = \theta(e_2) = z, \theta(c_2) = \theta(a) = \theta(b) = 1,$
(19) $\theta(c_2) = \theta(e_1) = \theta(e_2) = z, \theta(c_1) = \theta(a) = \theta(b) = 1,$
(20) $\theta(c_1) = \theta(e_1) = \theta(e_2) = \theta(a) = z, \theta(c_2) = \theta(b) = 1,$
(21) $\theta(c_2) = \theta(e_1) = \theta(e_2) = \theta(a) = z, \theta(c_1) = \theta(b) = 1,$
(22) $\theta(c_1) = \theta(e_1) = \theta(e_2) = \theta(b) = z, \theta(c_2) = \theta(a) = 1,$
(23) $\theta(c_2) = \theta(e_1) = \theta(e_2) = \theta(b) = z, \theta(c_1) = \theta(a) = 1,$
(24) $\theta(c_1) = \theta(e_1) = \theta(e_2) = \theta(a) = \theta(b) = z, \theta(c_2) = 1,$
(25) $\theta(c_2) = \theta(e_1) = \theta(e_2) = \theta(a) = \theta(b) = z, \theta(c_1) = 1,$

(26) $\theta(c_1) = \theta(a) = z, \theta(c_2) = \theta(b) = \theta(e_1) = \theta(e_2) = 1,$
(27) $\theta(c_2) = \theta(a) = z, \theta(c_1) = \theta(b) = \theta(e_1) = \theta(e_2) = 1,$
(28) $\theta(c_1) = \theta(b) = z, \theta(c_2) = \theta(a) = \theta(e_1) = \theta(e_2) = 1,$
(29) $\theta(c_2) = \theta(b) = z, \theta(c_1) = \theta(a) = \theta(e_1) = \theta(e_2) = 1,$
(30) $\theta(c_1) = \theta(a) = \theta(b) = z, \theta(c_2) = \theta(e_1) = \theta(e_2) = 1,$
(31) $\theta(c_2) = \theta(a) = \theta(b) = z, \theta(c_1) = \theta(e_1) = \theta(e_2) = 1.$

Applying theorem 4.12 and the Riemann-Hurwitz formula in each case we deduce that the signature of Γ_2 is

$$(1) \qquad (3, +, [\ ], \{\ \})$$

$$\left.\begin{array}{c}(2)\\ \mathrm{g}\\ \mathrm{g}\\ (7)\end{array}\right\} \quad (2, +, [\ ], \{(\ )^2\})$$

$$\left.\begin{array}{c}(8)\\ \\ (9)\\ \\ (10)\end{array}\right\} \quad (1, +, [\ ], \{(\ )^4\})$$

$$\left.\begin{array}{c}(11)\\ \mathrm{g}\\ \mathrm{g}\\ (17)\end{array}\right\} \quad (6, -, [\ ], \{\ \})$$

$$\left.\begin{array}{c}(18)\\ \mathrm{g}\\ \mathrm{g}\\ (25)\end{array}\right\} \quad (5, -, [\ ], \{(\ )\})$$

$$\left.\begin{array}{c}(26)\\ \mathrm{g}\\ \mathrm{g}\\ (31)\end{array}\right\} \quad (4, -, [\ ], \{(\ )^2\}).$$

As before homomorphisms giving the same signature for Γ_2 represent different double covers of S. Any connected unramified double cover of S will be isomorphic to one of the 31 Klein surfaces, U/Γ_2, represented above.

5) Let S be a Klein surface with genus g and r boundary components such that S – U/Γ where Γ is either a non-orientable surface group or a bordered surface group. Let (S_c, f, σ) be the complex double of S. S_c is orientable without boundary and because S has boundary if it is orientable, S_c is connected. It follows immediately from theorem 4.12 that the only way to form an orientable subgroup without boundary of index two in Γ is to take the kernel of the homomorphism which maps all the orientation reversing generators to the element of order two in Z_2 and all the orientation preserving generators to the identity. But the canonical Fuchsian group, Γ^+ of Γ is the subgroup of index two in Γ consisting of all elements which preserve orientation. Thus, as mentioned before,

$$Sc = U/\Gamma^+$$

F is the map

$$f([z]_{\Gamma^+}) = [z]_{\Gamma}$$

and if $\gamma \,\varepsilon\, \Gamma \setminus \Gamma^+$ then σ is defined by

$$\sigma([z]_{\Gamma^+}) = [\gamma z]_{\Gamma^+}.$$

(If we choose another element $\gamma' \varepsilon \Gamma \setminus \Gamma^+$ then $\gamma'\gamma^{-1} \varepsilon \Gamma^+$ so $[\gamma z]_{\Gamma^+} = [\gamma' z]_{\Gamma^+}$, which shows that σ is well-defined.) Γ^+ has signature (2g + r – 1, +, [ ], { }) if S is orientable and (g + r – 1, +, [ ], { }) if S is non-orientable.

In each of the examples 4.13, 4.14 and 4.15 the kernel of homomorphism (1) gives Γ^+.

Since S_c is a Riemann surface the algebraic genus of S_c (the non-negative integer that makes the algebraic version of the Riemann-Roch theorem work) is equal to the topological genus. If E and F are the fields of meromorphic functions on S and S_c respectively then F = E(I) (see [2]) and by a well-known classical result ([4]) the algebraic genus of S is equal to the algebraic genus of S_c, i.e. to the topological genus of $S_c = U/\Gamma^+$.

If S is a non-orientable Klein surface with genus g and r boundary components such that $S = U/\Gamma$ where Γ is a non-orientable bordered surface group, then the orienting double of S, S_o, is a connected orientable Klein surface with 2r boundary components.

Again it follows immediately from theorem 4.12 that the only way to form an orientable subgroup of index two in Γ with 2r boundary components, which we shall denote by Γ_o, is to take the kernel of the homomorphism which maps all the glide reflection generators of Γ to $z \varepsilon Z_2$ and all the hyperbolic and reflection generators of Γ to the identity. In example 4.14 this is a homomorphism (2). $S_o = U/\Gamma_o$ and from the Riemann-Hurwitz formula we see that the genus of Γ_o is g – 1 so Γ_o has signature

$$(g - 1, +, [\ ], \{(\ )^{2r}\}).$$

If we take the same non-orientable surface S with boundary then the Schottky double of S, S_s, is a connected non-orientable Klein surface without boundary. If Γ_s is the non-orientable subgroup of index two in Γ such that $S_s = U/\Gamma_s$ then from the Riemann – Hurwitz formula we see that the genus of Γ_s is 2g + 2r – 2, so Γ_s has signature

$$(2g + 2r - 2, -1, [\ ], \{\ \}).$$

However from example 4.14 we see that there is not a unique homomorphism whose kernel has the signature of Γ_s because in this example homomorphisms (5), (6) and (7) each have such a kernel.

Clearly from theorem 4.12 the homomorphism whose kernel is Γ_s must map all the reflection generators of Γ to the element of order two in Z_2.

As S_s is constructed by taking two copies of S and 'gluing' them together along their boundaries it should be clear that the homomorphism whose kernel of Γ_s is the one which maps all the generators of Γ to the identity except the reflections. In example 4.14 this is homomorphism (5).

6) Let us now consider normal n-sheeted coverings of Klein surfaces when n > 2.

If n is even it is clear from theorem 4.12 that the situation could be quite complex. However when n is odd we can obtain some general results.

Firstly when n = p prime, the only group of order p (upto isomorphism) is the cyclic group Z_p with presentation $\{z: z^p = 1\}$l. If Γ_1 is a bordered surface group we can extend the proof of part (i) of theorem 4.12 to ascertain the number of boundary components of a normal subgroup, Γ_2, of index p in Γ_1, i.e. the number of boundary components of U/Γ_2, a p-sheeted normal covering surface of U/Γ_1.

<u>Theorem 4.16</u>. Let Γ_1 be a bordered surface group with orbit-genus g and r boundary components. Let $\theta: \Gamma_1 \rightarrow Z_p$, for prime ; > 2, be defined on the canonical generators of Γ_1 (as described before theorem 4.12) so that θ , = Γ_2, is a normal subgroup of index p in Γ_1. Define a map τ_θ from $\{e_1, \ldots e_r\}$ (the set of generators of Γ_1 commuting with the generating reflections) to $\{1,p\}$ such that

$$\tau_\theta(e_i) = \begin{cases} p \text{ if } e_i \ \varepsilon \ \Gamma_2 \\ 1 \text{ if } e_i \not\varepsilon \ \Gamma_2 \end{cases} \qquad \text{for all } i = 1, \ldots r.$$

Then the number of boundary components of Γ_2 is

$$s = \sum_{i=1}^{r} \tau_\theta(e_i).$$

<u>Proof</u>.

Let g be one of the generators of Γ_1 such that g ε $\Gamma_1 \setminus \Gamma_2$. Because every element of Z_p which is not the identity generates the whole group it is easy to see that whichever generator of Γ_1 in $\Gamma_1 \setminus \Gamma_2$ we choose for g we can write

$$\Gamma_1 = \Gamma_2 + \Gamma_2^{g2} + \ldots + \Gamma_2^{gp-1}.$$

All reflections in Γ_1 are conjugate to one of the generating reflections C_1, … C_r. As in the proof of theorem 4.12 part (i) our aim is to count the number of conjugacy classes of reflections in Γ_2.

As before we define a reflection c' ε Γ_2 to be induced by the reflection c_i in Γ_1. By lemma 1.15 if c_i ε Γ_1 is also in Γ_2, c' is conujugate to $g^m c_i g^{-m}$ in Γ_2 for some m = 1,1,2, … p-1. As Z_p $(p \neq 2)$ contains no element of order two, c_i ε Γ_2 for all i = 1, … r. Therefore every c_i must induce one or more conjugacy classes of reflections in Γ_2. To determine the number of conjugacy classes of reflections in Γ_2 induced by c_i we only have to establish when $g^m c_i g^{-m}$ is conjugate to $g^n c_i g^{-n}$ in Γ_2, $m \neq n$, m,n ε {0,1, … p-1}.

If $g^m c_i g^{-m}$ is conjugate to $g^n c_i g^{-n}$ in Γ_2, $m \neq n$, then there exists $h \; \varepsilon \; \Gamma_2$ such that

$$H\,(g^m c_i g^{-m})h^{-1} = g^n c_i g^{-n},$$

Which implies that $g^{-n}hg^m$ is an element of the centralizer of ci in Γ_1, i.e.

$$g^{-n}hg^m \; \varepsilon \; \langle c_i, e_i \rangle$$

by theorem 1.16. So we can put

$$h = g^n x g^{-m}$$

for some $x \; \varepsilon \; \langle c_i, e_i \rangle$.

If $e_i \; \varepsilon \; \Gamma_2$ then $\langle c_i, e_i \rangle \subset \Gamma_2$ which implies $x \; \varepsilon \; \Gamma_2$, i.e. $\theta\,(x) = 1$. As Z_p is abelian

$$\theta(h) = \theta(g^n x g^{-m}) = (\theta(g)^{n-m}\theta(x) = (\theta(g))^{n-m}.$$

Since $g \not\varepsilon \; \Gamma_1$, $\theta(g) \neq 1$, also $n - m \neq$ o or p and so

$$(0\,(g))^{\,n-m} \neq 1.$$

This implies that $\theta\,(h) \neq 1$ so $h \not\varepsilon \; \Gamma_2$. Therefore $g^m c_i g^{-m}$ cannot be conjugate to $g^n c_i g^{-n}$ in Γ_2 if $m \neq n$.

If $e_i \not\varepsilon \; \Gamma_2$, we can choose $g = e_i$. Then, since $e_i c_i e_i^{-1} = c_i$,

$$g^m c_i g^{-m} = e_i^{\,m} c_i e_i^{\,-m} = c_I$$

and similarly $g^n c_i g^{-n} = c_i$.

So $g^m c_i g^{-m} = g^n c_i g^{-n}$ and are therefore trivially conjugate in Γ_2.

Therefore if $e_i \in \Gamma_2$, $g^m c_i g^{-m}$ is never conjugate to $g^n c_i g^{-n}$ in Γ_2 for $n \neq m$ and if $e_i \not\in \Gamma_2$, $g^m c_i g^{-m}$ is conjugate to $g^n c_i g^{-n}$ in Γ_2 for all m,n. So the number of conjugacy classes of reflections in Γ_2 induced by c_i is 1 or p depending on whether e_i is in Γ_2 or no. We can define a map $\tau_\theta : \{e_1, \ldots e_r\} \rightarrow \{l,p\}$ such that

$$\tau_\theta(e_i) = \begin{cases} p \text{ if } e_i \in \Gamma_2 \\ 1 \text{ if } e_i \not\in \Gamma_2 \end{cases} \qquad \text{for all } i = 1, \ldots r.$$

and the number of boundary components of Γ_2 is

$$s = \sum_{i=1}^{r} \tau_\theta(e_i).$$

Secondly when n is odd general results concerning the orientability of a subgroup of index n in an NEC group can be obtained.

Theorem 4.17. Let Γ_1 be an orientable NEC group with signature

$$(g, +, [m_1, \ldots m_k], \{(n_{11}, \ldots n_{1s_1}), \ldots (n_{r1}, \ldots n_{rs_r})\}) \text{ and let } \theta : \Gamma_1 \rightarrow G,$$

where G is any finite group with odd order n, be a homomorphism defined on the canonical generators of Γ_1 such that ker θ, = Γ_2, is a subgroup of index n in Γ_1. Then Γ_2 must be orientable.

Proof.

Γ_1 will have generators and relations as in (1.6). The generators a_i, b_i, x_i, e_i are all orientation preserving and the generators c_{ij} are reflections, i.e. orientation reversing.

The result is obvious if Γ_1 does not contain reflections as it is then a Fuchsian group. So we suppose that Γ_1 contains reflections.

Since G has odd order it has no element of order two, so for $\theta(c_{ij}) = 1$, for all i = 1, … r, j = 0,1, … s_i.

As Γ_2 is a normal subgroup of index n in Γ_1 there exist coset representatives $g_1, g_2, \ldots g_n$ in Γ_1 such that

$$\Gamma_1 = \Gamma_2 g_1 + \Gamma_2 g_2 + \ldots \Gamma_2 g_n$$

Without loss of generality we can assume that $g_1, \ldots g_n$ are orientation preserving because we can replace any coset $\Gamma_2 g$ by $\Gamma_2 (cg)$, where $c \in \Gamma_2$ is a reflection, if necessary. Also without loss of generality we can assume $g_1 = 1$.

Let F be the canonical fundamental region for Γ_1 associated with the canonical generators. Then

$$F' = FUg_2FU \ldots Ug_nF$$

Is a fundamental region for Γ_2.

The sides of F' are images of sides of F and fall into pairs congruent by transformations of Γ_2 (except the sides of reflection which are fixed by reflection generators and their conjugates)

If we can show that all pairs of congruent sides of F' are congruent by orientation preserving transformations then we shall have shown that U/Γ_2 is orientable, i.e. that Γ_2 is an orientable group. So let us assume that there is one pair of sides of F' which are congruent by an orientation reversing transformation in Γ_2, x say, and try to reach a contradiction.

Let the two sides congruent by x be p and q, where p is a side of g_iF and q is a side of g_jF, I,j = 1,2, . . . n. Then

$$p = g_i \eta, \quad \text{where } \eta \text{ is a side of F}$$

$$q = g_j \zeta, \quad \text{where } \zeta \text{ is a side of F.}$$

(If η and ζ then i $\neq$ j so that p and q are distinct.) so

$$x(g_i \eta) = g_j \zeta$$

which implies that

$$(g_j^{-1} x g_i)\ \eta = \zeta .$$

Thus η and ζ are congruent by $g_j^{-1}xg_i \ \varepsilon\ \Gamma_1$. But η and ζ are sides of F, the canonical fundamental region for Γ_1 associated with the canonical generators of Γ_1 and so if η and ζ are congruent by a transformation in Γ_1 that transformation must be one of the canonical generators of Γ_1 and is unique (upto inverse). Hence

$$g_j^{-1}xg_i = t, \text{ say}$$

where t is one of the generators of Γ_1 (or the inverse of one of the generators) and thus is either orientation preserving or a reflection. However g_i and g_j have been chosen so that they are both orientation preserving so for x to be orientation reversing t must be orientation reversing and therefore a reflection.

But if t is a reflection we have the following situation.

Since η and ζ are congruent by t in F, $\eta = \zeta = \xi$ say and

$$p = g_i \xi$$
$$q = g_j \xi$$

As t is one of the reflection generators of Γ_1, $t \varepsilon \Gamma_2$. Therefore $g_i t g_i^{-1}, g_j t g_j^{-1} \varepsilon \Gamma_2$ because $\Gamma_2 < \Gamma_1$. Now $g_i \xi$ is fixed pointwise by the reflection $g_i t g_i^{-1}$ and $g_j \xi$ is fixed by the reflection $g_j t g_j^{-1}$. Also $x(g_i \xi) = g_j \xi$, $x \varepsilon \Gamma_2$.

Consider any point on the N.E. line $g_i \xi$. Because of the continuity of elements in G, we can always find a small enough neighbourhood, V say, of this point such that a point $p \varepsilon V I g_i F$ is mapped just outside $g_i F$ by $g_i t g_i^{-1}$. The transformation x will map this point to a point just outside $g_j F$ within a small neighbourhood of some point on the N.E. line $g_j \xi$. This point just outside $g_j F$ will be mapped by $g_j t g_j^{-1}$ just inside $g_j F$. Therefore our original point p has been mapped by transformation in Γ_2 from just inside $g_i F$ to a point just inside $g_j F$ as illustrated in the following diagram.

So we have two points in the same Γ_2-orbit in the interior of a fundamental region for Γ_2 which is a contradiction.

Hence t cannot be a reflection and so Γ_2 must be orientable.

<u>Theorem 4.18</u>. Let Γ_1 be a non-orientable NEC group with signature

$(g, -, [m_1, \ldots m_k], \{(n_{11}, \ldots n_{1_{s_1}}), \ldots (n_{r1}, \ldots n_{rs_r})\})$

and let $\theta : \Gamma_1 \rightarrow G$ where G is any finite group with odd order n, be a homomorphism defined of the canonical generators of Γ_1 such that ker θ, = Γ_2, is a normal subgroup of index n in Γ_1. Then Γ_2 must be non-orientable.

Proof.

Γ_1 will have generators and relations as in (1.7.) The generators x_i, e_i are all orientation preserving, the c_ij's are reflection and the a_i's glide reflection (orientation reversing).

Since G has odd order it has no element of order two, so for θ to be a homomorphism $\theta(c_ij) = 1$, for all i = 1, . . . r, j = 0,1 . . . s_i. We choose coset representatives $g_1, \ldots g_n$ in Γ_1 such that

$$\Gamma_1 = \Gamma_2 g_1 + \Gamma_2 g_2 + \ldots + \Gamma_2 g_n.$$

Without loss of generality we can choose $g_1 = 1$.

Let F be the canonical fundamental region for Γ_1 associated with the canonical generators. Then

$$F' = FUg_2FU \ldots Ug_nF$$

Is a fundamental region for Γ_2.

Let us consider any one of the glide reflection generators in Γ_1, i.e. in the set $\{a_1, \ldots 1_g\}$. Call it a. Denote by α the side across which F is mapped by a and denote by α^* the side congruent to α in F by a, so

$$a(\alpha^*) = \alpha$$

Now if $a \in \Gamma_2$, $a \neq g_i$ for i = 1, . . . n and the two sides congruent by an in F (α and α^*) are still congruent in F' by a, which is orientation reversing. Thus we can embed a Mobius band in the surface U/Γ_2 and hence Γ_2 is non-orientable.

(Note: We can always choose coset representatives such that

$$g_iFng_jF = \varnothing, i,j = 1, . . . n, i \neq j,$$

so α and α^* are sides of F' and not interior to it.)

If $a \not\subset \Gamma_2$ then let the order of $\theta(a)$ in G be m. So $2 < m \not\subset n$. Then we can choose $g_i = a^i\text{-}1$ for i = 1, . . . m, so

$$F' = FUaFUa^2FU \ldots Ua^{m-1}Fug_{m+1}FU \ldots Ug_nF.$$

Now $\theta(a^m) = (\theta(a))^m = 1$ in G, i.e. $a^m \in \Gamma_2$ and since the order of $\theta(a)$ must divide n which is odd, m must be odd and hence a^m must be orientation reversing. Also

$$a^m(\alpha^*) = a^{m-1}(a\alpha^*) = a^{m-1}(\alpha).$$

Am-1(α) is a side of $a^{m-1}F$ and therefore a side of F'. α^* is a side of F and so also is a side of F'. Thus we have two sides of connected component of F' congruent by an orientation reversing transformation in Γ_2 and again Γ_2 must be non-orientable. (Note: When $a \not\subset \Gamma_2$ let FU aFU . . . $Ua^{m-1}F = F_a$. F_a is connected but α^* and $a^{m-1}\alpha$ are not sides of intersection with F_a. Because $\Gamma_2 < \Gamma_1$ it is always possible to find elements $\gamma_1 K \ \gamma_m, \varepsilon \Gamma_1$, where m' = n/m – 1, such that the set

$$\begin{Bmatrix} 1, & a, & a^2, & \ldots & a^{m-1} \\ \gamma_1, & \gamma_1 a, & \gamma_1 a^2, & \ldots & \gamma_1 a^{m-1} \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ \gamma_{m''} & \gamma_{m'} a, & \gamma_{m'} a^2, & \ldots & \gamma_{m'} a^{m-1} \end{Bmatrix}$$

$$F' = F_a \cup \gamma_1 F_a \cup \ldots \cup \gamma_{m'} F_a$$

Where $\gamma_i F_a = \gamma_i F \cup \gamma_i a F \cup \ldots \cup \gamma_i a^{m-1} F$. We can always choose the elements such that $F_a \cap \gamma_i F_a = \varnothing$, i = 1, . . . m' and $\gamma_i F_a \cap \gamma_j F_a = \varnothing$, i = 1, . . . m' and $\gamma_i F_a \cap \gamma_j F_a = \varnothing$, $i \neq j$, i,j = 1, . . . m'. So we can always choose a fundamental region for Γ_2 such that α^* and $a^{m-1}\alpha$ are sides of F' and not interior to it.)

In theorems 4.17 and 4.18 U/Γ_2 is a normal n-sheeted covering surface of U/Γ_1, possibly ramified. We have thus proved the following.

Theorem 4.19. Let S be a Klein surface such that $S = U/\Gamma_1$, where Γ_1 is an orientable (respectively non-orientable) NEC group. Then an n-sheeted covering surface of S of the form U/Γ_2, where $\Gamma_2 < \Gamma_1$, must be orientable (respectively non-orientable), provided m is odd.

7) To end this chapter we shall take a brief look at non-normal n-sheeted coverings. I.e. n-sheeted coverings of Klein surface U/Γ_1 (Γ_1 and NEC group) of the form U/Γ_2, where $\Gamma_2 \subset \Gamma_1$ is a non-normal subgroup of index n in Γ_1.

To find non-normal subgroups of index n in an NEC group Γ_1 we look at homomorphisms $\theta : \Gamma_1 \to G$, where G is a finite permutation group transitive on n points. If $G' \subset G$ is the stabilizer of a point then $\Gamma_2 = \theta^{-1}(G')$ is a subgroup of index n in Γ_1. The case when $\Gamma_2 < \Gamma_1$ is just a special case of this with $G = \Gamma_1/\Gamma_2$ acting in its right regular representation and $\theta : \Gamma_1 \to G$ the natural homomorphism.

The following example shows that we cannot extend theorem 4.17 to non-normal subgroups, in other words there exist non-normal odd-sheeted coverings of orientable surfaces which are non-orientable.

<u>Example 4.20</u> Let Γ_1 be an orientable bordered surface group with orbit-genus g – 1 and r = 1 boundary components. So Γ_1 has signature

$$(1, +, [\,], \{(\,)\})$$

with generators a, b, c, e and relations

$$c^2 = 1$$
$$ece^{-1} = c$$
$$eaba^{-1}b^{-1} = 1.$$

We define θ on the generators of Γ_1 so that θ is a homomorphism onto a permutation group transitive on three points. Let

$$\theta(a) = (1\ 2\ 3)$$
$$\theta(b) = (1)\ (2)\ (3)$$
$$\theta(c) = (1\ 2)\ (3)$$
$$\theta(e) = (1)\ (2)\ (3).$$

θ is easily verified to be a homomorphism. Let $\Gamma_2 = \theta^{-1}$ (Stab(1)), so $[\Gamma_1 : \Gamma_2] = 3$ and e,b ε Γ_2, c,a $\not\varepsilon$ Γ_2. Choose coset representatives 1, a, a^2 so

$$\Gamma_1 = \Gamma_2 + \Gamma_2 a + \Gamma_2 a^2.$$

Let F be the canonical fundamental region for Γ_1 associated with the canonical generators. Then

$$F' = F \cup aF \cup a^2F$$

is a fundamental region for Γ_2.

Denote by α, β, γ, ε the sides across which F is mapped by the transformations a, b, c, e. Then the canonical surface symbol for Γ_1 is $\varepsilon\gamma\varepsilon'\alpha\beta\alpha'\beta'$.

Now $\theta(ca^{-1}) = \theta(c)(\theta(a))^{-1} = (1)(2\ 3)$ so $(ca^{-1}) \varepsilon$ Stab(1) and $ca^{-1} \varepsilon \Gamma_2$. Also $a\gamma$ is a side of aF, γ is a side of F and

$$Ca^{-1}(a\gamma) = \gamma.$$

Therefore two sides of F' are congruent by an orientation reversing transformations in Γ_2. So we can embed a Mobius band in the surface U/Γ_2 and Γ_2 must be non-orientable.

We can also count the number of conjugacy classes of reflections induced in Γ_2 by the generating reflection $c \ \varepsilon \ \Gamma_1$. For

$$\theta(ac) = \theta(a)\ \theta(c) = (1)(2\ 3)$$

so $ac \ \varepsilon \ \Gamma_2$ and $\Gamma_2 = \Gamma_2 ac$. Since c has order two $\Gamma_2 c = \Gamma_2 a$ and as

$$\Gamma_1 = \Gamma_1 c = \Gamma_2 c + \Gamma_2 ac + \Gamma_2 a^2 c$$

we must have

$$\Gamma_2 a^2 c = \Gamma_2 a^2,$$

i.e. $a^2ca^{-2} \varepsilon \Gamma_2$. Now suppose there exists $g \varepsilon \Gamma_1$ such that $gcg^{-1} \varepsilon \Gamma_2$. This implies that $\Gamma_2 gc = \Gamma_2 g$, i.e. $\Gamma_2 g = \Gamma_2 a^2$ and $g = \gamma a^2$ where $\gamma \varepsilon \Gamma_2$. Then

$$gcg^{-1} = \gamma a^2ca^{-2}\gamma^{-1}$$

which implies that gcg^{-1} conjugate to a^2ca^{-2} in Γ_2, Thus, as $c \not\varepsilon \Gamma_2$, there is only one conjugacy class of reflections in Γ_2, i.e. Γ_2 has only one boundary component.

We can now use the Riemann-Hurwitz formula to deduce that the genus of Γ_2 is 2 and that Γ_2 has signature

$$(2, -, [\ ], \{(\ )\}).$$

REFERENCES

[1] Ahlfors L. and Sario L. Riemann surfaces, Princeton (1960).

[2] Alling N.L. and Greenleaf N. Foundations of the theory of Klein surfaces, Springer-Verlag (1971).

[3] Bers L. Quasiconformal mappings and Teichmuller's theorem, Princeton Conference on Analytic Functions(1960).

[4] Chevalley C. Introduction to the theory of algebraic functions of one variable Providence, Rhode Island (1951).

[5] Dickson L.E. Linear groups with an exposition of the Galois Field theore, Leipzig (1901), reprinted by Dover (1960).

[6] Griffiths H.B. Surfaces, Cambridge University Press (1976).

[7] Harvey W.J. Cyclic groups of automorphisms of compact Riemann surfaces, Quat. J. of Math. Vol 17, no. 66 (June1966) 86-97.

[8] Hurwitz A. Uber Algebraische Gebilde mit Eindeutigen Transformation in Sich, Math. Ann. 41 (1892) 401-442.

[9] Klein F. Uber Riemanns Theorie der algebraischen Funktionen und ihrer Integrale, Teubner, Leipzig (1882), English translation New York Dover Press (1963).

[10] Lefschetz S. Introduction to Topology, Princeton (1949).

[11] Lehner J. Discontinuous groups and automorphic functions, Rhode Island: Math. Surveys of the Amer. Math. Soc. (1964).

[12] Lehner J. and Newman M. Riemann surfaces with maximal automorphism groups, Glasgow J. of Math. (1967) 102-112.

[13] Macbeath A.M. On a theorem of Hurwitz, Glasgow J. of Math. Vol 5, Part 2 (1961) 90 – 96.

[14] Macbeath A.M. Fuchsian groups, Proc. Of Dundee Summer School, University of St. Andrews (1961).

[15] Macbeath A.M. On a curve of genus 7, Proc. Of the Lon. Math. Soc. Vol.XIV (1965) 527 – 542.

[16] Macbeath A.M. Generators of the linear fractional groups, Proc. Symp. Pure Math., Houston (1967).

[17] Macbeath A.M. The classification of non-Euclidean plane crystallographic groups, Canadian J. of Math. Vol. 19 (1967) 1192 – 1205.

[18] Maclachlan C. Abelian groups of automorphisms of a compact Riemann surface, Proc. Of the Lon. Math. Soc. 111 Vol. XV (1964) 669 –712.

[19] Massey W. Algebraic Topology. An Introduction, Harbrace College Math. Series (1967).

[20] May C.L. Automorphisms of compact Klein surfaces with boundary, Pacific J. of Math. Vol. 59, No. 1 (1975)

[21] May C.L. Large automrophism groups of compact Klein surfaces with boundary, (to appear). Glasgow Maths J. 18(1979) 1-10.

[22] May C.L. Cyclic automorphism groups of compact Klein surfaces with boundary, (to appear). Houston Journal of Mathsw. Vol 3 No3. (1977).

[23] Schiffer M. and Spencer D. Functionals of finite Riemann surfaces, Princeton (1954).

[24] Singerman D. Automorphisms of compact non-orientable Riemann surfaces, Glasgow Math. J. Vol. 12, Part 1 (1971) 50 –59.

[25] Singerman D. On non-Euclidean crystallographic groups, Proc. of the Camb, Phil. Soc. 76 (1974) 233 –240.

[26] Singerman D. Symmetries of Riemann surfaces with large automorphism group, Math. Ann. 210 (1974) 17 – 32.

[27] Springer G. Introduction to Riemann surfaces, Addison-Wesley (1957).

[28] Wilkie H.C. On non-Euclidean crystallographic groups, Math. Zeit. 91 (1966) 87 – 102.