GRAPH TOPOLOGIES, GAP METRICS AND ROBUST STABILITY FOR NONLINEAR SYSTEMS

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Abstract. Graph topologies for nonlinear operators which admit coprime factorisations are defined w.r.t. a gain function notion of stability in a general normed signal space setting. Several metrics are also defined and their relationship to the graph topologies are examined. In particular, relationships between nonlinear generalisations of the gap and graph metrics, Georgiou-type formulae and the graph topologies are established. Closed loop robustness results are given w.r.t. the graph topology, where the role of a coercivity condition on the nominal plant is emphasised.

Key words. gap metric, graph metric, graph topology, robust stability, nonlinear systems

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1. Introduction.

The theory of coprime factorisations of linear signal operators is well known to be a significant tool in the study of robustness of stability for linear feedback systems and has been extensively studied (see [5, 16, 20]). Perturbations to normalized co-prime factors form a good description of physically realistic deviations from nominal models, since they allow a unified treatment of both low and high frequency uncertainties [8]. In the linear theory, it is well known that the graph topology is the appropriate topological description for studying robustness of stability and that co-prime factor perturbations can be used to induce the graph topology. Furthermore, the graph topology is metrizable, and both the gap metric [3, 21] and the graph metric [20] provide suitable metrizations, the former being more suitable for calculations by standard H^∞ optimizations, (although both metrics are topologically equivalent) [5, 16, 21]. There is thus a rich set of equivalences between the notions of co-prime factorisations, gap/graph metrics and topologies and their attendant robust stability theorems. Moreover, this framework is a cornerstone of modern robust linear control theory.

Given the richness and importance of this framework in the linear setting, it is natural to seek extensions to the nonlinear case, and to alternative signal spaces. Indeed, by adopting a notion of stability corresponding to the existence of a linear gain, (typically either in an L^2 or L^∞ setting), a number of authors have previously considered a nonlinear theory of co-prime factorisation. Here we highlight three contributions of particular relevance to the context of this paper. In [18], Verma defined a notion of co-prime factorisation for nonlinear mappings and, presented a stability result for a nonlinear system. In [2], Anderson, James and Limebeer generalised the linear theory of normalized co-prime factor robustness optimisation to the case of affine input nonlinear systems and presented a optimal robustness margin. In [10], a new definition of "normalized" was introduced for left representation for the graph of a nonlinear system and different gap metrics were studied. Many further pointers to a growing literature on nonlinear co-prime factorisation can be found in the monograph [14] and the references therein.

On the other hand, the gap metric has also been generalised into a nonlinear setting in a fundamental contribution by Georgiou and Smith, [7]. In further recent papers [1], [22], [10], generalisations of Vinnicombe's ν-gap metric [21] to nonlinear

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operators have been considered (for linear systems the ν gap is always smaller than the gap, and has sharper properties, however, the nonlinear theory does not as yet reflect these extra properties).

The purpose of this paper is three-fold:

1. To further extend the existing robust stability theory by replacing the restrictive requirement of the existence of an induced gain by the weaker requirement of the existence of a gain function, see also [7].
2. To provide the topological descriptions underpinning the convergence and robust stability notions in both the case of linear gain and gain function stability; in particular to provide a topological characterisation of nonlinear gap topologies in terms of co-prime factor perturbations.
3. To establish links between the nonlinear graph topologies, the recent results on the nonlinear gap metric [7] and other metrizations (eg. graph metrics [20] and Georgiou-type formulae [4]).

In the context of the first and second items, directly related work of which the authors are aware can be found in [17], where a sufficient condition for the existence of coprime factorisation of nonlinear mappings was given in the sense of IOS; we will show much more of the theory for linear gains can be extended to this more general setting. In [7], robustness of stability results were given in a gain function setting using a generalisation of the gap metric. Interestingly whilst the results in the gain function setting given in [7] implicitly define a notion of plant convergence, the underlying topology is not explicit. In particular, in contrast to the case of the linear gain, a metric was not defined, hence a topology cannot be automatically induced. One contribution of this paper is to provide the underlying topology, and to provide explicit metrizations. In the case of stability of nonlinear operators defined via a linear gain, we show that the graph metric naturally generalizes and induces the graph topology. In the more general case of gain function stability, we only show that the gap topology is stronger than the (weighted) graph topology. The converse relationship remains open. However, we do establish many other relationships and equivalences between a variety of gap and graph metrics and topologies.

An outline of this paper is as follows. Section 2 is devoted to the preliminaries, in particular known results on coprime factorisation for nonlinear systems are briefly reviewed. The main results are arranged in three sections. In Section 3, we define *pointwise* and *weighted* graph topologies and study the associated convergence over a general subset of signal operators admitting coprime factorisations. In Section 4, we study the metrizability of the weighted graph topology. Seven gap metrics are considered. Equivalences and other relationships between the metrics and their associated topologies (including equivalence to the weighted graph topology) are presented. Finally in Section 5, we apply the graph topologies to study the robust stability of nonlinear feedback systems. A summary and discussion of future work is given in Section 6.

2. Background on Coprime Factorisation. The material in this section is mostly directly based on (and straightforward generalisations of) work of previous authors, [7, 17, 18, 19]. However, we need to present this material within the language of this paper and for completeness.

We let $\mathcal{U}, \mathcal{Y}$ be two signal spaces respectively representing the input and output signal spaces. These could be the spaces $L^\infty_n := L^\infty(\mathbb{R}_+, \mathbb{R}^n)$, $L^{\infty,e}_n$, L^p_n, $L^{p,e}_n$, l^p or even a general set on which a truncation can be defined and for which any truncated domain is a normed linear space and $\sup_{\tau>0} \|T_\tau x\| < \infty$ implies $x \in \mathcal{U}_s$. In particular, for one-dimensional continuous domains, we define the truncation operator and the truncated norm for a signal u, say $u \in L_n^{\infty,e}$, by

$$(T_\tau u)(t) = \begin{cases} u(t), & t \leq \tau \\ 0, & t > \tau \end{cases}, \quad \|u\|_\tau = \|T_\tau u\|.$$

where norm of a normed space X is denoted by $\|\cdot\|_X$ or $\|\cdot\|$ if the usage is unambiguous. Note however, that the notion of truncation and all the material in this paper equally applies to signal spaces with discrete domains, eg. $L^\infty(\mathbb{Z}_+, \mathbb{R}^n)$, and to multidimensional domains, eg. $L^\infty(\mathbb{R}_+^m, \mathbb{R}^n)$, under a suitably modified notion of truncation. Let $\mathcal{U}_s, \mathcal{Y}_s$ be the auxiliary normed subspaces which consist of all bounded signals in $\mathcal{U}, \mathcal{Y}$, respectively. In the case where $\mathcal{U}$ (resp. $\mathcal{Y}$) is a normed space, $\mathcal{U}_s = \mathcal{U}$ (resp. $\mathcal{Y}_s = \mathcal{Y}$). Typically $\mathcal{U}, \mathcal{Y}$ are taken to be extended spaces (eg. $L_n^{\infty,e}$), and $\mathcal{U}_s, \mathcal{Y}_s$ are their non-extended subspaces (eg. L_n^∞).

The identity operator on any space Y is denoted by I_Y or I if the usage is clear. Given a matrix operator (A, B), let $(A, B)^\top$ be its transpose, that is $(A, B)^\top = \begin{pmatrix} A \\ B \end{pmatrix}$. We also let $\mathcal{K}_\infty$ denote the set of functions $\omega : [0, \infty) \to [0, \infty)$ which are continuous, strictly increasing and $\omega(0) = 0, \omega(\infty) = \infty$.

Any signal operator $P\colon \operatorname{Dom}(P) \to \mathcal{Y}$ is assumed to be causal and its domain is denoted by

$$\operatorname{Dom}(P) = \{u \in \mathcal{U}_s : Pu \in \mathcal{Y}_s\}.$$

It is worthwhile to observe that unstable plant operators $\hat{P}$ are often thought of as operators $\mathcal{U} \to \mathcal{Y}$ for suitably large signal spaces $\mathcal{U}$, $\mathcal{Y}$. We will only have need to be interested in the relation between elements in $\operatorname{Dom}(P)$ and $\mathcal{Y}_s$ so do not consider the definition of P on the wider signal spaces. However, it should be noted that under extra assumptions such as causal extendibility [6] and for appropriate choices of signal space, the operator $P\colon \operatorname{Dom}(P) \to \mathcal{Y}_s$ uniquely extends to an operator $\hat{P}\colon \mathcal{U} \to \mathcal{Y}$, hence the topologies we will define on sets of operators $P\colon \operatorname{Dom}(P) \to \mathcal{Y}_s$ can be thought of as topologies on sets of operators $\hat{P}\colon \mathcal{U} \to \mathcal{Y}$.

Linear gains of operators $P\colon \operatorname{Dom}(P) \to \mathcal{Y}$ are defined by:

$$\|P\| := \sup\left\{ \frac{\|Pu\|}{\|u\|} : u \in \operatorname{Dom}(P) \text{ with } \|u\| \neq 0 \right\}.$$

If P is causal, one can prove that

$$\|P\| = \sup\left\{ \frac{\|Pu\|_\tau}{\|u\|_\tau} : \tau > 0, u \in \operatorname{Dom}(P) \text{ with } \|u\|_\tau \neq 0 \right\}$$

which is used in [7] as the definition of linear gain. When P is a linear operator, $\|P\|$ is the induced operator norm of P. In the nonlinear setting, in contrast to linear systems, it is often the case that $\operatorname{Dom}(P) = \mathcal{U}_s$ and yet no linear gain exists. Therefore a weaker notion of stability is adopted, namely that of the existence of a gain function. The gain function of an operator P is defined by:

$$\gamma(P)(r) := \sup\Big\{ \|Pu\|_\tau : \tau > 0, u \in \operatorname{Dom}(P) \text{ with } \|u\|_\tau \leq r \Big\} \quad \text{for} \quad r \geq 0.$$

In the case where P is causal, we also have

$$\gamma(P)(r) = \sup\Big\{ \|Pu\| : u \in \operatorname{Dom}(P), \|u\| \leq r \Big\} \quad \text{for} \quad r \geq 0.$$

We summarize elementary properties of the linear gain $\|P\|$ and the gain function $\gamma(P)$ in the following lemma:

LEMMA 2.1. *The linear gain and gain function have the following properties:*

1. $\gamma(P)(0) = 0$ *if* $P(0) = 0$;
2. $\gamma(P)(r_1) \leq \gamma(P)(r_2)$ *if* $r_1 \leq r_2$;
3. *For any two well-defined operators* P_1, P_2 *and any* $r > 0$, $\lambda \in \mathbb{R}$, *we have*

$$\begin{aligned}
&\|\lambda P_1\| = 0 \iff P_1 = 0, && \gamma(P_1) = 0 \iff P_1 = 0,\\
&\|\lambda P_1\| \leq |\lambda|\|P_1\|, && \gamma(\lambda P_1)(r) \leq |\lambda|\gamma(P_1)(r),\\
&\|P_1 + P_2\| \leq \|P_1\| + \|P_2\|, && \gamma(P_1 + P_2)(r) \leq \gamma(P_1)(r) + \gamma(P_2)(r),\\
&\|P_1 P_2\| \leq \|P_1\|\|P_2\|, && \gamma(P_1 P_2)(r) \leq \gamma(P_1)\big(\gamma(P_2)(r)\big).
\end{aligned}$$

4. $\gamma(P)(r) \leq r\|P\|$ *for all* $r > 0$. *In particular, if* P *is linear and bounded, then* $\gamma(P)(r) = r\|P\|$.

DEFINITION 2.2. *A signal operator* P *is said to be: (i) gain stable if* $\|P\| < \infty$; *(ii)* (gf)*-stable if* $\gamma(P)(r) < \infty$ *for each* $r \geq 0$.

We remark that gain stability implies (gf)-stability and both imply $P(\mathrm{Dom}((P)) \subset \mathcal{Y}_s$. In fact, a stable operator P maps bounded subsets of $\mathcal{U}_s$ into bounded subsets of $\mathcal{Y}_s$ (compare to [19]).As a shorthand, in the rest of this paper, a stable operator is taken to mean that the operator is stable in the sense of (gf)-stability unless specified otherwise.

DEFINITION 2.3. *A causal operator* $P : \mathrm{Dom}(P) \subset \mathcal{U}_s \to \mathcal{Y}$ *is said to admit a (right) coprime factorisation if and only if there exist causal stable operators* $N : \mathcal{U}_s \to \mathcal{Y}_s$ *and* $D : \mathcal{U}_s \to \mathcal{U}_s$ *such that*

i) D *is causally invertible with* $\mathrm{Dom}(D^{-1}) = \mathrm{Dom}(P)$;

ii) $P = ND^{-1}$;

iii) There exists a causal stable mapping $L : \mathcal{U}_s \times \mathcal{Y}_s \to \mathcal{U}_s$ *such that* $L(D, N)^\top = I$.

In that case, we also say that P *admits the coprime factorisation* (N, D) *and we write* $P = ND^{-1}$. *For convenience, we call* L *the associated operator to this coprime factorisation. The set of all coprime factorisations of* P *is denoted by* $\mathrm{rcf}(P)$.

In this definition, and henceforth, an operator $D : \mathcal{U}_s \to \mathcal{U}_s$ is said to be invertible with inverse D^{-1} if $D^{-1} : \mathrm{Dom}(D^{-1}) \subset \mathcal{U}_s \to \mathcal{U}_s$ is a well-defined operator and $DD^{-1}|_{\mathrm{Dom}(D^{-1})} = I, D^{-1}D|_{\mathcal{U}_s} = I$. Equivalently, D is required to be both left and right invertible.

DEFINITION 2.4. *Suppose* (N, D) *is a coprime factorisation of* P. *If*

$$\|(D, N)^\top u\| = \|u\| \text{ for all } u \in \mathcal{U},$$

we say that (N, D) *is a normalized right coprime factorisation of* P. *The set of all normalized right coprime factorisations is denoted by* $\mathrm{nrcf}(P)$.

Definitions 2.3 and 2.4 are generalizations of the coprime factorisation and normalized coprime factorisation for linear operators (see [20]) to the nonlinear case, as considered previously by various authors. Definition 2.3 is given by Verma and Hunt in [19] (see also [12, 18]) where the stability is in the sense of "bounded input implies bounded output" (resp. linear gain) between normed spaces. Sontag [17] also defined the concept in which L is required to be of the form (B, A) with $A : \mathcal{Y} \to \mathcal{U}, B : \mathcal{U} \to \mathcal{U}$. Others using this Bezout identity to define coprime factorisations for nonlinear systems include Hammer [9], James, Smith and Vinnicombe [10], Moore and Irlichet [11] etc. Whilst the Bezout identity $BN + AD = I$ always appears in the linear case, the more general form of L is less restrictive in the nonlinear setting. Generalizations of normalized coprime factorisation, including those for specific signal operators, can be found in [2, 10, 13, 14, 15] and the references therein.

Existence and construction of (normalized) coprime factorisations for certain classes of nonlinear systems have been considered previously. For example, in [2, 10, 13, 15], normalized coprime factorisations for stabilizable nonlinear affine systems

$$x' = f(x) + g(x)u, \quad y = C(x)$$

were constructed; Sontag [17] proved that, in the sense of IOS, if the above system with $C = I$ is smoothly input to state stabilizable by a controller of the form $u = k(x) + v$, then its input to state mapping $P : u \mapsto x$ admits a coprime factorisation with $L = (I, A)$, where $-A$ is the memoryless operator induced by the smooth state feedback controller $u = k(x)$, N is the input to state mapping $v \mapsto x$ of the closed-loop system and $D = I - AN$. Similar existence results were obtained by Verma and Hunt [19], in the sense of (gf)-stability, for the causal I/O mapping of the system

$$\begin{aligned} &x'(t) = f(x(t), u(t), t), \quad x(0) = x_0, \\ &y(t) = h(x(t), u(t), t) \end{aligned}$$

in the case when $\mathcal{U}, \mathcal{Y}$ are L^p spaces. Other references to the state space construction of co-prime factors can be found in [14] and the references therein.

The following two results can be found in [18] where the notion of stability is in the sense of a finite linear gain. However, the proofs remain valid in the context of (gf)-stability, hence we omit the proofs.

Proposition 2.5. *Suppose P admit coprime right factorisation (N, D). Then*

$$\mathrm{Graph}(P) := \{(u, Pu)^\top : u \in \mathrm{Dom}(P)\} = \{(Du, Nu)^\top : u \in \mathcal{U}_s\}.$$

Proof. See [18]. □

Proposition 2.6. *$(N, D), (N_1, D_1) \in \mathrm{rcf}(P)$ if and only if there exists an causally stable operator U on $\mathcal{U}_s$, where U^{-1} exists and is stable and is such that $N = N_1 U, D = D_1 U$.*

Proof. See [18]. □

If the coprime factorisations in Proposition 2.6 are also normalized, then we also have:

Proposition 2.7. *If $(N, D), (N_1, D_1) \in nrcf$, then the operator U in Proposition 2.6 is such that $\|Uu\| = \|U^{-1}u\| = \|u\|$ for all $u \in \mathcal{U}_s$.*

Proof. Let $u \in \mathcal{U}_s$. By Proposition 2.6 and the definition of normalized coprime factorisation, we see $\|U^{-1}u\| = \|(D, N)^\top U^{-1}u\| = \|(D_1, N_1)^\top U U^{-1}u\| = \|(D_1, N_1)^\top u\| = \|u\|$ and, similarly, $\|Uu\| = \|(D_1, N_1)^\top Uu\| = \|(D, N)^\top U^{-1}Uu\| = \|(D, N)^\top u\| = \|u\|$. This proves the proposition. □

3. Graph Topologies. In this section, we will study graph topologies on the set of certain signal operators having coprime factorisations. As in the linear case, we will show that the graph topologies play a natural role in the theory of closed loop robust stability.

In practice, the signal spaces, operators and associated coprime factorisations concerned are constrained to lie within certain classes for different control problems. For example, one may only interested in the case when $\mathcal{U} = L_n^{\infty,e}, \mathcal{Y} = L_m^{\infty,e}$ and where all operators considered lie in the subset of i/o operators of all affine (nonlinear) systems.

Here we list some particular categories that will be considered in this paper.

- Category *nor*: $\mathcal{U}, \mathcal{Y}$ are general signal spaces as assumed and all operators considered are those that admits normalized coprime factorisations in the sense defined in the last section.
- Category ω with $\omega \in \mathcal{K}_\infty$: $\mathcal{U}, \mathcal{Y}$ are general signal spaces and all operators F considered are such that $\mathrm{rcf}(F) \neq \emptyset$ and $\sup_{r>0} \frac{\gamma((N,D)^\top)(\omega(r))}{r} < \infty$ for all $(N, D) \in \mathrm{rcf}(F)$.
- Category $\mathbf{L}$: $\mathcal{U}, \mathcal{Y}$ are both the frequency domain Hardy spaces $\mathcal{H}_2$ (see [20]), operators are the real rational $p \times q$ transfer function matrices and the associated coprime factorisations are those linear factorisations over $\mathcal{RH}_\infty$[1] as in [20] or [21].

Since $F \equiv 0$ has normalized coprime factorisation $(0, I)$, we see that each category is non-empty.

The graph topologies will be defined on a general category although in the next section we mainly consider $\mathbf{N}_{nor}(\mathcal{U}, \mathcal{Y})$ and $\mathbf{N}_\omega(\mathcal{U}, \mathcal{Y})$. So we use Γ to represent the category concerned and write

$$\mathbf{N}_\Gamma(\mathcal{U}, \mathcal{Y}) := \left\{ P : \mathrm{Dom}(P) \subset \mathcal{U} \to \mathcal{Y} : \begin{array}{l} P \text{ and the associated rcf's} \\ N, D \text{ are with in category } \Gamma \end{array} \right\}.$$

Correspondingly, we have notations $\mathbf{N}_{nor}(\mathcal{U}, \mathcal{Y}), \mathbf{N}_\omega(\mathcal{U}, \mathcal{Y})$ and $\mathbf{N}_\mathbf{L}(\mathcal{H}_2, \mathcal{H}_2) =: \mathbf{N}_\mathbf{L}$. For example

$$\mathbf{N}_{nor}(\mathcal{U}, \mathcal{Y}) := \{P : \mathrm{Dom}(P) \subset \mathcal{U} \to \mathcal{Y} \text{ and } \mathrm{nrcf}(P) \neq \emptyset\}.$$

$$\mathbf{N}_\omega(\mathcal{U}, \mathcal{Y}) := \left\{ P : \mathrm{Dom}(P) \subset \mathcal{U} \to \mathcal{Y} : \begin{array}{l} \mathrm{rcf}(P) \neq \emptyset \text{ and for all} (N, D) \in \mathrm{rcf}(P), \\ \sup\limits_{r>0} \dfrac{\gamma((N,D)^\top)(\omega(r))}{r} < \infty \end{array} \right\}.$$

A graph topology for $\mathbf{N}_\mathbf{L}$, denoted by $\mathcal{T}_\mathbf{L}$, has been defined in [20] by the following local base for $P \in \mathcal{RH}_\infty$:

$$\mathcal{N}(N, D; \varepsilon) = \{N_1 D_1^{-1} : \|(N_1 - N, D_1 - D)^\top\|_\infty < \varepsilon, N, D, N_1, D_1 \in \mathcal{RH}_\infty\} \quad (3.1)$$

with $\varepsilon > 0, (N, D) \in \mathrm{rcf}(P), (N_1, D_1) \in \mathrm{rcf}(N_1 D_1^{-1})$, respectively. We will show that our L^2 topologies for $\mathbf{N}_\mathbf{L}$ are the same as $\mathcal{T}_\mathbf{L}$.

For notational ease in the sequel, any pairs N, D or N_k, D_k are always assumed to be coprime factorisations of $P = ND^{-1}$ and $P_k = N_k D_k^{-1}$ respectively, and P and P_k are taken to be well-defined operators from $\mathrm{Dom}(P) \to \mathcal{Y}_s$, $\mathrm{Dom}(P_k) \to \mathcal{Y}_s$ respectively.

3.1. Pointwise Graph Topology. Let $\Re$ be the vector space of all functions from $\mathbb{R}^+$ to $\mathbb{R}$. For any open subset $\Omega \subset \mathbb{R}$ and a finite subset $\{t_1, \cdots, t_n\} \subset \mathbb{R}^+$, let

$$\mathcal{V}(t_1, \cdots, t_n; \Omega) = \{f \in \Re : f(t_i) \in \Omega\}.$$

[1] $\mathcal{H}_2$ is the space of Fourier transforms of signals in $L^2(\mathbb{R}_+, \mathbb{R}^n)$ endowed with the norm $\|x\|_2^2 := \frac{1}{2\pi}\int_{-\infty}^{\infty} x^*(j\omega)x(j\omega)d\omega$. By Parseval's Theorem, it is isometrically isomorphic to $L^2(\mathbb{R}_+, \mathbb{R}^n)$ and, therefore, the two notations are not distinguished. $\mathcal{RH}_\infty$ is the space of rational transfer functions of stable linear, time-invariant, continuous time systems endowed with the norm $\|P\|_\infty := \sup_{\omega\in\mathbb{R}} \bar{\sigma}[P(j\omega)]$, where $\bar{\sigma}$ denotes the maximum singular value. Equivalently, $\|P\|_\infty := \sup\{\|Pu\|_{\mathcal{H}_2}/\|u\|_{\mathcal{H}_2} : u \in \mathcal{H}_2, u \neq 0\}$. So by Paserval's Theorem, the H_∞-norm in the frequency domain corresponds to the induced L^2 norm in the time domain.

It can be proved that $\{\mathcal{V}(t_1,\cdots,t_n;\Omega) : t_i \in \mathbb{R}^+, n>0, \Omega\subset\mathbb{R}, \Omega \ \text{ open}\}$ forms a subbase for a topology on $\Re$. Moreover, the family of subsets

$$\Re_0 = \{\mathcal{V}(t_1,\cdots,t_n;\varepsilon) := \mathcal{V}(t_1,\cdots,t_n;(-\varepsilon,\varepsilon)) : \varepsilon>0, t_i\in\mathbb{R}^+, n>0\}$$

is a base for the neighborhood of $f(t)\equiv 0$ in $\Re$ under such a topology.

For each $P\in \mathbf{N}_\Gamma(\mathcal{U},\mathcal{Y})$ with coprime factorisation (N,D) and each $V\in\Re_0$, we define

$$O(N,D;V) = \left\{P_1 = N_1D_1^{-1}\in \mathbf{N}_\Gamma(\mathcal{U},\mathcal{Y}) : \gamma((D-D_1,N-N_1)^\top)\in V\right\}.$$

Obviously, $P = ND^{-1}\in O(N,D;V)$ for each $V\in\Re_0$. Moreover, we have the following result:

Proposition 3.1. *If $ND^{-1}\in O(N_1,D_1;V_1)\cap O(N_2,D_2;V_2)$, then there exist $V\in\Re_0$ such that $O(N,D;V)\subset O(N_1,V_1)\cap O(N_2,D_2;V2)$.*

Proof. We may suppose $V_1 = \mathcal{V}(t_1,\cdots,t_n;\varepsilon_1), V_2=\mathcal{V}(s_1,\cdots,s_m;\varepsilon_2)$ with $t_i>0$, $s_j>0$, $\varepsilon_k>0$ $(i=1,\cdots,n, j=1,\cdots,m, k=1,2)$. Then $\varepsilon_1-\gamma((D-D_1,N-N_1)^\top)(t_i)$ and $\varepsilon_2-\gamma((D-D_2,N-N_2)^\top)(s_j)$ are all positive numbers. Let $\varepsilon>0$ such that

$$\varepsilon < \min\left\{\begin{array}{ll}\varepsilon_1-\gamma((D-D_1,N-N_1)^\top)(t_i), & i=1,\cdots,n,\\ \varepsilon_2-\gamma((D-D_2,N-N_2)^\top)(s_j), & j=1,\cdots,m.\end{array}\right\}$$

If $\tilde N\tilde D^{-1}\in O(N,D;V)$ with $V=\mathcal{V}(t_1,\cdots,t_n,s_1,\cdots,s_m;\varepsilon)$, then

$$\begin{aligned}\gamma((\tilde D-D_1,&\tilde N-N_1)^\top)(t_i)\le \gamma((\tilde D-D,\tilde N-N)^\top)(t_i)+\gamma((D-D_1,N-N_1)^\top)(t_i)\\ &<\varepsilon_1-\gamma((D-D_1,N-N_1)^\top)(t_i)+\gamma((D-D_1,N-N_1)^\top)(t_i)=\varepsilon_1\end{aligned}$$

for all $i=1,\cdots,n$. This gives $\tilde N\tilde D^{-1}\in O(N_1,D_1;V_1)$. Similarly, we can show $\tilde N\tilde D^{-1}\in O(N_2,D_2;V_2)$. Therefore, $\tilde N\tilde D^{-1}\in O(N_1,D_1;V_1)\cap O(N_2,D_2;V_2)$ which means $O(N,D;V)\subset O(N_1,D_1;V_1)\cap O(N_2,D_2;V_2)$. □

¿From the above result, it follows that a topology on $\mathbf{N}_\Gamma(\mathcal{U},\mathcal{Y})$ can be uniquely determined by the base $\mathbb{B}$, where

$$\mathbb{B} = \{O(N,D;V) : ND^{-1}\in\mathbf{N}_\Gamma(\mathcal{U},\mathcal{Y}), V\in\Re_0\},$$

and $\{O(N,D;V) : ND^{-1}=P, V\in\Re_0\}$ a local base of P. We denote this topology by $\mathcal{T}$ and call it the *pointwise (graph) topology* (see the preceding footnote). The following proposition provides alternative base for this topology.

Proposition 3.2. *Let Q_+ be the set of all positive rational numbers and*

$$O'(N,D;r,\varepsilon) = \left\{N_1D_1^{-1}\in\mathbf{N}_\Gamma(\mathcal{U},\mathcal{Y}) : \gamma((D-D_1,N-N_1)^\top)(r)<\varepsilon\right\}.$$

Then a base for the pointwise graph topology $\mathcal{T}$ is the family of subsets:

$$\mathbb{B}' = \{O'(N,D;r,\varepsilon) : ND^{-1}\in\mathbf{N}_\Gamma(\mathcal{U},\mathcal{Y}), r,\varepsilon\in Q_+\}.$$

Proof. Obviously $\mathbb{B}'\subset\mathbb{B}$. Suppose $O(N,D;V)\in\mathbb{B}$ with $V=\mathcal{V}(t_1,\cdots,t_n;\varepsilon)$. Let $r,\varepsilon_1\in Q$ such that $r>\max\{t_1,\cdots,t_n)$ and $\varepsilon_1<\varepsilon$. Then for each $N_1D_1^{-1}\in O'(N,D;r,\varepsilon_1)$, from Lemma 2.1 2), it follows

$$\gamma((D_1-D,N_1-N)^\top)(t_i)\le\gamma((D_1-D,N_1-N)^\top)(t)<\varepsilon_1<\varepsilon$$

for all $i = 1, \cdots, n$. This means $N_1 D_1^{-1} \in O(N, D; V)$ and, therefore, $O'(N, D; r, \varepsilon) \subset O(N, D; V)$. Hence $\mathbb{B}$ and $\mathbb{B}'$ are equivalent. □

If we restrict our consideration to $\mathbf{N_L}$, from Lemma 2.1 4) and (3.1), we see

$$\begin{aligned} O'(N, D; r, \varepsilon) &= \{N_1 D_1^{-1} : \gamma((D - D_1, N - N_1)^\top)(r) < \varepsilon, N, D, N_1, D_1 \in \mathcal{RH}_\infty\} \\ &= \left\{N_1 D_1^{-1} : \|(D - D_1, N - N_1)^\top\| < \frac{\varepsilon}{r} =: \varepsilon_1, N, D, N_1, D_1 \in \mathcal{RH}_\infty\right\} \\ &= \mathcal{N}(N, D; \varepsilon_1), \end{aligned}$$

hence we have

Corollary 3.3. *The pointwise graph topology $\mathcal{T}$ in the category $\mathbf{L}$ is the same as the graph topology $\mathcal{T}_\mathbf{L}$.*

Now we begin to consider the convergence of sequences under the pointwise topology. The following result shows that any convergent sequence has only one limit.

Proposition 3.4. *The pointwise graph topology $\mathcal{T}$ is Hausdorff. Therefore, the limit point of a convergent sequence is unique.*

Proof. Let $P_1 \neq P_2$ be two distinct plants. Then there exist $(N_1, D_1) \in \mathrm{rcf}(P_1)$, $(N_2, D_2) \in \mathrm{rcf}(P_2)$ with $(D_1, N_1)^\top \neq (D_2, N_2)^\top$. This shows

$$\varepsilon := \gamma((D_1 - D_2, N_1 - N_2)^\top)(r) > 0 \text{ for some } r > 0.$$

Consider the neighbourhoods $O(N_1, D_1; r, \varepsilon/3)$ of P_1 and $O(N_2, D_2; r, \varepsilon/3)$ of P_2. If there exists $P = ND^{-1} \in O(N_1, D_1; r, \varepsilon/3) \cap O(N_2, D_2; r, \varepsilon/3)$, since

$$\gamma((N_1 - N_2, D_1 - D_2)^\top)(r) \leq \gamma((N_1 - N, D_1 - D)^\top)(r) + \gamma((N_2 - N, D_2 - D)^\top)(r),$$

we see $\gamma((N_1 - N_2, D_1 - D_2)^\top)(r) < \varepsilon$. This is a contradiction. Hence we have that $O(N_1, D_1; r, \varepsilon/3) \cap O(N_2, D_2; r, \varepsilon/3) = \emptyset$ which proves the proposition. □

Suppose $\{P_n\} \subset \mathbf{N}_\Gamma(\mathcal{U}, \mathcal{Y})$ is a sequence. We let $P_n \xrightarrow{\mathcal{T}} P$ denote the convergence of the sequence $\{P_n\}_{n \geq 1}$ to P under the graph topology $\mathcal{T}$. From Proposition 3.2, we see that $P_n \xrightarrow{\mathcal{T}} P$ means that, for any $r > 0, \varepsilon > 0$ and each coprime factorisation ND^{-1} of P, there exist $n_0 > 0$ and coprime factorisation $N_n D_n^{-1}$ of P_n such that $N_n D_n^{-1} \in O(N, D; r, \varepsilon)$ for all $n \geq n_0$. Necessary and sufficient conditions for this convergence are given below.

Theorem 3.5. *The following statements are equivalent.*

i) $P_n \xrightarrow{\mathcal{T}} P$.

ii) For each $(N, D) \in \mathrm{rcf}(P)$*, there exists* $(N_n, D_n) \in \mathrm{rcf}(P_n)$ *such that*

$$\gamma((D_n - D, N_n - N)^\top)(r) \to 0 \quad \textit{for each } r > 0.$$

iii) There exists $(N, D) \in \mathrm{rcf}(P)$ *and, for each* n*, there exists* $(N_n, D_n) \in \mathrm{rcf}(P_n)$ *such that*

$$\gamma((D_n - D, N_n - N)^\top)(r) \to 0 \quad \textit{for all } r > 0.$$

Proof. ii) ⇒ i) and ii) ⇒ iii) are immmediate, we need only to prove i) ⇒ ii) and iii) ⇒ ii).

i) ⇒ ii). Let $r > 0$ and $P = ND^{-1}$ be given. According to the assumptions, for each $\varepsilon > 0$ and $n > 0$, there exists coprime factorisation $N_{n,\varepsilon} D_{n,\varepsilon}^{-1}$ of P_n and

$n_\varepsilon > 0$ such that $N_{n,\varepsilon}D_{n,\varepsilon}^{-1} \in O(N,D;r,\varepsilon)$ for all $n \geq n_\varepsilon$. Let $\varepsilon = 1, 1/2, \cdots, 1/2^k, \cdot$, respectively, to obtain the corresponding integers $n_k := n_{1/2^k}$. Define

$$N_n = N_{n,1/2^k} \quad \text{and} \quad D_n = D_{n,1/2^k} \quad \text{for} \quad n_k \leq n < n_{k+1}.$$

Then $N_nD_n^{-1}$ is a coprime factorisation of P_n and $N_nD_n^{-1} \in O(N,D;r,1/2^k)$ for $n \geq n_k$. Hence $\gamma((D_n - D, N_n - N)^\top)(r) \to 0$.

iii) $\Rightarrow$ ii). Suppose $(\tilde{N}, \tilde{D})$ is an arbitrary coprime factorisation of P. Then by Proposition 2.6, there exists stable operator U with $\tilde{N} = NU, \tilde{D} = DU$. Moreover, $(\tilde{N}_n, \tilde{D}_n) := (N_nU, D_nU)$ is a coprime factorisation of P_n due to the same proposition. Using Lemma 2.1, we have

$$\gamma((\tilde{D}_n - \tilde{D}, \tilde{N}_n - \tilde{N})^\top)(r) \leq \gamma((D_n - D, N_n - N)^\top)(\gamma(U)(r)).$$

The stability of U and the assumption ensure $\gamma((\tilde{D}_n - \tilde{D}, \tilde{N}_n - \tilde{N})^\top)(r) \to 0$ as $n \to \infty$ and, therefore, ii) has been established. This completes the proof. □

Because the continuity of a mapping from a first-countable topological space to another topological space can be described by the convergence of sequences, we have shown:

Corollary 3.6. *Let Λ be a first-countable topological space, $P_\lambda : \Lambda \to \mathbf{N}_\Gamma(\mathcal{U}, \mathcal{Y})$. Then $\lambda \mapsto P_\lambda$ is continuous at $\lambda = \lambda_0$ under the pointwise graph topology $\mathcal{T}$ if and only if there exist coprime factorisations $P_{\lambda_0} = N_0D_0^{-1}$ and $P_\lambda = N_\lambda D_\lambda^{-1}$ for each $\lambda \in \Lambda$ such that*

$$\gamma((D_0 - D_\lambda, N_0 - N_\lambda)^\top)(r) \to 0 \text{ for all } r \geq 0 \quad \text{as } \lambda \to \lambda_0.$$

3.2. Weighted Graph Topology. In this section, we consider another topology on the set $\mathbf{N}_\Gamma(\mathcal{U}, \mathcal{Y})$, which will be related to a given function $\omega \in \mathcal{K}_\infty$ and a weighted gain $\|\cdot\|_\omega$ defined by

$$\|\mathbf{P}\|_\omega = \sup_{r>0} \frac{\gamma(\mathbf{P})(\omega(r))}{r} \quad \text{for any signal operator } \mathbf{P}.$$

It is straightforward to prove that $\|\cdot\|_\omega$ is a norm. Moreover, if $\omega(r) \geq c_1 r$ with $c_1 > 0$ for all $r > 0$, then $\|\mathbf{P}\|_\omega \geq c_1\|\mathbf{P}\|$; If $\mathbf{P}0 = 0, c_2 > 0$ and $\omega(r) \leq c_2 r$ for all $r > 0$, then $\|\mathbf{P}\|_\omega \leq c_2\|\mathbf{P}\|$.

Let

$$\Sigma = \{\mathbf{P} : \mathcal{U} \to \mathcal{U} \times \mathcal{Y} \quad \text{with } \|\mathbf{P}\|_\omega < \infty\},$$

It can be seen from the basic properties of γ that Σ is a linear space and, therefore, $(\Sigma, \|\cdot\|_\omega)$ is a normed space. The norm induces a corresponding topology on Σ, of which a local base of open ball neighbourhoods of $\mathbf{P} \equiv 0$ is denoted by $\mathcal{B}$.

For each $P \in \mathbf{N}_\Gamma(\mathcal{U}, \mathcal{Y})$ with coprime factorisation $P = ND^{-1}$ and each $V \in \mathcal{B}$, we denote by

$$O_\omega(N,D;V) = \left\{N_1D_1^{-1} \in \mathbf{N}_\Gamma(\mathcal{U}, \mathcal{Y}) : (D - D_1, N - N_1)^\top \in V\right\}.$$

Obviously, $P = ND^{-1} \in O_\omega(N,D;V)$ for each $V \in \mathcal{B}$. Moreover, we have

Proposition 3.7. *If $ND^{-1} \in O_\omega(N_1,D_1;V_1) \cap O_\omega(N_2,D_2;V_2)$, then there exist $V \in \mathcal{B}$ such that $O_\omega(N,D;V) \subset O_\omega(N_1,V_1) \cap O_\omega(N_2,D_2;V_2)$.*

Proof. We may suppose that $V_i = \{\mathbf{P} \in \Sigma : \|\mathbf{P}\|_\omega < \varepsilon_i\}$ with $\varepsilon_i > 0,\ i = 1,\ 2$. Let

$$\alpha_i = \sup_{r>0} \frac{\gamma((D - D_i, N - N_i)^\top)(\omega(r))}{r},\ i = 1, 2$$

and let ε be a positive number such that $\varepsilon < \min\{\varepsilon_1 - \alpha_1, \varepsilon_2 - \alpha_2\}$. Then for each $\tilde{N}\tilde{D}^{-1} \in O_\omega(N, D; \varepsilon)$ and each $r > 0$, from the third property of γ, it follows

$$\frac{\gamma((\tilde{D} - D_i, \tilde{N} - N_i)^\top)(\omega(r))}{r} \leq \frac{\gamma((\tilde{D} - D, \tilde{N} - N)^\top)(\omega(r))}{r} + \frac{\gamma((D - D_i, N - N_i)^\top)(\omega(r))}{r} < \varepsilon + \alpha_i, \quad i = 1, 2.$$

Hence

$$\sup_{r>0} \frac{\gamma((\tilde{D} - D_i, \tilde{N} - N_i)^\top)(\omega(r))}{r} \leq \varepsilon + \alpha_i < \varepsilon_i - \alpha_i + \alpha_i = \varepsilon_i, \quad i = 1, 2.$$

This implies $\tilde{N}\tilde{D}^{-1} \in O_\omega(N_1, D_1; \varepsilon_1) \cap O_\omega(N_2, D_2; \varepsilon_2)$ and, therefore, $O_\omega(N, D; \varepsilon) \subset O_\omega(N_1, D_1; \varepsilon_1) \cap O_\omega(N_2, D_2; \varepsilon_2)$. □

Let

$$\mathbb{B}_\omega = \{O_\omega(N, D; V) : ND^{-1} \in \mathbf{N}_\Gamma(\mathcal{U}, \mathcal{Y}), V \in \mathcal{B}\}.$$

From the above result, it follows that a topology on $\mathbf{N}_\Gamma(\mathcal{U}, \mathcal{Y})$ can be uniquely determined with $\mathbb{B}_\omega$ its base. We denote this topology by $\mathcal{T}_\omega$ and call it the *weighted (graph) topology* related to function ω. Obviously, $\mathcal{T}_\omega$has countable local base.

If $\mathbf{P} \in \Sigma$ is linear and $\omega(t) \equiv t$, then from Lemma 2.1 4), we see $\|\mathbf{P}\|_\omega = \|\mathbf{P}\|$. Therefore, if we restrict attention to $\mathbf{N_L}$, then for each $P = ND^{-1} \in \mathcal{RH}_\infty$ and $V = \{\mathbf{P} : \mathcal{H}_2 \to \mathcal{H}_2 \times \mathcal{H}_2,\ \|\mathbf{P}\|_\omega < \varepsilon\}$, we have

$$O_\omega(N, D; V) = \{N_1 D_1^{-1} : \|(N_1 - N, D_1 - D)^\top\| < \varepsilon, N, D, N_1, D_1 \in \mathcal{RH}_\infty\}$$

and $O_\omega(N, D; V) = \mathcal{N}(N, D; \varepsilon)$. This fact yields the following corollary.

Corollary 3.8. *For $\omega(t) \equiv t$ and $\mathbf{N}_\Gamma(\mathcal{U}, \mathcal{Y}) = \mathbf{N_L}$, the weighted graph topology $\mathcal{T}_\omega$ is the same as the graph topology $\mathcal{T}_\mathbf{L}$ defined for $\mathbf{N_L}(\mathcal{U}, \mathcal{Y})$.*

¿From Proposition 3.7, we see that a sequence of operators $\{P_n\}_{n\geq 1}$ converges to P under this graph topology, denoted by $P_n \xrightarrow{\mathcal{T}_\omega} P$, means that, for any $\varepsilon > 0$ and each coprime factorisation ND^{-1} of P, there exist $n_0 > 0$ and coprime factorisation $N_n D_n^{-1}$ of P_n such that $\|(D_n - D, N_n - N)^\top\|_\omega < \varepsilon$ for all $n \geq n_0$.

Using a method similar to the one used in Proposition 3.4, we can also prove that the weighted topology is a Hausdorff topology. So a convergent sequence has unique limit.

Theorem 3.9. *$P_n \xrightarrow{\mathcal{T}_\omega} P$ if and only if for each coprime factorisation ND^{-1} of P, there exists coprime factorisation $N_n D_n^{-1}$ of P_n such that*

$$\sup_{r>0} \frac{\gamma((D_n - D, N_n - N)^\top)(\omega(r))}{r} \to 0.$$

Proof. The proof is omitted for brevity as it follows the same reasoning as used in the first part proof for Theorem 3.5. □

Two immediate corollaries are:

Corollary 3.10. *If $P_n \xrightarrow{\mathcal{T}_\omega} P$, then $P_n \to P$ under the pointwise graph topology $\mathcal{T}$.*

Corollary 3.11. *Let Λ be a first-countable topological space, $P_\lambda : \Lambda \to \mathbf{N}_\Gamma(\mathcal{U},\mathcal{Y})$. Then $\lambda \mapsto P_\lambda$ is continuous at $\lambda = \lambda_0$ under a weighted graph topology $\mathcal{T}_\omega$ if and only if for each coprime factorisations $P_{\lambda_0} = N_0 D_0^{-1}$, there exist coprime factorisation $P_\lambda = N_\lambda D_\lambda^{-1}$ for each $\lambda \in \Lambda$ such that*

$$\gamma((D_0 - D_\lambda, N_0 - N_\lambda)^\top)(r) \to 0 \textit{ for all } r \geq 0 \quad \textit{as } \lambda \to \lambda_0.$$

To conclude this section, we observe that given two functions $\omega_1, \omega_2 \in \mathcal{K}_\infty$, each generates a weighted graph topology $\mathcal{T}_{\omega_1}, \mathcal{T}_{\omega_2}$. If $\omega_1(r) \leq \omega_2(r)$ for all $r \in \mathbb{R}^+$, then $\|\mathbf{P}\|_{\omega_1} \leq \|\mathbf{P}\|_{\omega_2}$ for all $\mathbf{P} : \mathcal{U} \to \mathcal{U} \times \mathcal{Y}$. Therefore

$$O_{\omega_2}(N, D; V) \subset O_{\omega_1}(N, D; V) \quad \text{for all} \quad ND^{-1} \in \mathbf{N}_\Gamma(\mathcal{U},\mathcal{Y}), V \in \mathcal{B}.$$

This implies the following comparison theorem.

Theorem 3.12. *Suppose $\omega_1, \omega_2 \in \mathcal{K}_\infty$ satisfying $\omega_1(r) \leq \omega_2(r)$ for all $r > 0$. Then $\mathcal{T}_{\omega_2}$ is stronger than $\mathcal{T}_{\omega_1}$, (ie. any sequence converging under $\mathcal{T}_{\omega_2}$ will converge under $\mathcal{T}_{\omega_1}$). Additionally, $\mathcal{T}_{c\omega_1}$ and $\mathcal{T}_{\omega_1}$ are equivalent for any $c > 0$ (i.e. they induce the same convergence).*

In particular we have the following corollary:

Corollary 3.13. *The linear gain induces a graph topology (denoted by $\mathcal{T}_{lg}$) on $\mathbf{N}_\Gamma(\mathcal{U},\mathcal{Y})$. If $c_1 r \leq \omega(r) \leq c_2 r$ for all $r \geq 0$, then $\mathcal{T}_\omega$ and $\mathcal{T}_{lg}$ are equivalent.*

Hence it can be seen that the weighted graph topologies inherit the partial order given by the natural partial order on the weights.

4. Metrizability. The question addressed in this section is simply whether the nonlinear graph topologies introduced earlier can be sensibly metrized. In the linear case it is well known that the answer is affirmative. We will show that useful metrics can also be given for the weighted nonlinear graph topology. We will introduce a number of metrics on specific subsets of $\mathbf{N}(\mathcal{U},\mathcal{Y})$ and prove that some of them induce the weighted graph topology.

Throughout this section, we suppose $\omega \in \mathcal{K}_\infty$ is a given function, $\|\cdot\|_\omega$ is the weighted gain and $\mathcal{U}, \mathcal{Y}, \mathcal{U}_s, \mathcal{Y}_s$ are defined as before. Every signal operator P (say) is assumed to be causal and $P(0) = 0$.

4.1. The metric formulas. We define:

$$\begin{aligned}
\mathcal{Q} &= \{Q : \mathcal{U}_s \to \mathcal{U}_s \text{ is stable with } Q^{-1}\text{exist and also stable}\},\\
\mathcal{Q}^* &= \{Q : \mathcal{U}_s \to \mathcal{U}_s \text{ is stable with } Q^{-1} \text{ exist (bijective)}\},\\
\mathcal{Q}^s &= \{Q : \mathcal{U}_s \to \mathcal{U}_s \text{ is stable and surjective}\}.
\end{aligned}$$

The subsets of signal operators we will consider are $\mathbf{N}_\omega(\mathcal{U},\mathcal{Y})$ and $\mathbf{N}_{nor}(\mathcal{U},\mathcal{Y})$ as defined in the last section. Recall that

$$\begin{aligned}
\mathbf{N}_\omega(\mathcal{U},\mathcal{Y}) &= \{P \in \mathbf{N}(\mathcal{U},\mathcal{Y}) : \|(D, N)^\top\|_\omega < \infty \text{ for all } (N, D) \in \text{rcf}(P)\},\\
\mathbf{N}_{nor}(\mathcal{U},\mathcal{Y}) &= \{P \in \mathbf{N}_\omega(\mathcal{U},\mathcal{Y}) : \text{nrcf}(P) \neq \emptyset\}.
\end{aligned}$$

We now define seven functionals over the above sets:

$$d_1(P_1,P_2) = \max\{\vec{d}_1(P_1,P_2), \vec{d}_1(P_2,P_1)\}, \quad \text{for } P_1,P_2 \in \mathbf{N}_\omega(\mathcal{U},\mathcal{Y})$$
$$\text{where } \vec{d}_1(P_1,P_2) := \sup_{(N_1,D_1)\in \mathrm{rcf}(P_1)} \inf_{(N_2,D_2)\in \mathrm{rcf}(P_2)} \|(D_1-D_2, N_1-N_2)^\top\|_\omega;$$
$$d_2(P_1,P_2) = \max\{\vec{d}_2(P_1,P_2), \vec{d}_2(P_2,P_1)\} \quad \text{for } P_1,P_2 \in \mathbf{N}_{nor}(\mathcal{U},\mathcal{Y}),$$
$$\text{where } \vec{d}_2(P_1,P_2) = \inf_{\substack{Q\in\mathcal{Q}\\ \|Q\|\le 1}} \|(D_1-D_2Q, N_1-N_2Q)^\top\|_\omega,\ (N_i,D_i)\in \mathrm{nrcf}(P_i), i=1,2;$$
$$d_3(P_1,P_2) = \max\{\vec{d}_3(P_1,P_2), \vec{d}_3(P_2,P_1)\} \quad \text{for } P_1,P_2 \in \mathbf{N}_{nor}(\mathcal{U},\mathcal{Y}),$$
$$\text{where } \vec{d}_3(P_1,P_2) = \inf_{\substack{Q\in\mathcal{Q}^*\\ \|Q\|\le 1}} \|(D_1-D_2Q, N_1-N_2Q)^\top\|_\omega,\ (N_i,D_i)\in \mathrm{nrcf}(P_i), i=1,2;$$
$$d_4(P_1,P_2) = \log(1+\max\{\vec{d}_4(P_1,P_2), \vec{d}_4(P_2,P_1)\}) \text{ for any } P_1,P_2: \mathcal{U}\to\mathcal{Y}$$
$$\text{where } \vec{d}_4(P_1,P_2) = \begin{cases} \inf\left\{\|I-\Phi\|_\omega : \begin{array}{l}\Phi \text{ is a surjective mapping from}\\ \mathrm{Graph}(P_1) \text{ to } \mathrm{Graph}(P_2), \Phi(0)=0\end{array}\right\}, \\ \infty \text{ if no such operator } \Phi \text{ exists;}\end{cases}$$
$$d_5(P_1,P_2) = \log(1+\max\{\vec{d}_5(P_1,P_2), \vec{d}_5(P_2,P_1)\}) \text{ for any } P_1,P_2: \mathcal{U}\to\mathcal{Y}$$
$$\text{where } \vec{d}_5(P_1,P_2) = \begin{cases} \inf\left\{\|I-\Phi\|_\omega : \begin{array}{l}\Phi \text{ is a bijective mapping from}\\ \mathrm{Graph}(P_1) \text{ to } \mathrm{Graph}(P_2), \Phi(0)=0\end{array}\right\}, \\ \infty \text{ if no such operator } \Phi \text{ exists;}\end{cases}$$
$$d_6(P_1,P_2) = \log(1+\max\{\vec{d}_6(P_1,P_2), \vec{d}_6(P_2,P_1)\}) \quad \text{for } P_1,P_2 \in \mathbf{N}_{nor}(\mathcal{U},\mathcal{Y}),$$
$$\text{where } \vec{d}_6(P_1,P_2) = \inf_{Q\in\mathcal{Q}^*} \|(D_1-D_2Q, N_1-N_2Q)^\top\|_\omega,\ (N_i,D_i)\in \mathrm{nrcf}(P_i), i=1,2;$$
$$d_7(P_1,P_2) = \log(1+\max\{\vec{d}_7(P_1,P_2), \vec{d}_7(P_2,P_1)\}) \quad \text{for } P_1,P_2 \in \mathbf{N}_{nor}(\mathcal{U},\mathcal{Y}),$$
$$\text{where } \vec{d}_7(P_1,P_2) = \inf_{Q\in\mathcal{Q}^s} \|(D_1-D_2Q, N_1-N_2Q)^\top\|_\omega,\ (N_i,D_i)\in \mathrm{nrcf}(P_i), i=1,2.$$

Notice, when ω is the identity, d_3 is closely related to the graph metric studied in [20] for finite dimensional linear systems, whilst d_5 is the gap metric defined in [7] where $\vec{d}_5$ is extensively exploited for the robustness of stability of nonlinear systems. In most cases, $\vec{d}_5$ can be replaced by $\vec{d}_4$ as shown in Lemma 6.1 later. The functionals d_6 and d_7 are closely related to the Georgiou formula for the gap metric [4].

We will prove that $d_1, \cdots, d_7$ are metrics on suitable sets of signal operators and show relations between all seven functionals $d_1, \cdots, d_7$ and their induced topologies.

The results developed in this section are as follows:

1. The weighted topology $\mathcal{T}_\omega$ on $\mathbf{N}_{nor}(\mathcal{U},\mathcal{Y})$ can be metrized by graph metrics d_2, d_3 provided $cr \le \omega(r)$;
2. The weighted graph topology can also be metrized by Georgiou and Smith's gap metrics d_4, d_5 provided $r \le \omega(r)$;
3. The graph metrics $\vec{d}_2, \vec{d}_3$ and gap metric $\vec{d}_5$ are equivalent to each other, therefore, the graph metrics give the rise to the same robust stability margin as the gap metric [7].
4. The gap metrics d_4 and d_5 can be equivalently expressed by the Georgiou-type formulae, d_7 and d_6 respectively.

The following diagrams show the relations among the discussed topologies and (gap) metrics that will be established.

$$d_1 \geq d_6, \quad d_2 \geq d_3 \geq d_6 = d_5 \geq d_4 = d_7$$

Diagram 1 : Metric Relations.

$$\mathcal{T}_{d_1} \overset{\text{P4.1}}{\geq} \mathcal{T}_\omega \overset{\text{T4.4}}{=} \mathcal{T}_{d_2} \overset{\text{T4.7}}{=} \mathcal{T}_{d_3} \overset{\text{T4.4}}{=} \mathcal{T}_{d_5} \overset{\text{P4.6}}{=} \mathcal{T}_{d_6} \overset{\text{P4.6}}{\geq} \mathcal{T}_{d_4} \overset{\text{P4.6}}{=} \mathcal{T}_{d_7}$$

$$\mathcal{T}_\omega \overset{\text{C3.10}}{\geq} \mathcal{T}$$

Diagram 2: Topological Relations.

Here, the letters 'T', 'P', 'C' represent 'Theorem', 'Proposition' and 'Corollary', respectively, and $\mathcal{T}_{d_i}$ means the topology induced by d_i.

4.2. The gap metric d_1. The first result is:

PROPOSITION 4.1. *$d_1(\cdot,\cdot)$ is a metric on $\mathbf{N}_\omega(\mathcal{U},\mathcal{Y})$, whose topology, $\mathcal{T}_{d_1}$, is stronger than the weighted graph topology $\mathcal{T}_\omega$.*

Proof. From Lemma 2.1 and the definition of d_1, it follows that to prove d_1 is a metric we need only to verify $d_1(P_1,P_2)=0$ implies $P_1=P_2$.

In fact $d_1(P_1,P_2)=0$ implies that for each $(N_1,D_1)\in \mathrm{rcf}(P_1)$,

$$\inf_{(N_2,D_2)\in \mathrm{rcf}(P_2)} \|(D_1-D_2, N_1-N_2)^\top\|_\omega = 0.$$

So there exists a sequence $\{(N_{2,n},D_{2,n})\}\subset \mathrm{rcf}(P_2)$ with $\|(D_1-D_{2,n},N_1-N_{2,n})^\top\|_\omega\to 0$ as $n\to\infty$, from which it follows that, for each $r>0$, $\gamma((D_1-D_{2,n},N_1-N_{2,n})^\top)(r)\to 0$ as $n\to\infty$. By Theorem 3.5, $P_n \xrightarrow{\mathcal{T}} P$ and therefore $P_2=P_1$.

Now we suppose $P_n\in \mathbf{N}_\omega(\mathcal{U},\mathcal{Y})$ with $d_1(P_n,P)\to 0$ as $n\to\infty$. Then

$$\sup_{(N,D)\in \mathrm{rcf}(P)} \inf_{(N_n,D_n)\in \mathrm{rcf}(P_n)} \|(D-D_n,N-N_n)^\top\|_\omega \to 0.$$

So, for each $ND^{-1}\in \mathrm{rcf}(P), \inf_{(N_n,D_n)\in \mathrm{rcf}(P_n)}\|(D-D_n,N-N_n)^\top\|_\omega\to 0$ and therefore there exists $N_nD_n^{-1}\in \mathrm{rcf}(P_n)$ such that $\|(D_n-D,N_n-N)^\top\|_\omega\to 0$ as $n\to\infty$. This proves $P_n\xrightarrow{\mathcal{T}_\omega} P$ and hence $\mathcal{T}_{d_1}$ is stronger than $\mathcal{T}_\omega$. □

4.3. The graph metrics d_2 and d_3. In this subsection, we will show that both d_2 and d_3 are well-defined metrics on $\mathbf{N}_{nor}(\mathcal{U},\mathcal{Y})$ and the topologies induced are equivalent to the weighted graph topology $\mathcal{T}_\omega$ provided $\omega(r)\geq cr$ with $c>0$.

PROPOSITION 4.2. *$d_2(\cdot,\cdot)$ is a metric defined on $\mathbf{N}_{nor}(\mathcal{U},\mathcal{Y})$.*

Proof. First, we need to prove d_2 is well-defined, that is, $d_2(P_1,P_2)$ is independent of the choice of normalized coprime factorisations and is finite for all $P_1,P_2\in \mathbf{N}_{nor}(\mathcal{U},\mathcal{Y})$. So let $P_i\in \mathbf{N}_{nor}(\mathcal{U},\mathcal{Y})$, $(N_i,D_i),(\hat N_i,\hat D_i)\in \mathrm{nrcf}(P_i)$, $i=1,2$. By Propositions 2.6 and 2.7, there exist $Q_i\in\mathcal{Q}$ $(i=1,2)$ with $\|Q_iu\|=\|Q_i^{-1}u\|=\|u\|$ (for all $u\in\mathcal{U}_s$) such that $\hat D_i=D_iQ_i, \hat N_i=N_iQ_i$. Notice, for every stable operator A

$$\|AQ_1\|_\omega \leq \sup_{r>0}\frac{\gamma(A)(\gamma(Q_1)(\omega(r)))}{r} = \sup_{r>0}\frac{\gamma(A)(\omega(r))}{r} = \|A\|_\omega, \tag{4.1}$$

so we have

$$\begin{aligned}\inf_{\substack{Q\in\mathcal{Q}\\ \|Q\|\leq 1}} \|(\hat D_1-\hat D_2Q, \hat N_1-\hat N_2Q)^\top\|_\omega &= \inf_{\substack{Q\in\mathcal{Q}\\ \|Q\|\leq 1}} \|(D_1Q_1-D_2Q_2Q, N_1Q_1-N_2Q_2Q)^\top\|_\omega\\ &\leq \inf_{\substack{\hat Q\in\hat{\mathcal{Q}}\\ \|\hat Q\|\leq 1}} \|(D_1-D_2\hat Q, N_1-N_2\hat Q)^\top\|_\omega.\end{aligned}$$

Replacing Q_i by Q_i^{-1}, we see that the opposite inequality is also true and, therefore

$$\inf_{\substack{\hat{Q}\in\mathcal{Q}\\ \|\hat{Q}\|\leq 1}} \|(D_1 - D_2\hat{Q}, N_1 - N_2\hat{Q})^\top\|_\omega = \inf_{\substack{Q\in\mathcal{Q}\\ \|Q\|\leq 1}} \|(\hat{D}_1 - \hat{D}_2 Q, \hat{N}_1 - \hat{N}_2 Q)^\top\|_\omega.$$

This shows the value of $\vec{d}_2(P_1, P_2)$ is independent of the choice of normalized coprime factorizations. Similarly, we can prove $\vec{d}_2(P_2, P_1)$ is independent of the choice of normalized coprime factorisations and hence so is d_2. Also, for any $Q \in \mathcal{Q}$ with $\|Q\| \leq 1$, we have

$$\|(D_1 - D_2 Q, N_1 - N_2 Q)^\top\|_\omega \leq \|(D_1, N_1)^\top\|_\omega + \|(D_2, N_2)^\top\|_\omega \leq 2 < \infty.$$

Hence d_2 is well-defined.

Next we prove d_2 is a metric.

Obviously d_2 is symmetric and $d_2(P, P) = 0$ for any $P \in \mathbf{N}_{nor}(\mathcal{U}, \mathcal{Y})$. Conversely, suppose $d_2(P_1, P_2) = 0$ with $P = N_1 D_1^{-1}, P_2 = N_2 D_2^{-1}$. Then for all $n > 0$, there exist $Q_n \in \mathcal{Q}$ such that $\|(D_1 - D_2 Q_n, N_1 - N_2 Q_n)^\top\|_\omega \to 0 \quad$ as $n \to \infty$, from which we have

$$\gamma((D_1 - D_2 Q_n, N_1 - N_2 Q_n)^\top)(r) \to 0 \;\text{ for all } r > 0 \text{ as } n \to \infty.$$

By Proposition 2.6, $(N_2 Q_n, D_2 Q_n)$ is also a coprime factorisation of $P_n \equiv P_2$ for each n. From Theorem 3.5, it follows that $P_2 = P_n \xrightarrow{\mathcal{T}} P_1$ in $\mathbf{N}(\mathcal{U}, \mathcal{Y})$, which implies $P_1 = P_2$ since the pointwise graph topology is Hausdorff.

To prove the triangle inequality, we suppose $N_i D_i^{-1}$ are normalized coprime factorisations for P_i with $i = 1, 2, 3$. Then for each $\varepsilon > 0$, there exists $Q_1, Q_2 \in \mathcal{Q}$ with $\|Q_1\| \leq 1, \|Q_2\| \leq 1$ such that

$$\begin{aligned}
\|(D_1 - D_3 Q_1, N_1 - N_3 Q_1)^\top\|_\omega &\leq \vec{d}_2(P_1, P_3) + \varepsilon,\\
\|(D_3 - D_2 Q_2, N_3 - N_2 Q_2)^\top\|_\omega &\leq \vec{d}_2(P_3, P_2) + \varepsilon.
\end{aligned}$$

Since $Q_2 Q_1 \in \mathcal{Q}$, $\|Q_2 Q_1\| \leq 1$ and by using (4.1), we have

$$\begin{aligned}
\vec{d}_2(P_1, P_2) &\leq \|(D_1 - D_2 Q_2 Q_1, N_1 - N_2 Q_2 Q_1)^\top\|_\omega\\
&\leq \|(D_1 - D_3 Q_1, N_1 - N_3 Q_1)^\top + (D_3 Q_1 - D_2 Q_2 Q_1, N_3 Q_1 - N_2 Q_2 Q_1)^\top\|_\omega\\
&\leq \vec{d}_2(P_1, P_3) + \varepsilon + \|(D_3 - D_2 Q_2, N_3 - N_2 Q_2)^\top Q_1\|_\omega\\
&\leq \vec{d}_2(P_1, P_3) + \vec{d}_2(P_3, P_2) + 2\varepsilon \leq d_2(P_1, P_3) + d_2(P_3, P_2) + 2\varepsilon.
\end{aligned}$$

Since ε is arbitrary, we see that $\vec{d}_2(P_1, P_2) \leq d_2(P_1, P_3) + d_2(P_3, P_2)$. By changing the order of P_1, P_2 on the left hand side (they are arbitrary) and noticing that the right hand side is symmetric, we have $\vec{d}_2(P_2, P_1) \leq d_2(P_1, P_3) + d_2(P_3, P_2)$. This proves the triangle inequality and completes the proof. □

PROPOSITION 4.3. *Suppose $c > 0$ and $cr \leq w(r)$ for all $r \geq 0$. Then d_3 is a metric which is topologically equivalent to d_2 on $\mathbf{N}_{nor}(\mathcal{U}, \mathcal{Y})$.*

Proof. The proof for the well-definedness and the triangle inequality for d_3 is exactly the same as in Proposition 4.2.

Suppose $d_3(P_1, P_2) = 0$. Hence there exists a sequence $\{Q_n\} \subset \mathcal{Q}^*$ satisfying

$$\|(D_1 - D_2 Q_n, N_1 - N_2 Q_n)^\top\|_\omega \to 0 \quad \text{as } n \to \infty, \tag{4.2}$$

and therefore there exists $n_0 > 0$ such that

$$\sup_{\|u\|\le\omega(r)} \|(D_1 - D_2Q_n, N_1 - N_2Q_n)^\top u\| < \frac{c}{2}r \quad \text{for all } r > 0, n \ge n_0.$$

For any $u \in \mathcal{U}_s$, let $r = \|u\|/c$. Then $\|u\| \le cr \le \omega(r)$ and therefore,

$$\|(D_1 - D_2Q_n, N_1 - N_2Q_n)^\top u)\| < \frac{c}{2}\frac{\|u\|}{c} = \frac{1}{2}\|u\| \quad \text{for all } u \in \mathcal{U}_s, n \ge n_0.$$

Since $(N_1, D_1), (N_2, D_2)$ are normalized coprime factorisations, we see

$$\begin{aligned}\|Q_n u\| &= \|(D_2, N_2)^\top Q_n u\| \ge \|(D_1, N_1)^\top u\| - \|(D_1 - D_2Q_n, N_1 - N_2Q_n)^\top u)\| \\ &\ge \|u\| - \frac{1}{2}\|u\| = \frac{1}{2}\|u\|, \quad \text{for all } u \in \mathcal{U}_s, n \ge n_0.\end{aligned}$$

This means that $\|Q_n^{-1}\| \le 2$ and Q_n^{-1} is stable for all $n > n_0$. By Proposition 2.6, for all $n > n_0, (N_2Q_n, D_2Q_n)$ is a coprime factorisation of P_2. Also from (4.2), we see

$$\gamma((D_1 - D_2Q_n, N_1 - N_2Q_n)^\top)(r) \to 0 \quad \text{for all } r > 0, \text{ as } n \to \infty.$$

From Theorem 3.5, it follows that $P_2 \equiv P_n \xrightarrow{\mathcal{T}} P_1$ in $\mathbf{N}(\mathcal{U}, \mathcal{Y})$, which implies $P_1 = P_2$. Hence d_3 is a well-defined metric on $\mathbf{N}_{nor}(\mathcal{U}, \mathcal{Y})$.

To prove the equivalence between d_2 and d_3, we first notice that $d_3 \le d_2$. This yields that convergence under d_2 implies convergence under d_3. On the other hand, let $P_n, P \in \mathbf{N}_{nor}(\mathcal{U}, \mathcal{Y})$ with $(N, D) \in \text{nrcf}(P), (N_n, D_n) \in \text{nrcf}(P_n)$ and $d_3(P_n, P) \to 0$ as $n \to \infty$. Then $\vec{d_3}(P, P_n) \to 0$ which means

$$\inf_{Q\in\mathcal{Q}^*} \|(D - D_nQ_n, N - N_nQ_n)^\top\|_\omega \to 0 \quad \text{as } n \to \infty.$$

This shows that for each $\varepsilon \in (0, c/2]$, there exists $n_\varepsilon > 0$ such that

$$\|(D - D_nQ_n, N - N_nQ_n)^\top\|_\omega < \varepsilon \quad \text{for all } n \ge n_\varepsilon. \tag{4.3}$$

Without loss of generality, we may suppose that $n_{\varepsilon_1} \le n_{\varepsilon_2}$ if $\varepsilon_1 > \varepsilon_2$. By letting $\varepsilon = c/2$, we see that there exists $0 < n_0 \le n_\varepsilon$ such that, for each $n \ge n_0$, there is $Q_n \in \mathcal{Q}^*$ satisfying

$$\sup_{\|u\|\le\omega(t)} \|(D - D_nQ_n, N - N_nQ_n)^\top u\| \le \frac{c}{2}r \quad \text{for all } r > 0, n \ge n_0.$$

Using the same method as used in the first part (just replace (N_1, D_1) by (N, D) and (N_2, D_2) by (N_n, D_n), respectively), we can prove that Q_n^{-1} is stable for $n \ge n_0$. So from (4.3), it follows

$$\vec{d_2}(P, P_n) \le \|(D - D_nQ_n, N - N_nQ_n)^\top\|_\omega < \varepsilon \quad \text{for all } n \ge n_\varepsilon$$

and, therefore, $\vec{d_2}(P, P_n) \to 0$ as $n \to \infty$, Similarly, $\vec{d_2}(P_n, P) \to 0$ and, therefore, $d_2(P_n, P) \to 0$ as $n \to \infty$. This completes the proof. □

THEOREM 4.4. *Suppose $c > 0$ and $cr \le w(r)$ for all $r \ge 0$. Then the topology induced by either d_2 or d_3 is equivalent to the weighted graph topology $\mathcal{T}_\omega$ on $\mathbf{N}_{nor}(\mathcal{U}, \mathcal{Y})$.*

Proof. By Proposition 4.3, we only need to show $d_3(P_n, P) \to 0$ if and only if $P_n \xrightarrow{\mathcal{T}_\omega} P$, for any $P_n, P \in \mathbf{N}_{nor}(\mathcal{U}, \mathcal{Y})$.

First, suppose $d_3(P_n, P) \to 0$. Then for every normalized coprime factorisation $P = ND^{-1}, P_n = N_n D_n^{-1}$, there exist $Q_n \in \mathcal{Q}^*$ such that

$$\|(D - D_n Q_n, N - N_n Q_n)^\top\|_\omega \to 0 \text{ as } n \to \infty. \tag{4.4}$$

Let $(\hat{N}, \hat{D})$ be an arbitrary coprime factorisation of P. By Proposition 2.6, there exists stable operator Q on $\mathcal{U}_s$, with Q^{-1} also stable, such that $\hat{D} = DQ, \hat{N} = NQ$, from which we see

$$\|Q\|_\omega = \|(D, N)^\top Q\|_\omega = \|(\hat{D}, \hat{N})^\top\|_\omega < \infty.$$

Write $\hat{D}_n = D_n Q_n Q, \hat{N}_n = N_n Q_n Q$. Then from (4.4), it follows that

$$\begin{aligned} |(\hat{D} - \hat{D}_n, \hat{N} - \hat{N}_n)^\top\|_\omega &= \|(D - D_n Q_n, N - N_n Q_n)^\top Q\|_\omega \\ &\le \|(D - D_n Q_n, N - N_n Q_n)^\top\| \|Q\|_\omega \\ &= \frac{1}{c}\|(D - D_n Q_n, N - N_n Q_n)^\top\|_\omega \|Q\|_\omega \to 0 \text{ as } n \to \infty. \end{aligned}$$

Using the same method as used in Proposition 4.3, we can prove that $(\hat{N}_n, \hat{D}_n)$ is a coprime factorisation of P_n for all large n. Hence from Theorem 3.9, we see $P_n \xrightarrow{\mathcal{T}_\omega} P$.

Secondly, suppose $P_n, P \in \mathbf{N}_{nor}(\mathcal{U}, \mathcal{Y})$ with $P_n \xrightarrow{\mathcal{T}_\omega} P$. Let (N, D) be a normalized coprime factorisation of P. Then there exist coprime factorisations $N_n D_n^{-1}$ of P_n with $\|(D - D_n, N - N_n)^\top\|_\omega \to 0$ as $n \to \infty$. Therefore, $\{\|(D_n, N_n)^\top\|\}$ is bounded and $\|(D - D_n, N - N_n)^\top\| \to 0$. Hence

$$\|(D_n, N_n)^\top\| \to \|(D, N)^\top\| = 1 \text{ as } n \to \infty \tag{4.5}$$

and for each $\varepsilon > 0$, there exists $n_\varepsilon > 0$ such that

$$\|(D - D_n, N - N_n)^\top u\| \le \varepsilon\|u\| \quad \text{for all } u \in \mathcal{U}_s, n > n_\varepsilon. \tag{4.6}$$

Let $\hat{N}_n \hat{D}_n^{-1}$ be a normalized coprime factorisation of P_n. Then there exists stable operator U_n on $\mathcal{U}_s$, where U_n^{-1} exists and is stable, such that $D_n = \hat{D}_n U_n, N_n = \hat{N}_n U_n$. Since $\|U_n u\| = \|(\hat{D}_n, \hat{N}_n)^\top U_n u\| = \|(D_n, N_n)^\top u\|$ for any $u \in \mathcal{U}_s$, we see $\{\|U_n\|_\omega\}$ is bounded and, from (4.5), it follows

$$\|U_n\| = \|(\hat{D}_n, \hat{N}_n)^\top U_n\| = \|(D_n, N_n)^\top\| \to 1 \text{ as } n \to \infty.$$

From (4.6), it follows that for each $u \in \mathcal{U}_s$ and each $n > n_\varepsilon$

$$\|U_n u\| = \|(D_n, N_n)^\top u\| \ge \|(D, N)^\top u\| - \|(D_n - D, N_n - N)^\top u\| \ge (1 - \varepsilon)\|u\|,$$

which implies that $\|U_n^{-1} u\| \le \frac{1}{1-\varepsilon}\|u\|$ and therefore $\|U_n^{-1}\| \le \frac{1}{1-\varepsilon}$. Since $\|U_n^{-1}\| \ge 1/\|U_n\|$, we see $\|U_n^{-1}\| \to 1$ as $n \to \infty$.

Let $Q_n u = U_n u/\|U_n\|$ for each $u \in \mathcal{U}_s$. Then $\|Q_n\| \le 1$ and since $Q_n^{-1} = U_n^{-1} \cdot \|U_n\|$ exists and is stable, we have $Q_n \in \mathcal{Q}^*$. Also

$$\begin{aligned} \|(\hat{D}_n U_n - \hat{D}_n Q_n, \hat{N}_n U_n - \hat{N}_n Q_n)^\top\|_\omega &= \|(\hat{D}_n, \hat{N}_n)^\top (U_n - Q_n)\|_\omega \\ = \|(U_n - Q_n)\|_\omega &= \frac{\big|\|U_n\| - 1\big|}{\|U_n\|}\|U_n\|_\omega \to 0 \end{aligned}$$

which implies

$$\begin{aligned}\vec{d}_3(P,P_n) &\le \|(D-\hat{D}_nQ_n, N-\hat{N}_nQ_n)^\top\|_\omega \\ &\le \|(D-\hat{D}_nU_n, N-\hat{N}_nU_n)^\top\|_\omega \\ &\quad + \|(\hat{D}_nU_n-\hat{D}_nQ_n, \hat{N}_nU_n-\hat{N}_nQ_n)^\top\|_\omega \to 0 \text{ as } n\to\infty.\end{aligned}$$

Similarly, for $\tilde{Q}_n u = U_n^{-1}u/\|U_n^{-1}\|$, we can prove

$$\vec{d}_3(P_n,P) \le \|(\hat{D}_n - D\tilde{Q}_n, \hat{N}_n - N\tilde{Q}_n)^\top\|_\omega \to 0 \quad \text{as } n\to\infty.$$

This shows $d_3(P_n,P)\to 0$ and completes the proof. ◻

We remark that the first part of the proof shows, in the case $cr \le \omega(r)$, that $P_n \xrightarrow{\mathcal{I}_\omega} P$ in $\mathbf{N}_{nor}(\mathcal{U},\mathcal{Y})$ if and only if there exist normalized coprime factorization (N,D) of P and coprime factorisations (N_n,D_n) of P_n such that $\|(D-D_n, N-N_n)^\top\|_\omega \to 0$ as $n\to\infty$.

In the case of $\omega(r)=r$ and stability is taken to be in the sense of linear gain, the above theorem shows that the graph topology induced by the linear gain is metrizable.

4.4. The gap metrics d_4, d_5, d_6 and d_7. In this subsection, we present the metric properties of $d_4,\cdots,d_7$ over the subset $\mathbf{N}_{nor}(\mathcal{U},\mathcal{Y})$. In particular, the equivalence between the weighted graph topology and the topologies induced by either d_5 or d_6 will be established.

Using the same method as used in [7], we can prove that d_4, d_5 are pseudo-metrics on the set of signal operators from $\mathcal{U}$ to $\mathcal{Y}$ provided $\omega(r)\ge r$ for all $r>0$. Here pseudo-metric means that $d_4(P_1,P_2)=0$ (resp. $d_5(P_1,P_2)=0$) does not necessarily imply $P_1=P_2$ unless extra conditions are imposed. Moreover, as in [7], they are only "generalized" pseudo-metrics, which means that possibly (say) $d_5(P_1,P_2)=\infty$ for some P_1,P_2. The following comparison results show that they both become well-defined metrics if restricted to $\mathbf{N}_{nor}(\mathcal{U},\mathcal{Y})$ (no extra condition required).

We first give a key lemma.

LEMMA 4.5. *Suppose $P_i\in\mathbf{N}_{nor}(\mathcal{U},\mathcal{Y})$ with $(D_i,N_i)\in \mathrm{nrcf}(P_i), i=1,2$. Then there exists a mapping $\Phi:\mathrm{Graph}(P_1)\to\mathrm{Graph}(P_2)$ if and only if there exists a mapping $Q:\mathcal{U}_s\to\mathcal{U}_s$ such that*

$$\Phi\begin{pmatrix} D_1 \\ N_1\end{pmatrix} u = \begin{pmatrix} D_2 \\ N_2\end{pmatrix} Qu, \ \textit{for all } u\in\mathcal{U}_s. \tag{4.7}$$

Moreover,

(i) Φ is surjective if and only if Q is surjective;

(ii) $\|Q\|=\|\Phi\|$ and $\gamma(\Phi)(r)=\gamma(Q)(r)$ for any $r>0$ (so Φ stable if and only if Q stable);

(iii) Φ is injective if and only if Q is injective;

(iv) $\|Q^{-1}\|=\|\Phi^{-1}\| =: \|\Phi^{-1}|_{\mathcal{M}_2}\|$ and $\gamma(\Phi^{-1})(r)=\gamma(Q^{-1})(r)$ for any $r>0$;

(v) Φ is causal if and only if Q is causal, and $\Phi 0=0$ if and only if $Q0=0$.

Proof. Write $\mathcal{M}_i=\mathrm{Graph}(P_i)$ for $i=1,2$.

Let $\Phi:\mathcal{M}_1\to\mathcal{M}_2$ be a given mapping. Then, for any $u\in\mathcal{U}_s$, by Proposition 2.5, $\Phi(D_1,N_1)^\top u\in\mathcal{M}_2$ and therefore, there exists $v_u\in\mathcal{U}_s$ such that $\Phi(D_1,N_1)^\top u = (D_2,N_2)^\top v_u$. Since $(D_2,N_2)^\top$ is left invertible, such a point v_u is unique. This yields that the mapping

$$Qu = v_u$$

is well defined on $\mathcal{U}_s$ and satisfies (4.7).

Conversely, let Q be a given mapping on $\mathcal{U}_s$. For any $w \in \mathcal{M}_1$, let $\Phi w = (D_2, N_2)^\top Q L_1 w$ where L_1 is the left inverse of $(D_1, N_1)^\top$ which is stable by the definition of the coprime factorisation. Then, obviously, Φ is a well defined mapping from $\mathcal{M}_1$ to $\mathcal{M}_2$ satisfying (4.7).

Now we prove the other claims.

(i) First, we suppose $\Phi : \mathcal{M}_1 \to \mathcal{M}_2$ is given and surjective. Since $(D_1, N_1)^\top : \mathcal{U}_s \to \mathcal{M}_1$ is surjective, for any $v \in \mathcal{U}_s$ there exists $u \in \mathcal{U}_s$ with $\Phi(D_1, N_1)^\top u = (D_2, N_2)^\top v$. The left invertibility of $(D_2, N_2)^\top$ and (4.7) show $Qu = v$. Therefore, Q is surjective.

If Q is surjective on $\mathcal{U}_s$, then for any $w \in \mathcal{M}_2$, the surjectivity of $(D_2, N_2)^\top$ implies the existence of $u \in \mathcal{U}_s$ such that $(D_2, N_2)^\top Qu = w$. Hence $\Phi(D_1, N_1)^\top u = w$ which shows that Φ is surjective.

(ii) From (4.7), we see

$$\|Qu\| = \left\| \begin{pmatrix} D_2 \\ N_2 \end{pmatrix} Qu \right\| = \left\| \Phi \begin{pmatrix} D_1 \\ N_1 \end{pmatrix} u \right\| \quad \text{for all } u \in \mathcal{U}_s. \tag{4.8}$$

Since $\|u\| = \|(D_1, N_1)^\top u\|, \mathcal{M}_1 = (D_1, N_1)^\top \mathcal{U}_s$, the conclusions follows.

(iii) From (4.7) and the left invertibility of $(D_i, N_i)^\top (i = 1, 2)$, it follows

$$\begin{aligned}
\Phi \text{ is injective } &\Leftrightarrow \Phi w_1 = \Phi w_2 \text{ implies } w_1 = w_2 \text{ for any } w_1, w_2 \in \mathcal{M}_1 \\
&\Leftrightarrow \Phi \begin{pmatrix} D_1 \\ N_1 \end{pmatrix} u_1 = \Phi \begin{pmatrix} D_1 \\ N_1 \end{pmatrix} u_2 \text{ implies } u_1 = u_2 \text{ for any } u_1, u_2 \in \mathcal{U}_s \\
&\Leftrightarrow \begin{pmatrix} D_2 \\ N_2 \end{pmatrix} Qu_1 = \begin{pmatrix} D_2 \\ N_2 \end{pmatrix} Qu_2 \text{ implies } u_1 = u_2 \text{ for any } u_1, u_2 \in \mathcal{U}_s \\
&\Leftrightarrow Qu_1 = Qu_2 \text{ implies } u_1 = u_2 \text{ for any } u_1, u_2 \in \mathcal{U}_s \\
&\Leftrightarrow Q \text{ is injective .}
\end{aligned}$$

(iv) Since $\|w\| \le \|\Phi^{-1}\| \|\Phi w\|$ for any $w \in \mathcal{M}_1$, we have

$$\|u\| = \left\| \begin{pmatrix} D_1 \\ N_1 \end{pmatrix} u \right\| \le \|\Phi^{-1}\| \left\| \Phi \begin{pmatrix} D_1 \\ N_1 \end{pmatrix} u \right\| = \|\Phi^{-1}\| \left\| \begin{pmatrix} D_2 \\ N_2 \end{pmatrix} Qu \right\| = \|\Phi^{-1}\| \|Qu\|$$

for any $u \in \mathcal{U}_s$. So $\|Q^{-1}\| \le \|\Phi^{-1}\|$. Similarly, for any $w = (D_1, N_1)^\top u \in \mathcal{M}_1$,

$$\left\| \begin{pmatrix} D_1 \\ N_1 \end{pmatrix} u \right\| = \|u\| \le \|Q^{-1}\| \|Qu\| = \|Q^{-1}\| \left\| \begin{pmatrix} D_2 \\ N_2 \end{pmatrix} Qu \right\| = \|Q^{-1}\| \left\| \Phi \begin{pmatrix} D_1 \\ N_1 \end{pmatrix} u \right\|$$

which gives the reverse inequality. Hence $\|Q^{-1}\| = \|\Phi^{-1}\|$.

For any $r > 0$, (4.8) and the surjectivity of $\Phi, (D_i, N_i)^\top$ and Q yield

$$\gamma(Q^{-1})(r) = \sup_{\|Qv\| \le r} \|v\| = \sup_{\|\Phi(D_1,N_1)^\top v\| \le r} \left\| \begin{pmatrix} D_1 \\ N_1 \end{pmatrix} v \right\| = \sup_{\|w\| \le r} \|\Phi^{-1} w\| = \gamma(\Phi^{-1})(r).$$

(v) Let L_1, L_2 be the associated operators to $(D_1, N_1)^\top, (D_2, N_2)^\top$ respectively. By applying L_2 to (4.7), we have $Qu = L_2 \Phi (D_1, N_1)^\top$. By the definition of Φ, $\Phi w = (D_2, N_2)^\top Q L_1$. So, the conclusions follow from the preassumptions on signal operators. This completes the proof. ◻

Proposition 4.6. *$\vec{d}_5(P_1, P_2) = \vec{d}_6(P_1, P_2)$, $\vec{d}_4(P_1, P_2) = \vec{d}_7(P_1, P_2)$ for $P_i \in \mathbf{N}_{nor}(\mathcal{U}, \mathcal{Y})$, $i = 1, 2$.*

Proof. Let Q be a given stable bijective mapping on $\mathcal{U}_s$. Then there exists a stable and bijective map $\Phi : \mathcal{M}_1 \to \mathcal{M}_2$ satisfying (4.7), for which

$$\|\Phi - I\|_\omega = \|(D_1, N_1)^\top - (D_2, N_2)^\top Q\|_\omega. \tag{4.9}$$

Therefore, $\vec{d}_5(P_1, P_2) \le \|(D_1, N_1)^\top - (D_2, N_2)^\top Q\|_\omega$ and $\vec{d}_5(P_1, P_2) \le \vec{d}_6(P_1, P_2)$ as Q is arbitrary.

Since $\Phi_1(D_1, N_1)^\top u = (D_2, N_2)^\top u$ is a bijective operator from $\mathcal{M}_1$ to $\mathcal{M}_2$ and $\|\Phi_1 - I\|_\omega < \infty$, we have

$$\vec{d}_5(P_1, P_2) = \inf\{\|\Phi - I\|_\omega : \Phi : \mathcal{M}_1 \to \mathcal{M}_2 \text{ bijective } \|\Phi - I\|_\omega < \infty\}.$$

Notice that $\|\Phi - I\|_\omega < \infty$ implies the stability of Φ. So, given any bijective map $\Phi : \mathcal{M}_1 \to \mathcal{M}_2$ with $\|\Phi - I\|_\omega < \infty$, by Lemma 4.5, there exists a stable, bijective mapping Q on $\mathcal{U}_s$ satisfying (4.7) and, therefore, (4.9). Hence $\vec{d}_6(P_1, P_2) \le \|\Phi - I\|_\omega$ which indicates that $\vec{d}_6(P_1, P_2) \le \vec{d}_5(P_1, P_2)$. This proves that $\vec{d}_6(P_1, P_2) = \vec{d}_5(P_1, P_2)$. The equality $\vec{d}_4(P_1, P_2) = \vec{d}_7(P_1, P_2)$ can be proved similarly. □

THEOREM 4.7. *d_5, d_6 are well-defined metrics on $\mathbf{N}_{nor}(\mathcal{U}, \mathcal{Y})$ and the graph topology $\mathcal{T}_\omega$ is equivalent to the topology induced by either d_5 or d_6, provided $r \le \omega(r)$ for all $r \ge 0$.*

Proof. Using the same methods as in Propositions 4.2 and 4.3, we see that d_6 is well-defined and $d_6(P_1, P_2) = 0$ if and only if $P_1 = P_2$ on $\mathbf{N}_{nor}(\mathcal{U}, \mathcal{Y})$. By Proposition 4.6, d_5 satisfies the same property.

To prove the triangle inequality for d_5, we suppose $P_1, P_2, P_3 \in \mathbf{N}_{nor}(\mathcal{U}, \mathcal{Y})$ and $\Phi_1 : \text{Graph}(P_1) \to \text{Graph}(P_2), \Phi_i : \text{Graph}(P_2) \to \text{Graph}(P_3)$ are bijective mappings. Then $\Phi := \Phi_2\Phi_1$ is a bijective mapping from Graph(P_1) to Graph(P_3) and $\Phi - I = (\Phi_2 - I)\Phi_1 - I$. So

$$\begin{aligned}\|\Phi - I\|_\omega &\le \|\Phi_2 - I\|\|\Phi_1\|_\omega + \|\Phi_1 - I\|_\omega \le \|\Phi_2 - I\|_\omega\|\Phi_1\|_\omega + \|\Phi_1 - I\|_\omega \\ &\le \|\Phi_2 - I\|_\omega(\|\Phi_1\|_\omega - I)\| + 1) + \|\Phi_1 - I\|_\omega\end{aligned}$$

and, therefore

$$\hat{d}_5(P_1, P_3) \le \hat{d}_5(P_1, P_2) + \hat{d}_5(P_2, P_3).$$

This means that d_5 satisfies the triangle inequality. Hence d_5 is a well-defined metric on $\mathbf{N}_{nor}(\mathcal{U}, \mathcal{Y})$ and so is d_6 because of Proposition 4.6.

Since $d_6 \le d_3$ and by Theorem 4.4, the convergence of sequence under $\mathcal{T}_\omega$ implies the convergence under d_6. Conversely, if $d_6(P, P_n) \to 0$ as $n \to \infty$, then by using the same method as in Theorem 4.4 (see the Theorem's remark), we can prove that $P_n \xrightarrow{\mathcal{T}_\omega} P$. This shows the equivalence between $\mathcal{T}_\omega$ and the topology induced by either d_6 or d_5. □

Proposition 4.6 and Theorem 4.7 suggest that the two metrics d_3 and d_6 might be equivalent (we already know that $d_6 \le d_3$). In fact, Georgiou [4] has proved $d_3(P_1, P_2) \le 2d_6(P_1, P_2)$ in the linear setting. In the nonlinear setting and in the case where $(D_2, N_2)^\top$ is incrementally stable, that is where

$$\|(D_2, N_2)^\top\|_\triangle := \sup\left\{\frac{\|(D_2, N_2)^\top u_1 - (D_2, N_2)^\top u_2\|_\tau}{\|u_1 - u_2\|_\tau} : \tau > 0, u_1, u_2 \in \mathcal{U}_s\right\} < \infty,$$

this claim can be proved by exactly the same technique as in [4].

Finally we consider the relationship between d_1 and d_6. For $(N_i, D_i) \in \text{rcf}(P_i), i = 1, 2$, by Proposition 2.6, we have

$$\inf_{Q\in\mathcal{Q}} \left\| \begin{pmatrix} D_1 \\ N_1 \end{pmatrix} - \begin{pmatrix} D_2 \\ N_2 \end{pmatrix} Q \right\|_\omega = \inf_{(\tilde{N}_2,\tilde{D}_2)\in\text{rcf}(P_2)} \left\| \begin{pmatrix} D_1 \\ N_1 \end{pmatrix} - \begin{pmatrix} \tilde{D}_2 \\ \tilde{N}_2 \end{pmatrix} \right\|_\omega \le \vec{d}_1(P_1, P_2).$$

This gives a direct relation between d_1 and d_6 as below.

PROPOSITION 4.8. *For $P_i \in \mathbf{N}_{nor}(\mathcal{U}, \mathcal{Y})$ with $(D_i, N_i) \in \text{nrcf}(P_i), i = 1, 2$, $\vec{d}_1(P_1, P_2) \ge \vec{d}_6(P_1, P_2)$ and, therefore, $d_1(P_1, P_2) \ge d_6(P_1, P_2)$.*

5. Robustness of Stability of Nonlinear Feedback Systems. The importance of graph topology in the linear case is well known. In this section, we will show that it may also play a significant role in the nonlinear case by considering the system described by the configuration of Figure 5.1.

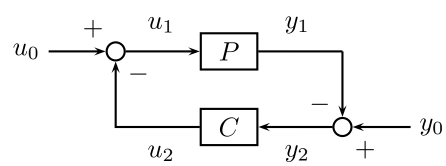

FIG. 5.1. *Standard Feedback Configuration.*

In this configuration, $u_i \in \mathcal{U}$, $y_i \in \mathcal{Y}$ for $i = 0, 1, 2$, and both the plant P and compensator C are, in general, causal and nonlinear. We suppose all systems in this section are well-posed, that is, for each $(u_0, y_0)^\top \in \mathcal{U}_s \times \mathcal{Y}_s$, there exist unique signals $u_1, u_2 \in \mathcal{U}$ and $y_1, y_2 \in \mathcal{Y}$ such that

$$u_0 = u_1 + u_2, \quad y_0 = y_1 + y_2, \quad y_1 = Pu_1, \quad u_2 = Cy_2.$$

and the feedback operator

$$H_{P,C} : \mathcal{W}_s \to \mathcal{W} \times \mathcal{W} : \begin{pmatrix} u_0 \\ y_0 \end{pmatrix} \mapsto \left(\begin{pmatrix} u_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} u_2 \\ y_2 \end{pmatrix} \right)$$

is causal. Here, $\mathcal{W}_s = \mathcal{U}_s \times \mathcal{Y}_s, \mathcal{W} = \mathcal{U} \times \mathcal{Y}$. The feedback stability of this system is the requirement that $H_{P,C}$ is stable in a suitable sense. We are concerning the robustness problem: when is $H_{P_\lambda,C}$ stable given that $H_{P,C}$ is stable and P_λ is a perturbation to P?

In [7], this problem has been studied using a gap metric. Particularly, in the case where the linear gain is considered, it is proved that if $H_{P,C}$ is gain stable and P_λ is close enough to P in the sense of gap metric, then $H_{P_\lambda,C}$ is gain stable. Similar results are also given when $H_{P,C}$ is (gf)-stable with super-linear growth. However in the (gf)-stability case, the notion of convergence was not made explicit as no topology was indicated. In this paper, we consider the robustness of (gf)-stability when the convergence of P_λ to P is in the sense of any of the two graph topologies defined in the previous sections.

We suppose Λ is a topological space and for each $\lambda \in \Lambda$, P_λ is a perturbation to the nominal plant $P = P_{\lambda_0}$. Define $\mathcal{M} = \text{Graph}(P), \mathcal{M}_\lambda = \text{Graph}(P_\lambda), \mathcal{N} = \text{Graph}(C)$, and let $\Pi_{\mathcal{M}//\mathcal{N}}$ be the parallel projection which maps $(u_0, y_0)^\top$ to $(u_1, y_1)^\top$ and

$\Pi_{\mathcal{N}//\mathcal{M}} = I - \Pi_{\mathcal{M}//\mathcal{N}}$. It is known that $H_{P,C}$ is (gf)-stable (resp. gain stable) if and only if $\Pi_{\mathcal{M}//\mathcal{N}}$ is (gf)-stable (resp. gain stable), see [7].

A signal operator $F : \mathcal{U} \to \mathcal{Y}$ is said to be causally extendable if, for each $u \in \mathcal{U}, y = Fu$ and each $\tau > 0$, there exists $u_\tau \in \mathrm{Dom}(F)$ such that $T_\tau(u,y)^\top = T_\tau(u_\tau, y_\tau)^\top$ with $y_\tau = Fu_\tau$. Henceforth, we suppose that P, C and each P_λ are causally extendable.

Lemma 5.1. *Suppose Φ is a surjective mapping from $\mathcal{M}$ to $\mathcal{M}_\lambda$, Then, for any $z \in \mathcal{W}_s$ and any $\tau > 0$, there exists $x_\tau \in \mathcal{W}_s$ such that*

$$T_\tau z = T_\tau x_\tau + T_\tau(\Phi - I)\Pi_{\mathcal{M}//\mathcal{N}} T_\tau x_\tau \quad \textit{and} \quad T_\tau \Pi_{\mathcal{M}_\lambda//\mathcal{N}} z = T_\tau \Phi \Pi_{\mathcal{M}//\mathcal{N}} T_\tau x_\tau.$$

Proof. Let $H_{P_\lambda,C} z = (z_1, z_2)$ with $z_1 = (u_1, P_\lambda u_1)^\top, z_2 = (Cy_2, y_2)^\top$ for some $u_1 \in \mathcal{U}, y_2 \in \mathcal{Y}$. Then $z = z_1 + z_2$ and $\Pi_{\mathcal{M}_\lambda//\mathcal{N}} z = z_1$. By the causal extendability, for each $\tau > 0$, there exist $z_1^\tau \in \mathcal{M}_\lambda, z_2^\tau \in \mathcal{N}$ such that $T_\tau z_1 = T_\tau z_1^\tau, T_\tau z_2 = T_\tau z_2^\tau$. Since Φ is surjective from $\mathcal{M}$ to $\mathcal{M}_\lambda$, there exists $z_3^\tau \in \mathcal{M}$ with $\Phi z_3^\tau = z_1^\tau$. Write $x_\tau = z_3^\tau + z_2^\tau$. Then $x_\tau \in \mathcal{W}_s$ and $\Pi_{\mathcal{M}//\mathcal{N}} x_\tau = z_3^\tau, \Pi_{\mathcal{N}//\mathcal{M}} x_\tau = z_2^\tau$. Hence

$$\begin{aligned} T_\tau z &= T_\tau z_1 + T_\tau z_2 = T_\tau z_1^\tau + T_\tau z_2^\tau = T_\tau \Phi z_3^\tau + T_\tau z_2^\tau \\ &= T_\tau \Phi \Pi_{\mathcal{M}//\mathcal{N}} x_\tau + T_\tau \Pi_{\mathcal{N}//\mathcal{M}} x_\tau = T_\tau \Phi \Pi_{\mathcal{M}//\mathcal{N}} T_\tau x_\tau + T_\tau \Pi_{\mathcal{N}//\mathcal{M}} T_\tau x_\tau \\ &= T_\tau (\Phi - I) \Pi_{\mathcal{M}//\mathcal{N}} T_\tau x_\tau + T_\tau x_\tau \end{aligned}$$

and

$$T_\tau \Pi_{\mathcal{M}_\lambda//\mathcal{N}} z = T_\tau z_1 = T_\tau \Phi z_3^\tau = T_\tau \Phi \Pi_{\mathcal{M}//\mathcal{N}} x_\tau = T_\tau \Phi \Pi_{\mathcal{M}//\mathcal{N}} T_\tau x_\tau.$$

□

For our main results, we will always require that the nominal plant satisfies a k-coercive condition as stated below, note that this assumption will be not imposed on the perturbed plant P_λ.

Definition 5.2. *A signal operator $P : \mathcal{U} \to \mathcal{Y}$ is said to be k-coercive, with $k \in \mathcal{K}_\infty$, if P has a coprime factorisation (N, D) such that*

$$\|(D,N)^\top u\| \geq k(\|u\|) \quad \textit{for all } u \in \mathcal{U}_s; \tag{5.1}$$

Notice that P is k-coercive if and only if

$$\|Lw\| \leq k^{-1}(\|w\|), \text{ for all } w \in \mathrm{Graph}(P), \tag{5.2}$$

where L is the associated operator of (N, D). Hence any operator P with coprime factors is $\gamma(L)^{-1}$-coercive, where L is the associated operator of a coprime factorisation, since (5.2) always holds with $k^{-1}(r) = \gamma(L)(r)$. It is of interest to observe that a linear operator with coprime factors is always k-coercive with $k(r) = cr, c > 0$. Also note that if P has a normalized coprime factorisation, then P is 1-coercive and, therefore, c-coercive for any $c > 0$.

In the case when $k(r) = cr$ is linear, (5.2) is required by James etc [10] in their definition of (right) coprime factorisation, while (5.1) is required by Verma [18] in one of his definitions and exploited for the stability of another system in the sense of linear gain.

Proposition 5.3. *Suppose that the nominal plant P is k-coercive and $\lambda \mapsto P_\lambda$ is continuous at λ_0 under a weighted topology $\mathcal{T}_\omega$ with $w \in \mathcal{K}_\infty$. Then, for each λ, there exists a surjective mapping $\Phi_\lambda : \mathcal{M} \to \mathcal{M}_\lambda$ such that*

$$\sup_{r>0} \frac{\gamma(I-\Phi_\lambda)(k(\omega(r)))}{r} \to 0, \text{ as } \lambda \to \lambda_0. \tag{5.3}$$

Proof. Let ND^{-1} be the coprime factorisation of $P = P_{\lambda_0}$ satisfying the coercive condition (5.1). From Corollary 3.11, it follows that P_λ has coprime factorisation $N_\lambda D_\lambda^{-1}$ such that

$$\sup_{r>0} \frac{\gamma((D-D_\lambda, N-N_\lambda)^\top)(\omega(r))}{r} \to 0, \text{ as } \lambda \to \lambda_0. \tag{5.4}$$

For each $\lambda > 0$ and each $u \in \mathcal{U}_s$, let

$$\Phi_\lambda((Du, Nu)^\top) = ((D_\lambda u, N_\lambda u)^\top). \tag{5.5}$$

Φ_λ is a well defined, causal and surjective mapping from $\mathcal{M}$ to $\mathcal{M}_\lambda$ since $\Phi_\lambda w = ((D_\lambda, N_\lambda)^\top L w)$ with L the left inverse of $(D,N)^\top$.

Now let $r > 0, (Du, Nu)^\top \in \mathcal{M}$ with $u \in \mathcal{U}_s$ and $\|(Du, Nu)^\top\| \le k(w(r))$. From (5.1), it follows that $\|u\| \le w(r)$, which implies

$$\|((D-D_\lambda)u, (N-N_\lambda)u)^\top\| \le \sup_{\|v\|\le w(r)} \|((D-D_\lambda)v, (N-N_\lambda)v)^\top\|.$$

Therefore

$$\begin{aligned} \gamma(I-\Phi_\lambda)(k(w(r))) &= \sup_{\substack{(Du,Nu)^\top \in \mathrm{Dom}(I-\Phi_\lambda) \\ \|(Du,Nu)^\top\| \le k(w(r))}} \|((D-D_\lambda)u, (N-N_\lambda)u)^\top\| \\ &\le \sup_{\|v\|\le w(r)} \|((D-D_\lambda)v, (N-N_\lambda)v)^\top\| \\ &= \gamma((D-D_\lambda, N-N_\lambda)^\top)(w(r)). \end{aligned} \tag{5.6}$$

By (5.4), we see (5.3) holds. □

Remark. From (5.3), we see that $\|I-\Phi_\lambda\|_{k\circ\omega} \to 0$ which implies $\vec{\delta}(P, P_\lambda) \to 0$ under a new weighted function $\omega_1 = k \circ \omega$. However, we cannot show whether $\vec{\delta}(P_\lambda, P) \to 0$ unless each P_λ is also k-coercive. Also notice here the l.a.c. assumption was not imposed. So (5.3) does not implies $P_\lambda \to P$ under d_4.

Similarly, for the pointwise continuity, we have

Proposition 5.4. *Suppose that P is k-coercive and $\lambda \mapsto P_\lambda$ is continuous at λ_0 under the pointwise topology $\mathcal{T}$. Then, the mapping $\Phi_\lambda : \mathcal{M} \to \mathcal{M}_\lambda$ defined in* (5.5) *is surjective and that $\gamma(I-\Phi_\lambda)(r) \to 0$ for each $r > 0$, as $\lambda \to \lambda_0$.*

Henceforth, we define the map Φ_λ to be as in Proposition 5.3 or 5.4 and give robustness results under each graph topology. The following results follow as consequences of Proposition 5.3 or 5.4 and the results of [7]; however, we give the entire proofs for completeness. First, we consider the case when weighted topology is involved.

Theorem 5.5. *Suppose P is k-coercive and $H_{P,C}$ is (gf)-stable. If $\lambda \mapsto P_\lambda$ is continuous at λ_0 under a weighted topology $\mathcal{T}_\omega$ with $\omega \in \mathcal{K}_\infty$ and for all $r > 0$*

$$\gamma(\Pi_{\mathcal{M}//\mathcal{N}})(r) \le k(w(r)), \tag{5.7}$$

then, for any $n > 0$, there exists a neighbourhood V_n of λ_0 such that $H_{P_\lambda,C}$ is (gf)-stable for $\lambda \in V_n$ and

$$\gamma(\Pi_{\mathcal{M}_\lambda//\mathcal{N}})(r) \leq \gamma(\Pi_{\mathcal{M}//\mathcal{N}})\left(\frac{n+1}{n}r\right) + \frac{1}{n}r.$$

Proof. Let $\tau > 0, r > 0$ and $z \in \mathcal{W}$ be given with $\|z\|_\tau \leq r$. By Proposition 5.3, for each λ, there exists a surjective mapping $\Phi_\lambda : \mathcal{M} \to \mathcal{M}_\lambda$ satisfying (5.3). From Lemma 5.1, it follows that for each λ, there exists $x_\lambda^\tau \in \mathcal{W}_s$ such that

$$T_\tau x_\lambda^\tau = T_\tau z - T_\tau(\Phi_\lambda - I)\Pi_{\mathcal{M}//\mathcal{N}} T_\tau x_\lambda^\tau, \quad T_\tau \Pi_{\mathcal{M}_\lambda//\mathcal{N}} z = T_\tau \Phi_\lambda \Pi_{\mathcal{M}//\mathcal{N}} T_\tau x_\lambda^\tau. \quad (5.8)$$

By (5.3) and the properties of γ, there exists a neighbourhood V_n of λ_0 such that

$$\frac{\gamma(\Phi_\lambda - I)(\gamma(\Pi_{\mathcal{M}//\mathcal{N}})(\|x_\lambda^\tau\|_\tau))}{\|x_\lambda^\tau\|_\tau} \leq \frac{\gamma(\Phi_\lambda - I)(k(w(\|x_\lambda^\tau\|_\tau)))}{\|x_\lambda^\tau\|_\tau} < \frac{1}{n+1}$$

for all $n > 0$ and $\lambda \in V_n$. So, from (5.8), it follows that

$$\begin{aligned}\|x_\lambda^\tau\|_\tau &\leq \|z\|_\tau + \|(\Phi_\lambda - I)\Pi_{\mathcal{M}//\mathcal{N}} T_\tau x_\lambda^\tau\|_\tau \\ &\leq \|z\|_\tau + \gamma(\Phi_\lambda - I)(\gamma(\Pi_{\mathcal{M}//\mathcal{N}})(\|x_\lambda^\tau\|_\tau)) \leq \|z\|_\tau + \frac{1}{n+1}\|x_\lambda^\tau\|_\tau,\end{aligned}$$

which implies $\|x_\lambda^\tau\|_\tau \leq (n+1)\|z\|_\tau/n \leq (n+1)r/n$ for all $\lambda \in V_n$. By (5.8), we have

$$T_\tau \Pi_{\mathcal{M}_\lambda//\mathcal{N}} z = T_\tau \Phi_\lambda \Pi_{\mathcal{M}//\mathcal{N}} T_\tau x_\lambda^\tau = T_\tau \Pi_{\mathcal{M}//\mathcal{N}} T_\tau x_\lambda^\tau + T_\tau(\Phi_\lambda - I)\Pi_{\mathcal{M}//\mathcal{N}} T_\tau x_\lambda^\tau$$

and therefore

$$\begin{aligned}\|\Pi_{\mathcal{M}_\lambda//\mathcal{N}} z\|_\tau &\leq \|\Pi_{\mathcal{M}//\mathcal{N}} T_\tau x_\lambda^\tau\|_\tau + \|(\Phi_\lambda - I)\Pi_{\mathcal{M}//\mathcal{N}} T_\tau x_\lambda^\tau\|_\tau \\ &\leq \gamma(\Pi_{\mathcal{M}//\mathcal{N}})\left(\frac{n+1}{n}r\right) + \frac{1}{n+1}\|x_\lambda\|_\tau \leq \gamma(\Pi_{\mathcal{M}//\mathcal{N}})\left(\frac{n+1}{n}r\right) + \frac{1}{n}r.\end{aligned}$$

Since τ is arbitrary, $\gamma(\Pi_{\mathcal{M}_\lambda//\mathcal{N}})(r) \leq \gamma(\Pi_{\mathcal{M}//\mathcal{N}})\left(\frac{n+1}{n}r\right) r + \frac{1}{n}r < \infty$ for $\lambda \in V_0$. ☐

We remark that condition (5.7) can be replaced by the weaker condition:

$$\gamma(\Pi_{\mathcal{M}//\mathcal{N}})(r) \leq k(cw(r)) \text{ with } c > 0$$

since P is also kc-coercive due to the remark made after Definition 5.2. This claim is also supported by Theorem 3.12 from which we see that $\lambda \mapsto P_\lambda$ is also continuous at λ_0 under a weighted topology $\mathcal{T}_{c\omega}$, so the ω in (5.7) can be replaced by $c\omega$. This replacement gives a weaker bound for $\gamma(\Pi_{\mathcal{M}//\mathcal{N}})(r)$.

In the case of the pointwise topology, we have:

Theorem 5.6. *Suppose that P is k-coercive, $H_{P,C}$ is (gf)-stable and $\lambda \mapsto P_\lambda$ is continuous at λ_0 under the pointwise topology $\mathcal{T}$. If, for each λ, $T_\tau(\Phi_\lambda - I)\Pi_{\mathcal{M}//\mathcal{N}}$ is continuous and compact as a mapping from any subset $S_r = \{w \in \mathcal{W} : \sup_{\tau>0}\|w\|_\tau \leq r\}$ to $\mathcal{W}$, then for each $r > 0$, there exists a neighbourhood V_r of λ_0 in Λ such that $\gamma(H_{P_\lambda,C})(r) < \infty$ for all $\lambda \in V_r$. Here Φ_λ is defined as in* (5.5).

Proof. Let $r > 0$ and $w \in \mathcal{W}$ be given with $\|w\|_\tau \leq r$. Consider the operator

$$\mathbf{A}_\lambda : \mathbf{A}_\lambda x = w + (\Phi_\lambda - I)\Pi_{\mathcal{M}//\mathcal{N}} x, \quad x \in \mathcal{W}.$$

Since $H_{P,C}$ is stable, $\gamma(\Pi_{\mathcal{M}//\mathcal{N}})(k) < \infty$ for all $k > 0$. Using Proposition 5.3, we see there exists a neighbourhood V_r of λ_0 such that

$$\|(\Phi_\lambda - I)\Pi_{\mathcal{M}//\mathcal{N}} x\|_\tau \leq \gamma(\Phi_\lambda - I)\big(\gamma(\Pi_{\mathcal{M}//\mathcal{N}})(2r)\big) < r \tag{5.9}$$

for all $x \in S_{2r}$ and $\lambda \in V_r$. This implies $\|\mathbf{A}_\lambda x\|_\tau < \|w\|_\tau + r < 2r$ for all $x \in S_{2r}$. Due to our assumption, we may suppose that $\mathbf{A}_\lambda$ is continuous and compact on S_{2r}. From Schauder's fixed point theorem, it follows that there exists $x_\lambda \in S_{2r}$ such that

$$x_\lambda = w + (\Phi_\lambda - I)\Pi_{\mathcal{M}//\mathcal{N}} x_n \quad \text{for } \lambda \in V_r,$$

ie.

$$w = \Pi_{\mathcal{N}//\mathcal{M}} x_\lambda + \Phi_\lambda \Pi_{\mathcal{M}//\mathcal{N}} x_\lambda.$$

Since $\Pi_{\mathcal{N}//\mathcal{M}} x_\lambda \in \mathcal{N}, \Phi_\lambda \Pi_{\mathcal{M}//\mathcal{N}} x_\lambda \in \mathcal{M}_\lambda$ and the perturbed system is well-posed, we have

$$\Pi_{\mathcal{M}_\lambda//\mathcal{N}} w = \Phi_\lambda \Pi_{\mathcal{M}//\mathcal{N}} x_\lambda = \Pi_{\mathcal{M}//\mathcal{N}} x_\lambda + (\Phi_\lambda - I)\Pi_{\mathcal{M}//\mathcal{N}} x_\lambda$$

and therefore

$$\begin{aligned}\|\Pi_{\mathcal{M}_\lambda//\mathcal{N}} w\|_\tau &= \|\Pi_{\mathcal{M}//\mathcal{N}} x_\lambda\|_\tau + \|(\Phi_\lambda - I)\Pi_{\mathcal{M}//\mathcal{N}} x_\lambda\|_\tau \\ &\leq \gamma(\Pi_{\mathcal{M}//\mathcal{N}})(2r) + \gamma(\Phi_\lambda - I)\big(\gamma(\Pi_{\mathcal{M}//\mathcal{N}})(2r)\big) \leq \gamma(\Pi_{\mathcal{M}//\mathcal{N}})(2r) + r.\end{aligned}$$

Hence $\gamma(\Pi_{\mathcal{M}_\lambda//\mathcal{N}})(r) \leq \gamma(\Pi_{\mathcal{M}//\mathcal{N}})(2r) + r < \infty$ for $\lambda \in V_r$. ☐

With the technical assumption of compactness of $T_\tau(\Phi_\lambda - I)\Pi_{\mathcal{M}//\mathcal{N}}$, this result states the boundedness of $H_{P_\lambda,C}(u_0, y_0)^\top$ for $\|(u_0, y_0)^\top\| \leq r$ and λ sufficiently close to λ_0. Obviously, if the neighbourhood V_r for (5.9) is independent of r, that is, if

$$\gamma(\Phi_\lambda - I)\big(\gamma(\Pi_{\mathcal{M}//\mathcal{N}})(2r)\big) < r, \quad \text{for all } \lambda \text{ sufficiently close to } \lambda_0$$

hold for all (large) r, then $H_{P_\lambda,C}$ would be (gf)-stable.

6. Conclusions. The main contributions of this paper are as follows. Natural generalisations of the graph topology w.r.t. to a gain function notion of stability for nonlinear systems in a general normed signal space setting were defined. Convergence in the graph topology was shown to have a natural application in robust stability results. Various metrizations of the graph topologies were given; in particular it was shown that the generalisations of the gap metric given by [7] and the natural generalisation of the graph metric both induce the graph topology when the stability notion is that of an (unweighted) induced gain, subject to certain assumptions on local asymptotic completeness and the existence of normalized coprime factorisations. Weaker results have been derived for the more general cases (including the weighted case). Georgiou-type formulas [4] have been derived and are shown to be equivalent to other alternative formulations of the gap metric.

There are many directions for future work. An important topic is the extension of the above results to the ν-gap setting; in particular the investigation of a coprime factor characterisation of the underlying induced topology of the nonlinear generalisations of the ν-gap. A more fundamental area for future research concerns the investigation of the continuity of the closed loop response w.r.t. to gap perturbations to the loop, probably involving greater regularity assumptions [7]. A final area of worthy future study concerns the explicit study of the numerical computation of the gap, possibly based on the Georgiou-type formula's, but with additional regularity assumptions on the minimiser Q, perhaps allowed by greater regularity assumptions on P and C. In this regard, nonlinear generalisations of the commutant lifting theory may be the appropriate tool.

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