DESIGN OF OVERCOMPLETE EXPANSIONS FOR CHANNEL CODING

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ABSTRACT

The redundancy afforded by overcomplete expansions have been recently been considered for channel coding. In this paper, we utilise this approach in order to propose a channel coder design to for a correlated additive Gaussian noise channel, of which the noise covariance matrix is assumed to be known. We demonstrate that this approach can lead to a significant reduction of the noise interference by exploiting both the correlation of the channel and the redundancy of the filter banks. Simulation results providing some insight into these mechanisms are provided.

1. INTRODUCTION

Overcomplete expansion offer redundancy and design freedom, which has recently been used for the reconstruction of erased or erroneously received expansion coefficients [1], or for the design of error correction codecs [2]. More recently in [3] a systematic parallelism between block codes and oversampled filter bank (OSFB) systems implementing overcomplete expansions for channel coding has been drawn, whereby the system design is based on unquantised "soft-input" signals [4]. The channel coding schemes in [2, 3, 4] are based on an encoding stage using a preset analysis filter bank. In the decoding stage an OSFB synthesis exploits the redundancy in order to reconstruct the signal — either perfectly or in the mean square error sense — while ideally projecting away from the noise. Most OSFBs consist of simple discrete Fourier transforms [2, 3, 4], which leads to low cost implementations, robustness towards burst-type errors and compatibility with OFDM-based modulation system.

If the additive channel noise is correlated, the projection in [3] is performed in the direction of the principle components of the noise subspace, which ideally is restricted such that a noise-free signal subspace exists. Also, in [2, 3, 4] the synthesis design is, despite some degrees of freedoms (DOFs) due to oversampling, limited by the a-priori choice of the analysis filter bank. In [5], the synthesis filter bank is given more flexibility by aiming the design at the suppression of the channel noise under the constraint of invertibility, such that an analysis filter bank encoder can be derived from the synthesis bank. However, the filter bank design in [5] is based on a crude iterative method that can prove the potential of the approach but is otherwise far from optimal.

Therefore, in this paper we follow the channel codec scheme in [5] for a correlated additive Gaussian noise channel, but apply a considerably improved constrained synthesis filter bank design method based on the second order sequential best rotation (SBR2) algorithm [6]. By linking the remaining noise variance after decoding to the covariance matrix of the channel noise in dependency of the synthesis filter bank, a suitable broadband eigenvalue decomposition using SBR2 leads to a paraunitary filter bank design that exploits both the correlation of the channel noise as well as the DOFs provided by the OSFBs.

The paper is organised as follows. Based on an analysis of the overall setup in Sec. 2, the codec design is presented in Sec. 3. Simulations discussed in Sec. 4 will provide some insight into the working and benefits of the proposed method.

2. SYSTEM MODEL

2.1. Overcomplete Expansions

For implementation and analysis purposes, OSFBs implementing overcomplete expansions can be conveniently represented by polyphase analysis and synthesis matrices [7]. A K channel OSFB decimated by $N < K$ is thus characterised by a polyphase analysis matrix $\mathbf{H}(z) \in \mathbb{C}^{K\times N}(z)$, while a polyphase synthesis matrix $\mathbf{G}(z) \in \mathbb{C}^{N\times K}(z)$ can denote the synthesis operation. Using a demultiplexing operation for the input signal $X(z)$ into N polyphase components

$$X(z) = \sum_{n=0}^{N-1} z^{-N+n-1} X_n(z^N) \tag{1}$$

as shown in 1, a vector $\underline{Y}(z)$ containing the filter bank outputs can be written as $\underline{Y}(z) = \mathbf{H}(z)\underline{X}(z)$, whereby $\underline{X}(z)$ hold the N polyphase components $X_n(z)$, $n = 0(1)N-1$. A filter bank system is perfectly reconstructing if

$$\mathbf{G}(z)\mathbf{H}(z) = z^{-\Delta}\mathbf{I}_N \quad . \tag{2}$$

The redundancy afforded by OSFBs has recently attracted attention for channel coding [2]. There, a coding rate $N/K < 1$ can ensure robustness against noise interference, with the aim of restoring noise corrupted samples due to the redundant format in which the data is transmitted. The analysis and synthesis filter banks function as encoder and decoder, while the filters in $\mathbf{H}(z)$ and $\mathbf{G}(z)$ are no longer limited to a bandpass design, but will rather be selected according to the characteristics of the interfering noise.

2.2. Channel Codec

The general channel coding system using OSFBs is given in Fig. 1. The polyphase components $\hat{X}_n(z)$ of the received signal in Fig. 1 can be collected similar to $\underline{X}(z)$ in a vector $\underline{\hat{X}}(z)$, which is given by

$$\underline{\hat{X}}(z) = \mathbf{G}(z)\left(\underline{Y}(z) + \underline{W}(z)\right) \tag{3}$$

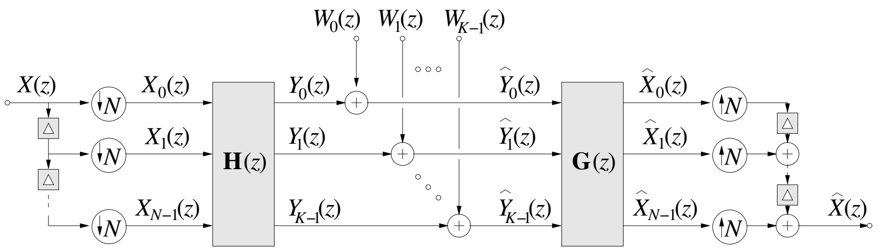

Fig. 1. General setup of a channel coder based on K channel analysis and synthesis filter banks, arranged around the transmission over K additive Gaussian noise channels.

whereby $\underline{Y}(z) = \mathbf{H}(z)\underline{X}(z) \in \mathbb{C}^K(z)$ and $\underline{W}(z) \in \mathbb{C}^K(z)$ contain the subband signal components of the transmitted data and the noise, respectively. Selecting perfect reconstruction filter banks $\mathbf{G}(z)\mathbf{H}(z) = \mathbf{I}_N$,

$$\underline{E}(z) = \underline{X}(z) - \underline{\hat{X}}(z) = -\mathbf{G}(z)\underline{W}(z) \tag{4}$$

is obtained.

To assess the total received noise variance σ_e^2 in $\hat{X}(z)$, let the N-element vector $\mathbf{e}[m]$ contain the N time series associated with the z-domain quantities in $\underline{E}(z) \bullet\!\!-\!\!\circ\, \mathbf{e}[m]$, which depend on the time index m in the decimated domain. Thus we have

$$\sigma_e^2 = \frac{1}{N}\mathrm{tr}\{\mathcal{E}\Big\{\mathbf{e}[m]\,\mathbf{e}^{\mathrm{H}}[m]\Big\}\} \quad , \tag{5}$$

where $\mathrm{tr}\{\cdot\}$ denotes trace and $\mathcal{E}\{\cdot\}$ is the expectation operator. Defining the auto-correlation matrix

$$\mathbf{R}_{ee}[\tau] = \mathcal{E}\Big\{\mathbf{e}[m]\,\mathbf{e}^{\mathrm{H}}[m-\tau]\Big\} \tag{6}$$

and its z-transform $\mathbf{R}_{ee}(z) \bullet\!\!-\!\!\circ\, \mathbf{R}_{ee}[\tau]$ denoting the power spectrum of the process $\mathbf{e}[m]$ [7], the noise variance is given by

$$\begin{aligned} \sigma_e^2 &= \frac{1}{N}\mathrm{tr}\{\mathbf{R}_{ee}[0]\} = \frac{1}{N}\mathrm{tr}\{\mathbf{R}_{ee}(z)\}\big|_{z=0} && (7)\\ &= \frac{1}{N}\mathrm{tr}\{\mathbf{G}(z)\mathbf{R}_{ww}(z)\tilde{\mathbf{G}}(z)\}\big|_{z=0} \quad , && (8) \end{aligned}$$

Note that (4) has been exploited to trace the noise variance back to the power spectrum $\mathbf{R}_{ww}(z)$, which is the z-transform of the covariance matrix of the channel noise,

$$\mathbf{R}_{ww}[\tau] = \mathcal{E}\Big\{\mathbf{w}[m]\,\mathbf{w}^{\mathrm{H}}[m-\tau]\Big\} \tag{9}$$

with $\mathbf{w}[m] \circ\!\!-\!\!\bullet\, \underline{W}(z)$ as defined in Fig. 1.

3. CODEC AND FILTER BANK DESIGN

Based on the idea of the channel codec outlined in Sec. 3.1, this section considers a suitable factorisation of the power spectrum at the decoder output in Sec. 3.2, admitting a useful codec design in Sec. 3.3. An algorithm to construct filter banks achieving this design is reviewed in Sec. 3.4.

3.1. Proposed Coding Approach

Note that the current standard approach [3] exploits the degrees of freedom in the design of $\mathbf{G}(z)$ for a given $\mathbf{H}(z)$ in order to minimise the noise power σ_e^2 in (5). The proposed method aims to minimise σ_e^2 by optimising $\mathbf{G}(z)$ without restrictions. The only condition placed on $\mathbf{G}(z)$ is that it admits a right inverse $\mathbf{G}^\dagger(z)$ such that $\mathbf{G}(z)\mathbf{G}^\dagger(z) = z^{-\Delta}$. A stronger restriction placed on $\mathbf{G}(z)$ is paraunitarity, which however has two important advantages: (i) the analysis filter banks is immediately given by $\mathbf{H}(z) = \tilde{\mathbf{G}}(z)$, and (ii) paraunitarity provides a minimum norm solution such that the transmit power is limited. As a counter example, an invertible $\mathbf{G}(z)$ might elicit an ill-conditioned $\mathbf{H}(z)$ which may encourage to transmit highly powered signals over subspaces associated with near rank deficiency.

3.2. Factorisation of the Noise Covariance Matrix

We approach the minimisation of (8) via a factorisation of the power spectrum

$$\mathbf{R}_{ww}(z) = \mathbf{U}(z)\mathbf{\Gamma}(z)\tilde{\mathbf{U}}(z) \tag{10}$$

such that $\mathbf{U}(z) \in \mathbb{C}^{K\times K}(z)$ is paraunitary and strongly decorrelates $\mathbf{R}_{ww}(z)$, i.e.

$$\mathbf{\Gamma}(z) = \mathrm{diag}\{\Gamma_0(z),\ \Gamma_1 z,\ \ldots\ \Gamma_{K-1}(z)\} \tag{11}$$

is a diagonal matrix with polynomial entries $\Gamma_k(z)$. This factorisation presents a broadband eigenvalue decomposition, which can be further specified by demanding $\mathbf{\Gamma}(z)$ to be spectrally majorised [6] such that the power spectral density of the kth noise component $\Gamma_k(e^{j\Omega}) = \Gamma_k(z)\big|_{z=e^{j\Omega}}$ evaluated on the unit circle obeys

$$\Gamma_k(e^{j\Omega}) \geq \Gamma_{k+1}(e^{j\Omega}) \qquad \forall\Omega \quad \text{and} \quad k = 0(1)K-2, \tag{12}$$

similar to the ordering of the singular values in a singular value decomposition. Note that paraunitarity or losslessness of $\mathbf{U}(z)$ conserves power, i.e. $\mathrm{tr}\{\mathbf{\Gamma}(z)\}|_{z=0} = \mathrm{tr}\{\mathbf{R}_{ww}(z)\}|_{z=0}$.

3.3. Codec Design

Using the redundancy $N < K$ due to oversampling, we can construct $\mathbf{G}(z)$ from $\mathbf{U}(z)$ to select the lower (and therefore smallest) N elements on the main diagonal of $\mathbf{\Gamma}(z)$. Let

$$\mathbf{U}(z) = [\underline{U}_0(z)\ \ \underline{U}_1(z)\ \ \cdots\ \ \underline{U}_{K-1}(z)] \quad , \tag{13}$$

then

$$\mathbf{G}(z) = \begin{bmatrix} \underline{\tilde{U}}_{K-N}(z) \\ \underline{\tilde{U}}_{K-N+1}(z) \\ \vdots \\ \underline{\tilde{U}}_{K}(z) \end{bmatrix} \in \mathbb{C}^{N\times K}(z) \quad , \tag{14}$$

such that $\mathbf{G}(z)\mathbf{U}(z) = [\mathbf{0}_{N\times K-N}\ \ \mathbf{I}_N]$. If

$$\mathbf{\Gamma}(z) = \begin{bmatrix} \mathbf{\Gamma}_{00}(z) & \mathbf{\Gamma}_{01}(z) \\ \mathbf{\Gamma}_{10}(z) & \mathbf{\Gamma}_{11}(z) \end{bmatrix} \tag{15}$$

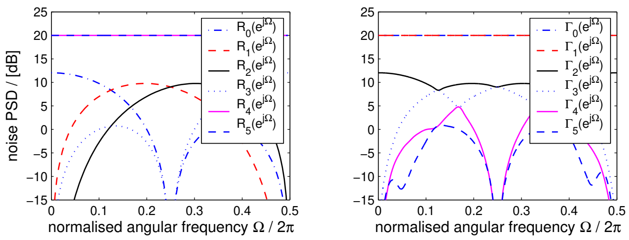

Fig. 2. PSDs on the main diagonals of (top) the power spectrum $\mathbf{R}_{ww}(z)$ of the channel noise consisting of the $R_k(e^{j\Omega})$ of (18) and (bottom) $\mathbf{\Gamma}(z)$ after application of the SBR2 algorithm.

with $\mathbf{\Gamma}_{11}(z) \in \mathbb{C}^{N\times N}$ and the remaining sub-matrices of appropriate dimension, then the noise power at the decoder output becomes

$$\sigma_e^2 = \frac{1}{N}\mathrm{tr}\{\mathbf{\Gamma}_{11}(z)\}\big|_{z=0} \tag{16}$$

$$= \frac{1}{N}\sum_{i=K-N}^{K-1}\int_0^{2\pi}\Gamma_i(e^{j\Omega})\,d\Omega\ . \tag{17}$$

Therefore, the spectral majorisation in the broadband eigenvalue decomposition (10) is essential to the success of the proposed channel coder design.

3.4. Sequential Best Rotation Algorithm

In order to achieve the factorisation in (10) fulfilling spectral majorisation according to (12), we use the second order sequential best rotation (SBR2) algorithm [6]. SBR2 is an iterative broadband singular value decomposition technique, which finds a sequence of paraunitary matrix operations, each eliminating the largest off-diagonal element of the remaining power spectrum matrix. In extensive simulations, SBR2 has proven very robust and stable in achieving both a diagonalisation and spectral majorisation of any given covariance matrix, whereby the algorithm is stopped either after reaching a certain measure for suppressing off-diagonal terms or after exceeding a defined number of iteration [6].

4. SIMULATIONS AND RESULTS

4.1. Multichannel Transmission

In a first transmission scenario as shown in Fig. 1, K sub-channels are corrupted by Gaussian noise processes $w_k[m]$, $k = 0(1)K-1$, such that

$$\mathcal{E}\{w_k[m]w_j[m-\tau]\} = \begin{cases} 0 & \text{for } k \neq j; \\ r_k[\tau] \circ\!\!-\!\!\bullet\ R_k(e^{j\Omega}) & \text{for } k = j. \end{cases} \tag{18}$$

Specifically, power spectrum $\mathbf{R}_{ww}(z)$ is diagonal with power spectral densities (PSD) $R_k(e^{j\Omega})$ as defined in (18) and shown in Fig. 2(top).

In order to run the SBR2 algorithm on $\mathbf{R}_{ww}(z)$ requires the application of an arbitrary paraunitary matrix to perturb its purely diagonal structure. Thereafter, running SBR2 results in a diagonalisation after approximately 250 iteration, whereby a ratio of 10^{-3} between the energy of off-diagonal and on-diagonal terms is

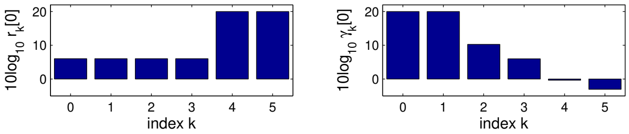

Fig. 3. Variances of (left) uncoded noise, $r_k[0]$, and (right) coded sub-channels, $\gamma_k[0] = \frac{1}{2\pi}\int_0^{2\pi}\Gamma_k(e^{j\Omega})d\Omega$.

reached. The PSDs of the main diagonal elements, $\Gamma_k(e^{j\Omega})$, are depicted in Fig. 2(bottom). Except for a low-power region of the bands $\Gamma_4(e^{j\Omega})$ and $\Gamma_5(e^{j\Omega})$ near $\Omega = \pi$, spectral majorisation has been achieved in the sense of (12).

Integrating over the PSDs in Fig. 2 provides the noise variance of the various sub-channels, which are illustrated in Fig. 3 for $\mathbf{R}_{ww}(z)$ and $\mathbf{\Gamma}(z)$ without and with coding, respectively. The coder would then be utilise the N coded sub-channels represented in$\mathbf{\Gamma}(z)$ that carry the lowest noise power. These N coded sub-channels then convey the N polyphase components of the transmitted signal $X(z)$, which according to Fig. 3 are subject to different levels of noise. Note that the polyphase component transmitted over the lowest sub-channel provides the best protection again noise, while noise introduced on higher sub-channels increases in power. This fact can be exploited for unequal error protection for, for example, high quality high-speed video transmission.

If a decimation factor of $N = 2$ is chosen for the filter banks, only the two coded sub-channels with the lowest noise variance in Fig. 3(right) will be utilised. The reduction in noise power results in an SNR enhancement of the coded scheme with respect to a transmission scenario of identical symbol throughput based on maximum ratio combining of the $K = 6$ channels in Fig. 3(left) of 7.5 dB. Note that a maximum ratio combiner uses a zero order diagonal $\mathbf{G}(z)$ and accordingly $\mathbf{H}(z)$, with the elements inversely proportional to the noise variance in the sub-channels.

Some insight into how the reduction of noise power is gained by the proposed coding method for the case $N = 2$ is demonstrated in Fig. 4, where the resulting characteristics of a $K = 6$ channel filter bank decimated by $N = 2$ are shown. The displayed magnitude response of the filter bank are plotted against the PSDs of the channel noise after $N = 2$ fold expansion. Fig. 4 very clearly underlines the functioning of the coder, which effectively excludes the two sub-channels with high noise power from transmission, while in all other sub-channels the transmitted power is concentrated in frequency bands where the noise PSD takes on its lowest values.

4.2. Time-Multiplexed Transmission

Through a time multiplexed transmission of the K encoded symbols over the same channel, the additive Gaussian noise $w[m]$ on the system results in polyphase components which are likely to be correlated. Hence the covariance $\mathbf{R}_{ww}(z)$ will not be of diagonal form, but can be shown to be pseudo-circulant.

We assume that the innovation filter is a 24th order finite impulse response (FIR) system with coefficients drawn from a zero mean unit variance white Gaussian process, resulting in a channel noise PSD of $w[m]$ as shown in Fig. 5. The noise and PSD within each of the sub-channels described by $\mathbf{R}_{ww}(z)$ for any given K is identical. Therefore, different from Sec. 4.1 it is additionally the inter-channel correlation that the coder has to exploit. After ap-

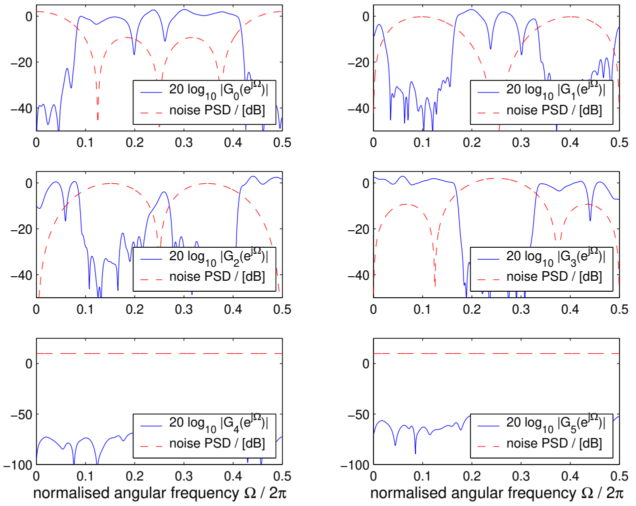

Fig. 4. PSDs of channel noise processes $w_k[m]$, $k = 0(1)K-1$, decimated by $N = 2$ and magnitude responses of the filters $|G_k(e^{j\Omega})| = |H_k(e^{j\Omega})|$.

plication of the SBR2 algorithm, the reduction in noise power — the ratio between the output power of the coder to the power of the channel noise process $w[m]$ — for various choices of K and N is depicted in Fig. 6. In comparison to maximum ratio combining with identical symbol throughput, the proposed coder in general perform consistently and considerably better, whereby an increase in K permits both a finer resolution to exploit spatial correlation as well as the use of more flexible code rates N/K.

5. CONCLUSIONS

In this paper we have proposed a channel coding approach based on OSFBs by first designing a decoder that minimises the influence of correlated channel noise in the receiver, and thereafter derives the encoder. By demanding paraunitarity for the decoding OSFB, the latter step is trivial and ensures a strict bound on the transmitted power. An OSFB design method has been proposed, which is based on a broadband eigenvalue decomposition and demonstrates good performance in suppressing the correlated channel noise. Some insight into the effects of the design have been given by considering transmission scenarios over K independent channels or by time multiplex transmission, where the exploitation of spatial or spectral correlations can bring substantial benefits over a transmission of identical symbol throughput using maximum ratio combining of the sub-channels.

The SNR enhancement gained from the proposed coding architecture can be utilised in conjunction with the transmission of quantised data such as found in binary phase shift keying or multilevel quadrature amplitude modulation symbols, such that the occurrence of symbol or bit errors is reduced.

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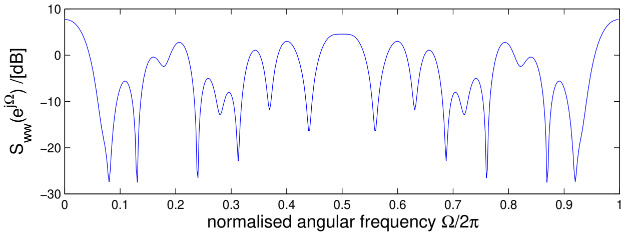

Fig. 5. Channel noise PSD in time multiplex channel I.

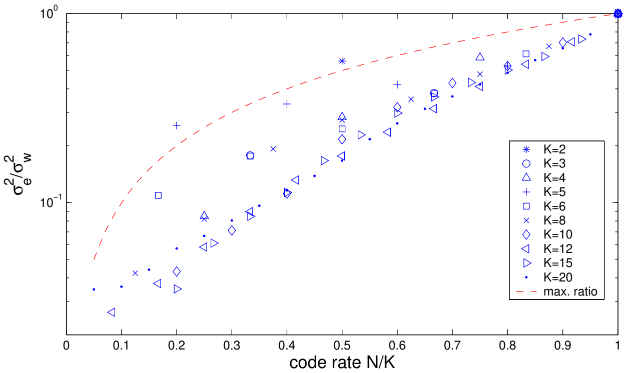

Fig. 6. Noise reduction achieved by the proposed coding scheme over channel I characterised in Fig. 5.

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