Efficient Force Calculations Based on Continuum Sensitivity Analysis

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Using continuum design sensitivity analysis (CDSA), in conjunction with the virtual work principle, equations have been derived for calculating forces without the need to solve the adjoint system. The resultant expressions are similar to the Maxwell stress tensor, but have the important advantage of the integration taking place on the surface of material rather than in the air outside. Implementation of the scheme leads to efficient calculations and improved accuracy.

***Index Terms*—Electromagnetic force calculation, finite elements, sensitivity analysis.**

I. Introduction

ALMOST all the commonly used methods for force calculation in low frequency devices including current driven excitations, permanent magnets and nonlinear materials, are based on one of two approaches. These are the Maxwell stress tensor (MST) and the virtual work principle (VWP). These two approaches are derived from somewhat different starting points. The MST is, classically, derived by starting from the Lorentz force expression describing the interactions of currents and magnetic fields, i.e., the $\mathbf{J} \times \mathbf{B}$ force. Virtual work, however, is based on the mechanical concepts of forces being related to the change in stored energy as a body changes its position in a magnetic field. A comprehensive review of force formulations, and their implications, is given in [1].

Unfortunately, both these methods have significant problems if they are implemented directly as they are derived. The MST describes the forces in terms of the flux density distributions on a contour surrounding the body. Theoretically, the force is computed by integrating over a surface (or around a contour in two dimensions) enclosing the volume of interest and the surface can be placed outside of the body *as long as it does not enclose any more bodies on which there is an electromagnetic force*. In a numerical solution, the flux density distributions are discretized, i.e., they are not smooth, and the value of the integral depends on where the contour is drawn. This leads to a large number of variants of the basic MST which try to overcome the problems inherent in applying it to a real numerical modeling system. However, its major benefit is that it requires only a single solution of the problem.

Virtual work, on the other hand, computes the force on a body by a virtual displacement of the body and computing the change in the co-energy of the system, i.e., the force is given by $\partial W'/\partial x$, where x is a displacement in the direction if the force. In a traditional numerical computation where the system produces the value at one point in the solution space, the gradient of the co-energy function is not easily available. In this case, the "virtual" displacement is simulated by a small physical displacement and a second problem is solved. The differential can then be computed through a finite difference approach. The disadvantage of such a system is that it requires two solutions.

Variations of the two previous approaches have been considered by many authors in an attempt to improve the accuracy and reduce the computation. Of major note is the work of Coulomb [2], which can be viewed as an implementation of a discrete sensitivity approach. However, such an approach is developed around the numerical method used to solve the problem. This work was further developed by McFee *et al.* [3] who showed that an "average" Maxwell stress approach is equivalent to virtual work and the approach given in [2]. A related approach for the computation of forces at a point on the surface of a body has been presented by Kameari [4]. More recently, there has been a discussion of force distributions within ferromagnetic bodies [5].

Continuum sensitivity analysis, however, allows the computation of the sensitivity of any global quantity to a perturbation in a parameter to be computed without reference to the underlying numerical computation scheme [6]. In effect, it allows a virtual work calculation to be performed without the need for a physical displacement and thus could allow for a more accurate force calculation. This paper focuses on using this analysis to implement a new virtual work based, single solution method of electromagnetic force calculation. In addition, it can be shown that this formulation includes the more traditional approaches such as the MST.

II. Derivation

A. Material Derivative

Consider a domain Ω, shown schematically in Fig. 1, where the domain shape is treated as the design variables. When dealing with variation of the shape, it is very convenient to think of Ω as a continuous medium and utilize the material derivative idea of continuum mechanics [7].

The process of deforming Ω to Ω_t during a time interval from 0 to t is given by a following mapping T:

$$T = \mathbf{x} \rightarrow \mathbf{x}_t = T(\mathbf{x}, t) \quad \mathbf{x} \in \Omega, \quad \mathbf{x}_t \in \Omega_t \qquad (1)$$

Digital Object Identifier 10.1109/TMAG.2005.844343

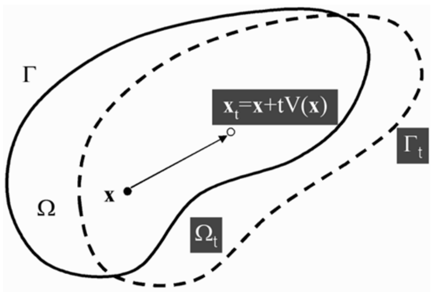

Fig. 1. Variation of domain.

Along the trajectory of the point x, a design velocity can be defined as

$$\mathbf{V}(\mathbf{x}_t, t) \equiv \frac{d\mathbf{x}_t}{dt} = \frac{\partial T(\mathbf{x}, t)}{\partial t}. \tag{2}$$

From the point of view of the VWP, it can be considered that the design velocity corresponds to a virtual displacement. In the neighborhood of an initial time $t = 0$, T can be expressed by

$$T(\mathbf{x}, t) = \mathbf{x} + t\mathbf{V}(\mathbf{x}) \tag{3}$$

where $\mathbf{V}(\mathbf{x}) = \mathbf{V}(\mathbf{x}, 0)$ and higher order approximate functions are ignored. Therefore, using the transformation T, the material derivative of the magnetic vector potential $\mathbf{A}$ belonging to Ω can be defined as

$$\dot{\mathbf{A}} = \frac{d}{dt}\mathbf{A}_t(\mathbf{x} + t\mathbf{V}(\mathbf{x}))\bigg|_{t=0} = \mathbf{A}' + \nabla\mathbf{A}\cdot\mathbf{V}(\mathbf{x}) \tag{4}$$

where is A' the partial derivative of $\mathbf{A}_t$. Consider a function P defined as an integral of h_t over the domain Ω_t

$$P = \int_{\Omega_t} h_t(\mathbf{x}_t)\, d\Omega_t \tag{5}$$

where h_t is a differentiable function. The derivative of P with respect to its material parameters is given by

$$\boldsymbol{d}P/\boldsymbol{d}p = \int_{\Omega} h'(\mathbf{x})\, d\Omega + \int_{\Gamma} h(\mathbf{x})(\mathbf{V}\cdot\mathbf{n})\, d\Gamma. \tag{6}$$

B. Sensitivity Formula

When dealing with design optimization of magnetostatic systems an objective function O mathematically expressed by (7) is usually encountered

$$O = \int_{\Omega} g(\mathbf{A}(\mathbf{p}), \nabla\times\mathbf{A}(\mathbf{p}))\, d\Omega \tag{7}$$

where g is a scalar function differentiable with respect to the magnetic vector potential, $\mathbf{A}$, and $\nabla\times\mathbf{A}$ that are themselves implicit functions of the design variable vector $\mathbf{p}$. In order to deduce the design sensitivity and the adjoint system equation systematically, the variational form of Maxwell's equation, referred to as the primary system, is added to (7) based on the augmented Lagrangian method

$$\bar{O} = \int_{\Omega} g(\mathbf{A}, \nabla\times\mathbf{A})\, d\Omega + \int_{\Omega} \boldsymbol{\lambda}[-\nabla\times(\upsilon\nabla\times\mathbf{A} - \mathbf{M}) + \mathbf{J}]\, d\Omega \tag{8}$$

where λ is the Lagrange multiplier vector interpreted as the adjoint variable. In a nonlinear system, the solution produced results in a variation of reluctivity across the problem domain. However, from the point of view of the virtual work perturbation, the solution could be considered as equivalent to a linear problem with the reluctivities in the problem frozen at the incremental value determined by the nonlinear solution. In this situation, each point in the nonlinear material has a reluctivity equal to its incremental reluctivity and a magnetization equal to the effective coercive force for the linear B–H relationship. Thus, the spatial distribution of the reluctivity υ in (8) can be assumed to be already known under the given source, permanent magnetization $\mathbf{M}$ and current density $\mathbf{J}$, at an arbitrary time t.

To obtain an explicit expression for the variation of the interface boundary between different materials, Ω_1 and Ω_2, the second integral on the right-hand side of (8) is split into the two regions. Then, we take the material derivative on both sides of (8) as

$$\begin{aligned}\dot{\bar{O}} = &\int_{\Omega_1}[\mathbf{g}_{\mathbf{A}}\cdot\bar{\boldsymbol{\lambda}}_1 + \mathbf{g}_{\nabla\times\mathbf{A}}\cdot\nabla\times\bar{\boldsymbol{\lambda}}_1]\, d\Omega\\ &-\int_{\Omega_1}\upsilon_1[\nabla\times\bar{\boldsymbol{\lambda}}_1\cdot\nabla\times\boldsymbol{\lambda}_1 + \nabla\times\mathbf{A}_1\cdot\nabla\times\bar{\mathbf{A}}_1]\, d\Omega\\ &-\int_{\Omega_2}\upsilon_2[\nabla\times\bar{\boldsymbol{\lambda}}_2\cdot\nabla\times\boldsymbol{\lambda}_2 + \nabla\times\mathbf{A}_2\cdot\nabla\times\bar{\mathbf{A}}_2]\, d\Omega\\ &+\int_{\Omega_1}[\mathbf{J}_1\cdot\bar{\mathbf{A}}_1 + \mathbf{M}_1\cdot\nabla\times\bar{\mathbf{A}}_1]\, d\Omega\\ &+\int_{\Omega_2}[\mathbf{J}_2\cdot\bar{\mathbf{A}}_2 + \mathbf{M}_2\cdot\bar{\mathbf{A}}_2]\, d\Omega\\ &+\int_{\gamma}[\upsilon_1\nabla\times\mathbf{A}_1\cdot\nabla\times\boldsymbol{\lambda}_1 - \upsilon_2\nabla\times\mathbf{A}_2\cdot\nabla\times\boldsymbol{\lambda}_2]V_n\, d\gamma\\ &-\int_{\gamma}[\mathbf{M}_1\cdot\nabla\times\boldsymbol{\lambda}_1 - \mathbf{M}_2\cdot\nabla\times\boldsymbol{\lambda}_2\\ &+\mathbf{J}_1\cdot\boldsymbol{\lambda}_1 - \mathbf{J}_2\cdot\boldsymbol{\lambda}_2]V_n\, d\gamma\end{aligned} \tag{9}$$

where $\mathbf{g}_{\mathbf{A}} = \partial g/\partial A$, $\mathbf{g}_{\nabla\times\mathbf{A}} = \partial g/\partial(\nabla\times A)$, $\bar{\mathbf{A}} \equiv (\dot{\boldsymbol{\lambda}} - \nabla\lambda\cdot\mathbf{V})$ and $\bar{\boldsymbol{\lambda}} \equiv (\dot{\mathbf{A}} - \nabla\mathbf{A}\cdot\mathbf{V})$. The notation γ denotes the part of the interface boundary that is allowed to move. The integrands related to $\bar{\mathbf{A}}$ in (9) vanish because they have the same variational form as the primary system (and are thus equal to zero). Then, the variational equation of the adjoint system corresponding to the primary system is defined by the integrands related to $\bar{\boldsymbol{\lambda}}$ in (9):

$$\begin{aligned}-\int_{\Omega_1}\upsilon_1\nabla\times\boldsymbol{\lambda}_1\cdot\nabla\times\bar{\boldsymbol{\lambda}}_1\, d\Omega - \int_{\Omega_2}\upsilon_2\nabla\times\boldsymbol{\lambda}_2\cdot\nabla\times\bar{\boldsymbol{\lambda}}_2\, d\Omega\\ +\int_{\Omega_1}[\mathbf{g}_{\mathbf{A}}\cdot\bar{\boldsymbol{\lambda}}_1 + \mathbf{g}_{\nabla\times\mathbf{A}}\cdot\nabla\times\bar{\boldsymbol{\lambda}}_1]\, d\Omega = 0\end{aligned} \tag{10}$$

where the pseudo-sources, $\mathbf{g}_{\mathbf{A}}$ and $\mathbf{g}_{\nabla\times\mathbf{A}}$, play the roles of the current density and permanent magnetization, respectively.

Finally, the sensitivity formula applicable to linear and nonlinear magnetostatic problems is given by

$$\begin{aligned}\frac{d\bar{O}}{d\mathbf{p}} = &\int_{\gamma}[(\upsilon_1 - \upsilon_2)\nabla\times\mathbf{A}_1\cdot\nabla\times\boldsymbol{\lambda}_2\\ &-(\mathbf{M}_1 - \mathbf{M}_2)\cdot\nabla\times\boldsymbol{\lambda}_2 - (\mathbf{J}_1 - \mathbf{J}_2)\cdot\boldsymbol{\lambda}_2]V_n\, d\gamma\end{aligned} \tag{11}$$

where the integral represents the variation of magnetic quantities experienced over the interface.

C. Self-Adjoint System

When dealing with the objective functions that are related to the system energy, the dual system consisting of the primary and the adjoint systems is self-adjoint. This is because the pseudo-sources coincide precisely with those of the current density and magnetization. Let the objective function be described by

$$\bar{O} = W = W_m + W'_m = \int_\Omega \mathbf{B}\cdot\mathbf{H}\,d\Omega = \int_\Omega \left[\int \mathbf{H}\cdot d\mathbf{B} + \int \mathbf{B}\cdot d\mathbf{H}\right] d\Omega = \int_\Omega \mathbf{J}\cdot\mathbf{A}\,d\Omega \quad (12)$$

where W_m and W'_m are referred to as the stored magnetic energy and co-energy, respectively. In this case, the pseudo-source is given by

$$\mathbf{g_A} = \partial(\mathbf{J}\cdot\mathbf{A})/\partial\mathbf{A} = \mathbf{J}, \quad \mathbf{g}_{\nabla\times\mathbf{A}} = 0. \quad (13)$$

Applying (13) to (10), the variational of the adjoint system is same as that of the primary system. Thus, $\mathbf{A} = \boldsymbol{\lambda}$ *and there is no need to solve the adjoint problem*. Based on the aforementioned relation, the expression of the variation of W (interpreted as electrical energy supplied to the system and split into the energy stored and the energy which appears as mechanical output) can now be written as

$$\frac{dW}{d\mathbf{p}} = \int_\gamma [(\upsilon_1 - \upsilon_2)\mathbf{B}_1\cdot\mathbf{B}_2 - (\mathbf{M}_1 - \mathbf{M}_2)\cdot\mathbf{B}_2 - (\mathbf{J}_1 - \mathbf{J}_2)\cdot\mathbf{A}_2]V_n\,d\gamma \quad (14)$$

where the first integral expresses the variation of the total input energy over γ.

D. Expression of Force

In order to make a connection between the energy sensitivity (14) and the mechanical force $\mathbf{F}$, we assume the constant current condition and a virtual displacement $\mathbf{p}$. $\mathbf{F}$ is then given by

$$\mathbf{F} = \left.\frac{\partial \boldsymbol{W}'_m(\mathbf{p},\mathbf{I})}{\partial\mathbf{p}}\right|_{\mathbf{I}=\mathrm{cons\,tan}\,t} = \frac{\partial \boldsymbol{W}}{\partial\mathbf{p}} - \frac{\partial \boldsymbol{W}_m}{\partial\mathbf{p}} \cong \frac{\boldsymbol{d}W}{\boldsymbol{d}\mathbf{p}} - \frac{\boldsymbol{d}W_m}{\boldsymbol{d}\mathbf{p}}. \quad (15)$$

If the domain Ω shown in Fig. 1 is occupied by magnetic materials of μ, $\mathbf{M}$ and $\mathbf{J}$, respectively, the magnetic force can be expressed over the interface γ in terms of three components—the force due to the magnetization in the material expressed in terms of the reluctivity difference across the interface

$$\mathbf{F}_{\mathrm{iron}} = \int_\gamma [(\upsilon_1 - \upsilon_2)\mathbf{B}_1\cdot\mathbf{B}_2]V_n\,d\gamma - \frac{1}{2}\int_\gamma \left[\int \mathbf{H}_1\cdot d\mathbf{B}_1 - \int \mathbf{H}_2\cdot d\mathbf{B}_2\right] V_n\,d\gamma. \quad (16)$$

The force due to permanent magnet magnetizations on either side of the interface

$$\mathbf{F}_{\mathrm{magnet}} = \int_\gamma [(\mathbf{M}_2 - \mathbf{M}_1)\cdot\mathbf{B}_2]V_n\,d\gamma - \frac{1}{2}\int_\gamma \left[\int \mathbf{M}_2\cdot d\mathbf{B}_2 - \int \mathbf{M}_1\cdot d\mathbf{B}_1\right] V_n\,d\gamma \quad (17)$$

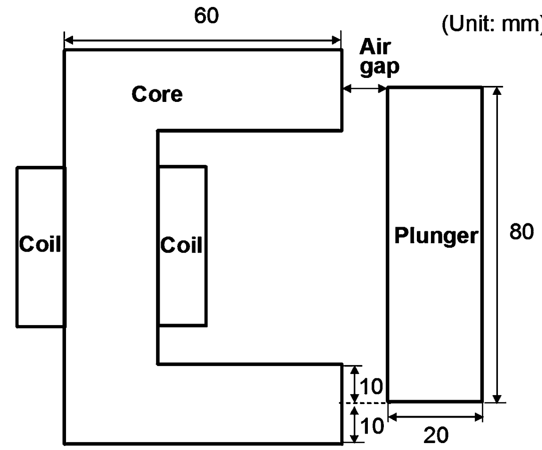

Fig. 2. C-core actuator structure.

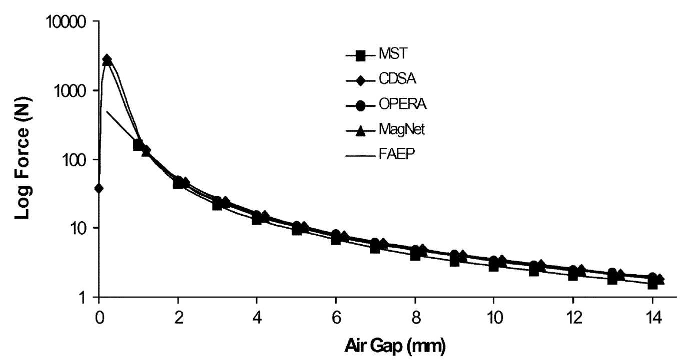

Fig. 3. Force versus distance in the C-core actuator for linear materials.

and the force due to the currents on either side of the interface

$$\mathbf{F}_{\mathrm{conductor}} = \int_\gamma [(\mathbf{J}_2 - \mathbf{J}_1)\cdot\mathbf{A}_2]V_n\,d\gamma. \quad (18)$$

III. Implications of the Analysis

The expressions developed in (16)–(18) have several interesting implications. Equation (16) gives the force on an interface between materials of two different reluctivities (or permeabilities). As will be shown in the examples below, when the material outside the body is air, this expression is in agreement with an MST calculation in the air surrounding the body. However, unlike the MST derivation, the expression can also evaluate the forces between two touching bodies. Additionally, if the velocity V_n is defined on only one edge segment, then the force computed applies only to that edge. This would seem to suggest that the approach can produce force distributions, as well as global forces. Equation (17) provides the forces due to the presence of permanent magnets only and (18) represents the force due to current carrying conductors only, i.e., it is a surface integral equivalent of the volume integral of $\mathbf{J}\times\mathbf{B}$. Unlike the MST approach, these equations clearly illustrate the contributions to the global force on a body in terms of each source of the magnetic field.

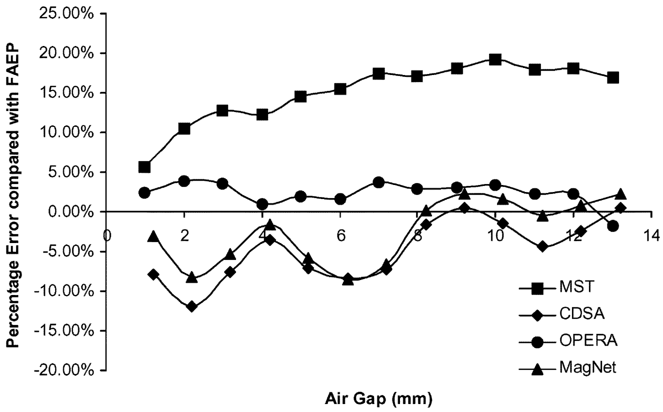

Fig. 4. Normalized errors for linear materials.

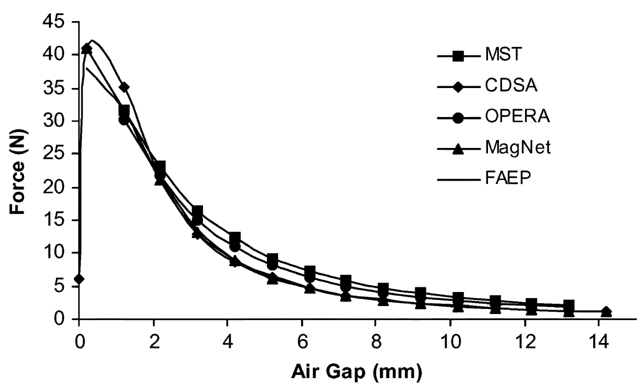

Fig. 5. Force versus distance in the C-core actuator for a nonlinear material (M19).

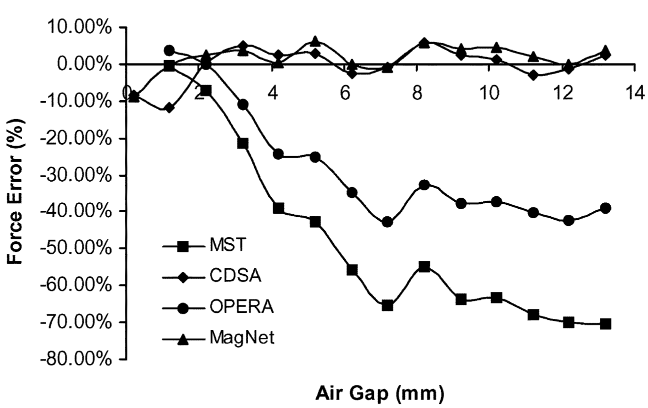

Fig. 6. Normalized errors for a nonlinear material.

IV. Examples

The formulation given in (16)–(18) has been tested against other calculation methods based on either traditional virtual work or variants of the MST. Several models have been considered. The first is a simple C-core actuator shown in Fig. 2. The core is 60-mm wide by 100-mm high and the poles are 20-mm wide. The "plunger" has dimensions of 20 mm by 80 mm. The air gap ranges from 14.2 to 0 mm. In the first example, the core and plunger were set to be constructed from a linear material with a relative permeability of 1000. The force results are shown in Fig. 3 and the errors [based on the differentiation of the co-energy curve (FAEP)] are shown in Fig. 4. As can be seen, the results match those from the other methods and have much the same accuracy.

Figs. 5 and 6 show the force results and errors for the same problem but with the core and plunger constructed from a typical, nonlinear electrical steel. Again, good agreement is shown between the new method and the more conventional approaches. In this case, some of the error is probably due to the inaccuracy of the interpolation used on the magnetization curve for the material to compute the incremental reluctivities.

V. Conclusion

This paper has discussed the derivation of a force computation algorithm based on continuum design sensitivity analysis. The approach can generate global forces as well as force distributions over the surface of a body and can work in a situation where the air gap has been reduced to zero. In addition, the force expression clearly indicates the contributions to the global force from each source of magnetic field. The implementation is simple and is independent of the numerical analysis approach taken. The method has been compared with well tested implementations of traditional approaches and has been shown to agree well with them.

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Manuscript received June 8, 2004.