

A Forward Regression Algorithm based on M-estimators

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Abstract

This paper introduces an orthogonal forward regression (OFR) model structure selection algorithm based on the M-estimators. The basic idea of the proposed approach is to incorporate an IRLS inner loop into the modified Gram-Schmidt procedure. In this manner the OFR algorithm is extended to bad data conditions with improved performance due to M-estimators' inherent robustness to outliers. An illustrative example is included to demonstrate the effectiveness of the proposed algorithm.

1 Introduction

The orthogonal forward regression (OFR) is an efficient algorithm to determine a parsimonious model structure [3]. Driven by requirements for improved model generalization, a few variants of OFR have been introduced in order to tackle ill-conditioning problem by modifying the selective criteria in forward regression [5]-[10]. Although these methods do not generally need the assumption of a normal error distribution, the parameter estimator may not be statistically optimal if the data exhibit bad conditions such as outliers, or are heavy tailed compared to normal distribution. As a generalization of maximum-likelihood estimation method for data with outliers, the general method of M-estimation [12] is well established to tackle outliers in observational data. Computationally M-estimator can be derived using an iterative reweighted least squares (IRLS) algorithm. M-estimation has been applied successfully to time series prediction, image processing and pattern recognition [6, 2, 7]. This paper presents a new model identification algorithm that combines the M-estimator with forward regression. Based on the modified Gram-Schmidt procedure for orthogonal forward regression (OFR), the proposed algorithm incorporates an IRLS inner loop into the modified Gram-Schmidt procedure to derive a M-estimator of model parameters. In combination with D-optimality for model

structure selection[10], the proposed algorithm simultaneously derive robust model structure and parameter estimates for bad data conditions.

2 Preliminaries

A linear regression model (RBF neural network, B-spline neurofuzzy network) can be formulated as [8, 1]

$$y(t) = \sum_{k=1}^M p_k(\mathbf{x}(t))\theta_k + \xi(t) \quad (1)$$

where $t = 1, 2, \dots, N$, and N is the size of the estimation data set. $y(t)$ is system output variable, $\mathbf{x}(t) = [x_1(t), \dots, x_n(t)]^T$ is system input vector with an assumed known dimension of n . $p_k(\bullet)$ is a known nonlinear basis function, such as RBF, or B-spline fuzzy membership functions. $\xi(t)$ is an uncorrelated model residual sequence with zero mean and variance of σ^2 . θ_k is model parameter, and M is the number of regressors.

Eq.(1) can be written in the matrix form as

$$\mathbf{y} = \mathbf{P}\Theta + \Xi \quad (2)$$

where $\mathbf{y} = [y(1), \dots, y(N)]^T$ is the output vector. $\Theta = [\theta_1, \dots, \theta_M]^T$ is parameter vector, $\Xi = [\xi(1), \dots, \xi(N)]^T$ is the residual vector, and \mathbf{P} is the regression matrix

$$\mathbf{P} = \begin{bmatrix} p_1(1) & p_2(1) & \dots & p_k(1) & \dots & p_M(1) \\ p_1(2) & p_2(2) & \dots & p_k(2) & \dots & p_M(2) \\ \dots & \dots & \dots & \dots & \dots & \dots \\ p_1(N) & p_2(N) & \dots & p_k(N) & \dots & p_M(N) \end{bmatrix}$$

with $p_k(t) = p_k(\mathbf{x}(t))$. Denote the column vectors in \mathbf{P} as $\mathbf{p}_k = [p_k(1), \dots, p_k(N)]^T$, $k = 1, \dots, M$. An orthogonal decomposition of \mathbf{P} is

$$\mathbf{P} = \mathbf{W}\mathbf{A} \quad (3)$$

where $\mathbf{A} = \{\alpha_{ij}\}$ is an $M \times M$ unit upper triangular matrix and \mathbf{W} is an $N \times M$ matrix with orthogonal columns that

satisfy

$$\mathbf{W}^T \mathbf{W} = \text{diag}\{\kappa_1, \dots, \kappa_M\} \quad (4)$$

with

$$\kappa_k = \mathbf{w}_k^T \mathbf{w}_k, \quad k = 1, \dots, M \quad (5)$$

so that (2) can be expressed as

$$\mathbf{y} = (\mathbf{P}\mathbf{A}^{-1})(\mathbf{A}\Theta) + \Xi = \mathbf{W}\Gamma + \Xi \quad (6)$$

where $\Gamma = [\gamma_1, \dots, \gamma_M]^T$ is an auxiliary vector. The above orthogonal decomposition can be realized by the modified Gram-Schmidt algorithm [3], in which least squares parameter estimates are usually used. Based on the modified Gram-Schmidt algorithm, a few variants of forward OLS algorithms have been introduced to improve model generalization capability based on the concepts from Bayesian regularization/basis pursuit [9], experimental design and leave-one-out (LOO) score respectively [4, 11]. Although these methods do not generally need normality error distribution assumption, the parameter estimator may not be statistically optimal if the data exhibit bad conditions such as outliers, or are heavy tailed. The general method of tackling this problem is well established as M-estimation [12], which is a generalization of maximum-likelihood estimation method for data with outliers. The M-estimator [12] is described in the following section.

2.1 M-estimators

The M-estimators have been well studied [12]. Considering the linear regression model given by (1), M-estimator minimizes the cost function

$$V_M = \sum_{t=1}^N \rho(\xi(t)) \quad (7)$$

where the function $\rho(\xi(t))$ is some predetermined nonnegative functionals for different types of estimators, e.g. for least squares $\rho(\xi(t)) = \rho_L(\xi(t)) = \xi^2(t)$. Typically $\rho(\xi(t))$ is an even function and nondecreasing with respect to the absolute value of $\xi(t)$. The problem of least squares estimator is that V_M will be influenced by any outlier typified by a large absolute value $\xi(t)$, assuming that if any outlier has yet been detected and removed in the estimation data set. The general M-estimator can tolerate undetected outliers by assigning a smaller weight to observations with residuals with large absolute values, so the parameter estimates are less vulnerable to unusual data. The most common types of M-estimators are the *Huber* estimator given by [12]

$$\rho_H(\xi) = \begin{cases} \frac{1}{2}\xi^2 & \text{for } |\xi| \leq \tau \\ \tau|\xi| - \frac{1}{2}\tau^2 & \text{for } |\xi| > \tau \end{cases} \quad (8)$$

or the Turkey *bisquare* estimator, given by

$$\rho_B(\xi) = \begin{cases} \frac{\tau^2}{6} \{1 - [1 - (\frac{\xi}{\tau})^2]^3\} & \text{for } |\xi| \leq \tau \\ \frac{1}{6}\tau^2 & \text{for } |\xi| > \tau \end{cases} \quad (9)$$

where the parameter τ is called a tuning constant, e.g. it is common to choose $\tau = 1.345\sigma$ for the *Huber* estimator and $\tau = 4.685\sigma$ for the Turkey *bisquare* estimator. These values offer robustness against outliers, but yet produce 95% efficiency when the errors are normal [12].

The M-estimator can be derived by setting

$$\frac{\partial V_M}{\partial \Theta} \Big|_{\Theta = \hat{\Theta}_M} = \mathbf{0} \quad (10)$$

to yield

$$\frac{\partial V_M}{\partial \Theta} = \mathbf{P}^T \boldsymbol{\psi} = \mathbf{0} \quad (11)$$

where $\mathbf{0}$ is zero vector.

$$\begin{aligned} \boldsymbol{\psi} &= \left[\frac{\partial V_M}{\partial \xi(1)}, \dots, \frac{\partial V_M}{\partial \xi(N)} \right]^T \\ &= [\psi(\xi(1)), \dots, \psi(\xi(N))]^T \end{aligned} \quad (12)$$

where $\psi(\xi)$ is the derivative of $\rho(\xi)$ with respect to ξ . Define the weight function

$$\omega(t) = \frac{\psi(\xi(t))}{\xi(t)}, \quad \text{for } t = 1, \dots, N. \quad (13)$$

Equation (11) can be written as

$$\mathbf{P}^T \Omega \Xi = \mathbf{0} \quad (14)$$

where $\Omega = \text{diag}\{\omega(1), \omega(2), \dots, \omega(N)\}$, whose solution is given as the weighted least squares

$$\hat{\Theta}_M = \{\mathbf{P}^T \Omega \mathbf{P}\}^{-1} \mathbf{P}^T \Omega \mathbf{y} \quad (15)$$

Because $\omega(t)$'s are *a priori* unknown, an iteratively reweighted least square (IRLS) is required. The M-estimator IRLS procedure is as follows:

Denote m as the iteration step. Initially set $m = 1$, $\Omega^{(1)} = \mathbf{I}$ (i.e. least squares) to derive an initial model residuals $\xi^{(1)}(t)$, then for $m = 2, \dots, m_w$,

$$\omega^{(m)}(t) = \frac{\psi(\xi^{(m-1)}(t))}{\xi^{(m-1)}(t)}, \quad \text{for } t = 1, \dots, N. \quad (16)$$

From (8) and (9), the weight functions of *Huber* and the Turkey *bisquare* estimator can be explicitly given by

$$\omega_H^{(m)}(t) = \begin{cases} 1 & \text{for } |\xi^{(m-1)}(t)| \leq \tau \\ \frac{\tau}{|\xi^{(m-1)}(t)|} & \text{for } |\xi^{(m-1)}(t)| > \tau \end{cases} \quad (17)$$

and

$$\omega_B^{(m)}(t) = \begin{cases} [1 - (\frac{\xi^{(m-1)}(t)}{\tau})^2]^2 & \text{for } |\xi^{(m-1)}(t)| \leq \tau \\ 0 & \text{for } |\xi^{(m-1)}(t)| > \tau \end{cases} \quad (18)$$

respectively. Let $\Omega^{(m)} = \text{diag}\{\omega^{(m)}(1), \omega^{(m)}(2), \dots, \omega^{(m)}(N)\}$, then

$$\hat{\Theta}_M^{(m)} = \{\mathbf{P}^T \Omega^{(m)} \mathbf{P}\}^{-1} \mathbf{P}^T \Omega^{(m)} \mathbf{y} \quad (19)$$

$$\Xi^{(m)} = \mathbf{y} - \mathbf{P} \hat{\Theta}_M^{(m)} \quad (20)$$

where $\Xi^{(m)} = [\xi^{(m)}(1), \dots, \xi^{(m)}(N)]^T$ are ready for next iteration step. The above procedure iterates until the parameter estimator $\hat{\Theta}_M$ converges at $m = m_w$.

$$\hat{\Theta}_M = \{\mathbf{P}^T \Omega^{(m_w)} \mathbf{P}\}^{-1} \mathbf{P}^T \Omega^{(m_w)} \mathbf{y} \quad (21)$$

3 Model identification algorithm using forward regression with M-estimation

The modified Gram-Schmidt procedure can be used to perform the orthogonalization and parameter estimation, usually with parameters derived as least squares parameters. In this section a new model identification algorithm that combines M-estimator with forward regression is introduced based on the modified Gram-Schmidt procedure. Geometrically the system output vector \mathbf{y} , is projected onto a set of orthogonal basis vectors, $\{\mathbf{w}_1, \dots, \mathbf{w}_k, \dots\}$. For the modified Gram-Schmidt algorithm, the model residual is decreased by projecting the system output vector \mathbf{y} onto a new basis \mathbf{w}_k at step k . Denote model residual vector as $\Xi_{(k)}$, where the subscript denotes forward regression step k . Initially model residuals $\Xi_{(0)}$ is \mathbf{y} . The procedure at forward regression step k , can be explicitly interpreted as fitting the previous model residual vector $\Xi_{(k-1)}$ (as derived from forward regression step $(k-1)$) using a single variable \mathbf{w}_k to solve a new model residual vector $\Xi_{(k)}$. Because M-estimator can enhance model parameter robustness in bad data conditions such as outliers, the proposed algorithm in this work is a variant of modified Gram-Schmidt procedure that includes the IRLS inner loop so as to derive the M-estimators of the auxiliary vector Γ .

Starting from $k = 1$, the columns \mathbf{p}_j , $k + 1 \leq j \leq M$ are made orthogonal to the k th column at the k th stage. The D-optimality criterion [10] for each of \mathbf{p}_j , $k + 1 \leq j \leq M$ columns is evaluated, and the most relevant column is selected to be interchanged with the k th column. The M-estimator for the k th regressor (the selected regressor) is then derived, as shown below, via the proposed Re-weighted least squares (IRLS) inner loop. The operation is repeated for $1 \leq k \leq n_\theta < (M - 1)$.

The following IRLS algorithm inner loop aims to derive either *Huber* or *bisquare* M-estimator for the k th element of the auxiliary vector Γ , which is initialized as the ordinary least squares parameter estimator $\gamma_k^{(1)} = \frac{\mathbf{w}_k^T \Xi_{(k-1)}}{\mathbf{w}_k^T \mathbf{w}_k} \neq 0$.

Iterated Re-weighted least squares (IRLS) inner loop:

i. Initialize $m = 2$. Note that model residual vector is initialized as $\Xi_{(k)}^{(1)}$ based on the parameter $\gamma_k^{(1)}$.

ii. For *Huber* M-estimator, set $\tau = \tau_{(k)}^H = 1.345 \text{std}(\Xi_{(k)}^{(m-1)})$, where $\text{std}(\bullet)$ denotes standard deviation. Use (17) to construct

$$\Omega_H^{(m)} = \text{diag}\{\omega_H^{(m)}(\xi_{(k)}^{(m-1)}(1)), \omega_H^{(m)}(\xi_{(k)}^{(m-1)}(2)), \dots, \omega_H^{(m)}(\xi_{(k)}^{(m-1)}(N))\}. \quad (22)$$

or for *bisquare* M-estimator, set $\tau = \tau_{(k)}^B = 4.685 \text{std}(\Xi_{(k)}^{(m-1)})$. Then use (18) to construct

$$\Omega_B^{(m)} = \text{diag}\{\omega_B^{(m)}(\xi_{(k)}^{(m-1)}(1)), \omega_B^{(m)}(\xi_{(k)}^{(m-1)}(2)), \dots, \omega_B^{(m)}(\xi_{(k)}^{(m-1)}(N))\} \quad (23)$$

iii. Denote

$$\Omega^{(m)} = \begin{cases} \Omega_H^{(m)} & \text{for Huber M-estimator} \\ \Omega_B^{(m)} & \text{for bisquare M-estimator} \end{cases} \quad (24)$$

and

$$\gamma_k^{(m)} = \frac{\mathbf{w}_k^T \Omega^{(m)} \Xi_{(k-1)}}{\mathbf{w}_k^T \Omega^{(m)} \mathbf{w}_k} \quad (25)$$

$$\Xi_{(k)}^{(m)} = \Xi_{(k-1)} - \gamma_k^{(m)} \mathbf{w}_k \quad (26)$$

where $\Xi_{(k)}^{(m)} = [\xi_{(k)}^{(m)}(1), \xi_{(k)}^{(m)}(2), \dots, \xi_{(k)}^{(m)}(N)]^T$.

(NB. The orthogonal forward regression can be explicitly interpreted as fitting the previous model residual vector $\Xi_{(k-1)}$ using the selected orthogonal basis \mathbf{w}_k . While $\gamma_k^{(1)}$ is derived as least squares parameter estimates associated with \mathbf{w}_k , (25)-(26) are the direct application of (19)-(20) to derive Re-weighted least square parameter estimates for M-estimators.)

iv. If $\|\gamma_k^{(m)} - \gamma_k^{(m-1)}\| \geq \delta$, where δ is arbitrarily small number, then set $m = m + 1$, and goto step ii. Otherwise, set $\Xi_{(k)} = \Xi_{(k)}^{(m)}$, $\gamma_k = \gamma_k^{(m)}$. Finish the IRLS inner loop.

4 An illustrative example

Consider using an RBF network to approximate the 'sinc' function

$$z(x) = \frac{\sin(x)}{x}, \quad -10 \leq x \leq 10 \quad (27)$$

Table 1. RMS errors and model size of derived models with respect to true function z

		β					
		0	0.03	0.05	0.10	0.15	0.20
OFR with D-optimality and least squares	Training set	0.0102	0.0138	0.0143	0.0157	0.0175	0.0249
	Test set	0.0102	0.0135	0.0139	0.0158	0.0175	0.0254
	Model size	22	22	22	22	22	21
OFR with D-optimality and Huber M-estimator	Training set	0.0131	0.0139	0.0141	0.0129	0.0140	0.0219
	Test set	0.0131	0.0135	0.0136	0.0126	0.0137	0.0219
	Model size	22	22	22	22	22	21
OFR with D-optimality and Bisquare M-estimator	Training set	0.0128	0.0131	0.0137	0.0124	0.0135	0.0218
	Test set	0.0128	0.0128	0.0132	0.0121	0.0133	0.0217
	Model size	22	22	22	22	22	21

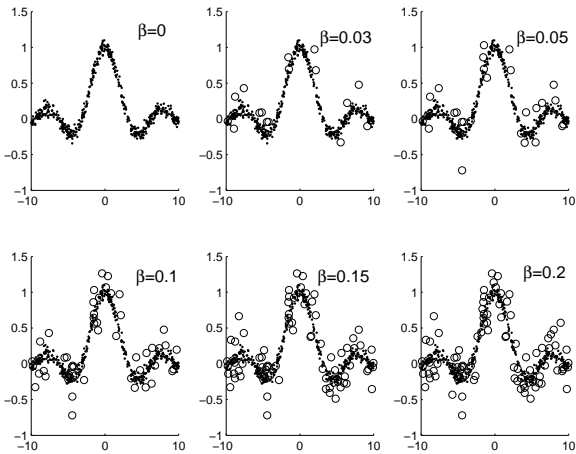


Figure 1. Data generated by 'sinc' function with additive noise of various levels of outliers; (Dotted – $N(0, 0.05^2)$ (normal) and Circle – $N(0, 0.2^2)$ (outliers))

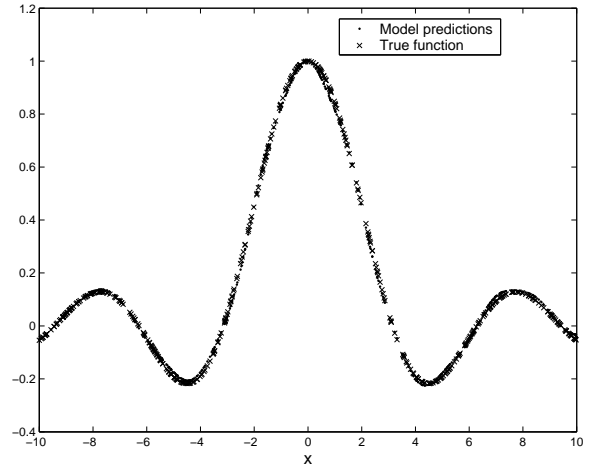


Figure 2. The Bisquare M-estimator model predictions with $\beta = 0.1$ and true functions.

1000 training data $y(x)$ were generated from $y(x) = z(x) + \xi$, using uniformly distributed random $x \in [-10, 10]$. The additive noise ξ is a Gaussian mixture that mixes two types of noises, a larger portion of normal noise with smaller variance and a smaller portion of noise with higher variance. i.e. $\xi \sim \beta N(0, 0.2^2) + (1 - \beta)N(0, 0.05^2)$, where $0 < \beta < 0.2$ as a small number to denote the contamination ratio, such that ξ has the probability $(1 - \beta)$ of being drawn from $N(0, 0.05^2)$ (as “normal”), and a probability β of $N(0, 0.2^2)$ (as “outliers”).

For various levels of contamination ratio β , 1000 noisy observations were generated and divided into a training data set of 500 data points and a test data set of 500 data points. The 500 training data points is shown in Fig.1 for different β . For each case, the proposed algorithm is applied based on the RBF network. All the training data points are used as the candidate centre set c_i 's, with $p_k(\mathbf{x}(t))$ constructed using Gaussian function $p_k = \phi(x, c_k) = \exp\{-\|x -$

$c_k\|^2/h^2\}$. The width $h = 1$ is fixed for simplicity. Note that by removing the IRLS inner loop of the algorithm, the procedure simply reduces to OFR with D-optimality algorithm [10]. With various values of β as different level of bad data conditions, the proposed algorithm is compared with OFR with D-optimality algorithm using only least squares estimates and SVM regression. All of the derived models based on OFR algorithm have the number of centers in the range of $n_\theta = 21 \sim 22$. The root of mean squares (RMS) errors of a range of data conditions are listed in Table 1. It is seen that the proposed algorithm is most robust to outliers when the data contains approximately 10% outliers. To achieve better performance for M-estimators, it is useful to slightly adjust tuning constants because these are set for 95% efficiency when data is normal. As data distribution is unknown these values can be adjusted via iterations and cross-validation. For the training data set with $\beta = 0.1$, the model predicted output by using the proposed algorithm with Turkey bisquare M-estimators is shown in Fig.2.

5 Conclusions

In this paper a new orthogonal forward regression (OFR) model identification algorithm has been introduced. The orthogonal forward regression (OFR), often based on the modified Gram-Schmidt procedure, is an efficient method incorporating structure selection and parameter estimation simultaneously. The proposed algorithm includes M-estimator by using an iterative re-weighted least squares (IRLS) algorithm inner loop based on the modified Gram-Schmidt procedure. D-optimality as a model structure robustness criterion is used in model selection. In this manner the proposed approach extends the use of the OFR algorithm for parsimonious model structure determination even in bad data conditions via the derivation of parameter M-estimators with inherent robustness to outliers.

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