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Quasi-Particles, Conformal Field Theory, and q-Series

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Abstract

We review recent results concerning the representation of conformal field theory characters in terms of fermionic quasi-particle excitations, and describe in detail their construction in the case of the integrable three-state Potts chain. These fermionic representations are q-series which are generalizations of the sums occurring in the Rogers-Ramanujan identities.

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1. Introduction

The fundamental problem of all condensed matter physics is the explanation of macroscopic phenomena in many-body systems in terms of a microscopic quantum mechanical description of the system. For all practical applications there is no dispute that the microscopic description of the world is as a collection of electrons and nuclei which interact by electromagnetic forces (which may usually be well thought of as non-relativistic Coloumb interactions and possibly a spin-orbit coupling). The problem is to extract macroscopic collective properties from this microscopic interaction.

The importance and difficulty of this problem is revealed in the question of the origins of organic chemistry. All organic molecules of biological significance, such as DNA, are optically active and rotate light in a preferred direction. This rotation clearly violates parity. Nevertheless, the underlying microscopic interaction is parity invariant. This vividly illustrates the fact that the physics of the collective excitations may be qualitatively different from that of the underlying microscopic system.

It is thus no surprise that the study of collective excitations in macroscopic systems is far from understood. It is also not surprising that approximate methods have only limited utility in building insight into these phenomena. Thus it is that ever since the invention of quantum mechanics there has been constant attention to the problem of finding and studying simplified microscopic model systems for which exact, nontrivial computations can be done which give insight into the relation of the collective to the microscopic.

In this paper we will discuss two such approaches which have proven exceedingly fruitful: integrable models of statistical mechanics and conformal field theory. We will discuss these in relation to what is one of the most simple of macroscopic properties: the low-temperature behavior of the specific heat. It is one of the loveliest discoveries of the past decade that this most simple of collective properties has profound connections to the theory of representations of affine Lie algebras and the mathematical study of q-series and generalized Rogers-Ramanujan identities.

2. Specific Heat and Quasi-Particles

Perhaps the most fundamental quantity used in the study of macroscopic systems is the partition function defined as

$$Z = \mathrm{Tr}\ e^{-H/k_B T} \tag{2.1}$$

where H is the hamiltonian, the trace is over all states of the system, k_B is Boltzmann's constant and T is the temperature. More explicitly this may be written as

$$Z = e^{-E_{GS}/k_BT} \sum_n e^{-(E_n - E_{GS})/k_BT} \tag{2.2}$$

where the sum is over all the eigenvalues E_n of H and we have explicitly factored out the contribution of the ground state energy E_{GS}.

For a macroscopic system we are usually more interested in the free energy per site f in the thermodynamic limit, defined as

$$f = -k_BT \lim_{M\to\infty} \frac{1}{M} \ln Z \ , \tag{2.3}$$

where M is the size of the system, and for concreteness we will think of H as the hamiltonian of a spin system of a linear chain of M sites. The thermodynamic limit is defined as

$$\text{fixed} \quad T > 0 \qquad \text{and} \qquad M \to \infty \ , \tag{2.4}$$

and the specific heat is given as

$$C \ = \ -T \ \frac{\partial^2 f}{\partial T^2} \quad . \tag{2.5}$$

The low-temperature behavior of the specific heat is now obtained by taking $T \to 0$.

To evaluate the sum (2.2) and thus to study the specific heat (2.5) we need to study the energy levels of H (which are obtained from Schrödinger's equation) in the $M \to \infty$ limit. We will further make the assumption that H is translationally invariant with periodic boundary conditions so that the momentum P is a good quantum number. It is then almost universally found that if $E - E_{GS}$ is finite and non-zero as $M \to \infty$ then the energy levels may be expressed in terms of single-particle levels $e_\alpha(P_i^\alpha)$, with α labelling the type of excitation, which depend on a momentum P_i^α and a set of combination rules as

$$E - E_{GS} \ = \ \sum_{\alpha,\text{rules}} \ \sum_{i=1}^{m_\alpha} e_\alpha(P_i^\alpha) \ , \tag{2.6}$$

and that the total momentum is given as

$$P \ \equiv \ \sum_{\alpha,\text{rules}} \ \sum_{i=1}^{m_\alpha} P_i^\alpha \ (\text{mod } 2\pi). \tag{2.7}$$

Energy levels of a many-body system of this form are said to be a quasi-particle spectrum. When one of the rules of composition is the fermi exclusion rule

$$P_i^\alpha \neq P_j^\alpha \qquad \text{if} \qquad i \neq j \ , \tag{2.8}$$

the spectrum is said to be fermionic.

If $e_\alpha(P)$ is positive for all P the system is said to have a mass gap, and the specific heat vanishes exponentially as $T \to 0$. However, in many spin chains one or more $e_\alpha(P)$ vanish as $P \to 0$ as

$$e(P) \ \sim \ v|P| \ , \tag{2.9}$$

where v is positive. These systems are said to be massless and v is called the speed of sound. If this massless single-particle energy is used in (2.6) and (2.2) and the momenta P_i are taken to have a uniform distribution it is a familiar result that (with a single species of excitation) the specific heat vanishes linearly when $T \to 0$ as

$$C \ \sim \ \frac{\pi k_B \tilde{c}}{3v} \, T \ , \tag{2.10}$$

where $\tilde{c}$ is a constant which is equal to $\frac{1}{2}$ in this case.

This argument, however, is not complete as is apparent from the observation that any energy level with $\lim_{M\to\infty} e(P) > 0$ will contribute only a term exponentially small in T to the specific heat. Thus the order one excitations which are of the form (2.6) do not contribute to the linear behavior (2.10). Instead, it is the levels with the property that $\lim_{M\to\infty} e(P) = 0$ which contribute to the leading behavior.

3. Conformal Field Theory

In contrast to the condensed matter description of quasi-particles of the previous section, the study of the $\frac{1}{M}$ excitation energies is much more recent and, in particular, the most remarkable progress has been made only in the last decade starting with the seminal work of Belavin, Polyakov and Zamolodchikov [1] on conformal field theory.

In the statistical mechanics context, the work of [1] applies directly to the continuum limit of two-dimensional lattice models which are assumed to exhibit conformal invariance at criticality. A fundamental object [2] in that framework is the finite-size (classical) partition function $\hat{Z}_{2d}$ of the critical system. Namely, consider the partition function at T_c of the (possibly anisotropic) system on an M by M' periodic lattice

$$Z_{2d}(M,M') = \sum_{\text{states}} e^{-\mathcal{E}/k_B T_c} = \sum_j (\Lambda_j(M))^{M'} \ , \tag{3.1}$$

which we expressed in terms of the eigenvalues $\Lambda_j(M)$ of the transfer matrix $\mathcal{T}_M$ in one of the directions. Defining the bulk free energy $f_{2d} \equiv -k_B T_c \lim_{M,M'\to\infty} \frac{1}{MM'} \ln Z_{2d} = -k_B T_c \lim_{M\to\infty} \frac{1}{M} \ln \Lambda_{\max}(M)$, the finite-size partition function is defined by scaling out the bulk free energy via

$$\hat{Z}_{2d} = \lim_{M,M'\to\infty} e^{MM' f_{2d}/k_B T_c} Z_{2d} = \lim_{M,M'\to\infty} \sum_j \left(\frac{\Lambda_j(M)}{\Lambda_{\max}(M)} \right)^{M'} \ . \tag{3.2}$$

the limit being taken with M'/M held fixed. $\hat{Z}_{2d}$ is a finite function of $q_{2d} = e^{\alpha M'/M}$, where α (possibly complex) depends on the anisotropy.

The analogous object in the context of the gapless spin chain which is of interest to us here, is the (quantum) partition function (2.2) in the limit

$$M \to \infty, \quad T \to 0 \qquad \text{with} \qquad MT \quad \text{fixed,} \tag{3.3}$$

which focuses directly on the order $\frac{1}{M}$ energy levels of the hamiltonian. More precisely, introducing $e_0 \equiv \lim_{M\to\infty} \frac{1}{M} E_{GS}$, define

$$\hat{Z} = \lim e^{M e_0/k_B T} Z \tag{3.4}$$

in the limit (3.3), so that $\hat{Z}$ is a finite function of

$$q = \exp\left(-\frac{2\pi v}{M k_B T}\right) \ . \tag{3.5}$$

For a hamiltonian obtained from a family of commuting transfer matrices $\mathcal{T}_M(u)$ of an integrable critical lattice model via $H = \frac{d}{du} \ln \mathcal{T}_M(u)\big|_{u=u_0}$, where u_0 is a special value of the spectral parameter where $\mathcal{T}_M$ becomes the identity, $\hat{Z}$ coincides as a function with the corresponding $\hat{Z}_{2d}$.

The limit (3.3) is not the same as the limit (2.4) which defines the specific heat. However, if no additional length scale appears in the system, it is expected that the behavior of the specific heat computed using the prescription (3.3) will agree when $q \to 1$ with the

$T \to 0$ behavior computed using the prescription (2.4). We are therefore led to discuss the $q \to 1$ behavior of $\hat{Z}(q)$.

An important feature of conformal field theory is [2] that $\hat{Z}$ can be expressed in the factorized form

$$\hat{Z}(q) \;=\; \sum_{k,l} N_{kl}\ \chi_k(q)\ \chi_l(\bar{q})\ , \tag{3.6}$$

where the $\chi_k(q)$ are characters of a chiral algebra [3], with the N_{kl} non-negative integers. (In so-called coset models of conformal field theory [4], the characters are known to be branching functions [5][6][7] of some affine Lie algebras.) In the two-dimensional context $\bar{q}$ in (3.6) is the complex conjugate of q, while in the one-dimensional one q and $\bar{q}$ are real and equal and are associated with contributions from right- and left-movers, respectively. We will restrict attention to rational conformal field theories, where the sum in (3.6) is finite. The characters take the form

$$\chi_k(q) = q^{\Delta_k - \frac{c}{24}}\hat{\chi}_k(q)\ \ , \qquad \hat{\chi}_k(q) = 1 + \sum_{n=1}^{\infty} a_n q^n\ \ , \tag{3.7}$$

with the a_n non-negative integers. Here c and the Δ_k are the central charge and conformal dimensions, respectively, of the conformal field theory.

The partition function $\hat{Z}_{2d}$ of the two-dimensional system must clearly have the property

$$\hat{Z}_{2d}(q) = \hat{Z}_{2d}(\tilde{q})\ , \tag{3.8}$$

where

$$\tilde{q} = e^{-2\pi i/\tau} \qquad \text{when} \qquad q = e^{2\pi i\tau}\ , \tag{3.9}$$

simply by symmetry in M and M' combined with an appropriate change in the anisotropy, when present. If (3.8) holds for $\hat{Z}$ as well, then one concludes from (3.6) and (3.7) that

$$\hat{Z}(q)\ \sim\ \tilde{q}^{-(c-12d_{\min})/12} \qquad \text{as} \quad q \to 1^-\ , \tag{3.10}$$

where $d_{\min}$ is the minimal $\Delta_k + \Delta_l$ such that $N_{kl} > 0$. This shows that the $q \to 0$ behavior of $\hat{Z}(q)$, determining the finite-size corrections to the ground state energy [8][9][10]

$$-k_B \lim_{T\to 0} T \ln Z \;=\; E_{GS} - Me_0 \;=\; -\frac{\pi(c-12d_{\min})v}{6M} + o(M^{-1})\ , \tag{3.11}$$

is related to the $q \to 1$ behavior which is relevant for the specific heat. Namely, from (3.10) and (3.5) we conclude that $\tilde{c}$ in (2.10) is given by

$$\tilde{c} = c - 12 d_{\rm min} \ , \tag{3.12}$$

where the rhs is called the effective central charge.

One of the objectives in this work is to point out an alternative method to compute the low-temperature specific heat, which is based on an analysis of the order one energy levels of the hamiltonian and bypasses the use of (3.8) which is a property not a priori obvious from the viewpoint of a generic one-dimensional chain. We will demonstrate (in two particular models) how the full partition function $\hat{Z}$ — or at least the "normalized characters" $\hat{\chi}_k(q)$ — can be obtained from the quasi-particle description of the spectrum discussed in sect. 2 (where the specifics of a model are encoded in the "rules" in (2.6)). The specific heat is then deduced using (3.6) from the $q \to 1$ behavior of the $\hat{\chi}_k(q)$, which can be determined by a steepest descent calculation. The leading behavior, which is the same for all characters in a given model, is

$$\hat{\chi}_k(q) \ \sim \ \tilde{q}^{-\tilde{c}/24} \qquad \text{as} \quad q \to 1^- \ , \tag{3.13}$$

where $\tilde{c}$ agrees with (3.12).

Let us emphasize that in this computation of the low-temperature specific heat no use of modular covariance [11] of the characters is made. The approach pioneered in [1] relies on the existence of conformal symmetry in the system, which severely constrains the order $\frac{1}{M}$ spectrum in terms of representations of some infinite-dimensional chiral algebra [3]. The characters of these chiral algebra representations are computed either abstractly [12] or by the Feigin-Fuchs-Felder construction [13][14]. Using these methods, the explicit expressions obtained for the characters usually involve modular forms, and therefore the characters $\chi_k(q)$ of a given model (regarded as functions of a complex variable q) can be seen to form [2][5][15] a representation of the modular group, generated by S: $q \to \tilde{q}$ and T: $q \to e^{2\pi i} q$. In particular, they satisfy a linear transformation law

$$\chi_k(\tilde{q}) \ = \ \sum_l S_{kl} \ \chi_l(q) \ , \tag{3.14}$$

from which (3.13) can be obtained.

However, the detailed connection between the above-mentioned expressions for the characters in terms of modular forms to a hamiltonian spectrum is rather obscure. In the approach of this paper, alternative expressions — which we call (fermionic) quasi-particle representations — for the characters are obtained from the spectrum, and thus a direct understanding of the conformal field theory partition function $\hat{Z}$ in terms of the underlying spin chain is gained.

We will now provide some more details in a few examples. In the past 10 years there has been an immense effort to discover and classify conformal field theories, compute the corresponding characters and partition functions, and identify the underlying statistical mechanics models. The earliest example is the series of minimal models $\mathcal{M}(p,p')$ [1], specified by pairs of coprime positive integers p and p', where the central charge is

$$c = 1 - \frac{6(p-p')^2}{pp'} \tag{3.15}$$

and the conformal dimensions are

$$\Delta_{r,s}^{(p,p')} = \frac{(rp'-sp)^2-(p-p')^2}{4pp'} \qquad (r=1,\ldots,p-1;\quad s=1,\ldots,p'-1). \tag{3.16}$$

The corresponding characters are [13][14][16]

$$q^{c/24}\chi_{r,s}^{(p,p')} = \frac{q^{\Delta_{r,s}^{(p,p')}}}{(q)_\infty}\sum_{k=-\infty}^{\infty}(q^{k(kpp'+rp-sp')}-q^{(kp'+s)(kp+r)}) \tag{3.17}$$

where

$$(q)_n = \prod_{k=1}^{n}(1-q^k)\ . \tag{3.18}$$

The unitary [17] minimal conformal field theories $\mathcal{M}(p,p+1)$ (with the A-series partition function [18]) were identified [19] as describing the continuum limit of the RSOS models of Andrews, Baxter and Forrester [20] at the critical point between regimes III and IV.

A second widely studied class of theories comprises the coset models [4]

$$\frac{(G_r^{(1)})_k \times (G_r^{(1)})_l}{(G_r^{(1)})_{k+l}}\quad , \tag{3.19}$$

where $(G_r^{(1)})_k$ is the affine Lie algebra at level k [12] based on the simply-laced Lie algebra G_r of rank r. (The unitary minimal models $\mathcal{M}(p,p+1)$ are obtained [4] from (3.19) by specializing to $G_r = A_1$, $k = p-2$, and $l = 1$.) For the case $k = l = 1$ and $G_r = A_{N-1}$ the model (3.19) is identical by level-rank duality [21] to the coset model $\frac{(A_1^{(1)})_N}{U(1)}$, known as $\mathbf{Z}_N$-parafermionic conformal field theory [22]. The central charge is

$$c = \frac{2(N-1)}{N+2} \quad , \tag{3.20}$$

and the characters are branching functions given by Hecke indefinite forms of [5][6] (or an equivalent form [23])

$$\begin{aligned} &q^{c/24} b^l_m = \\ &\frac{q^{h^l_m}}{(q)^2_\infty}\Bigg[\Bigg(\sum_{s\geq 0}\sum_{n\geq 0} - \sum_{s<0}\sum_{n<0}\Bigg)(-1)^s q^{s(s+1)/2+(l+1)n+(l+m)s/2+(N+2)(n+s)n} \\ &\quad + \Bigg(\sum_{s>0}\sum_{n\geq 0} - \sum_{s\leq 0}\sum_{n<0}\Bigg)(-1)^s q^{s(s+1)/2+(l+1)n+(l-m)s/2+(N+2)(n+s)n}\Bigg] , \end{aligned} \tag{3.21}$$

where the dimensions h^l_m are

$$h^l_m = \frac{l(l+1)}{4(N+2)} - \frac{m^2}{4N} \quad . \tag{3.22}$$

Here $l = 0, 1, \ldots, N-1$, $l - m$ is even, and the formulas are valid for $|m| \leq l$ while for $|m| > l$ one uses the symmetries

$$b^l_m = b^l_{-m} = b^l_{m+2N} = b^{N-l}_{N-m} \ . \tag{3.23}$$

For the more general cosets of (3.19) the branching functions can be found in [24][25][26]. The statistical mechanical models underlying the theories (3.19) are discussed in [24][27][28][29].

The above expressions for the characters, from which their modular properties can be derived, all have the feature that there are several powers of $(q)_\infty$ in the denominator, corresponding to the fact that the Feigin-Fuchs-Felder construction from which they can be obtained is based on bosonic Fock spaces (which are then truncated in a particular way, encoded by the "numerator"). We will call such representations bosonic.

But there are other forms in which the characters may be expressed. Most notable is the equivalent form of the branching functions (3.21) obtained by Lepowsky and Primc [30]

$$q^{c/24} b^l_{2Q-l} \;=\; q^{\frac{l(N-l)}{2N(N+2)}} \sum_{\substack{m_1,\ldots,m_{N-1}=0 \\ \text{restrictions}}}^{\infty} \frac{q^{\mathbf{m} C^{-1}_{N-1} \mathbf{m}^t - \mathbf{A}_l \cdot \mathbf{m}}}{(q)_{m_1} \ldots (q)_{m_{N-1}}}, \tag{3.24}$$

where $\mathbf{m} = (m_1, \ldots, m_{N-1})$ is subject to the restriction

$$\sum_{\alpha=1}^{N-1} \alpha m_\alpha \equiv Q \ (\mathrm{mod}\ N), \tag{3.25}$$

C_{N-1} is the Cartan matrix of the Lie algebra A_{N-1} in the basis where we explicitly have

$$\mathbf{m}C_{N-1}^{-1}\mathbf{m}^t \ = \ \frac{1}{N}\left(\sum_{\alpha=1}^{N-1} \alpha(N-\alpha)m_\alpha^2 + 2\sum_{1\le\alpha<\beta\le N-1} \alpha(N-\beta)m_\alpha m_\beta\right) \ , \tag{3.26}$$

and

$$\mathbf{A}_l \cdot \mathbf{m} = -(\mathbf{m}C_{N-1}^{-1})_l = -\left(\frac{N-l}{N}\sum_{\alpha=1}^{l} \alpha m_\alpha + \frac{l}{N}\sum_{\alpha=l+1}^{N-1} (N-\alpha)m_\alpha\right) \ . \tag{3.27}$$

This representation is of the form of a q-series which generalizes the sum-side of the Rogers-Ramanujan identities [31][32][33] to multiple sums, such as appear in the Andrews-Gordon identities [34][35]. For reasons that will become clear in the next sections we refer to such a representation as fermionic.

4. Three state Potts chain

The general discussion of specific heat and quasi-particles of sect. 2 and the sketch of conformal field theory of the previous section do not rely on any microscopic hamiltonian. There are, however, a large number of integrable spin chains and corresponding two-dimensional classical statistical mechanics systems which are closely related to conformal field theories. These spin chains have eigenvalue spectra which can be studied by means of functional and Bethe's equations. It is thus natural to attempt to compute the conformal field theory characters from the spin chain.

This program has recently been carried out [36]-[40] for the 3-state Potts chain. We will here summarize the results of this study to illustrate the relations which both the Rocha-Caridi (3.17) and the Lepowsky-Primc (3.24) character formulae have to the spin chain and to the order one excitations (2.6) of condensed matter physics. This investigation will lead to a physical interpretation of (3.24) and a new representation for (3.17).

4.1. The hamiltonian and Bethe's equations.

The 3-state Potts chain is specified by the hamiltonian

$$H \;=\; \frac{\pm 2}{\sqrt{3}} \sum_{j=1}^{M} \left(X_j + X_j^\dagger + Z_j Z_{j+1}^\dagger + Z_j^\dagger Z_{j+1} \right) \;, \tag{4.1}$$

where

$$X_j = I \otimes I \otimes \cdots \otimes \underbrace{X}_{j^{th}} \otimes \cdots \otimes I \;, \qquad Z_j = I \otimes I \otimes \cdots \otimes \underbrace{Z}_{j^{th}} \otimes \cdots \otimes I \;. \tag{4.2}$$

Here I is the 3×3 identity matrix,

$$X = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \;, \qquad Z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} \;, \qquad \omega = e^{2\pi i/3} \;, \tag{4.3}$$

and we impose periodic boundary conditions $Z_{M+1} \equiv Z_1$. If the $-$ $(+)$ sign is chosen in (4.1), the spin chain is called ferromagnetic (anti-ferromagnetic).

This spin chain is invariant under translations and under $\mathbf{Z}_3$ spin rotations. Thus the eigenvalues may by classified in terms of P, the total momentum of the state, and Q, where $e^{2\pi i Q/3}$ is the eigenvalue of the spin rotation operator. Here $P = 2\pi n/M$ where n is an integer $0 \le n \le M-1$, and $Q = 0, \pm 1$. Furthermore, because H is invariant under complex conjugation there is a conserved C parity of ± 1 in the sector $Q = 0$, and the sectors $Q = \pm 1$ are degenerate.

This spin chain is integrable because of its connection with the two-dimensional 3-state Potts model at the critical point, which is integrable. The eigenvalues of the transfer matrix satisfy functional equations [36][41]- [43] which are solved in terms of Bethe equations [36]

$$(-1)^{M+1} \left[\frac{\sinh(\lambda_j - iS\gamma)}{\sinh(\lambda_j + iS\gamma)} \right]^{2M} \;=\; \prod_{k=1}^{L} \frac{\sinh(\lambda_j - \lambda_k - i\gamma)}{\sinh(\lambda_j - \lambda_k + i\gamma)} \tag{4.4}$$

with

$$\gamma = \frac{\pi}{3} \;, \qquad S = \frac{1}{4} \;, \qquad L = 2(M - |Q|) \quad \text{for} \quad Q = 0, \pm 1 \;. \tag{4.5}$$

In terms of these λ_k, the eigenvalues of the transfer matrix of the statistical model are

$$\Lambda(\lambda) = \left[\frac{\sinh(\frac{\pi i}{6}) \sinh(\frac{\pi i}{3})}{\sinh(\frac{\pi i}{4} - \lambda) \sinh(\frac{\pi i}{4} + \lambda)} \right]^{M} \prod_{k=1}^{L} \frac{\sinh(\lambda - \lambda_k)}{\sinh(\frac{\pi i}{12} + \lambda_k)} \;\;, \tag{4.6}$$

the eigenvalues of the hamiltonian (4.1) are

$$E = \sum_{k=1}^{L} \cot(i\lambda_k + \frac{\pi}{12}) - \frac{2M}{\sqrt{3}} \quad , \tag{4.7}$$

and the corresponding momentum is

$$e^{iP} = \Lambda(-i\pi/12) = \prod_{k=1}^{L} \frac{\sinh(\lambda_k + \frac{\pi i}{12})}{\sinh(\lambda_k - \frac{\pi i}{12})} \quad . \tag{4.8}$$

4.2. Order one excitations.

These equations have been recently solved to obtain the order one excitation energies [38]. The computations are discussed in detail in the article in these proceedings [44]. The results are as follows (we describe them in detail only the sector $Q = 0$):

- Ferromagnetic case:

The order one excitation energies and momenta are

$$E - E_{GS} \; = \; \sum_{j=1}^{m_+} e(P_j^+) \quad , \qquad P \; \equiv \; \sum_{j=1}^{m_+} P_j^+ \ (\text{mod } 2\pi), \tag{4.9}$$

where

$$m_+ = 2m_{ns} + 3m_- + 4m_{-2s} \tag{4.10}$$

with m_{ns}, m_-, m_{-2s} arbitrary non-negative integers, and

$$e(P_j^+) = 6\sin(\frac{P_j^+}{2}) \qquad 0 \le P_j^+ \le 2\pi \ , \qquad P_j^+ \neq P_k^+ \quad \text{for} \quad j \neq k \ . \tag{4.11}$$

Each state has a degeneracy [37], which for $Q = 0$ is

$$\binom{m_- + m_{-2s}}{m_-} \binom{2m_- + 2m_{-2s} + m_{ns}}{m_{ns}} \ . \tag{4.12}$$

The speed of sound v is found to be 3, since

$$e(P^+) \ \sim \ 3|P^+| \qquad \text{for} \quad P^+ \ \sim \ 0. \tag{4.13}$$

• Anti-ferromagnetic case:

We restrict our attention to M even. The order one excitation energies and momenta in the sector $Q = 0$ are

$$E - E_{GS} \;=\; \sum_{\alpha=2s,-2s,ns} \; \sum_{j=1}^{m_\alpha} e_\alpha(P_j^\alpha) \ , \qquad P - P_{GS} \;=\; \sum_{\alpha=2s,-2s,ns} \; \sum_{j=1}^{m_\alpha} P_j^\alpha \ , \tag{4.14}$$

where P is defined modulo 2π, and

$$P_{GS} \equiv \frac{M}{2}\pi \ (\mathrm{mod}\ 2\pi) \ , \tag{4.15}$$

$$m_{2s} + m_{-2s} \quad \text{is} \quad \text{even}. \tag{4.16}$$

The single-particle momenta are subject to the fermi exclusion rule (2.8), and the single-particle energies are

$$\begin{aligned} e_{2s}(P) \;&=\; 3\{\sqrt{2}\cos(\frac{|P|}{2} - \frac{3\pi}{4}) + 1\} \qquad && 0 \le P \le 3\pi \\ e_{-2s}(P) \;&=\; 3\{\sqrt{2}\cos(\frac{|P|}{2} - \frac{\pi}{4}) - 1\} \qquad && 0 \le P \le \pi \\ e_{ns}(P) \;&=\; 3\sin(\frac{|P|}{2}) \qquad && 0 \le P \le 2\pi \ . \end{aligned} \tag{4.17}$$

The speed of sound is $\frac{3}{2}$ for all three excitations:

$$e_\alpha(P) \;\sim\; \frac{3}{2}|P| \qquad \text{for} \qquad P \;\sim\; 0 \ . \tag{4.18}$$

4.3. Conformal field theory predictions.

We turn now to the conformal field theory predictions for the partition functions of both the ferromagnetic and the anti-ferromagnetic cases.

• Ferromagnetic case:

The conformal field theory in this case was identified by Dotsenko [45] to be the minimal model $\mathcal{M}(5,6)$ of central charge $c = \frac{4}{5}$ (cf. (3.15)), and the partition function was argued by Cardy [2] to be the modular-invariant non-diagonal combination of characters

$$\begin{aligned} \hat{Z}_F \;&=\; [\chi_0(q) + \chi_3(q)][\chi_0(\bar{q}) + \chi_3(\bar{q})] + [\chi_{2/5}(q) + \chi_{7/5}(q)][\chi_{2/5}(q) + \chi_{7/5}(\bar{q})] \\ &\quad + 2\chi_{1/15}(q)\chi_{1/15}(\bar{q}) + 2\chi_{2/3}(q)\chi_{2/3}(\bar{q}) \ . \end{aligned} \tag{4.19}$$

Here we use the notation $\chi_\Delta = \chi^{(5,6)}_{\Delta_{r,s}}$ for the characters whose first few terms are obtained from (3.17) as

$$\begin{aligned}
q^{\frac{c}{24}}\chi_0 &= 1+q^2+q^3+2q^4+2q^5+4q^6+4q^7+7q^8+8q^9+12q^{10}\ldots \\
q^{\frac{c}{24}}\chi_{2/5} &= q^{\frac{2}{5}}(1+q+q^2+2q^3+3q^4+4q^5+6q^6+8q^7+11q^8+15q^9\ldots) \\
q^{\frac{c}{24}}\chi_{7/5} &= q^{\frac{7}{5}}(1+q+2q^2+2q^3+4q^4+5q^5+8q^6+10q^7+15q^8+19q^9\ldots) \\
q^{\frac{c}{24}}\chi_{3} &= q^{3}(1+q+2q^2+3q^3+4q^4+5q^5+8q^6+10q^7+14q^8+18q^9\ldots) \\
q^{\frac{c}{24}}\chi_{1/15} &= q^{\frac{1}{15}}(1+q+2q^2+3q^3+5q^4+7q^5+10q^6+14q^7+20q^8+26q^9\ldots) \\
q^{\frac{c}{24}}\chi_{2/3} &= q^{\frac{2}{3}}(1+q+2q^2+2q^3+4q^4+5q^5+8q^6+10q^7+15q^8+19q^9\ldots).
\end{aligned} \tag{4.20}$$

- Anti-ferromagnetic case:

In this case the conformal field theory was identified by Pearce [47] to be that of $\mathbf{Z}_4$ parafermions, of central charge c=1 (cf. (3.20) with N=4), with the non-diagonal partition function [46]

$$\hat{Z}_{AF} = [b^0_0(q)+b^0_4(q)][b^0_0(\bar{q})+b^0_4(\bar{q})] + 4b^0_2(q)b^0_2(\bar{q}) + 2b^2_0(q)b^2_0(\bar{q}) + 2b^2_2(q)b^2_2(\bar{q}) \ , \tag{4.21}$$

in terms of the branching functions b^l_m, which are obtained from (3.21) or (3.24) with N=4 as

$$\begin{aligned}
q^{\frac{1}{24}}b^0_0 &= (1+q^2+2q^3+4q^4+5q^5+9q^6+12q^7+19q^8+25q^9+37q^{10}\ldots) \\
q^{\frac{1}{24}}b^0_2 &= q^{\frac{3}{4}}(1+q+2q^2+3q^3+5q^4+7q^5+12q^6+16q^7+24q^8+33q^9\ldots) \\
q^{\frac{1}{24}}b^0_4 &= q(1+q+3q^2+3q^3+6q^4+8q^5+13q^6+17q^7+27q^8+35q^9\ldots) \\
q^{\frac{1}{24}}b^2_0 &= q^{\frac{1}{3}}(1+2q+3q^2+5q^3+8q^4+13q^5+19q^6+28q^7+41q^8+58q^9\ldots) \\
q^{\frac{1}{24}}b^2_2 &= q^{\frac{1}{12}}(1+q+3q^2+4q^3+8q^4+11q^5+18q^6+25q^7+38q^8+52q^9\ldots).
\end{aligned} \tag{4.22}$$

4.4. Characters from Bethe's equations.

In order to obtain the characters (4.20) and (4.22) from the formalism of Bethe's equation (4.4), the order one computations of [38] and [44] must be extended to order $\frac{1}{M}$. We consider the ferromagnetic and the anti-ferromagnetic cases separately, simpler case first.

- Anti-ferromagnetic characters:

To extend the analysis of [38] to order $\frac{1}{M}$ it is natural to use the order one energies (4.17) in the region $P \sim \frac{1}{M}$ where the linear form (4.18) holds. However, we must in addition (i) add a possible P-independent contribution of order $\frac{1}{M}$ to the energy and (ii) specify the allowed values of P^α_j, as we now explain.

Both of these questions are investigated in detail in [39]. A principal result of that paper is that the $\frac{1}{M}$ spectrum decouples into a spectrum of right- and left-movers, namely

$$E - E_{GS} \;=\; \sum_{\alpha=2s,-2s,ns} \;\sum_{h=r,l} \;\sum_{j_{\alpha(h)}=1}^{m_{\alpha(h)}} e_\alpha(P^{\alpha(h)}_{j_{\alpha(h)}}) \tag{4.23}$$

with all $e_\alpha(P) = 3|P|$, and (in the $Q = 0$ sector)

$$P^{\alpha(h)}_{j_{\alpha(h)}} \;=\; \pm\frac{2\pi}{M}\Big[\frac{1}{2}\big(m_{ns(h)} + \frac{m_{2s(h)} + m_{-2s(h)}}{2} + 1\big) + k^{\alpha(h)}_{j_{\alpha(h)}}\Big] \tag{4.24}$$

for $\alpha = 2s, -2s$, and

$$P^{ns(h)}_{j_{ns(h)}} \;=\; \pm\frac{2\pi}{M}\Big[\frac{1}{2}\big(m_{ns(h)} + m_{2s(h)} + m_{-2s(h)} + 1\big) + k^{ns(h)}_{j_{ns(h)}}\Big] \;\;, \tag{4.25}$$

where the $+,-$ applies to $h = r, l$, respectively, and the $k^{\alpha(h)}_{j_{\alpha(h)}}$ are distinct non-negative integers for each $\alpha(h)$.

It is significant that the lower limits on the three momentum ranges depend on the number of quasi-particles present in the state. It is this exclusion of states in the infrared that causes the specific heat of this system to be less than that of 3 free fermions, namely less than $\frac{3}{2}$.

From this order $\frac{1}{M}$ energy spectrum we may construct the branching functions b^0_0, b^0_4 and b^0_2 for $N = 4$ by using (4.24) and (4.25) in (2.2) with $m_{\alpha(l)} = 0$ and $m_{\alpha(r)}$ satisfying the following restrictions (where the subscript r is dropped for convenience):

$$\begin{aligned}
b^0_0:&\quad m_{2s}+m_{-2s} \text{ is even, and } m_{ns}+m_{-2s}+\frac{m_{2s}+m_{-2s}}{2} \text{ is even;}\\
b^0_4:&\quad m_{2s}+m_{-2s} \text{ is even, and } m_{ns}+m_{-2s}+\frac{m_{2s}+m_{-2s}}{2} \text{ is odd;}\\
b^0_2:&\quad m_{2s}+m_{-2s} \text{ is odd, and } m_{2s}<m_{-2s}.
\end{aligned} \tag{4.26}$$

The branching functions are now evaluated from

$$q^{c/24} b^0_m \;=\; \sum_{m_{2s(r)},m_{-2s(r)},m_{ns(r)}} e^{-(E-E_{GS})/k_B T} \;\;, \tag{4.27}$$

using the relation

$$\sum_{N=0}^{\infty} Q_m(N) q^N = \frac{q^{m(m-1)/2}}{(q)_m} \tag{4.28}$$

where $Q_m(N)$ is the number of distinct additive partitions of N into m non-negative integers. We find that

$$\sum_{m_{2s}, m_{-2s}, m_{ns}=0}^{\infty} \frac{q^{\frac{1}{4}(3m_{2s}^2+3m_{-2s}^2+4m_{ns}^2+4m_{ns}m_{2s}+4m_{ns}m_{-2s}+2m_{2s}m_{-2s})}}{(q)_{m_{2s}}(q)_{m_{-2s}}(q)_{m_{ns}}} = q^{1/24} b_m^0 \tag{4.29}$$

with the m_α restricted by (4.26). The lhs is obtained directly from Bethe's equation (4.4). However, if we set $m_1 = m_{2s}$, $m_2 = m_{ns}$ and $m_3 = m_{-2s}$, we see that it is exactly the rhs of (3.24) obtained by Lepowsky and Primc [30], and thus the equality in (4.29) follows.

The sector $Q = \pm 1$ is more complicated and for details the reader is referred to [39]. The analysis there shows that each of the branching functions b_0^2 and b_2^2 is represented in terms of two types of spectra with non-trivial lower bounds. The final result is that these branching functions are given as the sum of two 3-dimensional sums as follows:

$$\begin{aligned}
&\sum_{\substack{m_1,m_2,m_3=0\\ m_1+m_3 \text{ even}}}^{\infty} \frac{q^{\mathbf{m}C_3^{-1}\mathbf{m}^t+m_1+m_2+m_3-\frac{1}{4}}}{(q)_{m_1}(q)_{m_2}(q)_{m_3}} + \sum_{\substack{m_1,m_2,m_3=0\\ m_1+m_3 \text{ odd}}}^{\infty} \frac{q^{\mathbf{m}C_3^{-1}\mathbf{m}^t+\frac{1}{2}(m_1+m_3)}}{(q)_{m_1}(q)_{m_2}(q)_{m_3}} = q^{\frac{1}{24}-\frac{1}{3}} b_0^2 \\
&\sum_{\substack{m_1,m_2,m_3=0\\ m_1+m_3 \text{ odd}}}^{\infty} \frac{q^{\mathbf{m}C_3^{-1}\mathbf{m}^t+m_1+m_2+m_3-\frac{1}{4}}}{(q)_{m_1}(q)_{m_2}(q)_{m_3}} + \sum_{\substack{m_1,m_2,m_3=0\\ m_1+m_3 \text{ even}}}^{\infty} \frac{q^{\mathbf{m}C_3^{-1}\mathbf{m}^t+\frac{1}{2}(m_1+m_3)}}{(q)_{m_1}(q)_{m_2}(q)_{m_3}} = q^{\frac{1}{24}-\frac{1}{12}} b_2^2 \; .
\end{aligned} \tag{4.30}$$

Unlike the case of the $Q = 0$, the lhs's here are not of the form (3.24) of [30]. Nevertheless, we have verified to order q^{200} that the identities (4.30) hold.

- Ferromagnetic characters:

The extension of the ferromagnetic order one spectrum (4.9) to the order $\frac{1}{M}$ region is complicated by the degeneracy factor (4.12). At order one this degeneracy may be thought of as additional excitations which must be included in the sum (4.9) but have zero energy and zero momentum. However, at order $\frac{1}{M}$ such excitations can have dispersion relations linear in P just as long as the number of allowed momentum states is finite as $M \to \infty$. It is also not instantly obvious that the speed of sound of these finite-momentum-range excitations should be the same as the speed of sound of the quasi-particle of (4.13). These questions have been investigated in [40] where we find that the characters can be computed from the following expressions for the energies

$$E - E_{GS} = \sum_{a=+,-2s,ns} \sum_{j_a=1}^{m_a} e_a(P^a_{j_a}) \quad , \tag{4.31}$$

where (in the $Q = 0$ sector)

$$P^+_{j_+} = \frac{2\pi}{M}\Big[-\frac{1}{2}(m_- + m_{-2s} - 1) + k^+_{j_+}\Big] \tag{4.32}$$

$$P^{-2s}_{j_{-2s}} = \frac{2\pi}{M}\Big[-\frac{1}{2}(m_- + m_{-2s} - 1) + k^{-2s}_{j_{-2s}}\Big] \tag{4.33}$$

$$P^+_{j_{ns}} = \frac{2\pi}{M}\Big[-\frac{1}{2}(m_{ns} + 2m_- + 2m_{-2s} - 1) + k^{ns}_{j_{ns}}\Big] \ , \tag{4.34}$$

with the $k^a_{j_a}$ distinct non-negative integers for each a, which for $a = -2s, ns$ also have an upper bound,

$$k^{-2s}_{j_{-2s}} \leq m_- + m_{-2s} - 1 \quad , \qquad k^{ns}_{j_{ns}} \leq m_{ns} + 2m_- + 2m_{-2s} - 1 \quad . \tag{4.35}$$

Here

$$m_+ = 2m_{ns} + 3m_- + 4m_{-2s} \tag{4.36}$$

and

$$e_a(P) = 3P \quad , \qquad a = +, -2s, ns. \tag{4.37}$$

We emphasize that (4.37) differs from (2.9) in that P occurs instead of $|P|$, which is significant since the lower limit in (4.32) is in general negative. Note also that the number of states allowed by (4.33)-(4.35) is finite as $M \to \infty$ for any given m_+.

Expressions for the characters χ_0 and χ_3 are constructed using these rules with the further restriction

$$m_- \quad \text{is} \quad \text{even (odd)} \qquad \text{for} \quad \chi_0 \quad (\chi_3). \tag{4.38}$$

The characters $\chi_{2/5}$ and $\chi_{7/5}$ may also be constructed from these rules provided we add an additional term

$$\frac{\pi v}{M}(m_- + m_{-2s} - 1) \tag{4.39}$$

to the energy, set $m_+ = 2m_{ns} + 3m_- + 4m_{-2s} - 1$ and use the restrictions

$$m_- \quad \text{is} \quad \text{even (odd)} \qquad \text{for} \quad \chi_{2/5} \quad (\chi_{7/5}). \tag{4.40}$$

The evaluation of the sum for the characters is done as in the anti-ferromagnetic case, except that here, because of the finite momentum ranges dictated by (4.35), it is not sufficient to use (4.28). In addition, we must introduce $Q_m(N;N')$, the number of partitions of $N \geq 0$ into m non-negative integers which are smaller or equal to $N' > 0$. Then the sums may be simplified using [48]

$$\sum_{N=0}^{\infty} Q_m(N;N')\, q^N \;=\; q^{\frac{1}{2}m(m-1)} \begin{bmatrix} N'+1 \\ m \end{bmatrix}_q \;, \tag{4.41}$$

where the q-binomial is defined (for integers m,n) by

$$\begin{bmatrix} n \\ m \end{bmatrix}_q \;=\; \begin{cases} \frac{(q)_n}{(q)_m (q)_{n-m}} & \text{if} \quad 0 \leq m \leq n \\ 0 & \text{otherwise.} \end{cases} \tag{4.42}$$

Thus we may directly find, for example,

$$q^{c/24}\chi_{0,3} \;=\; \sum_{\substack{m_{ns},m_{-2s},m_0=0 \\ \text{restrictions}}}^{\infty} \frac{q^{F(\mathbf{m})}}{(q)_{m_+}} \begin{bmatrix} m_{-2s}+m_- \\ m_- \end{bmatrix}_q \begin{bmatrix} 2(m_- + m_{-2s}) + m_{ns} \\ m_{ns} \end{bmatrix}_q \tag{4.43}$$

where

$$F(\mathbf{m}) = 2m_{ns}^2 + 3m_-^2 + 6m_{-2s}^2 + 4m_{ns}m_- + 6m_{ns}m_{-2s} + 8m_-m_{-2s} \;, \tag{4.44}$$

m_+ is given by (4.36), and the restrictions on the sum are given by (4.40).

For $Q = \pm 1$ considerations similar to those of the anti-ferromagnetic case give a representation of $\chi_{1/15}$ as the sum of five 3-fold sums with an additional linear term in the exponent.

However, in all cases $Q = 0, \pm 1$ the structure of the result is much more transparent if we set

$$m_1 = 2m_{ns} + 3m_- + 4m_{-2s} \;, \qquad m_2 = 2m_- + 2m_{-2s} \;, \qquad m_3 = m_-. \tag{4.45}$$

Then we find the following set of results

$$\begin{aligned} \hat{\chi}_\Delta(q) \;=\; & \sum_{\substack{m_1,m_2,m_3 \geq 0 \\ \text{restrictions}}} q^{\frac{1}{4}(2m_1^2+2m_2^2+2m_3^2-2m_1m_2-2m_2m_3)-\frac{1}{2}L(\mathbf{m})} \\ & \times \frac{1}{(q)_{m_1}} \begin{bmatrix} \frac{1}{2}(m_1+m_3+u_2) \\ m_2 \end{bmatrix}_q \begin{bmatrix} \frac{1}{2}(m_2+u_3) \\ m_3 \end{bmatrix}_q \end{aligned} \tag{4.46}$$

where the quadratic form in the exponential is recognized as $\frac{1}{4}\mathbf{m}C_3\mathbf{m}^t$ where C_3 is the Cartan matrix of the Lie algebra A_3. The restrictions differ according to the character, (below e=even, o=odd, and possibly several possibilities for obtaining a given character are listed):

Δ	m_1	m_2	m_3	u_2	u_3	$L(\mathbf{m})$
0	e	e	e	0	0	0
2/5	o	e	e	1	0	1
	o	o	o	0	1	1
7/5	e	e	o	1	0	1
	e	o	e	0	1	1
3	o	e	o	0	0	6
1/15	o	e	o	2	0	m_2+2
	e	e	e	2	0	m_2
	e	o	o	1	1	m_2
	o	o	e	1	1	m_2
	{e	o	e			
	+o	o	e}	1	-1	m_1-m_3
2/3	e	e	o	1	0	m_2+1
	o	e	e	1	0	m_2+1
	{e	o	e			
	+o	o	o}	0	-1	m_1-m_3+1

There are a few comments to be made about this summary of the results of [39] and [38]. Firstly, for $Q=\pm 1$ the form (3.24) for the anti-ferromagnetic characters and (4.46) for the ferromagnetic characters have not been derived from Bethe's equation (4.4) but have been verified to hold to order q^{200}. Secondly, the crucial factorization property (3.6) has only been shown for the anti-ferromagnetic case. Thus the momentum restrictions and the energy formula which give the characters by restricting to right-movers only as in (4.27) do not seem to be sufficient to give the full partition function in the ferromagnetic case. This is still under investigation, and the resolution presumably lies in the fact that the limited range excitations (4.33) and (4.34) can have many different forms at the order of $\frac{1}{M}$ which will all be degenerate at order 1.

5. Generalizations

The Lepowsky-Primc form (3.24) for the anti-ferromagnetic 3-state Potts characters and the expressions for the ferromagnetic characters (4.46) are written in terms of the Cartan and inverse Cartan matrix of A_3 and are extremely suggestive for generalizations. We have recently conjectured such generalizations [49][50] for many conformal field theories, including all those mentioned in sect. 3, and found that the conjectures agree with the previously known results to order q^{200} in many cases. Furthermore, by reversing the process of the previous section, each of the characters can be given an interpretation in terms of fermionic quasi-particles with momentum restrictions. We will here summarize both these conjectures and other recent results for fermionic sum representations.

5.1. $\frac{(G_r^{(1)})_1\times(G_r^{(1)})_1}{(G_r^{(1)})_2}$ *where G_r is a simply-laced Lie algebra of rank r.*

Let us first define the general sum

$$S_B^Q(q) \;\equiv\; \sum_{\substack{m_1,\ldots,m_n=0\\ \text{restrictions}}}^{\infty} \frac{q^{\frac{1}{2}\mathbf{m}B\mathbf{m}^t}}{(q)_{m_1}\cdots(q)_{m_n}} \quad , \tag{5.1}$$

where B is a real positive-definite $n\times n$ symmetric matrix, and the restrictions are generically of the form

$$\sum_{\alpha=1}^{n} m_\alpha Q_\alpha \equiv Q \pmod{\ell} \; . \tag{5.2}$$

The sum (5.1) is the partition function of a set of n types of (right-moving, say) fermionic quasi-particles with momenta specified by

$$P_{j_\alpha}^{\alpha} \;=\; P_{\min}^{\alpha}(\mathbf{m}) + \frac{2\pi}{M}k_{j_\alpha}^{\alpha} \; , \tag{5.3}$$

where the $k_{j_\alpha}^{\alpha}$ are distinct non-negative integers for each α and

$$P_{\min}^{\alpha}(\mathbf{m}) \;=\; \frac{2\pi}{M}\left[\frac{1}{2} \;+\; \frac{1}{2}\sum_{\beta=1}^{n}(B_{\alpha\beta}-\delta_{\alpha\beta})\,m_\beta\right] . \tag{5.4}$$

The interpretation of a restriction (5.2) is that each quasi-particle of type α carries a $\mathbf{Z}_\ell$ charges Q_α, and so S_B^Q is the partition function of the sector of total charge Q.

To obtain characters for the coset conformal field theory $\frac{(G_r^{(1)})_1\times(G_r^{(1)})_1}{(G_r^{(1)})_2}$ we take $n=r$ and $B=2C_{G_r}^{-1}$, namely twice the inverse Cartan matrix of G_r. The results in the various cases are as follows:

$\mathbf{G_r = A_n}$: This is the original case of Lepowsky and Primc [30]: the sum (5.1) with $B = 2C^{-1}_{A_{N-1}}$ is (3.24) with $l = 0$. All the characters of the corresponding $\mathbf{Z}_{n+1}$-parafermionic conformal field theory are given by (3.24). We merely note here that the linear shift term $\mathbf{A}_l \cdot \mathbf{m}$ of (3.27) can be obtained from the form (3.24) with $\mathbf{A} = 0$ by replacing in the quadratic form m_l by $m_l + \frac{1}{2}$.

$\mathbf{G_r = D_n}$ $(\mathbf{n \geq 3})$: The corresponding conformal field theories are special points on the c=1 gaussian line (specified by the radius $\sqrt{\frac{n}{2}}$ in the conventions of [51]), where the characters are given by

$$f_{n,j}(q) \;=\; \frac{q^{-1/24}}{(q)_\infty} \sum_{k=-\infty}^{\infty} q^{n(k+\frac{j}{2n})^2} \;, \qquad j = 0, \ldots, n. \tag{5.5}$$

The inverse Cartan matrix is

$$\begin{aligned} \mathbf{m}C^{-1}_{D_n}\mathbf{m}^t \;=\; & \sum_{\alpha=1}^{n-2} \alpha m_\alpha^2 \;+\; \frac{n}{4}(m_{n-1}^2 \;+\; m_n^2) \;+\; 2 \sum_{1\leq\alpha<\beta\leq n-2} \alpha m_\alpha m_\beta \\ & + \sum_{\alpha=1}^{n-2} \alpha m_\alpha (m_{n-1} + m_n) \;+\; \frac{n-2}{2} m_{n-1} m_n \;, \end{aligned} \tag{5.6}$$

and we obtain

$$S^Q_{D_n}(q) \;=\; q^{1/24}\, f_{n,nQ}(q) \tag{5.7}$$

with $Q = 0, 1$, when summation in (5.1) is restricted to

$$m_{n-1} + m_n \equiv Q \pmod 2. \tag{5.8}$$

Note that due to the coincidence $D_3 = A_3$ the expressions (3.24) and (5.7) are related when $n = 3$ by (cf. [7][39]) $S^0_{D_3} = S^0_{A_3} + S^2_{A_3}$ and $S^1_{D_3} = 2S^1_{A_3}$.

$\mathbf{G_r = E_6}$: Here the conformal field theory is the minimal model $\mathcal{M}(6,7)$ of central charge $c = \frac{6}{7}$ with the D-series partition function. With a suitable labeling of roots we have

$$C^{-1}_{E_6} \;=\; \begin{pmatrix} 4/3 & 2/3 & 1 & 4/3 & 5/3 & 2 \\ 2/3 & 4/3 & 1 & 5/3 & 4/3 & 2 \\ 1 & 1 & 2 & 2 & 2 & 3 \\ 4/3 & 5/3 & 2 & 10/3 & 8/3 & 4 \\ 5/3 & 4/3 & 2 & 8/3 & 10/3 & 4 \\ 2 & 2 & 3 & 4 & 4 & 6 \end{pmatrix} , \tag{5.9}$$

and we find (cf. (3.17))

$$S^0_{E_6}(q) \;=\; q^{c/24}\,[\chi^{(6,7)}_{1,1}(q)+\chi^{(6,7)}_{5,1}(q)]\;\;,\qquad S^{\pm1}_{E_6}(q)\;=\;q^{c/24}\,\chi^{(6,7)}_{3,1}(q)\;\;, \tag{5.10}$$

with the restrictions

$$m_1-m_2+m_4-m_5\equiv Q \pmod 3. \tag{5.11}$$

$\mathbf{G_r = E_7}$: The conformal field theory is $\mathcal{M}(4,5)$ of central charge $c=\frac{7}{10}$. Now

$$C^{-1}_{E_7} \;=\; \begin{pmatrix} 3/2 & 1 & 3/2 & 2 & 2 & 5/2 & 3 \\ 1 & 2 & 2 & 2 & 3 & 3 & 4 \\ 3/2 & 2 & 7/2 & 3 & 4 & 9/2 & 6 \\ 2 & 2 & 3 & 4 & 4 & 5 & 6 \\ 2 & 3 & 4 & 4 & 6 & 6 & 8 \\ 5/2 & 3 & 9/2 & 5 & 6 & 15/2 & 9 \\ 3 & 4 & 6 & 6 & 8 & 9 & 12 \end{pmatrix} \tag{5.12}$$

and we find

$$S^0_{E_7}(q)\;=\;q^{c/24}\,\chi^{(4,5)}_{1,1}(q)\;\;,\qquad S^1_{E_7}(q)\;=\;q^{c/24}\,\chi^{(4,5)}_{3,1}(q)\;\;, \tag{5.13}$$

when the restrictions are

$$m_1+m_3+m_6\equiv Q \pmod 2. \tag{5.14}$$

$\mathbf{G_r = E_8}$: The coset in this case is equivalent to the Ising conformal field theory $\mathcal{M}(3,4)$ of central charge $c=\frac{1}{2}$. Here

$$C^{-1}_{E_8} \;=\; \begin{pmatrix} 2 & 2 & 3 & 3 & 4 & 4 & 5 & 6 \\ 2 & 4 & 4 & 5 & 6 & 7 & 8 & 10 \\ 3 & 4 & 6 & 6 & 8 & 8 & 10 & 12 \\ 3 & 5 & 6 & 8 & 9 & 10 & 12 & 15 \\ 4 & 6 & 8 & 9 & 12 & 12 & 15 & 18 \\ 4 & 7 & 8 & 10 & 12 & 14 & 16 & 20 \\ 5 & 8 & 10 & 12 & 15 & 16 & 20 & 24 \\ 6 & 10 & 12 & 15 & 18 & 20 & 24 & 30 \end{pmatrix} \tag{5.15}$$

and, without any restrictions in the sum (5.1),

$$S_{E_8}(q)\;=\;q^{c/24}\,\chi^{(3,4)}_{1,1}(q)\;\;. \tag{5.16}$$

We further note that if m_1 in the quadratic form in (5.1) is replaced by $m_1-\frac{1}{2}$ then one obtains (up to a power of q) $\hat\chi^{(3,4)}_{1,1}+\hat\chi^{(3,4)}_{1,2}$, and similarly replacing m_2 by $m_2-\frac{1}{2}$ the combination $\hat\chi^{(3,3)}_{1,1}+\hat\chi^{(3,4)}_{1,2}+\hat\chi^{(3,4)}_{1,3}$ is obtained.

5.2. The cosets of $\frac{(G_r^{(1)})_{n+1}}{U(1)^r}$.

This case has been considered in [52] and [53] where the identity characters in the corresponding generalized parafermion conformal field theory [54] are given by (5.1) (with suitable restrictions on the summation variables) by taking $B = C_{G_r} \otimes C_{A_n}^{-1}$, which is explicitly written in a double index notation as

$$B_{ab}^{\alpha\beta} = (C_{G_r})_{\alpha\beta}(C_{A_n}^{-1})_{ab} \qquad \alpha, \beta = 1, \ldots, r, \quad a, b, = 1, \ldots, n. \tag{5.17}$$

When $G_r = A_1$, this reduces to the result (3.24) of [30].

5.3. The non-unitary minimal models $\mathcal{M}(2, 2n+3)$.

This case has been discussed in [55] and [56]. Here one takes $B = 2(C_n')^{-1}$, where C_n' is the Cartan matrix of the tadpole graph with n nodes, namely it differs from C_{A_n} only in one entry which is $(C_n')_{nn} = 1$. The sum $S_B(q)$, with no restrictions, gives the (normalized) character $\hat{\chi}_{1,n}^{(2,2n+3)}(q)$ corresponding to the lowest dimension in the theory. All the other characters are obtained [55] by adding suitable linear terms to the quadratic form in (5.1), leading to the full set of sums appearing in the Gordon-Andrews identities [34][35].

5.4. Unitary minimal models $\mathcal{M}(p, p+1) = \frac{(A_1^{(1)})_{p-2} \times (A_1^{(1)})_1}{(A_1^{(1)})_{p-1}}$.

For this and subsequent cases we must extend the form (5.1) to

$$S_B \begin{bmatrix} \mathbf{Q} \\ \mathbf{A} \end{bmatrix} (\mathbf{u}|q) \equiv \sum_{\substack{\mathbf{m} \\ \text{restrictions}}} q^{\frac{1}{2}\mathbf{m}B\mathbf{m}^t - \frac{1}{2}\mathbf{A}\cdot\mathbf{m}} \prod_{a=1}^{n} \begin{bmatrix} (\mathbf{m}(1-B) + \frac{\mathbf{u}}{2})_a \\ m_a \end{bmatrix}_q , \tag{5.18}$$

where $\mathbf{A}$ and $\mathbf{u}$ are n-dimensional vectors of integers and the argument $\mathbf{Q}$ indicates certain restrictions on $\mathbf{m}$ (such that, in particular, the upper entries of the q-binomials are integers). We note that if $u_a = \infty$ then $\begin{bmatrix} (\mathbf{m}(1-B)+\frac{\mathbf{u}}{2})_a \\ m_a \end{bmatrix}_q = \frac{1}{(q)_{m_a}}$. Thus if all $u_a = \infty$ the form (5.1) is obtained, while if only $u_1 = \infty$ a form similar to (4.46) is obtained.

Generalizing the discussion leading to (4.43), the sum (5.18) can be shown [50] to be the partition function of a set of n quasi-particles having the same dispersion relation $e_a(P_{j_a}^a) = vP_{j_a}^a$ for all $a = 1, \ldots, n$, and the $P_{j_a}^a$ ($j_a = 1, 2, \ldots, m_a$ with the m_a restricted according to $\mathbf{Q}$) obey the exclusion principle (2.8) but are otherwise freely chosen from the sets

$$P_{j_a}^a \in \left\{ P_{\min}^a(\mathbf{m}),\ P_{\min}^a(\mathbf{m}) + \frac{2\pi}{M},\ \ldots,\ P_{\max}^a(\mathbf{m}) \right\} . \tag{5.19}$$

The vectors $\mathbf{P}_{\min,\max} = \{P^a_{\min,\max}\}$ here are

$$\mathbf{P}_{\min}(\mathbf{m}) \;=\; -\frac{2\pi}{M}\,\frac{1}{2}\Big(\mathbf{m}(1-B)+\mathbf{A}-\boldsymbol{\rho}\Big) \tag{5.20}$$

where $\boldsymbol{\rho}$ denotes the n-dimensional vector $(1,1,\ldots,1)$,

$$P^a_{\max}(\mathbf{m}) \;=\; -P^a_{\min}(\mathbf{m}) + \frac{2\pi}{M}\Big(\frac{\mathbf{u}}{2}-\mathbf{A}\Big)_a \;\;, \tag{5.21}$$

and we note that if some $u_a=\infty$ the corresponding $P^a_{\max}=\infty$.

For the present case of $\mathcal{M}(p,p+1)$ the $\mathbf{Q}$-restriction is taken to be $m_a \equiv Q_a \pmod 2$, and

$$B \;=\; \frac{1}{2}C_{A_{p-2}} \;, \qquad\qquad u_1=\infty \;. \tag{5.22}$$

Defining

$$\mathbf{Q}_{r,s} \;=\; (s-1)\boldsymbol{\rho} + (\mathbf{e}_{r-1}+\mathbf{e}_{r-3}+\ldots) + (\mathbf{e}_{p+1-s}+\mathbf{e}_{p+3-s}+\ldots) \tag{5.23}$$

where $(\mathbf{e}_a)_b = \delta_{ab}$ for $a=1,\ldots,p-2$ and 0 otherwise, the conjecture for the (normalized) Virasoro characters (3.17) is [50]

$$\hat\chi^{(p,p+1)}_{r,s}(q) \;=\; q^{-\frac14(s-r)(s-r-1)} S_B \begin{bmatrix}\mathbf{Q}_{r,s}\\ \mathbf{e}_{p-s}\end{bmatrix}(\mathbf{e}_r+\mathbf{e}_{p-s}|q) \;. \tag{5.24}$$

Due to the symmetry $(r,s)\leftrightarrow(p-r,p+1-s)$ of the conformal grid, another representation must also exist, namely

$$\hat\chi^{(p,p+1)}_{r,s}(q) = \; q^{-\frac14(s-r)(s-r-1)}\; S_B \begin{bmatrix}\mathbf{Q}_{p-r,p+1-s}\\ \mathbf{e}_{s-1}\end{bmatrix}(\mathbf{e}_{p-r}+\mathbf{e}_{s-1}|q) \;\;. \tag{5.25}$$

5.5. Cosets $\frac{(G_r^{(1)})_k\times(G_r^{(1)})_l}{(G_r^{(1)})_{k+l}}$ *with* G_r *simply-laced.*

In this case $B = C^{-1}_{G_r}\otimes C_{A_{k+l-1}}$, and the infinite entries of the vector $\mathbf{u}$ are u^α_l for all $\alpha=1,\ldots,r$, in the double index notation used in subsect. 5.2.

As an example with both k and l greater than 1, consider the case $G=A_1$ with l=2, the resulting series of theories labeled by k being the unitary N=1 superconformal series whose characters are given in [4]. We find that the character corresponding to the identity superfield in these models is obtained by summing over $m_1\in\mathbf{Z}$, $m_a\in 2\mathbf{Z}$ for $a=2,\ldots,k+1$.

Another example is the coset $\frac{(E_8^{(1)})_2\times(E_8^{(1)})_1}{(E_8^{(1)})_3}$ of central charge $c=\frac{21}{22}$, which is identified as the minimal model $\mathcal{M}(11,12)$ (with the partition function of the E_6-type). The corresponding sum (5.18), with $\mathbf{A}$=0, u^α_2=0 for all $\alpha=1,\ldots,8$, and all 16 summations running over all non-negative integers, gives $\hat\chi^{(11,12)}_{1,1}(q)+q^8\hat\chi^{(11,12)}_{1,7}(q)$, which is the (extended) identity character of this model.

5.6. Non-unitary minimal models $\mathcal{M}(p,p+2)$ (p odd).

The normalized character $\hat{\chi}^{(p,p+2)}_{(p-1)/2,(p+1)/2}(q)$ (see (3.17)) corresponding to the lowest conformal dimension $\Delta^{(p,p+2)}_{(p-1)/2,(p+1)/2} = -\frac{3}{4p(p+2)}$ in this model is given by (5.18) with $B = \frac{1}{2}C'_{(p-1)/2}$ (where C'_n is defined in subsect. 5.3), $\mathbf{A}$=0, u_1=∞ and u_a=0 for $a = 2,\ldots,\frac{p-1}{2}$, and the m_a are summed over all even non-negative integers.

5.7. Minimal models $\mathcal{M}(p,kp+1)$.

For k=1 these models are the ones considered in sect. 5.4, while for p=2 they were discussed in sect. 5.3. Here we consider the general case. The character $\hat{\chi}^{(p,kp+1)}_{1,k}(q)$ corresponding to the lowest conformal dimension in the model is obtained from (5.18) with B a $(k+p-3)\times(k+p-3)$ matrix whose nonzero elements are given by $B_{ab} = 2(C'^{-1}_{k-1})_{ab}$ and B_{ka}=B_{ak}=a for $a,b = 1,2,\ldots,k-1$, and $B_{ab} = \frac{1}{2}[(C_{A_{p-2}})_{ab} + (k-1)\delta_{ak}\delta_{bk}]$ for $a,b = k,k+1,\ldots,k+p-3$. Summation is restricted to even non-negative integers for $m_k,\ldots,m_{k+p-3}$, the other $m_1,\ldots,m_{k-1}$ running over all non-negative integers, and u_a=∞ for $a = 1,\ldots,k$ and 0 otherwise.

The case p=3 is special in that the fermionic sums are of the form (5.1) for any k. A slight modification of the matrix B appropriate for $\mathcal{M}(3,3k+1)$, namely just setting $B_{kk} = \frac{k}{2}$ while leaving all other elements unchanged, gives the normalized character $\hat{\chi}^{(3,3k+2)}_{1,k}$ of $\mathcal{M}(3,3k+2)$.

5.8. Unitary N=2 superconformal series.

Expressions for the characters of these models, of central charge $c = \frac{3k}{k+2}$ where k is a positive integer, can be found in [57]. The identity character, given by $\chi^{0(0)}_0(q)+\chi^{0(2)}_0(q)$ in the notation of [57], can be obtained from (5.18) if one takes $B = \frac{1}{2}C_{D_{k+2}}$, u_k=∞ (in the basis used in (5.6)) and all other u_a set to zero, and m_{k+1},m_{k+2} running over all non-negative integers while all other m_a summed only over the even non-negative integers.

5.9. $\mathbf{Z}_N$ parafermions.

The characters of these models are the branching functions b^l_m given by (3.21), or by the fermionic representation (3.24) of [30]. In sect. 4.4 we found another fermionic representation for the case N=3 which coincides with the minimal model $\mathcal{M}(5,6)$ with the D-series partition function. (The b^l_m in this case are linear combinations of the χ_Δ of (4.46).) Here we generalize the latter form to arbitrary N. For instance, b^0_0 is obtained from (5.18) by setting $B=\frac{1}{2}C_{D_N}$, $u_N{=}\infty$ (in the basis used in (5.6)) and all other u_a set to zero, and m_{N-1}, m_N running over all non-negative integers such that $m_{N-1}+m_N$ is even, while all other m_a are restricted to be even.

6. The $q\to 1$ behavior

We can now return to the discussion of specific heat and conformal field theory of sections 2 and 3 by computing the effective central charge (3.10) directly from the q-series (5.18). The computation will use the steepest descent method of [58] and [56]. We follow closely the presentation of [50].

It is easily seen that the $q\to 1$ behavior of (5.18) is independent of the restrictions $\mathbf{Q}$ and the linear terms $\mathbf{A}$. This is consistent with the k-independence of (3.13). Thus without loss of generality we set $\mathbf{A}=0$ and let all the sums run from 0 to ∞. The resulting unrestricted sum will be denoted by $S_B(\mathbf{u}|q)$.

Let $q=e^{2\pi i\tau}$ and $\tilde{q}=e^{-2\pi i/\tau}$, with $\mathrm{Im}\tau>0$. Then if the coefficients in the series for $S_B(\mathbf{u}|q)=\sum s_M q^M$ behave for large M like $s_M\sim e^{2\pi\sqrt{\gamma M/6}}$, $\gamma>0$, the series $S_B(\mathbf{u}|q)$ diverges like

$$S_B(\mathbf{u}|q)\ \sim\ \tilde{q}^{-\gamma/24} \qquad \text{as} \quad q\to 1^-\ . \tag{6.1}$$

Here γ must equal the effective central charge (3.12) of the corresponding conformal field theory.

The large M behavior of s_M is found by considering

$$s_{M-1}\ =\ \oint\frac{dq}{2\pi i}\,q^{-M}\ S_B(\mathbf{u}|q)\ =\ \sum_{\mathbf{m}\geq\mathbf{0}}\oint\frac{dq}{2\pi i}\,q^{-M}\ S_B^{\mathbf{m}}(\mathbf{u}|q)\ \ , \tag{6.2}$$

where the contour of integration is a small circle around 0. The behavior of the integral is now analyzed using a saddle point approximation. We first approximate

$$\begin{aligned}\ln\Bigl(q^{-M}S_B^{\mathbf{m}}(\mathbf{u}|q)\Bigr)\ &\simeq\ \bigl(\frac{1}{2}\mathbf{m}B\mathbf{m}^t-M\bigr)\ln q\\ &+\sum_{a=1}^{n}\left(\int_0^{(\mathbf{m}(1-B)+\frac{\mathbf{u}}{2})_a}-\int_0^{(-\mathbf{m}B+\frac{\mathbf{u}}{2})_a}-\int_0^{m_a}\right)dt\ \ln(1-q^t)\end{aligned} \tag{6.3}$$

for large $\mathbf{m}$, and set the derivatives of this expression with respect to the m_a to zero in order to find the saddle point. Introducing $x_a=\frac{(1-w_a)v_a}{1-v_aw_a}$ and $y_a=\frac{1-w_a}{1-v_aw_a}$ where $v_a=q^{m_a}$ and $w_a=q^{(-\mathbf{m}B+\frac{\mathbf{u}}{2})_a}$, these extremum conditions reduce to

$$1-x_a\ =\ \prod_{b=1}^{n}x_b^{B_{ab}}\ , \qquad 1-y_a\ =\ \sigma_a\prod_{b=1}^{n}y_b^{B_{ab}}\ \ , \tag{6.4}$$

where we define $\sigma_a{=}0$ if $u_a{=}\infty$ and 1 otherwise, ensuring $y_a{=}1$ for $u_a{=}\infty$.

At the extremum point with respect to the m_a we have

$$\begin{aligned}\ln\bigl(q^{-M} S_B^{\mathbf{m}}(\mathbf{u}|q)\bigr)\Big|_{\text{ext}} \simeq & -M\ln q \\ + \frac{1}{\ln q}\Biggl\{ & \frac{1}{2}\ln\mathbf{v}\ B\ \ln\mathbf{v}^t - \sum_{a=1}^{n}\bigl[\mathcal{L}(1-v_a)+\mathcal{L}(1-w_a)-\mathcal{L}(1-z_a)\bigr] \\ & -\frac{1}{2}\bigl[\ln\mathbf{v}\cdot\ln(1-\mathbf{v})+\ln\mathbf{w}\cdot\ln(1-\mathbf{w})-\ln\mathbf{z}\cdot\ln(1-\mathbf{z})\bigr]\Biggr\}\end{aligned} \tag{6.5}$$

with $(\ln\mathbf{v})_a=\ln v_a$ and $z_a=v_a w_a$, where

$$\mathcal{L}(z)=-\frac{1}{2}\int_0^z dt\left[\frac{\ln t}{1-t}+\frac{\ln(1-t)}{t}\right]=-\int_0^z dt\ \frac{\ln(1-t)}{t}+\frac{1}{2}\ln z\ln(1-z) \tag{6.6}$$

is the Rogers dilogarithm function [59]. Now using (6.4) we see that the first term inside the braces in (6.5) cancels against the last. Then using the five-term relation for the dilogarithm [59]

$$\mathcal{L}(1-v)+\mathcal{L}(1-w)-\mathcal{L}(1-vw) \ = \ \mathcal{L}(1-x)-\mathcal{L}(1-y) \quad , \tag{6.7}$$

where $x=\frac{(1-w)v}{1-vw}$ and $y=\frac{1-w}{1-vw}$, we obtain

$$\ln\left(q^{-M} S_B^{\mathbf{m}}(\mathbf{u}|q)\right)\Big|_{\text{ext}} \simeq -M\ln q-\frac{\pi^2\tilde{c}}{6\ln q} \tag{6.8}$$

with

$$\tilde{c} \ = \ \frac{6}{\pi^2}\sum_{a=1}^{n}\left[\mathcal{L}(1-x_a)-\mathcal{L}(1-y_a)\right] \quad . \tag{6.9}$$

Finally the value of q at the saddle point is determined by extremizing (6.8) with respect to q, which leads to $s_M\sim e^{2\pi\sqrt{\tilde{c}M/6}}$ and consequently to (6.1) with $\gamma=\tilde{c}$ of (6.9).

This computation of the $q\to 1$ behavior of (5.18) is completely general in that it is valid for all matrices B, and presumably for an arbitrary B no simplification of (6.4) and (6.9) is possible. Nevertheless, for the conformal field theories considered in sect. 5 there is one final simplification which occurs. Namely, there is a remarkable set of sum rules for the dilogarithms [28][29][60]-[66] which reduces (6.9) to rational numbers. These sum rules must be regarded as a vital piece of the theory, but are outside the scope of this article and we refer the reader to the original papers for details.

Finally, it must be pointed out that for the models corresponding to the conformal field theories of sect. 5 the specific heats have been derived in a completely independent fashion using the thermodynamic Bethe ansatz [28][29][60][67] which uses the definition of specific heat discussed in sect. 2. The agreement of these two procedures establishes the one length-scale scaling discussed in sect. 3.

7. Discussion

It is clear from sect. 5 that the existence of fermionic quasi-particle representations for conformal field theory characters is a very general feature which goes beyond the specific models discussed in sect. 4, where these representations were obtained from the spectrum of the hamiltonian. In these representations the focus is on the momentum selection rules (5.19)-(5.21). On the other hand, in most of the previously known expressions for the characters, obtained using conformal field theory or representation theory methods, the focus is on the modular transformation properties of the characters. It would be interesting to directly relate these two aspects.

This question can be made explicit by focusing on the quadratic-form matrix B of (5.18). If this matrix is considered as coming from the momentum restrictions for the fermionic quasi-particles there appears to be nothing to distinguish one matrix B from another. However, from the point of view of conformal field theory the general form (5.18) can only represent a character if it is possible to find some (possibly fractional) power of q which, when multiplied by the q-series (5.18), gives a function which transforms properly under the modular group. The mathematical structure of these q-series cannot be said to be fully understood until these modular properties are found directly from the series, which generalize the sum-side of the Rogers-Ramanujan identities.

A further property of great importance is the fact that there are often several completely different fermionic q-series representations for the same conformal field theory characters. As a particular example, we note that the characters obtained as (4.46) from the study of the ferromagnetic 3-state Potts hamiltonian can also be written in the Lepowsky-Primc form (3.24) with N=3. More generally, the representations of the $\mathbf{Z}_N$-parafermion characters of section 5.9 and (3.24) are of different forms with different quasi-particle interpretations, but nevertheless they are equal. This is representative of a general phenomenon. A full discussion is beyond the scope of this article, but we remark that these inequivalent fermionic representations for the characters are related to different integrable perturbations of the model.

Finally, there is the question of obtaining proofs of the several conjectures of sect. 5. One method is to find certain finitizations of the q-series in question into polynomials, whose properties can then be studied using recursion relations. Such a finitization exists for the characters of the unitary minimal models (5.24), which matches the finitization of the Rocha-Caridi formula (3.17) employed by Andrews, Baxter and Forrester [20] in their corner transfer matrix analysis of the underlying RSOS models. The details will be presented elsewhere.

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