A Distributed Kripke Semantics

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Abstract. An intuitionistic, hybrid modal logic suitable for reasoning about distribution of resources was introduced in [10]. We extend the Kripke semantics of intuitionistic logic, enriching each possible Kripke state with a set of places, and show that this semantics is both sound and complete for the logic. In the semantics, resources of a distributed system are interpreted as atoms, and placement of atoms in a possible state corresponds to the distribution of the resources. The modalities of the logic allow us to validate properties in a *particular place*, in *some* place and in *all* places. We extend the logic with disjunctive connectives, and refine our semantics to obtain soundness and completeness for extended logic. The extended logic can be seen as an instance of *Hybrid IS5* [2, 18].

1 Introduction

In current computing paradigm, distributed resources spread over and shared amongst different nodes of a computer system is very common. For example, printers may be shared in local area networks, or distributed data may store documents in parts at different locations. The traditional reasoning methodologies are not easily scalable to these systems as they may lack implicitly trustable objects such as a central control.

This has resulted in the innovation of several reasoning techniques. A popular approach in the literature has been the use of algebraic systems such as process algebra [13, 8, 5]. These algebras have rich theories in terms of semantics [13], logics [7, 15, 4, 3], or types [8]. Another approach is logically-oriented [9, 10, 19, 14]: intuitionistic modal logics are used as foundations of type systems by exploiting the *propositions-as-types, proofs-as-programs* paradigm [6]. An instance of this was introduced in [9, 10] and the logic introduced there is the focus of our study.

The formulae in this logic include names, called *places*. Assertions in the logic are associated with places, and are validated in places. In addition to considering *whether* a formula is true, we are also interested in *where* a formula is true. The three modalities of the logic allow us to infer whether a property is validated in a specific place of the system ($@p$), or in an unspecified place of the system ($\Diamond$), or in any part of the system($\Box$). The modality $@p$ internalises the model in the logic and hence can be classified as a hybrid logic [1, 16, 2]. An intuitionistic natural deduction for the logic is given in [9, 10], and judgements in the logic mention the places under consideration. The natural deduction rules for $\Diamond$ and $\Box$ resemble those for existential and universal quantification of first-order intuitionistic logic.

As noted in [9, 10], the logic can also be used to reason about distribution of resources in addition to serving as the foundation of a type system. The papers [9, 10], however, lack a model to match the usage of the logic as a tool to reason about distributed resources. In this report, we bridge the gap by presenting a Kripke-style semantics [12] for the logic of [9, 10]. In Kripke-style semantics, formulae are considered valid if they remain valid when the atoms mentioned in the formulae change their value from false to true. This is achieved by using a partially ordered set of *possible states*. Informally, more atoms are true in larger states.

We extend the Kripke semantics of the intuitionistic logic [12], enriching possible states with a *fixed* set of places. In each possible state, different places satisfy different formulae. For the intuitionistic connectives, the satisfaction of formulae at a place in a possible state follows the standard definition [12]. The enrichment of the model with places reveals the true meaning of the modalities in the logic. The modality $@p$ expresses a property in a named place. The modality $\Box$ corresponds to a weak form of universal quantification and expresses a common property, and the modality $\Diamond$ corresponds to a weak form of existential quantification and expresses a property valid somewhere in the system.

In the model, we interpret atomic formulae as resources of a distributed system, and placement of atoms in a possible state corresponds to the distribution of resources. As in intuitionistic logic [12], we need not evaluate all the formulae of the language, since the interpretation follows inductively from the structure of formulae.

In order to give semantics to a logical judgment, we allow models with more places than those mentioned in the judgement. This admits the possibility that a user may be aware of only a certain subset of names in a distributed system. As we shall see, this is crucial in the proof of soundness and completeness.

In the model, we can duplicate places in a conservative way. This fact is the key to the proof of soundness of introduction of $\Box$, and the elimination of $\Diamond$. The proof of completeness follows closely the standard proof of completeness of intuitionistic logic with one important difference: in addition to witnesses for the existential ($\Diamond$), we need witnesses for the universal ($\Box$) too.

The logic in [9, 10] did not have disjunctive connectives. We extend the logic with disjunctive connectives, and refine our Kripke semantics in order to obtain completeness. In the refined semantics, the set of places in Kripke states are not fixed. Different possible Kripke states may have *different* set of places. However, the set of places vary in a conservative way: larger Kripke states contain larger set of places.

We show that the refined semantics is both sound and complete for the extended logic. The proof of soundness once again depends on duplication of places. The proof of completeness follows closely the standard proofs of completeness of intuitionistic modal logics. The extended logic can be seen as hybridization of the well-known intuitionistic modal system *IS5* [2, 18].

The rest of the paper is organised as follows. In Section 2, we present the logic in [9, 10]. In Section 3 we present the distributed Kripke model used to interpret the logic, and prove soundness and completeness of the semantics. We present the extension of logic with logical connectives in Section4. The refined semantics is given in Section 5, where we also show soundness and completeness of the refined logic. We discuss related work in Section 6, and we summarise our results in Section 7.

2 Logic

We now introduce, through examples, the logic presented in [9, 10]. The logic is used to reason about heterogeneous distributed systems. To gain some intuition, consider a *distributed peer to peer database* where the information is partitioned over multiple communicating nodes (peers).

Informally, the database has a set of nodes, or *places*, and a set of resources (data) distributed amongst these places. The nodes are chosen from the elements of a fixed set, denoted by $p, q, r, s, \ldots$. Resources are represented by atomic formulae $A, B, \ldots \in Atoms$. Intuitively, an atom A is verified in a place p if that place can access the resource identified by A.

Were we reasoning about a particular place, the logic connectives of the intuitionistic framework would be sufficient. For example, assume that a particular document, doc, is partitioned in two parts, doc_1 and doc_2, and in order to access to the document a place has to access both of its parts. This can be formally expressed as the logical formula: $(\mathsf{doc}_1 \wedge \mathsf{doc}_2) \rightarrow \mathsf{doc}$, where $\wedge$ and $\rightarrow$ are the logical conjunction and implication. particular place, then the usual intuitionistic rules allow to infer that the place can access the entire document.

The intuitionistic framework is extended in [10] in order to reason about different places. An assertion in such a logic takes the form "φ ***at*** p", meaning that formula φ is valid at place p. The construct " ***at*** " is a meta-linguistic symbol and points to the place where the reasoning is located. For example, doc_1 ***at*** p and doc_2 ***at*** p formalises the notion that the parts doc_1 and doc_2 are located at the node p. If in addition, the assertion $((\mathsf{doc}_1 \wedge \mathsf{doc}_2) \rightarrow \mathsf{doc})$ ***at*** p is valid, we can conclude that the document doc is available at p. A formula φ may itself use three modalities to accommodate reasoning about the properties valid at different locations.

In order to internalise resources at a single location, the modality $@p$, one for every place in the system, is used. The modality @ casts the meta-linguistic " ***at*** " on the language level, and in fact the two constructs will have the same interpretation in the semantics. The modal formula $\varphi@p$ means that the property φ is valid at p, and not necessarily anywhere else. An assertion of the form

$\varphi@p$ ***at*** p' means that in the place p' we are reasoning about the property φ valid at the place p. For example, suppose that the place p has got the first half of the document, i.e., doc_1 ***at*** p, and p' has got the second one, i.e., doc_2 ***at*** p'. In the logic we can formalise the fact that p' can send the part doc_2 to p by using the assertion $(\mathsf{doc}_2 \rightarrow (\mathsf{doc}_2@p))$ ***at*** p'. The rules of the logic will conclude doc_2 ***at*** p and so doc ***at*** p.

Knowing exactly where a property holds is a strong ability, and we may only know that the property holds somewhere without knowing the specific location where it holds. In order to deal with this, the logic has the $\Diamond$ modality: $\Diamond\varphi$ means that the formula φ holds in some place. In the example above, the location of doc_2 is not important as long as we know that this document is located in some place that can send it to p. Formally, this can be expressed by the formula $\Diamond(\mathsf{doc}_2 \wedge (\mathsf{doc}_2 \rightarrow (\mathsf{doc}_2@p)))$ ***at*** p'. By assuming this formula, we can infer doc_2 ***at*** p, and hence the document doc is available at p.

Even if we deal with resources distributed in heterogeneous places, we cannot avoid the fact that certain properties are valid everywhere. For this purpose, the logic has the $\Box$ modality: $\Box\varphi$ means that the formula φ is valid everywhere. In the example above, p can access the document doc, if there is a place that has the part doc_2 and can send it everywhere. This can be expressed by the formula $\Diamond(\mathsf{doc}_2 \wedge (\mathsf{doc}_2 \rightarrow \Box\mathsf{doc}_2))$ ***at*** p'. The rules of the logic would allow us to conclude that doc_2 is available at p.

We now define the logic in [10] formally. For the rest of the paper, we shall assume a fixed countable set of atomic formulae *Atoms* and we will vary the set of places. Given a countable set of places *Pl*, let *Frm*(*Pl*) be the set of formulae built from the following grammar:

$$\varphi ::= \top \mid A \mid \varphi \wedge \varphi \mid \varphi \rightarrow \varphi \mid \varphi@p \mid \Box\varphi \mid \Diamond\varphi.$$

Here the syntactic category p stands for elements from *Pl*, and the syntactic category A stands for elements from *Atoms*. The elements in *Frm*(*Pl*) are said *pure formulae*, and are denoted by small Greek letters $\varphi, \psi, \mu \ldots$ An assertion of the form φ ***at*** p is called *sentence*. We denote by capital Greek letters $\Gamma, \Gamma_1, \ldots$ (possibly empty) finite sets of pure formulae, and by capital Greek letters $\Delta, \Delta_1, \ldots$ (possibly empty) finite sets of sentences.

Each judgement in this logic is of the form

$$\Gamma; \Delta \vdash^P \varphi \textbf{\textit{ at }} p.$$

where

- the *global context* Γ is a (possibly empty) finite set of pure formulae, and represents the properties assumed to hold at every place of the system;
- the *local context* Δ is a (possibly empty) finite set of sentences; since a sentence is a pure formula associated to a place, Δ represents what we assume

to be valid in any particular place.

- the sentence φ ***at*** p says that φ is derived to be valid in the place p by assuming $\Gamma; \Delta$.

In the judgement, it is assumed that the places mentioned in Γ and Δ are drawn from the set P. In order to be more formal, we define the function $\mathsf{PL}(X)$, which denotes the set of places that appear in X, for any syntactic object X. It is defined as follow

Definition 1 (Places in a formula). *We define inductively the operator* $\mathsf{PL}(\,)$ *on any syntactic object of the logic as:*

$$
\begin{array}{llll}
\mathsf{PL}(A) & \stackrel{\text{def}}{=} \emptyset; & \mathsf{PL}(\top) & \stackrel{\text{def}}{=} \emptyset; \\
\mathsf{PL}(\varphi_1 \wedge \varphi_2) & \stackrel{\text{def}}{=} \mathsf{PL}(\varphi_1) \cup \mathsf{PL}(\varphi_2); & \mathsf{PL}(\varphi_1 \rightarrow \varphi_2) & \stackrel{\text{def}}{=} \mathsf{PL}(\varphi_1) \cup \mathsf{PL}(\varphi_2); \\
\mathsf{PL}(\Box\varphi) & \stackrel{\text{def}}{=} \mathsf{PL}(\varphi); & \mathsf{PL}(\Diamond\varphi) & \stackrel{\text{def}}{=} \mathsf{PL}(\varphi); \\
\mathsf{PL}(\varphi @ p) & \stackrel{\text{def}}{=} \mathsf{PL}(\varphi) \cup \{p\}; & \mathsf{PL}(\varphi\ \mathbf{at}\ p) & \stackrel{\text{def}}{=} \mathsf{PL}(\varphi) \cup \{p\};
\end{array}
$$

$$
\begin{array}{ll}
\mathsf{PL}(\varphi_1, \ldots, \varphi_m) & \stackrel{\text{def}}{=} \mathsf{PL}(\varphi_1) \cup \ldots \cup \ldots \mathsf{PL}(\varphi_m); \\
\mathsf{PL}(\varphi_1\ \mathbf{at}\ p_1, \ldots, \varphi_n\ \mathbf{at}\ p_n) & \stackrel{\text{def}}{=} \mathsf{PL}(\varphi_1\ \mathbf{at}\ p_1) \cup \ldots \cup \mathsf{PL}(\varphi_n\ \mathbf{at}\ p_n); \\
\mathsf{PL}(\Gamma; \Delta) & \stackrel{\text{def}}{=} \mathsf{PL}(\Gamma) \cup \mathsf{PL}(\Delta).
\end{array}
$$

When we write a judgment of the form $\Gamma; \Delta \vdash^P \varphi$ ***at*** p, then it must be the case that $\mathsf{PL}(\Gamma) \cup \mathsf{PL}(\Delta) \cup \mathsf{PL}(\varphi\ \textit{\textbf{at}}\ p) \subseteq P$. Any judgment not satisfying this condition is assumed to be undefined.

In Fig. 1 we give the natural deduction for the judgements as defined in [10]. The most interesting of these rules are $\Diamond E$, the elimination of $\Diamond$, and $\Box I$, the introduction of $\Box$. In these rules, we use $P + p$ to denote the disjoint union $P \cup \{p\}$, and witness the fact that the place p does not occur in Γ and Δ. If $p \in P$, then $P + p$, and any judgment containing such notation, is assumed to be undefined in order to avoid a side condition stating this requirement.

The rule $\Diamond E$ explains how we can use the formulae validated at some unspecified location: we introduce a new place and extend the local context by assuming that the formula is validated there. If any assertion that does not mention the new place is validated thus, then it is also validated using the old local context. The rule $\Box I$ says that if a formula is validated in some new place, without any local assumption on that new place, then that formula must be valid everywhere.

The rules $\Diamond I$ and $\Box E$ are reminiscent of the introduction of the existential quantification, and the elimination of universal quantification in first-order intuitionistic logic. This analogy, however has to be taken carefully. For example, if $\Gamma; \Delta \vdash^P \Diamond\psi$ ***at*** p, then we can show using the rules of the logic that $\Gamma; \Delta \vdash^P \Box\Diamond\psi$ ***at*** p.

$$L\quad \frac{}{\Gamma;\Delta,\varphi\ \textit{\textbf{at}}\ p \vdash^{P} \varphi\ \textit{\textbf{at}}\ p}
\qquad G\quad \frac{}{\Gamma,\varphi;\Delta \vdash^{P} \varphi\ \textit{\textbf{at}}\ p}
\qquad \top I\quad \frac{}{\Gamma;\Delta \vdash^{P} \top\ \textit{\textbf{at}}\ p}$$

$$\wedge I\quad \frac{\Gamma;\Delta \vdash^{P} \varphi_1\ \textit{\textbf{at}}\ p \qquad \Gamma;\Delta \vdash^{P} \varphi_2\ \textit{\textbf{at}}\ p}{\Gamma;\Delta \vdash^{P} \varphi_1 \wedge \varphi_2\ \textit{\textbf{at}}\ p}
\qquad \wedge E_i\ (i{=}1,2)\quad \frac{\Gamma;\Delta \vdash^{P} \varphi_1 \wedge \varphi_2\ \textit{\textbf{at}}\ p}{\Gamma;\Delta \vdash^{P} \varphi_i\ \textit{\textbf{at}}\ p}
\qquad \rightarrow I\quad \frac{\Gamma;\Delta,\varphi\ \textit{\textbf{at}}\ p \vdash^{P} \psi\ \textit{\textbf{at}}\ p}{\Gamma;\Delta \vdash^{P} \varphi \rightarrow \psi\ \textit{\textbf{at}}\ p}$$

$$@I\quad \frac{\Gamma;\Delta \vdash^{P} \varphi\ \textit{\textbf{at}}\ p}{\Gamma;\Delta \vdash^{P} \varphi@p\ \textit{\textbf{at}}\ p'}
\qquad @E\quad \frac{\Gamma;\Delta \vdash^{P} \varphi@p\ \textit{\textbf{at}}\ p'}{\Gamma;\Delta \vdash^{P} \varphi\ \textit{\textbf{at}}\ p}
\qquad \rightarrow E\quad \frac{\Gamma;\Delta \vdash^{P} \varphi \rightarrow \psi\ \textit{\textbf{at}}\ p \qquad \Gamma;\Delta \vdash^{P} \varphi\ \textit{\textbf{at}}\ p}{\Gamma;\Delta \vdash \psi\ \textit{\textbf{at}}\ p}$$

$$\Diamond I\quad \frac{\Gamma;\Delta \vdash^{P} \varphi\ \textit{\textbf{at}}\ p}{\Gamma;\Delta \vdash^{P} \Diamond\varphi\ \textit{\textbf{at}}\ p'}
\qquad \Diamond E\quad \frac{\Gamma;\Delta \vdash^{P} \Diamond\varphi\ \textit{\textbf{at}}\ p' \qquad \Gamma;\Delta,\varphi\ \textit{\textbf{at}}\ q \vdash^{P+q} \psi\ \textit{\textbf{at}}\ p''}{\Gamma;\Delta \vdash^{P} \psi\ \textit{\textbf{at}}\ p''}$$

$$\Box I\quad \frac{\Gamma;\Delta \vdash^{P+q} \varphi\ \textit{\textbf{at}}\ q}{\Gamma;\Delta \vdash^{P} \Box\varphi\ \textit{\textbf{at}}\ p}
\qquad \Box E\quad \frac{\Gamma;\Delta \vdash^{P} \Box\varphi\ \textit{\textbf{at}}\ p \qquad \Gamma,\varphi;\Delta \vdash^{P} \psi\ \textit{\textbf{at}}\ p'}{\Gamma;\Delta \vdash^{P} \psi\ \textit{\textbf{at}}\ p'}$$

Figure 1. Natural deduction.

3 Kripke Semantics

There are a number of semantics for intuitionistic logic and intuitionistic modal logics that allow for a completeness theorem [2, 11, 18]. In this section we concentrate on the semantics introduced by Kripke [12, 20], as it is convenient for applications and fairly simple. This would provide a formalisation of the intuitive concepts introduced in Section 2.

In Kripke semantics for intuitionistic propositional logic, logical assertions are interpreted over Kripke models. The validity of an assertion depends on its behaviour as the truth values of its atoms change from false to true according to a Kripke model. A Kripke model consists of a *partially ordered* set of *Kripke states*, and an *interpretation*, I, that maps atoms into states. The interpretation tells which atoms are true a state. It is required that if an atom is true in a state, then it must remain true in all larger states. Hence, in a larger state more atoms may become true. Consider a logical assertion built from the atoms $A_1, \cdots, A_n$. The assertion is said to be valid in a state if it continues to remain valid in all larger state.

In order to express the full power of the logic introduced in Section 2, we need to enrich the model by introducing places. We achieve this by associating a fixed set of places *Pls* to each Kripke state. The interpretation, I, in our model maps atoms into places in each state. Since we consider atoms to be resources, the map I tells how resources are distributed in a Kripke state. We require that if I maps an atom into a place in a state, then it would map the atom into that place in all larger states. In terms of resources, it means that places in larger states have possibly more resources. The addition of places makes the Kripke model *distributed* in the obvious sense. We are ready to define Kripke model formally.

Definition 2 (Distributed Kripke Model). *A* distributed Kripke model *is a quadruple* $\mathcal{K} = (K, \leq, Pls, I)$*, where*

- K *is a (non empty) set;*
- $\leq$ *is a partial order on* K*;*
- *Pls is a (non empty) set of places;*
- $I : Atoms \to Pow(K \times Pls)$ *is such that if* $(k, p) \in I(A)$ *then* $(l, p) \in I(A)$ *for all* $l \geq k$*.*

for $Pow()$ *the powerset operator.*

The set K is the set of Kripke states, whose elements are denoted by $k, l, \ldots$ Relation $\leq$ is the partial order on the states and I is the interpretation of atoms. The definition tells only how resources, i.e. atoms, are distributed in the system.

In order to give semantics to the whole set of formulae $Frm(Pls)$, we need to extend I. The interpretation of a formula depends on its composite parts and if it is valid in a given state, then it remains valid at the same places in all larger states. For example, the formula $\varphi \wedge \psi$ is valid in a state k at place p, if both φ and ψ are true at place p in all states $l \geq k$.

The introduction of places in the model allows the interpretation of the spatial modalities of the logic. Formula $\varphi @ p$ is satisfied at a place in a state k, if it is true at p in all states $l \geq k$; $\Diamond\varphi$ and $\Box\varphi$ are satisfied at a place in state k, if φ is true respectively at some or at every place in all states $l \geq k$.

We extend now the interpretation of atoms to interpretation of formulae, we use induction on the structure of the formulae.

Definition 3 (Kripke Semantics). *For* $\mathcal{K} = (K, \leq, Pls, I)$ *a distributed Kripke model, the relation* $\models$ *between couples* (k, p) *and pure formulae is inductively* *defined by*

$$
\begin{array}{lll}
(k,p) \models A & \textit{iff } (k,p) \in I(A); \\
(k,p) \models \top & \textit{for all } (k,p) \in K \times Pls; \\
(k,p) \models \varphi \wedge \psi & \textit{iff } (k,p) \models \varphi \textit{ and } (k,p) \in I'(\psi); \\
(k,p) \models \varphi \to \psi & \textit{iff } l \geq k \textit{ and } (l,p) \models \varphi \textit{ imply } (l,p) \models \psi; \\
(k,p) \models \varphi @ q & \textit{iff } (k,q) \models \varphi; \\
(k,p) \models \Box\varphi & \textit{iff } (k,q) \models \varphi \textit{ for all } q \in Pls; \\
(k,p) \models \Diamond\varphi & \textit{iff there exists } q \in Pls \textit{ such that } (q,k) \models \varphi.
\end{array}
$$

We pronounce $(k,p) \models \varphi$ *it as* (k,p) forces φ, *or* (k,p) satisfies φ. *We write* $k \models \varphi$ **at** p *if* $(k,p) \models \varphi$.

Please note that in this extension, except for logical implication, we have not considered larger states in order to interpret a modality or a connective. It turns out that the satisfaction of a formula in a state implies the satisfaction in all larger states.

Lemma 1 (Kripke Monotonicity). *Given* $\mathcal{K} = (K, \leq, Pls, I)$ *distributed Kripke model,* $\models$ *preserves the partial order in* K*, that is for each* $p \in Pls$ *and each* $\varphi \in Frm(Pls)$*, if* $l \geq k$ *then* $(k,p) \models \varphi$ *implies* $(l,p) \models \varphi$.

Proof: We proceed by induction on the structure of formulae.

Base case. If $\varphi \in Atoms$ or $\varphi = \top$, the lemma holds by Definitions 2 and 3.

Inductive Hypothesis. We consider a formula $\varphi \in Frm(Pls)$. We assume that for every sub-formula φ_i of φ and for every $p \in Pls$: if $l \geq k$ then $(k,p) \models \varphi_i$ implies $(l,p) \models \varphi_i$. We refer to Definition 3. Cases $\varphi = \varphi_1 \wedge \varphi_2$ and $\varphi = \varphi_1 \to \varphi_2$ are treated as in [20]. Cases $\varphi = \varphi_1 @ q$, $\varphi = \Box\varphi_1$ and $\varphi = \Diamond\varphi_1$ are similar. We show only the case $\varphi = \varphi_1 @ q$. Assume $(p,k) \models \varphi_1 @ q$, then $(q,k) \models \varphi_1$ by definition, hence $(q,l) \models \varphi_1$ for every $l \geq k$ by inductive hypothesis, and so we conclude that $(p,l) \models \varphi_1 @ q$. ■

Consider now the distributed database described in Section 2. We can express the same properties that we inferred in Section 2 by using a distributed Kripke model. Fix a Kripke state k. The assumption that the two parts, $\mathsf{doc}_1, \mathsf{doc}_2$, can be combined in p in a state k to give the document doc can be expressed as $(k,p) \models (\mathsf{doc}_1 \wedge \mathsf{doc}_2) \to \mathsf{doc}$. If the resources doc_1 and doc_2 are assigned to the place p, i.e., $(k,p) \models \mathsf{doc}_1$ and $(k,p) \models \mathsf{doc}_2$, then, since $(k,p) \models \mathsf{doc}_1 \wedge \mathsf{doc}_2$, it follows that $(k,p) \models \mathsf{doc}$.

Let us consider a slightly more complex situation. Suppose that $k \models \Diamond(\mathsf{doc}_2 \wedge (\mathsf{doc}_2 \to \Box\mathsf{doc}_2))$ ***at*** p'. According to the semantics of $\Diamond$, there is some place r such that $(k,r) \models \mathsf{doc}_2 \wedge (\mathsf{doc}_2 \to \Box\mathsf{doc}_2)$. The semantics of $\wedge$ tells us that $(k,r) \models \mathsf{doc}_2$ and $(k,r) \models (\mathsf{doc}_2 \to \Box\mathsf{doc}_2)$. Since $(k,r) \models \mathsf{doc}_2$, we know from the semantics of $\to$ that $(k,r) \models \Box\mathsf{doc}_2$, and from $\Box$ that $(k,p) \models \mathsf{doc}_2$. Therefore, if doc_1 is placed at p in the state k, then the whole document doc would becomes available at place p in state k.

3.1 Some useful properties

In order to prove soundness of our semantics, we shall need some important properties of the distributed Kripke models. We state and prove those properties in this section.

Lemma 2 says that if we add a new place which duplicates a specific place in all Kripke states, then the set of valid properties does not change. Moreover, the new place mimics the duplicated place. In order to state this lemma, we first prove that duplication gives us a distributed Kripke model.

PROPOSITION 1 (p-DUPLICATED EXTENSION $\mathcal{K}\langle p,q\rangle$). *Let* $\mathcal{K} = (K, \leq, Pls, I)$ *be a distributed Kripke model. For* $p \in Pls$ *and* $q \notin Pls$ *a new place, let* $\mathcal{K}\langle\rangle p, q = (K', \leq', Pls', I')$ *where*

- K' *is* K*;*
- $\leq'$ *is* $\leq$*;*
- Pls' *is* $Pls \cup \{q\}$*;*
- $I' : Atoms \longrightarrow Pow(K' \times Pls')$ *is defined as*

$$(k,r) \in I'(A) \text{ iff } \begin{cases} (k,r) \in I(A) \ \ (r \in Pls); \\ (k,p) \in I(A) \ (r = q). \end{cases}$$

Then $\mathcal{K}\langle p,q\rangle$ *is a distributed Kripke model, and* $\mathcal{K}\langle p,q\rangle$ *is said to be a* p-duplicated extension of $\mathcal{K}$.

Proof: We just need to check that I' satisfies the monotonicity condition on atoms which follows immediately from definition. ∎

We show that p-duplicated extension is conservative over all the formulae that do not mention the added place. Moreover, for all such formulae, the new place mimics the duplicated one.

LEMMA 2 ($\mathcal{K}\langle p,q\rangle$ IS CONSERVATIVE). *Let* $\mathcal{K}$ *be a distributed Kripke model, and* $\mathcal{K}\langle p,q\rangle$ *be its* p*-duplicated extension. Let* $\models$ *and* $\models'$ *extend the interpretation of atoms in* $\mathcal{K}$ *and* $\mathcal{K}_{p,q}$ *respectively. For every* $k \in K$ *and formula* $\varphi \in Frm(Pls)$*, we have:*

1. *if* $r \in Pls$*, then* $(k,r) \models' \varphi$ *if and only if* $(k,r) \models \varphi$*; and*
2. *if* $r = q$*, then* $(k,q) \models' \varphi$ *if and only if* $(k,p) \models \varphi$*.*

Proof: We prove both of the properties simultaneously by induction on the structure of formulae in $Frm(Pls)$.

Base case. The two properties are verified on atoms by the definition of I', and on $\top$ by Definition 3.

Inductive hypothesis. We consider a formula $\varphi \in Frm(Pls)$ and we assume the points hold for each of its sub-formulae φ_i. In particular we assume that:

1. *if $r \in Pls$, then $(k,r) \models '\varphi_i$ if and only if $(k,r) \models \varphi_i$; and*
2. *if $r = q$, then $(k,q) \models '\varphi_i$ if and only if $(k,p) \models \varphi_i$.*

We consider $r \in Pls$ and fix it. We prove only property *1*, as the treatment of point *2* is analogous. Now, we consider several possibilities for ϕ.

Case $\varphi = \varphi_1 \wedge \varphi_2$. The assertion $(k,r) \models '\varphi_1 \wedge \varphi_2$ iff $(k,r) \models '\varphi_1$ and $(k,r) \models '\varphi_2$. By inductive hypothesis, this is equivalent to $(k,r) \models \varphi_1$ and $(k,r) \models \varphi_2$, which is equivalent to $(k,r) \models \varphi_1 \wedge \varphi_2$ by Definition 3.

Case $\varphi = \varphi_1 \rightarrow \varphi_2$. $(k,r) \models '\varphi_1 \rightarrow \varphi_2$ iff $(l,r) \models '\varphi_1$ implies $(l,r) \models '\varphi_2$ for every $l \geq k$. By inductive hypothesis, this is equivalent to $(l,r) \models \varphi_1$ implies $(l,r) \models \varphi_2$ for every $l \geq h$, and this is equivalent to $(k,r) \models \varphi_1 \rightarrow \varphi_2$.

Case $\varphi = \varphi_1 @ s$. $(k,r) \models '\varphi_1 @ s$ iff $(k,s) \models '\varphi_1$. Moreover, we know that $s \in Pls$ as $\varphi_1 @ s \in Frm(Pls)$. By inductive hypothesis $(k,s) \models '\varphi_1$ iff $(k,s) \models \varphi_1$. By definition, $(k,s) \models \varphi_1$ iff $(k,r) \models \varphi_1 @ s$.

Case $\varphi = \Diamond \varphi_1$. Suppose $(k,r) \models '\Diamond \varphi_1$, then there exists $s \in Pls' = Pls \cup \{q\}$ such that $(k,s) \models '\varphi_1$. If $s \in Pls$, then we use inductive hypothesis (property *1*) to obtain $(k,s) \models \varphi_1$, and therefore $(k,r) \models \Diamond \varphi_1$. Otherwise if $s = q$, then we use inductive hypothesis (property 2) *2* to obtain $(k,p) \models \varphi_1$, and therefore $(k,r) \models \Diamond \varphi_1$.

Vice versa, if $(k,r) \models \Diamond \varphi_1$ then there exists $s \in Pls$ such that $(k,s) \models \varphi_1$. Hence by inductive hypothesis (property *1*) $(k,s) \models '\varphi_1$, and we conclude $(k,r) \models '\Diamond \varphi_1$.

Case $\varphi = \Box \varphi_1$. Suppose that $(k,r) \models '\Box \varphi_1$. This means that $(k,s) \models '\varphi_1$ for every $s \in Pls \cup \{q\}$. We can conclude that $(k,r) \models \Box \varphi_1$ by considering every $s \in Pls$ and applying inductive hypothesis (property *1*).

Vice versa, if $(k,r) \models \Box \varphi_1$ then $(k,s) \models \varphi_1$ for every $s \in Pls$. By inductive hypothesis (property *2)* $(k,s) \models '\varphi_1$ for every $s \in Pls$. Also, since $(k,s) \models \varphi_1$ for every $s \in Pls$, we get $(k,p) \models \varphi_1$. Hence by inductive hypothesis (property *2*), $(k,q) \models '\varphi_1$. We conclude $(k,t) \models '\varphi_1$ for every $t \in Pls'$, which implies $(k,r) \models '\Box \varphi_1$. ■

Another property of distributed Kripke models is the possibility to rename the places in the model. The property says that if we rename a place in the model, then we do not modify the set of valid properties not involving the renamed place. First we prove that the renamed model is still a distributed Kripke model, then we formalise the property in Lemma 3.

Proposition 2 (p-Renaming $\mathcal{K}\langle q/p \rangle$). *Given a distributed Kripke model $\mathcal{K} = (K, \leq, Pls, I)$, where $Pls = P + \{p\}$. For a new place $q \notin P$, we define $\mathcal{K}\langle q/p \rangle = (K', \leq', Pls', I')$ where*

- *K' is K;*

- *$\leq'$ is $\leq$;*
- *Pls' is $P \cup \{q\}$;*
- *I' : $Atoms \longrightarrow Pow(K' \times Pls')$ is defined as*

$$(k,r) \in I_{q/p)}(A) \text{ iff } \begin{cases} (k,r) \in I(A) \ (r \in P); \\ (k,p) \in I(A) \ (r = q). \end{cases}$$

*Then $\mathcal{K}\langle q/p\rangle$ is a distributed Kripke model, and $\mathcal{K}\langle q/p\rangle$ is said to be a p-*renaming *of $\mathcal{K}$.*

Proof: We just need to check that I' satisfies the monotonicity condition on atoms, which follows immediately from definition and the monotonicity of I. ■

By mimicking the proof of Lemma 2, we show that $\mathcal{K}\langle q/p\rangle$ is conservative with respect to $\mathcal{K}$ and the renamed place behaves like the original one.

LEMMA 3 ($\mathcal{K}\langle q/p\rangle$ IS CONSERVATIVE). *Let $\mathcal{K}$ be a distributed Kripke model such that $Pls = P + \{p\}$ and $\mathcal{K}\langle q/p\rangle$ be its p-renaming. Let $\models$ and $\models'$ extend the interpretation of atoms in $\mathcal{K}$ and $\mathcal{K}\langle q/p\rangle$ respectively. For every $k \in K$ and formula $\varphi \in Frm(P)$, we have:*

1. *if $r \in P$, then $(k,r) \models' \varphi$ if and only if $(k,r) \models \varphi$; and*
2. *if $r = q$, then $(k,q) \models' \varphi$ if and only if $(k,p) \models \varphi$.*

Proof: We proceed as in the proof of Lemma 2, and prove both of the properties simultaneously by induction on the structure of formulae in $Frm(Pls)$.

Base case. The properties are verified on atoms and $\top$ by definition.

Inductive hypothesis. As for Lemma 2, we consider a formula $\varphi \in Frm(Pls)$ and we assume that the two properties hold for each of its sub-formulae φ_i. Inductive cases deal with connectives and modalities. Here we consider only the two most significant cases and prove property *1*. The other cases can be dealt with easily.

Case $\varphi = \Diamond\varphi_1$. Let $r \in P$ and suppose $(k,r) \models' \Diamond\varphi_1$. Then, by definition there exists $s \in Pls' = P \cup \{q\}$ such that $(k,s) \models' \varphi_1$. If $s \in P$, we use inductive hypothesis (property *1*) to obtain $(k,s) \models \varphi_1$, and in that case $(k,r) \models \Diamond\varphi_1$ by definition. In the case $s = q$, we use inductive hypothesis (property *2*) to obtain $(k,p) \models \varphi_1$ and so $(k,r) \models \Diamond\varphi_1$. The opposite direction is analogous.

Case $\varphi = \Box\varphi_1$. Suppose $(k,r) \models' \Box\varphi_1$. Then the definition says that $(k,s) \models' \varphi_1$ for every $s \in P \cup \{q\}$. We get by using inductive hypothesis

- $(k,s) \models \varphi_1$ for every $s \in P$, and
- $(k,p) \models \varphi_1$

We conclude that $(k,t) \models \varphi_1$ for every $t \in P + \{p\}$, and hence $(k,r) \models \Box\varphi_1$. The opposite direction is analogous. ■

3.2 Soundness

We shall now give a semantics of the judgments introduced in §2 using distributed Kripke models. We shall then show that the semantics is both sound and complete. In order to introduce the semantics, we extend the definition of *validity* for pure formulae to sets of pure formula and sets of sentences.

DEFINITION 4 (FORCING EXTENSION). Let $\mathcal{K} = (K, \leq, Pls, I)$ be a distributed Kripke model. Given Γ, a finite set of pure formulae and Δ, a finite set of sentences, such that $\mathsf{PL}(\Gamma; \Delta) \subseteq Pls$, we say that the Kripke state $k \in K$ forces the couple $\Gamma; \Delta$, (and we write $k \models \Gamma; \Delta$) if

1. $(k, p) \models \varphi$ for every $\varphi \in \Gamma$ and $p \in Pls$;
2. $k \models \psi$ ***at*** q for every ψ ***at*** $q \in \Delta$.

A judgment is respected by a distributed Kripke model, if whenever its assumptions are valid in a Kripke state, then its conclusion is also valid in that state. We are now ready to define the satisfaction of a judgement.

DEFINITION 5 (SATISFACTION FOR A JUDGMENT). We say that $\Gamma; \Delta \models^P \mu$ ***at*** p, and we read it as "$\Gamma; \Delta \vdash^P \mu$ **at** p *is valid*", if

- $PL(\Gamma) \cup PL(\Delta) \cup \{p\} \subseteq P$; and
- for every distributed Kripke model $\mathcal{K} = (K, \leq, Pls, I)$ with $P \subseteq Pls$, it is the case that for every $k \in K$, whenever $k \models \Gamma; \Delta$, then $(k, p) \models \mu$ too.

We prove that the semantics is sound for the judgements of the logic. The proof of soundness depends on Lemma 2 and Lemma 3. We need to show that if a judgement is provable in the natural deduction system, then it is also valid.

THEOREM 1 (SOUNDNESS). *If* $\Gamma; \Delta \vdash^P \mu$ **at** p *is derivable in the logic, then it is valid.*

Proof: The proof proceeds by induction on the number n of inference rules applied in the derivation of the judgement $\Gamma; \Delta \vdash^P \mu$ ***at*** p. The most interesting cases are $\Box I$, the introduction of $\Box$, and $\Diamond E$, the elimination of $\Diamond$.

Base Case ($n = 1$). Suppose the judgment is proved by using axiom L, or the axiom G, or the axiom $\top I$. We consider a model $(K, \leq, Pls, I)$ such that $P \subseteq Pls$. We need to show that for every $k \in K$ if $k \models \Gamma; \Delta$ then $(k, p) \models \mu$.

Suppose the derivation consists of just the axiom L, then the assertion μ ***at*** p is in Δ. Hence, by definition, for every $k \in K$ if $k \models \Gamma; \Delta$ then $(k, p) \models \mu$.

If the derivation consists of just the axiom G, then the formula μ is in Γ, and so $k \models \Gamma; \Delta$ implies $(k, r) \models \mu$ for every $r \in Pls$. In particular $(k, p) \models \mu$.

Finally If the derivation is the application of $\top I$, then μ is $\top$ and the result holds by definition.

Inductive hypothesis ($n > 1$). We assume the theorem holds for any judgment that is deducible by applying less than n instances of inference rules. In particular we assume that:

If the judgment $\Gamma; \Delta \vdash^P \mu$ **at** p *is deducible in the logic by using less than n instances of the rules, then* $\Gamma; \Delta \models {}^P \mu$ **at** p.

We consider a judgment $\Gamma; \Delta \vdash^P \mu$ ***at*** p which is derivable in the logic by using exactly n instances of inference rules. We fix a model $\mathcal{K} = (K, \leq, Pls, I)$ such that $P \subseteq Pls$, and let $\models$ be the extension of I on $Frm(Pls)$. We fix $k \in K$ such that $k \models \Gamma; \Delta$. We need to prove $(k, p) \models \mu$. We consider the last rule applied to obtain $\Gamma; \Delta \vdash^P \mu$ ***at*** p, and proceed by cases. In most cases, we apply the inductive hypothesis on the model $\mathcal{K}$ only. However, for $\Box I$ and $\Diamond E$ we will use inductive hypothesis on an extension of $\mathcal{K}$.

Cases $\wedge I$ and $\wedge E$ follow from Definition 3 and are treated as in [20].

Case $\rightarrow I$. Then $\mu = \varphi \rightarrow \psi$ and we can derive $\Gamma; \Delta, \varphi$ ***at*** $p \vdash^P \psi$ ***at*** p by applying $n - 1$ instances of the rules. The inductive hypothesis says that for every $l \in K$: $l \models \Gamma; \Delta, \varphi$ ***at*** p implies $l \models \psi$ ***at*** p.

Let $l \geq k$. Then $l \models \Gamma; \Delta$ by Kripke Monotonicity (Lemma 1). If we assume $(l, p) \models \varphi$, then the inductive hypothesis says that $(l, p) \models \psi$ too. Hence, we have that for all $l \geq k$, if $l \models \varphi$ then $l \models \psi$ also. We conclude that $(k, p) \models \varphi \rightarrow \psi$ by definition of $\models$.

Case $\rightarrow E$. Then, we have that $\Gamma; \Delta \vdash^P \varphi \rightarrow \mu$ and $\Gamma, \Delta \vdash^P \varphi$ for some φ. The inductive hypothesis says that $(k, p) \models \varphi \rightarrow \mu$ and $(k, p) \models \varphi$. Hence, we get $(k, p) \models \mu$ according to Definition 3.

Case @I. Then μ is of the form $\varphi@q$, and $\Gamma; \Delta \vdash^P \varphi$ ***at*** q. The inductive hypothesis says that $(k, q) \models \varphi$, and hence $(k, p) \models \varphi@q$.

Case @E. Then we have that $\Gamma; \Delta \vdash^P \mu@p$ ***at*** q for some $q \in P$. The inductive hypothesis says that $(k, q) \models \varphi@p$, and therefore $(k, p) \models \varphi$.

Case $\Box I$. Then μ is of the form $\Box\varphi$. Moreover $\Gamma; \Delta \vdash^{P+q} \varphi$ ***at*** p_1 for some $p_1 \notin P$ by using $n - 1$ instances of the inference rules. By inductive hypothesis we know that $\Gamma; \Delta \models {}^{P+p_1}\varphi$ ***at*** q. Please note that since $\Gamma; \Delta \vdash^P \mu atp$, we also have $\mathsf{PL}(\Gamma; \Delta) \cup \mathsf{PL}(\varphi) \subseteq P$. Let Pls be $P + p_1$.

First, consider the case when $p_1 \notin Pls$. We need to show that $k \models \Box\varphi$ ***at*** p. According to semantics of $\Box$, it suffices to show that $k \models \varphi$ ***at*** r, for all $r \in Pls$. Fix one $r \in Pls$, and consider the r-duplicated extension $\mathcal{K}_{q(r)}$. Let $\models_{q(r)}$ be the extension of $I_{q(r)}$. We get $k \models_{q(r)} \Gamma; \Delta$ by using Lemma 2 (since $k \models \Gamma; \Delta$).

Now, we have that $\Gamma; \Delta \vdash^{P+q} \varphi$ ***at*** p_1 .and $P + p_1 \subseteq Pls_{q(r)}$. Since $k \models_{q(r)} \Gamma; \Delta$, we get by using inductive hypothesis on $\mathcal{K}_{q(r)}$ that $(k, p_1) \models_{q(r)} \varphi$. Now, we can conclude $(k, r) \models \varphi$ ***at*** r by using Lemma 2.

Since r was arbitrary, we deduce $k \models \Box\varphi$ ***at*** p.

If $p_1 \in Pls$, then $Pls = Pls' + \{p_1\}$ with $\mathsf{PL}(\Gamma; \Delta) \cup \mathsf{PL}(\varphi) \subseteq P \subseteq Pls'$. We choose $t \notin Pls$ and consider $\mathcal{K}_{t/p_1}$ to be the p_1-renaming of $\mathcal{K}$, as defined in Proposition 2. Let $\models_{t/p_1}$ be the extension of I_{t/p_1}. By following the above reasoning we derive $k \models_{t/p_1} \Box\varphi$ ***at*** p, hence $k \models \Box\varphi$ ***at*** p by Lemma 3.

Case $\Box E$. Then we have that there is some formula φ such that $\Gamma; \Delta \vdash^P \Box\varphi$ ***at*** p_1 and $\Gamma, \varphi; \Delta \vdash^P \Box\mu$ ***at*** p by using less than n instances of inference rules. The inductive hypothesis on $\Gamma; \Delta \vdash^P \Box\varphi$ ***at*** p implies $(k, p_1) \models \Box\varphi$, and this means that $(k, q) \models \varphi$ for every $q \in Pls$. By definition, we obtain $k \models \Gamma, \varphi; \Delta$ and using inductive hypothesis on $\Gamma, \varphi; \Delta \vdash^P \Box\mu$ ***at*** p we conclude $(k, p') \models \psi$.

Case $\Diamond I$. Then we have that μ is of the form $\Diamond\varphi$ for some formula φ, and $\Gamma; \Delta \vdash^P \varphi$ ***at*** p_1 for some p_1. The inductive hypothesis says that $(k, p_1) \models \varphi$, so we conclude $(k, p) \models \Diamond\varphi$.

Case $\Diamond E$. Then for some $p' \in P$ and $\varphi \in Frm(P)$ we can derive $\Gamma; \Delta \vdash^P \Diamond\varphi$ ***at*** p' and $\Gamma; \Delta, \varphi$ ***at*** $q \vdash^{P+q} \mu$ ***at*** p by using less than n instances of the rules. Hence by inductive hypothesis: $\Gamma; \Delta \models^P \Diamond\varphi$ ***at*** p' and $\Gamma; \Delta, \varphi$ ***at*** $q \models^{P+q} \mu$ ***at*** p.

As in the case for $\Box I$, first assume $q \notin Pls$. We need to show that $(k, p) \models \mu$. Since $k \models \Gamma; \Delta$ we get $(k, p') \models \Diamond\varphi$, and this means that there exists $r \in Pls$ such that $(k, r) \models \varphi$.

Consider now the r-duplicated extension $\mathcal{K}_{q(r)}$ of $\mathcal{K}$. Let $\models_{q(r)}$ be the extension of $I_{q(r)}$. By Lemma 2 we have $(k, q) \models_{q(r)} \varphi$, and $k \models_{q(r)} \Gamma; \Delta$. Hence, we get $k \models_{q(r)} \Gamma; \Delta, \varphi$ ***at*** q. Since $\Gamma; \Delta, \varphi$ ***at*** $q \models^{P+q} \mu$ ***at*** p, we get $(k, p) \models_{q(r)} \mu$. As $\mathsf{PL}(\mu) \subseteq P \subseteq Pls$ and $p \in P \subseteq Pls$, we obtain $(k, p) \models \mu$ by Lemma 2.

In the case that $\mathcal{K}$ is such that $q \in Pls$, we can rename q by a fresh as we did in $\Box I$, and obtain the desired result. ∎

3.3 Completeness

We shall show that our semantics is complete for the natural deduction in Section 2. First, we extend the notion of provability to possible non-finite sets Σ of sentences by saying that $\Gamma; \Sigma \vdash^P \varphi$ ***at*** q, if and only if, there exists a finite set $\Delta \subseteq \Sigma$ such that $\Gamma; \Delta \vdash^P \varphi$ ***at*** q.

As in standard proofs of completeness of intuitionistic logics[20, 18, 2], the proof of completeness is based on the construction of a particular distributed Kripke model: the *canonical model*. We will prove that a sequent is valid in the canonical model if and only if it is derivable in the logic. In the construction of the canonical model, we consider particular kinds of sets of formulae.

Definition 6 (Prime Set). *Given a set of places Pls and a finite set* Γ *of pure formulae in* $Frm(Pls)$*, a (possibly non-finite) set* Σ *of sentences with* $\mathsf{PL}(\Sigma) \subseteq Pls$*, is said to be* (Γ, Pls)-prime *if for every formula* $\varphi \in Frm(Pls)$*:*

1. $\Gamma; \Sigma \vdash^{Pls} \Diamond\varphi$ **at** p*, implies that there exists* $q \in Pls$ *s.t.* $\Gamma; \Sigma \vdash^{Pls} \varphi$ **at** q*;*
2. $\Gamma; \Sigma \vdash^{Pls} \varphi$ **at** r *for all* $r \in Pls$*, implies* $\Gamma; \Sigma \vdash^{Pls} \Box\varphi$ **at** p *for all* $p \in Pls$*.*

The canonical model will be built by choosing the prime sets of formulae as Kripke states. We would show that given Γ and Δ, we can construct a set of places *Pls* and a prime set $\Sigma \supseteq \Delta$ such that Σ is (Γ, Pls)-prime. Before we proceed, we first state a proposition proved in [9]:

Proposition 3. *Let $P \subseteq P'$ and suppose $\mathsf{PL}(\Gamma) \cup \mathsf{PL}(\Delta) \cup \mathsf{PL}(\varphi \textbf{ at } p) \subseteq P$, then $\Gamma; \Delta \vdash^{P'} \varphi \textbf{ at } p$ if and only if $\Gamma; \Delta \vdash^{P} \varphi \textbf{ at } p$.*

Now, we show the existence of prime extensions:

Lemma 4 (Prime Extension). *Let P be a set of places and Γ be a finite set of pure formulae in $Frm(P)$. For every finite set Δ of sentences such that $\mathsf{PL}(\Delta) \subseteq P$, there exists a set of places P' extending P and a (Γ, P')-prime set of sentences Σ containing Δ, such that given $\varphi \in Frm(P)$ and $p \in P$:*

$$\Gamma; \Delta \vdash^{P} \varphi \textbf{ at } p \textit{ if and only if } \Gamma; \Sigma \vdash^{P'} \varphi \textbf{ at } p.$$

Proof: We enrich the set of places by introducing two kind of places: $\mathbf{q}_i$, which will be the witnesses for the formulae $\Diamond\varphi$, and $\mathbf{p}_j$, which will be the new places used to introduce $\Box\psi$ in the case ψ is provable for every place.

The set of places P' is obtained by a series of extensions $P = P_0 \subseteq P_1 \subseteq P_2 \ldots$ The sets P_{n+1} are constructed as $P_{n+1} = P_n \cup \{\mathbf{q}_{n+1}, \mathbf{p}_{n+1}\}$, where the places $\mathbf{q}_{n+1}, \mathbf{p}_{n+1}$ are new, i.e., $\mathbf{q}_{n+1}, \mathbf{p}_{n+1} \notin P_n$. Also, $\mathbf{q}_{n+1}$ is different from $\mathbf{p}_{n+1}$. The set P' is taken as $P' = \bigcup_{n \geq 0} P_n$.

Before we proceed with the construction, we pick up an enumeration of the pure formulae $Frm(P')$, and fix it. The set Σ is obtained by series of extensions $\Delta = \Sigma_0 \subseteq \Sigma_1 \subseteq \Sigma_2 \ldots$ that verify the following:

Property 1. For every $n \geq 0$:

1. $\mathsf{PL}(\Sigma_n) \subseteq P_n$.
2. *Given $\varphi \in Frm(P_n)$ and $p \in P_n$, we have $\Gamma; \Delta \vdash^{P} \varphi \textbf{ at } p$ if and only if $\Gamma; \Sigma_n \vdash^{P_n} \varphi \textbf{ at } p$.*

The series is constructed inductively. In the induction, we will create witnesses for the formulae of the type $\Diamond\psi$. We shall also construct a set, $treated_n$, of formulae of the sort $\Diamond\psi$. This set, initialised to be the empty set, will be the set of the formulae for which we have already created witnesses.

We put $treated_0 = \emptyset$, $P_0 = P$ and $\Sigma_0 = \Delta$. It is clear that $\mathsf{PL}(\Sigma_0) \subseteq P_0$, and $\Gamma; \Delta \vdash^{P} \varphi \textbf{\textit{ at }} p$ if and only if $\Gamma; \Sigma_0 \vdash^{P_0} \varphi \textbf{\textit{ at }} p$.

Now, we proceed inductively. Let Σ_n $(n \geq 0)$ extend Δ and satisfying Property 1. In step $n + 1$, we pick the first formula $\Diamond\psi$ in the enumeration such that

- $\Diamond\psi$ is in $Frm(P_n)$, i.e., all the places in $\Diamond\psi$ are taken from P_n;
- $\Diamond\psi \notin treated_n$; and
- $\Gamma; \Sigma_n \vdash^{P_n} \Diamond\psi \textbf{\textit{ at }} q$, for some $q \in P_n$.

We define $\Sigma_{n+1} = \Sigma_n \cup \{\psi \ \boldsymbol{at}\ \mathbf{q}_{n+1}\}$ and $\mathit{treated}_{n+1} = \mathit{treated}_n \cup \{\Diamond\psi\}$. The place $\mathbf{q}_{n+1}$ witnesses the existential $\Diamond$. Clearly $\mathsf{PL}(\Sigma_{n+1}) \subseteq P_{n+1}$. Now we prove the following:

Claim. For any $\varphi \in \mathit{Frm}(P_n)$ and $p \in P_n$, $\Gamma; \Sigma_n \vdash^{P_n} \varphi \ \boldsymbol{at}\ p$ if and only if $\Gamma; \Sigma_{n+1} \vdash^{P_{n+1}} \varphi \ \boldsymbol{at}\ p$.

The direction from left to right is a consequence of inference rule L, and Proposition 3. In order to prove the converse, assume $\Gamma; \Sigma_{n+1} \vdash^{P_{n+1}} \varphi \ \boldsymbol{at}\ p$. Now let ψ be the formula chosen at step $n+1$. We have by construction, $\Gamma; \Sigma_n \vdash^{P_n} \Diamond\psi \ \boldsymbol{at}\ q$. Also since $\Gamma; \Sigma_{n+1} \vdash^{P_{n+1}} \varphi \ \boldsymbol{at}\ p$, we get by using the inference rule L and Proposition 3 that $\Gamma; \Sigma_n, \psi \ \boldsymbol{at}\ \mathbf{q}_n \vdash^{P_n + \mathbf{q}_{n+1}} \varphi \ \boldsymbol{at}\ p$. Hence, we get $\Gamma; \Sigma_n \vdash^{P_n} \varphi \ \boldsymbol{at}\ p$ by application of the inference rule $\Diamond E$.

Suppose now that $\varphi \in \mathit{Frm}(P)$ and $p \in P$. We can assert using the claim above that $\Gamma; \Delta \vdash^{P} \varphi \ \boldsymbol{at}\ p$ if and only if $\Gamma; \Sigma_{n+1} \vdash^{P_{n+1}} \varphi \ \boldsymbol{at}\ p$. We have just proved Property 1 for the inductive step n.

Finally, we define $\Sigma = \bigcup_{n \geq 0} \Sigma_n$. Clearly $\Gamma; \Delta \vdash^{P} \varphi \ \boldsymbol{at}\ p$ implies $\Gamma; \Sigma \vdash^{P'} \varphi \ \boldsymbol{at}\ p$, by definition and Proposition 3.

In the other direction, suppose $\Gamma; \Sigma \vdash^{P'} \varphi \ \boldsymbol{at}\ p$ with $\varphi \in \mathit{Frm}(P)$ and $p \in P$. According to the definition, there exists a finite sequence $\Lambda \subseteq \Sigma$ such that $\Gamma; \Lambda \vdash^{P'} \varphi \ \boldsymbol{at}\ p$. We can then choose $n \geq 0$ big enough to have $\Lambda \subseteq \Sigma_n$ and so $\Gamma; \Sigma_n \vdash^{P'} \varphi \ \boldsymbol{at}\ p$ by the inference rule L. Using Proposition 3 once again, we have $\Gamma; \Sigma_n \vdash^{P_n} \varphi \ \boldsymbol{at}\ p$. Since $\mathsf{PL}(\Gamma), \mathsf{PL}(\varphi), \{p\} \subseteq P \subseteq P_n$, we conclude $\Gamma; \Delta \vdash^{P} \varphi \ \boldsymbol{at}\ p$ using Proposition 1.

All we need to prove now is that Σ is (Γ, P')-prime.

1. If $\Gamma; \Sigma \vdash^{P'} \Diamond\varphi \ \boldsymbol{at}\ p$, let n be the least such that $\Diamond\varphi \in \mathsf{PL}(P_n)$ and $p \in P_n$. By construction, there is some $m \geq n$, such that $\Diamond\varphi$ is picked in the construction of Σ_m. Hence $\varphi \ \boldsymbol{at}\ \mathbf{q}_m \in \Sigma_m \subseteq \Sigma$, and we conclude that $\Gamma; \Sigma \vdash^{P'} \varphi \ \boldsymbol{at}\ \mathbf{q}_m$.

2. Let $\psi \in \mathit{Frm}(P')$ and suppose $\Gamma; \Sigma \vdash^{P'} \psi \ \boldsymbol{at}\ p$ for all $p \in P'$. In particular, consider the place $\mathbf{p}_n$, with n such that $\psi \in \mathit{Frm}(P_n)$. We have that $\Gamma; \Sigma \vdash^{P'} \psi \ \boldsymbol{at}\ \mathbf{p}_n$.

   Using Proposition 3, we can find $m \geq 0$ such that $\Gamma; \Sigma_m \vdash^{P_m} \psi \ \boldsymbol{at}\ \mathbf{p}_n$. If $m > n$ then we use the above claim iteratively to conclude $\Gamma; \Sigma_n \vdash^{P_n} \psi \ \boldsymbol{at}\ \mathbf{p}_n$. In the case $m \leq n$ we obtain the same conclusion by the inference rule L.

   Since $\mathbf{p}_n \notin \mathsf{PL}(\Sigma_n)$ by construction, we can infer that $\Gamma; \Sigma_n \vdash^{P_n \setminus \{\mathbf{p}_n\}} \Box\psi \ \boldsymbol{at}\ p$ for all $p \in P_n$ by the inference rule $\Box I$. Hence $\Gamma; \Sigma \vdash^{P'} \Box\psi \ \boldsymbol{at}\ p$ for all $p \in P_n \setminus \{\mathbf{p}_n\}$ by Proposition 3.

   We conclude by extending $\Gamma; \Sigma \vdash^{P'} \Box\psi \ \boldsymbol{at}\ r$ to any $r \in (P' \setminus P_n) \cup \{\mathbf{p}_n\}$ in the following way (here z' is chosen to be a place $\notin P'$):

$$\dfrac{\Gamma;\Sigma \vdash^{P'} \Box\psi\ \textbf{\textit{at}}\ p \qquad \dfrac{\dfrac{}{\Gamma,\psi;\Sigma \vdash^{P'+z} \psi\ \textbf{\textit{at}}\ z}\ G}{\Gamma,\psi;\Sigma \vdash^{P'} \Box\psi\ \textbf{\textit{at}}\ r}\ \Box I}{\Gamma;\Sigma \vdash^{P'} \Box\psi\ \textbf{\textit{at}}\ r}\ \Box E$$

■

We are ready to define the canonical model for a finite set of pure formulae Γ and places Pls. In this model, the worlds will be (Γ, Pls)-prime sets. The partial order will be subset inclusion, and the atoms will be placed in a specific place p in a world Σ if $\Gamma;\Sigma \vdash^{Pls} A\ \textbf{\textit{at}}\ p$.

Definition 7 (Canonical Model). *Given a set of places Pls and a finite set Γ of pure formulae in $Frm(Pls)$, we define the* (Γ, Pls)-canonical *to be the quadruple* $\mathcal{M}_{(\Gamma,Pls)} = (M, \subseteq, Pls, I_\Gamma)$*, where:*

- *M is composed by all the (Γ, Pls)-prime sets;*
- *$\subseteq$ is set inclusion;*
- *$I_\Gamma : Atoms \longrightarrow Pow(M \times Pls)$ is defined by: $(\Sigma, p) \in I_\Gamma(A)$ iff $\Gamma;\Sigma \vdash^{Pls} A\ \textbf{at}\ p$.*

We now show that the model is a distributed Kripke model. We will also demonstrate that the extension of I_Γ to interpretation of formulae corresponds exactly to the provability in the logic, i.e., $(\Sigma, q) \models \psi$ in the canonical model if and only if $\Gamma;\Sigma \vdash^{Pls} \psi\ \textbf{\textit{at}}\ q$.

Lemma 5 (Canonical Evaluation). *Given a set of places Pls and a finite set Γ of pure formulae in $Frm(Pls)$, we have:*

1. *the (Γ, Pls)-canonical model $\mathcal{M}_{(\Gamma,Pls)} = (M, \subseteq, Pls, I_\Gamma)$ is a distributed Kripke model;*
2. *for all $\varphi \in Frm(Pls)$, $\Sigma \in M$ and $q \in Pls$: $(\Sigma, q) \models \varphi$ if and only if $\Gamma;\Sigma \vdash^{Pls} \varphi\ \textbf{at}\ q$.*

Proof: Clearly the inclusion among sets $\subseteq$ is a partial order on M and I_Γ is monotone on M, since if $\Sigma_1 \subseteq \Sigma_2$ then $\Gamma;\Sigma_1 \vdash^{Pls} A\ \textbf{\textit{at}}\ p$ implies $\Gamma;\Sigma_2 \vdash^{Pls} A\ \textbf{\textit{at}}\ p$ by definition. All we have to prove is the part *2* of the proposition. We proceed by induction on the structure of the formula φ and we prove that for every $\Sigma \in M$ and $q \in Pls$: $(\Sigma, q) \models \varphi$ if and only if $\Gamma;\Sigma \vdash^{Pls} \varphi\ \textbf{\textit{at}}\ q$.

Base Case. The property is verified on *Atoms*, by the definition of I_Γ, and on $\top$, by Definition 3.

Inductive hypothesis. We assume the property holds for any sub-formula of the formula φ we are considering. In particular we assume that:

Given φ_i *sub-formula of* $\varphi \in Frm(Pls)$*, then for every* $\Sigma \in M$ *and* $q \in Pls$*:* $(\Sigma, q) \models \varphi_i$ *if and only if* $\Gamma; \Sigma \vdash^{Pls} \varphi_i$ **at** q*.*

We need to show that $(\Sigma, q) \models \varphi$ if and only if $\Gamma; \Sigma \vdash^{Pls} \varphi$ ***at*** q. We proceed by cases on structure of φ. The cases in which φ is $\varphi_1 \wedge \varphi_2$, and φ is $\varphi_1 \rightarrow \varphi_2$ are fairly standard. We just consider the three modalities.

Case $\varphi_1@p$. Suppose that $(\Sigma, q) \models \varphi_1@p$. By definition, we have $(\Sigma, p) \models \varphi_1$. We get $\Gamma; \Sigma \vdash^{Pls} \varphi_1$ ***at*** p by inductive hypothesis. We can conclude $\Gamma; \Sigma \vdash^{Pls} \varphi_1@p$ ***at*** q by using the inference rule $@I$.

In the other direction, the fact $\Gamma; \Sigma \vdash^{Pls} \varphi_1@p$ ***at*** q implies $\Gamma; \Sigma \vdash^{Pls} \varphi_1$ ***at*** p by using the inference rule $@E$. Hence $(\Sigma, p) \models \varphi_1$ by inductive hypothesis, and therefore $(\Sigma, q) \models \varphi_1@p$.

Case $\Box\varphi_1$. $(\Sigma, q) \models \Box\varphi_1$ implies $(\Sigma, p) \models \varphi_1$ for all $p \in Pls$. By inductive hypothesis, this is $\Gamma; \Sigma \vdash^{Pls} \varphi_1$ ***at*** p for all $p \in Pls$. Since Σ is (Γ, Pls) prime, we can conclude $\Gamma; \Sigma \vdash^{Pls} \Box\varphi_1$ ***at*** q

In the other direction, let us assume that $\Gamma; \Sigma \vdash^{Pls} \Box\varphi_1$ ***at*** q. We apply the inference rule $\Box E$ to obtain $\Gamma; \Sigma \vdash^{Pls} \varphi_1$ ***at*** p for every $p \in Pls$. Hence $(\Sigma, p) \models \varphi_1$ for every $p \in Pls$, and therefore $(\Sigma, q) \models \Box\varphi_1$.

Case $\Diamond\varphi_1$. $(\Sigma, q) \models \Diamond_1\varphi$ says that there exists $p \in Pls$ such that $(\Sigma, p) \models \varphi_1$. Using inductive hypothesis, we get $\Gamma; \Sigma \vdash^{Pls} \varphi_1$ ***at*** p. We conclude $\Gamma; \Sigma \vdash^{Pls} \Diamond\varphi_1$ ***at*** q by $\Diamond I$.

In the other direction, assume $\Gamma; \Sigma \vdash^{Pls} \Diamond\varphi_1$ ***at*** q. Since Σ is (Γ, Pls) prime, there exists $p \in Pls$ such that $\Gamma; \Sigma \vdash^{Pls} \varphi_1$ ***at*** p. Using inductive hypothesis, we obtain $(\Sigma, p) \models \varphi_1$. We get $(\Sigma, q) \models \Diamond\varphi_1$ according to Definition 3. ■

Finally we use the canonical model to prove completeness.

THEOREM 2 (COMPLETENESS). $\Gamma; \Delta \models^{P} \mu$ **at** $p \Longrightarrow \Gamma; \Delta \vdash^{P} \mu$ **at** p.

Proof: Assume $\Gamma; \Delta \models^{P} \varphi$ ***at*** p. This means that

- $PL(\Gamma) \cup PL(\Delta) \cup \{p\} \subseteq P$; and
- for every distributed Kripke model $\mathcal{K} = (K, \leq, Pl, I)$ with $P \subseteq Pl$, it is the case that for every $k \in K$, whenever $k \models \Gamma; \Delta$ then $(k, p) \models \mu$ also.

We need to show that $\Gamma; \Delta \vdash^{P} \varphi$ ***at*** p.

Using Lemma 4, construct a set of places $Pls \supseteq P$, and a (Γ, Pls)-prime set of sentence Σ such that: *for every* $\varphi \in Frm(P)$ *and* $p \in P$ $\Gamma; \Delta \vdash^{P} \varphi$ **at** p *if and only if* $\Gamma; \Sigma \vdash^{Pls} \varphi$ **at** p*.*

Consider now the (Γ, Pls)-canonical model, as stated in Definition 7. In the canonical model, the worlds are the (Γ, Pls)-prime sets and the set of places is Pls. We focus our attention on the world Σ.

First we claim that in the canonical model $\Sigma \models \Gamma; \Delta$. In order to show this, we need the following:

- For every $\psi \in \Gamma, q \in Pls$, we need to show that $\Sigma \models \psi$ ***at*** q. Given $\psi \in \Gamma$, an application of inference rule G (see figure 1) gives us $\Gamma; \Sigma \vdash^{Pls} \psi$ ***at*** q. By Lemma 5, $\Sigma \models \psi$ ***at*** q if and only if $\Gamma; \Sigma \vdash^{Pls} \psi$ ***at*** q. Hence, we get $\Sigma \models \psi$ ***at*** q.

- For every ψ ***at*** $q \in \Delta$, we need to show that $\Sigma \models \psi$ ***at*** q. Given ψ ***at*** $q \in \Delta$, an application of the L rule (see figure 1), gives us that $\Gamma; \Delta \vdash^{Pls} \psi$ ***at*** q. Σ extends Δ, and hence we get $\Gamma; \Sigma \vdash^{Pls} \psi$ ***at*** q. By Lemma 5 once again, we get $\Sigma \models \psi$ ***at*** q.

So we have a model in which $\Sigma \models \Gamma; \Delta$. By assumption, this implies $\Sigma \models \mu$ ***at*** p. Using Lemma 5, we get that $\Gamma, \Sigma \vdash^{Pls} \mu$ ***at*** p. Since Σ is a prime extension of Δ constructed through Lemma 4, we conclude $\Gamma; \Delta \vdash^{P} \mu$ ***at*** p. ■

4 Hybrid IS5

We now extend the logic in [9, 10] with disjunctive connectives, thus achieving the full set of intuitionistic connectives. Given a set of places, *Pl*, the new set of pure formulae (see section 2), *Frm*(*Pl*), is the set of formulae built from the following grammar:

$$\varphi ::= \top \mid \bot \mid A \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \to \varphi \mid \varphi @ p \mid \Box\varphi \mid \Diamond\varphi.$$

To account for the new connectives, we extend the natural deduction presented in Figure 1 with rules for the disjunctive connectives. These rules are given in Figure 2. Please note that the rule $\bot E$ as stated has a local flavour: from $\bot$ ***at*** p, we can infer any other property in the same place, p. However, the rule has a "global" consequence. If we have $\bot$ ***at*** p, then we can infer $\bot @ q$ ***at*** p. Using $@E$, we can then infer $\bot$ ***at*** q. Hence if a set of assumptions make a place to be inconsistent, then it will make all places to be inconsistent.

As we shall see in section 5, the Kripke semantics of this extended logic would be similar to the one given for intuitionistic system *S5* [18]. Hence this logic can be seen as an instance of *Hybrid IS5* [2].

5 Refined Kripke Semantics

We were unable to prove completeness for the extended logic using the semantics defined in Section 3. We had to change the semantics in order to obtain a completeness result, and we present the semantics in this section. The difference from the model of Section 3, is that the set of places in Kripke states are not fixed and may vary. However, they change in a conservative way in that the set of places in a Kripke state is always contained in larger Kripke states. We now present the extended Kripke models which we shall call *Refined Distributed Kripke models*.

$\bot E$

$$\frac{\Gamma; \Delta \vdash^P \bot \textbf{\textit{ at }} p}{\Gamma; \Delta \vdash^P \psi \textbf{\textit{ at }} p}$$

$\vee I$ (i=1,2)

$$\frac{\Gamma; \Delta \vdash^P \varphi_i \textbf{\textit{ at }} p}{\Gamma; \Delta \vdash^P \varphi_1 \vee \varphi_2 \textbf{\textit{ at }} p}$$

$\vee E$

$$\frac{\Gamma; \Delta, \varphi_1 \textbf{\textit{ at }} p \vdash^P \psi \textbf{\textit{ at }} p \quad \Gamma; \Delta, \varphi_2 \textbf{\textit{ at }} p \vdash^P \psi \textbf{\textit{ at }} p \quad \Gamma; \Delta \vdash^P \varphi_1 \vee \varphi_2 \textbf{\textit{ at }} p}{\Gamma; \Delta \vdash^P \psi \textbf{\textit{ at }} p}$$

FIGURE 2. Disjunctive rules

DEFINITION 8 (REFINED DISTRIBUTED KRIPKE MODEL). *A quadruple* $\mathcal{K}_{ref} = (K, \leq, \{P_k\}_{k\in K}, \{I_k\}_{k\in K})$ *is called* refined distributed Kripke model *if*

- K *is a (non empty) set;*
- $\leq$ *is a partial order on* K*;*
- P_k *is a* non-empty *set of places for all* $k \in K$*;*
- $P_k \subseteq P_l$ *if* $k \leq l$*;*
- $I_k : Atoms \rightarrow Pow(P_k)$ *is such that if* $p \in I_k(A)$ *then* $p \in I_l(A)$ *for all* $l \geq k$.

Let $Pls = \cup_{k\in K} P_k$. *We shall say that* Pls *is the set of places of* $\mathcal{K}_{ref}$.

We extend the forcing relation of Def. 3. The difference from that relation is that the interpretation for $\Box$ changes. This is because larger Kripke states may have more places. Hence when interpreting $\Box\phi$ at a place in particular Kripke state, we have to account for places that may exist in a larger Kripke state. If we stick to the old interpretation, then Kripke monotonicity would fail. The interpretation of $\Box$ is similar to those used for modal intuitionistic logic [2, 18].

DEFINITION 9 (REFINED SEMANTICS). *Let* $\mathcal{K}_{ref} = (K, \leq, \{P_k\}_{k\in K}, \{I_k\}_{k\in K})$ *be a refined distributed Kripke model with set of places, Pls. Given* $k \in K$, $p \in P_k$, *a pure formula* φ *with* $\mathsf{PL}(\varphi) \subseteq Pls$, *we define* $(k, p) \models \varphi$ *inductively as:*

$(k,p) \models A$ *iff* $p \in I_k(A)$;
$(k,p) \models \top$ *iff* $p \in P_k$;
$(k,p) \models \bot$ *never;*
$(k,p) \models \varphi \wedge \psi$ *iff* $(k,p) \models \varphi$ *and* $(k,p) \models \psi$;
$(k,p) \models \varphi \vee \psi$ *iff* $(k,p) \models \varphi$ *or* $(k,p) \models \psi$;
$(k,p) \models \varphi \to \psi$ *iff* $l \geq k$ *and* $(l,p) \models \varphi$ *imply* $(l,p) \models \psi$;
$(k,p) \models \varphi@q$ *iff* $q \in P_k$ *and* $(k,q) \models \varphi$;
$(k,p) \models \Box\varphi$ *iff* $l \geq k$ *and* $q \in P_l$ *imply* $(l,q) \models \psi$;
$(k,p) \models \Diamond\varphi$ *iff* *there exists* $q \in P_k$ *such that* $(q,k) \models \varphi$.

We pronounce $(k,p) \models \varphi$ *as* (k,p) ref-forces φ, *or* (k,p) ref-satisfies φ. *We write* $k \models \varphi$ **at** p *if* $(k,p) \models \varphi$.

It is clear from the definition that if $k \models \varphi$ ***at*** p, then $\mathsf{PL}(\varphi$ ***at*** $p) \subseteq P_k$. Moreover, the usual Kripke monotonicity still holds.

Lemma 6 (Kripke Monotonicity). *Let* $\mathcal{K}_{ref} = (K, \leq, \{P_k\}_{k\in K}, \{I_k\}_{k\in K})$ *be a refined distributed Kripke model with set of places, Pls. The relation* $\models$ *preserves the partial order on* K, i.e.*, for each* $k,l \in K$, $p \in P_k$, *and* $\varphi \in Frm(P_k)$, *if* $l \geq k$ *then* $(k,p) \models \varphi$ *implies* $(l,p) \models \varphi$.

Proof: By induction on the structure of formulae, and is similar to the proof for Lemma1. ■

Now, we are ready to extend the definition of forcing to judgements. First, we extend the definition to contexts.

Definition 10 (Forcing on Contexts). *Let* $\mathcal{K}_{ref} = (K, \leq, \{P_k\}_{k\in K}, \{I_k\}_{k\in K})$ *be a refined distributed Kripke model. Given* $k \in K$, *a finite set of pure formulae* Γ, *and a finite set of sentences* Δ, *such that* $\mathsf{PL}(\Gamma;\Delta) \subseteq P_k$, *we say that* k *ref-forces the context* $\Gamma;\Delta$ *(and we write* $k \models \Gamma;\Delta$*) if*

1. *for every* $\varphi \in \Gamma$ *and any* $p \in P_k$: $(k,p) \models \Box\varphi$;
2. *for every* ψ **at** $q \in \Delta$: $q \in P_k$ *and* $(k,q) \models \psi$.

Finally, we extend the definition of forcing to judgements.

Definition 11 (Satisfaction for a Judgment). *Let* $\mathcal{K}_{ref} = (K, \leq, \{P_k\}_{k\in K}, \{I_k\}_{k\in K})$ *be a refined distributed Kripke model. We say that the judgement* $\Gamma;\Delta \vdash^P \mu$ ***at*** p is valid in $\mathcal{K}_{ref}$, *if*

- $PL(\Gamma) \cup PL(\Delta) \cup \{p\} \subseteq P$;
- *for every* $k \in K$ *such that* $P \subseteq P_k$, *if* $k \models \Gamma;\Delta$ *then* $k \models \mu$ **at** p.

Moreover we say that $\Gamma;\Delta \vdash^P \mu$ ***at*** p is ref-valid *(and we write* $\Gamma;\Delta \models \mu$ **at** p*) if it is valid in every refined distributed Kripke model.*

5.1 Soundness

In this section we shall prove the soundness of the extended logic in refined distributed Kripke models. The proof of soundness will follow the proof of the soundness in section 3.2. We start by defining the p-duplicated extension of a refined distributed Kripke model.

PROPOSITION 4 (p-DUPLICATED EXTENSION $\mathcal{K}_{ref}\langle p, q\rangle$). *Consider a refined distributed model $\mathcal{K}_{ref} = (K, \leq, \{P_k\}_{k\in K}, \{I_k\}_{k\in K})$, with Pls as set of places. Choose two places p, q such that $p \in Pls$, and $q \notin Pls$. Define $\mathcal{K}_{ref}\langle p, q\rangle$ to be the quadruple $(K', \leq', \{P'_k\}_{k\in K'}, \{I'_k\}_{k\in K'})$, where*

- *K' is K;*
- *$\leq'$ is $\leq$;*
- *P'_k is $P_k \cup \{q\}$ if $p \in P_k$, and P_k otherwise;*
- *$I'_k : Atoms \to Pow(P'_k)$ is defined as*

$$r \in I'_k(A) \text{ iff } \begin{cases} r \in I_k(A) \ (\text{for } r \in P_k); \\ p \in I_k(A) \ (\text{for } r = q). \end{cases}$$

*Then $\mathcal{K}_{ref}\langle p, q\rangle$ is a refined distributed Kripke model, and is said to be a p-*duplicated extension of $\mathcal{K}$.

Proof: We just need to check that $\{P'_k\}_{k\in K'}$ and $\{I'_k\}_{k\in K'}$ satisfy the monotonicity conditions of Def. 8. They follow immediately from the definition of P'_k and I'_k. ∎

We now show that the refined p-duplicated extension is conservative over all the formulae that do not mention the added place. Moreover, for all such formulae, the new place mimics the duplicated one.

LEMMA 7 ($\mathcal{K}_{ref}\langle p, q\rangle$ IS CONSERVATIVE). *Let $\mathcal{K}_{ref}$ be a refined distributed Kripke model with set of places, Pls, and $\mathcal{K}_{ref}\langle p, q\rangle$ be its p-duplicated extension. Let $\models$ and $\models'$ extend the interpretation of atoms in $\mathcal{K}$ and $\mathcal{K}_{ref}\langle p, q\rangle$ respectively. For every $k \in K$ and formula $\varphi \in Frm(Pls)$, we have:*

1. *for every $r \in P_k$, $(k, r) \models' \varphi$ if and only if $(k, r) \models \varphi$; and*
2. *if $q \in P'_k$, then $(k, q) \models' \varphi$ if and only if $(k, p) \models \varphi$.*

Proof: The proof is similar to the proof of Lemma 2 and we prove both properties simultaneously by induction on the structure of formulae in $Frm(Pls)$.

Base case. The two properties are easily verified on atoms and on $\top$ by the definition of p-duplicated extension.

Inductive hypothesis. We consider a formula $\varphi \in Frm(Pls)$ and assume that the two properties hold for every sub-formula of φ. In particular, we assume that if φ_i is a subformula of φ then *for every $k \in K$:*

1. *if $r \in P_k$, then $(k, r) \models' \varphi_i$ if and only if $(k, r) \models \varphi_i$; and*

2. *if $q \in P'_k$, then $(k, q) \models' \varphi_i$ if and only if $(k, p) \models \varphi_i$.*

The inductive cases for the connectives and modality @ have the same treatment as in Lemma 2. Here we show the most interesting cases, $\Box$ and $\Diamond$, by considering only property *1*. The treatment of property *2* is analogous. Pick $k \in K$ and $r \in P_k$, and fix them.

Case $\varphi = \Diamond\varphi_1$. Suppose $(k, r) \models' \Diamond\varphi_1$, then there is some $s \in P'_k$ such that $(k, s) \models \varphi_1$. In the case $s \in P_k$ we use induction to obtain $(k, s) \models \varphi_1$ and therefore $(k, r) \models \Diamond\varphi_1$. In the case $s = q$ we use induction to obtain $(k, p) \models \varphi_1$ and therefore $(k, r) \models \Diamond\varphi_1$. Vice versa, if $(k, r) \models \Diamond\varphi_1$ then there exists $s \in P_k$ such that $(k, s) \models \varphi_1$. Hence $(k, s) \models' \varphi_1$ by induction and we conclude $(k, r) \models' \Diamond\varphi_1$.

Case $\varphi = \Box\varphi_1$. Suppose that $(k, r) \models' \Box\varphi_1$. This means that $(l, s) \models' \varphi_1$ for every $l \geq k$ and every $s \in P'_l$. Since P'_l contains P_l, we obtain $(l, s) \models' \varphi_1$ for every $l \geq k$ and every $s \in P_l$. Hence, by induction $(l, s) \models \varphi_1$ for every $l \geq k$ and every $s \in P_l$, and we conclude that $(k, r) \models \Box\varphi_1$.

Vice versa if $(k, r) \models \Box\varphi_1$ then $(l, s) \models \varphi_1$ for every $l \geq k$ and every $s \in P_l$. By inductive hypothesis, we get that for every $l \geq k$ and $s \in P_l$, $(l, s) \models' \varphi_1$. If $q \notin Pk'_l$ for all $l \geq k$, then $P_l = P'_l$. In this case we conclude that $(k, r) \models' \Box\varphi$. On the other hand, if $q \in P'_l$ for some $l \geq k$, then it means that $p \in P_l$ and hence $(l, p) \models \varphi_1$. By induction (see property 2 of the proposition) $(l, q) \models' \varphi_1$, and we conclude $(k, r) \models' \Box\varphi_1$. ■

We now show that by renaming a place in a Kripke model, we do not change the set of valid formulae as long as the formulae do not mention renamed place or the fresh name.

Proposition 5 (p-Renaming $\mathcal{K}_{ref}\langle q/p\rangle$). *Let $\mathcal{K}_{ref} = (K, \leq, \{P_k\}_{k\in K}, \{I_k\}_{k\in K})$ be a refined distributed Kripke model with set of places Pls. For a place $q \notin \mathcal{K}_{ref}$, define $\mathcal{K}_{ref}\langle q/p\rangle = (K', \leq, \{P'_k\}_{k\in K'}, \{I'_k\}_{k\in K'})$ where*

- *K' is K;*
- *$\leq'$ is $\leq$;*
- *P'_k is $(P_k \setminus \{p\}) \cup \{q\}$ if $p \in P_k$, and P_k otherwise;*
- *$I'_k : Atoms \to Pow(P'_k)$ is defined[1] as*

$$r \in I'_k(A) \text{ iff } \begin{cases} r \in I_k(A) \ \ (\text{if } r \in P_k); \\ p \in I_k(A) \ (\text{if } r = q). \end{cases}$$

$\mathcal{K}_{ref}\langle q/p\rangle$ is a refined distributed Kripke model, and is said to be a p-renaming *of $\mathcal{K}_{ref}$.*

[1]Note that it cannot be the case that $r = p$, since $p \notin P'_k$.

Proof: As for Proposition 4, we just need to check that $\{P'_k\}_{k\in K'}$ and $\{I'_k\}_{k\in K'}$ satisfy the monotonicity conditions. They follow immediately by definition. ■

LEMMA 8 ($\mathcal{K}_{ref}\langle q/p\rangle$ IS CONSERVATIVE). *Let $\mathcal{K}_{ref} = (K, \leq, \{P_k\}_{k\in K}, \{I_k\}_{k\in K})$ be a refined distributed Kripke model on $\mathcal{K}_{ref}\langle q/p\rangle$ be its p-renaming. Let $\models$ and $\models'$ extend the interpretation of atoms in $\mathcal{K}_{ref}$ and $\mathcal{K}_{ref}\langle q/p\rangle$ respectively. For every $k \in K$, formula $\varphi \in Frm(Pls)$, and $r \in Pk_k$ we have:*

1. *if $r \neq p$, then $(k, r) \models \varphi$ if and only $(k, r) \models' \varphi$; and*
2. *if $r = p$, then $(k, p) \models \varphi$ if and only if $(k, q) \models' \varphi$.*

Proof: The proof is by induction on the structure of formulae in $Frm(Pls)$, and is similar to the proof for Lemma 7. ■

We are now ready to prove that the semantics is sound for the judgements of the logic. We need to show that if a judgement is provable in the extended natural deduction system, then it is also valid with respect to refined distributed Kripke models.

THEOREM 3 (SOUNDNESS). *If $\Gamma; \Delta \vdash^P \mu$ **at** p is derivable in the logic, then it is ref-valid.*

Proof: The proof is by induction on the number n of inference rules used in the derivation of the judgement of $\Gamma; \Delta \vdash^P \mu$ ***at*** p. The proof is similar to the proof of Theorem 1.

Base case ($n = 1$). If the derivation consists of either the axiom L, or the axiom G, or rule $\top I$ we use the same argument as in the proof of Theorem 1. The case $\bot E$ follows by definition of the forcing relation.

Inductive hypothesis ($n > 1$). We assume that the theorem holds for any judgment that is deducible by applying less than n instances of inference rules. We consider a judgment $\Gamma; \Delta \vdash^P \mu$ ***at*** p which is derivable in the logic by using exactly n instances of inference rules.

We fix a model $\mathcal{K}_{ref} = (K, \leq, \{P_k\}_{k\in K}, \{I_k\}_{k\in K})$ with set of places Pls such that $P \subseteq Pls$, and let $\models$ be the extension of I_k. Let $k \in K$ be an arbitrary state such that $k \models \Gamma; \Delta$. Fix k. We need to show $(k, p) \models \mu$. For this we consider the last inference rule used to obtain $\Gamma; \Delta \vdash^P \mu$ ***at*** p and proceed by cases. The treatment of logical connectives is standard. The modalities @ and $\Diamond$ are teated as in Theorem 1. If the last inference rule used is $\Box E$, then the result follows from a simple application of the definition. The most interesting case is when $\Box I$ is the last inference used, and we discuss this case below.

Case $\Box I$. It must be case that μ is of the form $\Box\varphi$. Moreover $\Gamma; \Delta \vdash^{P+q} \varphi$ ***at*** q for some $q \notin P$ by using $n-1$ instances of the rules, and $\mathsf{PL}(\Gamma; \Delta) \cup \mathsf{PL}(\varphi) \subseteq P$. By induction we know that $\Gamma; \Delta \vdash^{P+q} \varphi$ ***at*** q is ref-valid. Without loss of generality, we can assume that $q \notin Pls$ (otherwise, we can rename q in Pls, using Lemma 8).

We prove that $k \models \Box\varphi$ ***at*** p. The semantics of $\Box$ says that we need to show that $l \models \varphi$ ***at*** r, for all $l \geq k$ and $r \in P_l$. Fix one $l \geq k$ and one $r \in P_l$, and consider the refined r-duplicated extension $\mathcal{K}_{ref}\langle r, q\rangle$. $\mathcal{K}_{ref}\langle r, q\rangle$ is a refined distributed Kripke model with set of places, $Pls \cup \{q\}$. Let $\models'$ be the forcing relation on $\mathcal{K}_{ref}\langle r, q\rangle$.

From the hypothesis $k \models \Gamma; \Delta$ and by Kripke monotonicity (Lemma 6) we get $l \models \Gamma; \Delta$. Therefore, since $\mathcal{K}_{ref}\langle r, q\rangle$ is a r-duplicated extension, we get $l \models' \Gamma; \Delta$ by using Lemma 7. Now, since $P + q \subseteq Pls \cup \{q\}$ we can use inductive hypothesis on $\mathcal{K}_{ref}\langle r, q\rangle$ to obtain $k \models' \varphi$ ***at*** q. Using Lemma 7 once again, we conclude that $l \models \varphi$ ***at*** r. Since l and r are arbitrary, we conclude that $k \models \Box\varphi$ ***at*** p. ∎

5.2 Completeness

In this section, we will show that the refined semantics is complete for the natural deduction presented in Section 4. The proof will follow the standard proofs of completeness for intuitionistic modal logic [18]. In the proof, we construct a canonical model. If a judgement is not provable, then it will be invalidated in one of the Kripke states of the canonical model.

Please note that the notion of provability can be extended on possible non-finite sets Σ of sentences, as in Section 3.3. We say that $\Gamma; \Sigma \vdash^P \varphi$ ***at*** p, if and only if, there exists a finite subset $\Delta \subseteq \Sigma$ such that $\Gamma; \Delta \vdash^P \varphi$ ***at*** p. Also, note that Proposition 3 stated in Section 3.3 can be extended to the logic with disjunctive connectives. The canonical model is defined by considering a particular kind of set of sentences.

Definition 12 (Refined Prime Set). *Let P be a set of places and Γ be a set of pure formulae in $Frm(P)$. A (possibly non-finite) set Σ of sentences with* $\mathsf{PL}(\Sigma) \subseteq P$*, is said to be* (Γ, P)-refined prime *if it satisfies the following four properties.*

1. *If* $\Gamma; \Sigma \vdash^P \varphi$ **at** p *then* φ **at** $p \in \Sigma$ (Deductive Closure).
2. $\Gamma; \Sigma \nvdash^P \bot$ **at** p *for any* $p \in P$ (Consistency).
3. *If* $\Gamma; \Sigma \vdash^P \varphi \vee \psi$ **at** p *then either* φ **at** $p \in \Sigma$ *or* ψ **at** $p \in \Sigma$ (Disjunction Property).
4. *If* $\Gamma; \Sigma \vdash^P \Diamond\varphi$ **at** p *then there exists* $q \in P$ *such that* φ **at** $q \in \Sigma$ (Diamond Property).

As in [18, 2] we first show that every set of sentences can be extended to a prime set, that respects the non-provability with respect to a particular sentence.

Lemma 9 (Refined Prime Extension). *Let P be a set of places and Γ be a finite set of pure formulae in $Frm(P)$. Let φ be a pure formula, p be a place, and Δ be a set of sentences such that*

- $\mathsf{PL}(\varphi$ **at** $p) \cup \mathsf{PL}(\Delta) \subseteq P$*, and*

- $\Gamma; \Delta \nvdash^{P} \varphi$ **at** p.

Then there is a set of places P' extending P and a (Γ, P')- refined prime set of sentences Σ containing Δ, such that $\Gamma; \Sigma \nvdash^{P'} \varphi$ **at** *p.*

Proof: We enrich the set of places by introducing a denumerable set of new places: $\mathbf{q}_1, \mathbf{q}_2, \ldots$. They will be the witnesses for the formulae $\Diamond\varphi$ and are introduced in order to satisfy the diamond property.

The set of places P' is obtained by a series of extensions $P = P_0 \subseteq P_1 \subseteq P_2 \ldots$. Before we proceed with the construction, we pick up an enumeration of the pure formulae $Frm(P')$ and fix it. The set Σ is obtained by series of extensions $\Delta = \Sigma_0 \subseteq \Sigma_1 \subseteq \Sigma_2 \ldots$ that verify the following:

Property 2. For every $n \geq 0$:

1. $\mathsf{PL}(\Sigma_n) \subseteq P_n$;
2. $\Gamma; \Sigma_n \nvdash^{P_n} \varphi$ **at** p.

The series is constructed inductively. In the induction, at an odd step we will create a witness for a formula of the type $\Diamond\psi$. At an even step we deal with disjunction property. We shall also construct two sets:

- $treated_n^{\Diamond}$, that will be the set of the formulae $\Diamond\varphi$ for which we have already created a witness.
- $treated_n^{\vee}$, that will be the set of the formulae $\varphi \vee \psi$ ***at*** p which satisfy the disjunction property.

We start $treated_0^{\Diamond} = \emptyset$, $treated_0^{\vee} = \emptyset$, $P_0 = P$ and $\Sigma_0 = \Delta$. It is clear that $\mathsf{PL}(\Sigma_0) \subseteq P_0$, and $\Gamma; \Sigma_0 \nvdash^{P_0} \varphi$ ***at*** p.

Then we proceed inductively, and assume that P_n, Σ_n ($n \geq 0$) have been constructed satisfying Property 2. In step $n+1$, we consider two cases:

1. If $n+1$ is odd, pick the first formula $\psi_1 \vee \psi_2$ in the enumeration such that
   - $\psi_1 \vee \psi_2$ is in $Frm(P_n)$, i.e., all the places in $\psi_1 \vee \psi_2$ are taken from P_n;
   - $\Gamma; \Sigma_n \vdash^{P_n} \psi_1 \vee \psi_2$ ***at*** q, for some $q \in P_n$;
   - $\psi_1 \vee \psi_2$ ***at*** $q \notin treated_n^{\vee}$.

   Please note that if both $\Gamma; \Sigma_n, \psi_1$ ***at*** $q \vdash^{P_n} \varphi$ ***at*** p and $\Gamma; \Sigma_n, \psi_2$ ***at*** $q \vdash^{P_n} \varphi$ ***at*** p, then we can deduce $\Gamma; \Sigma_n \vdash^{P_n} \varphi$ ***at*** p. However, we have that Σ_n, P_n satisfy Property 2. Hence, it must be the case that either $\Gamma; \Sigma_n, \psi_1$ ***at*** $q \nvdash^{P_n} \varphi$ ***at*** p, or $\Gamma; \Sigma_n, \psi_2$ ***at*** $q \nvdash^{P_n} \varphi$ ***at*** p.

   We define $\Sigma_{n+1} = \Sigma_n \cup \{\psi_1$ ***at*** $q\}$ if $\Gamma; \Sigma_n, \psi_1$ ***at*** $q \nvdash^{P_n} \varphi$ ***at*** p, and $\Sigma_{n+1} = \Sigma_n \cup \{\psi_2$ ***at*** $q\}$ otherwise. We define $P_{n+1} = P_n$. We get by construction that P_{n+1}, Σ_{n+1} satisfy Property 2. Finally, we let $treated_{n+1}^{\vee} = treated_n^{\vee} \cup \{\psi_1 \vee \psi_2$ ***at*** $q\}$ and $treated_{n+1}^{\Diamond} = treated_n^{\Diamond}$.

2. If $n+1$ is even, pick the first formula $\Diamond\psi$ in the enumeration such that
   - $\Diamond\psi$ is in $Frm(P_n)$, i.e., all the places in $\Diamond\psi$ are taken from P_n;
   - $\Gamma; \Sigma_n \vdash^{P_n} \Diamond\psi$ ***at*** q, for some $q \in P_n$;
   - $\Diamond\psi \notin treated^{\Diamond}_n$.

   Let $P_{n+1} = P_n + q_{(n+1)/2}$, $\Sigma_{n+1} = \Sigma_n \cup \{\psi \text{ } \textbf{\textit{at}} \text{ } \mathbf{q}_{(n+1)/2}\}$, $treated_{n+1} = treated_n \cup \{\Diamond\psi\}$ and $treated^{\vee}_{n+1} = treated^{\vee}_n$. We claim that $\Gamma; \Sigma_{n+1} \nvdash^{P_{n+1}} \varphi$ ***at*** p.

   If $\Gamma; \Sigma_{n+1} \vdash^{P_{n+1}} \varphi$ ***at*** p, then $\Gamma; \Sigma, \psi$ ***at*** $\mathbf{q}_{(n+1)/2} \vdash^{P+\mathbf{q}_{(n+1)/2}} \varphi$ ***at*** p. Since $\Gamma; \Sigma_n \vdash^{P_n} \Diamond\psi$ ***at*** q, we get $\Gamma; \Sigma_n \vdash^{P_n} \varphi$ ***at*** p by the inference rule $\Diamond E$. This contradicts the hypothesis on P_n, Σ_n. Hence $\Gamma; \Sigma_{n+1} \nvdash^{P_{n+1}} \varphi$ ***at*** p.

Therefore, we get by construction that P_n, Σ_n satisfy Property 2. We define $P' = \bigcup_{n\geq 0} P_n$, and $\Sigma = \bigcup_{n\geq 0} \Sigma_n$. Clearly $P \subseteq P'$, and $\Delta \subseteq \Sigma$. Moreover, using Property 2, we can easily show that $\Gamma; \Sigma \nvdash^{P'} \varphi$ ***at*** p. Finally, we show that Σ is a (Γ, P')-refined prime set.

1. (Disjunction Property) If $\Gamma; \Sigma \vdash^{P'} \psi_1 \vee \psi_2$ ***at*** q, then let n be the least number such that $\Gamma; \Sigma_n \vdash^{P_n} \psi_1 \vee \psi_2$ ***at*** q. Clearly, $\psi_1 \vee \psi_2$ ***at*** $q \notin treated^{\vee}_n$, and $\Gamma; \Sigma_m \vdash^{P_m} \psi_1 \vee \psi_2$ ***at*** q for every $m \geq n$. Eventually $\psi_1 \vee \psi_2$ ***at*** q has to be treated at some stage $h \geq n$. Hence, either ψ_1 ***at*** $q \in \Sigma_{h+1}$ or ψ_2 ***at*** $q \in \Sigma_{h+1}$. Therefore, ψ_1 ***at*** $q \in \Sigma$ or ψ_2 ***at*** $q \in \Sigma$.

2. (Diamond Property) If $\Gamma; \Sigma \vdash^{P'} \Diamond\psi$ ***at*** q, then let n be the least number such that $\Gamma; \Sigma_n \vdash^{P_n} \Diamond\psi$ ***at*** q. As in the previous case, we assert that $\Diamond\psi$ ***at*** q is treated for some even number $h \geq n$. We get ψ ***at*** $\mathbf{q}_{h/2} \in \Sigma$ by construction.

3. (Deductive Closure) If Σ $\Gamma; \Sigma \vdash^{P'} \psi$ ***at*** q, then $\Gamma; \Sigma \vdash^{P'} \psi \vee \psi$ ***at*** q. The first case then gives us that ψ ***at*** $q \in \Sigma$.

4. (Consistency) If $\Sigma; \Gamma \vdash^{P'} \bot$ ***at*** q, then $\Sigma; \Gamma \vdash^{P'} \varphi@p$ ***at*** q by the inference rule $\bot E$. Therefore, $\Gamma; \Sigma \vdash^{P'} \varphi$ ***at*** p by $@E$, which contradicts our construction. Hence, $\Sigma; \Gamma \nvdash^{P'} \bot$ ***at*** q.

We conclude that Σ is a (Γ, P')- refined prime extending Δ such that $\Gamma; \Sigma \nvdash^{P'} \varphi$ ***at*** p. ∎

Now, we define the refined canonical model. In the refined canonical model, Kripke states are prime sets of sentences.

Definition 13 (Refined Canonical Model). *Given a finite set* Γ *of pure formulae, we define the* Γ-refined canonical model *to be the quadruple* $\mathcal{M}_{\Gamma Ref} = (M, \leq, \{P_l\}_{l\in M}, \{I_l\}_{l\in M})$*, where:*

- *M is set of all pairs (Σ, P) such that P is a set of places, and Σ is a (Γ, P)-refined prime set.*

- $(\Sigma_1, P_1) \leq (\Sigma_2, P_2)$ *if and only if* $\Sigma_1 \subseteq \Sigma_2$ *and* $P_1 \subseteq P_2$.
- $P_{(\Sigma,P)} \stackrel{\text{def}}{=} P$.
- $I_{(\Sigma,P)} : Atoms \longrightarrow Pow(P_{(\Sigma,P)})$ *is defined by:* $p \in I_{(\Sigma,P)}(A)$ *iff* A **at** $p \in \Sigma$.

We now show that in the canonical model a sentence is forced by a Kripke state (Γ, Σ) if and only if it is contained in Σ.

Lemma 10 (Refined Canonical Evaluation). *Let* Γ *be a finite set of pure formulae.*

1. *The* Γ*-refined canonical model* $\mathcal{M}_{\Gamma Ref} = (M, \leq, \{P_l\}_{l\in M}, \{I_l\}_{l\in M})$ *is a refined distributed Kripke model.*
2. *Let Pls be the set of places of* $\mathcal{M}_{\Gamma Ref}$*, and* $\models$ *be the forcing relation in* $\mathcal{M}_{\Gamma Ref}$*. For every* $(\Sigma, P) \in \mathcal{M}_{\Gamma Ref}$*, every formula* $\varphi \in Frm(Pls)$*, and every place* $p \in Pls$*,* $(\Sigma, P) \models \varphi$ **at** p *if and only if* φ **at** $p \in \Sigma$*.*

Proof: Clearly all the properties required for a refined distributed Kripke model are verified. All we have to prove is the part *2* of the proposition. The proof is standard, and we proceed by induction on the structure of the formula $\varphi \in Frm(Pls)$. Here, we just illustrate the inductive case in which φ is $\Box\varphi_1$. In the inductive hypothesis, we assume that part*2* is valid on all subformulae of φ.

Case $\Box\varphi_1$. Assume that $(\Sigma, P) \models \Box\varphi_1$ ***at*** p. By definition, this means that for every (Σ', P') greater than (Σ, P) and for every $r \in P'$, it is the case that $(\Sigma', P') \models \varphi$ ***at*** r (and therefore φ ***at*** $r \in \Sigma'$ by inductive hypothesis).

Chose a new place $q \notin P$. We claim that $\Gamma; \Sigma \vdash^{P+q} \varphi_1$ ***at*** q. Suppose $\Gamma; \Sigma \nvdash^{P+q} \varphi_1$ ***at*** q. Then by Lemma 4, there is a set of places Q extending $P + q$ and a (Γ, Q)-refined prime set Σ' extending Σ such that $\Gamma; \Sigma' \nvdash^{Q} \varphi_1$ ***at*** q. That means φ_1 ***at*** $p \notin \Sigma'$. Since (Σ', Q) is greater than (Σ, P), we obtain a contradiction. Therefore we conclude that $\Gamma; \Sigma \vdash^{P+q} \varphi_2$ ***at*** q.

Using the inference rule $\Box I$, we get $\Gamma; \Sigma \vdash^{P} \Box\varphi_1 p$. Since Σ is a (Γ, P)-prime set, we get that means $\Box\varphi_1$ ***at*** $p \in \Sigma$.

Vice-versa, let $\Box\varphi_1$ ***at*** $p \in \Sigma$. Pick (Σ', Q) greater than (Σ, P). We need to show $(\Sigma', Q) \models \Box\varphi_1$ ***at*** p. We have that $\Sigma \subseteq \Sigma'$, and therefore $\Box\varphi_1$ ***at*** $p \in \Sigma'$. We can apply $\Box E$ to prove that $\Gamma, \Sigma' \vdash^{Q} \varphi$ ***at*** q for every $q \in Q$. By definition of the canonical model, Σ' is (Γ, Q)-prime set. Therefore, we obtain φ_1 ***at*** $q \in \Sigma'$ for every $q \in Q$. Hence by inductive hypothesis, $(\Sigma', Q) \models \varphi_1$ ***at*** q for every $q \in Q$. Since $P \subseteq Q$, we get $(\Sigma', Q) \models \Box\varphi_1$ ***at*** p. ■

We are now ready to prove completeness.

Theorem 4 (Refined Completeness). $\Gamma; \Delta \models^{P} \varphi$ **at** $p \Longrightarrow \Gamma; \Delta \vdash^{P} \varphi$ **at** p

Proof: Assume that $\Gamma; \Delta \models^{P} \varphi$ ***at*** $p \Longrightarrow \Gamma; \Delta \vdash^{P} \varphi$ ***at*** p. We have:

1. $\mathsf{PL}(\Gamma) \cup \mathsf{PL}(\Delta) \cup \{p\} \subseteq P$.
2. If $\mathcal{K}_{ref} = (K, \leq, \{P_k\}_{k\in K}, \{I_k\}_{k\in K})$ is a refined distributed Kripke model, then for every $k \in K$ such that $P \subseteq P_k$, $k \models \varphi$ ***at*** p whenever $k \models \Gamma; \Delta$.

We need to show that $\Gamma; \Delta \vdash^P \varphi$ ***at*** p.

Assume that $\Gamma; \Delta \nvdash^P \varphi$ ***at*** p. Then by Lemma 9, there is a set of places $P' \supseteq P$, and a (Γ, P')-refined prime set of sentences Σ containing Δ such that $\Gamma; \Sigma \nvdash^{P'} \varphi$ ***at*** p. We get φ ***at*** $p \notin \Sigma$.

Now consider the Γ-canonical model $\mathcal{M}_{\Gamma Ref}$, and let $\models$ be the forcing relation in $\mathcal{M}_{\Gamma Ref}$. Consider the Kripke state (Σ, P'). Δ is contained in Σ, and therefore $(\Sigma, P') \models \Gamma; \Delta$ by Lemma10. By our assumption, we get $(\Sigma, P') \models \varphi$ ***at*** p. By Lemma 10, we get φ ***at*** $p \in \Sigma$. We have just reach a contradiction. Therefore, we can conclude that $\Gamma; \Delta \vdash^P \varphi$ ***at*** p. ■

6 Related Work

The logic studied in Section 2 was introduced in [9, 10], where it was used as the foundation of a type system for a distributed λ-calculus in the *propositions-as-types* paradigm. Although the authors of [9, 10] do discuss how the logic could be useful in distribution of resources, they have no corresponding model. The proof terms corresponding to modalities have computational interpretation in terms of remote procedure calls ($@p$), commands to broadcast computations to all nodes ($\Box$), and commands to use portable code ($\Diamond$). In [9], the authors also introduce a sequent calculus for the logic and prove that it enjoys cut elimination.

From a logical point of view, this logic can be viewed as a hybrid modal logic [16, 1]. A hybrid logic internalises the model in the logic by using modalities built from pure names [16, 1]. In [9, 10], the modality $@p$ gives the logic a hybrid flavour. Work on hybrid logics has been usually carried out in a classical setting, see the hybrid logics web page (`http://hylo.loria.fr/`). More recently, a first intuitionistic version of hybrid logics were investigated in [2].

There are several intuitionistic modal logics in the literature, and [18] is a good source on them. The modalities in [18] have a temporal flavour, and the spatial interpretation was not recognised then. There are no places in the Kripke states, and there is an accessibility relation on states that expresses the next step of a computation.

The work in [2] introduces the first intuitionistic version of hybrid logics. It investigates how to add names in constructive logics resulting in hybrid versions. A modal logic is hybridised by adding a new kind of propositional symbols: *nominals*. The nominals are the names in the logic. The authors extend the modal system of [18] by introducing nominals. They give a natural deduction system and a Kripke semantics for this logic. They prove soundness and completeness for the semantics, and also give a normalisation result for the natural deduction.

The extension given in Section 4 is a hybrid version of the intuitionistic modal system *IS5* [18]. In the modal system *IS5*, the accessibility relation among places is total. Hence, the logic in Section 4 can be seen as an instance of the hybrid modal logic in [2]. The only difference is that names in our logic only occur in the modality $@p$. In [2], names also occur as propositions.

Other work on logics in resources can be related to the separation logics [17], or the logic of bunched implications [15]. In [15], the authors give a Kripke model founded on a monoidal structure. In the logic, the formulae are the resources, and are interpreted as elements of the monoid. The focus of this work is the sharing of resources and not their distribution. There is no notion of places, and the logic has no modalities.

In the classical setting, there are also a number of logics used to study spatial properties. In [4, 3], for example, the authors use process calculi as their models. They have a classical modal logic to study spatial, temporal and security properties of the processes.

7 Conclusions and Future Work

We study the hybrid modal logic presented in [9, 10]. Formulae in the logic contain names, also called places. The logic may be used to reason about placement of resources in a distributed system. An intuitionistic natural deduction for this logic is presented in [10], and judgements mention the places under consideration.

We interpret the judgements in the logic in Kripke-style models [12]. Typically Kripke models [12] consists of partially ordered Kripke states. In our case the models are obtained from the Kripke models by adding a fixed set of places to each possible Kripke state. In each Kripke state, different places may satisfy different formulae. The satisfaction of atoms corresponds to placement of resources. The modalities of the logic allow formulae to be satisfied in a named place ($@p$), some place ($\Diamond$) and every place ($\Box$). We show that the interpretation of judgments in these models is both sound and complete.

We add disjunctive connectives to the modal logic in [9, 10], and refine our semantics to obtain soundness and completeness results. In the new Kripke models, larger Kripke states may contain bigger set of places. The refined semantics can be seen as an instance of hybrid *IS5* [2, 18].

As future work, we are currently investigating decidability of the extended logic. The intuitionistic modal systems in [18] are decidable. In order to prove decidability of those systems, [18] uses *birelational models*. These models are sound and complete, and enjoy finite model property: if a judgement is not valid in the logic, then there is a finite birelational model which invalidates the judgement. The finite model property is not enjoyed by the Kripke models in [18]. We are investigating if we can adapts the proofs in [18].

We are also considering other extensions of the logic. A major limitation of the logic presented in [10] is that if a formula φ is validated at some named place, say p, then the formula $\varphi @ p$ can be inferred at every other place. Similarly if $\Diamond\varphi$ or $\Box\varphi$ can be inferred at one place, then they can be inferred at any other place. In a large distributed system, we may want to restrict the rights of accessing information in a place. This can be done by adding an accessibility relation as in [18, 2]. We are currently investigating the computational interpretation of this extended logic. This would result in an extension of λ-calculus presented in [9, 10].

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