ω-Inductive Completion of Monoidal Categories and Infinite Petri Net Computations

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Abstract. There exists a KZ-doctrine on the 2-category of the locally small categories whose algebras are exactly the categories which admits all the colimits indexed by ω-chains. The paper presents a wide survey of this topic. In addition, we show that this chain cocompletion KZ-doctrine lifts smoothly to KZ-doctrines on (many variations of) the 2-categories of monoidal and symmetric monoidal categories, thus yielding a universal construction of colimits of ω-chains in those categories. Since the processes of Petri nets may be axiomatized in terms of symmetric monoidal categories this result provides a universal construction of the algebra of infinite processes of a Petri net.

Introduction

The idea of completing a mathematical stucture by adding to it some desirable limit 'points' is indeed a very natural one and it arises in many different fields of mathematics, particularly in topology and partial order theory. Since categories are a generalization of the notion of partial orders, the issue of completing categories for a given class of limits or colimits arose rather early in the development of the theory (see [21] and references therein).

As far as computer science is concerned, the theory of complete partial orders and the associated completion techniques have assumed great relevance since the pioneering work on semantics by Scott [33]. In the last few years, however, many computing systems have been given a semantics through the medium of category theory, the general pattern being to look at objects as representing states and at arrows as representing computations. It is therefore natural to expect that the theory of *cocompletion of categories* may play an interesting role in this kind of semantics. The main purpose of this paper is to illustrate how this theory fits well with the issue of infinite computations and, therefore, to make it more easily available to the computer science community. In a sense, by viewing categories as generalized posets, this view of infinite computations is very natural and indeed generalizes to categories similar constructions for adding limits to posets [28]. We motivate this further in terms of processes of Petri nets in Section 1.

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Petri nets [31] are probably the most clear examplification of the categorical semantics pattern discussed above. They are unanimously considered among the most representative *models for concurrency*, since they are a fairly simple and natural model of *concurrent* and *distributed* computation. Notwithstanding their 'naturality'—perhaps because of that—Petri nets are, in our opinion, still far from being completely understood. In recent works, Degano, Meseguer and Montanari [30, 5] have shown that the semantics of Petri nets can be understood in terms of *symmetric monoidal categories*—where objects are states, arrows processes, and the tensor product and the arrow composition model respectively the operations of parallel and sequential composition of processes. This yields an axiomatization of the causal behaviour of nets as an *essentially algebraic theory*.

However, when modeling perpetual systems, describing finite processes is not enough: we need to consider also *infinite behaviours*. Actually, infinite computations of Petri nets have occasionally been considered in connection with acceptors of ω-languages (see [14] and references therein). These approaches, of course, focus just on sequential computations and treat nets simply as generalized automata. Our interest, instead, resides on *processes*, i.e., on structures able to describe concurrent computations more intensionally, taking into account causality. More precisely, we aim at defining an *algebra* of net computations which includes *infinite processes* as well. To the best of our knowledge, this issue is still completely unexplored.

In order to fulfill our programme, we first address the general issue of *completion of categories* by colimits of ω-chains. Since ω-chain cocompleteness coincides with the completeness by colimits taken over countable filtered index categories and for technical reasons countable filtered colimits are also needed, we present the theory of cocompletion of categories by such kind of colimits. More precisely, for $\underline{\underline{\mathsf{CAT}}}$ the 2-category of locally small categories, we define a *Kock-Zöberlain (KZ-)doctrine* [20, 35] $\mathtt{Ind}_\omega(\_)\colon \underline{\underline{\mathsf{CAT}}} \to \underline{\underline{\mathsf{CAT}}}$ which associates to each locally small category its completion by countable filtered colimits, or its ω-*ind*-completion (ind standing for *ind*uctive), and such that the countable filtered cocomplete categories with functors preserving countable filtered colimits are exactly the *algebras* for the doctrine.[1] Although related results have already appeared in several different forms in the literature, e.g. [13, 20, 35, 15], the presentation here is a rather complete survey which integrates the best features of the existing approaches and explores the application of these ideas to computer science.

Then, we show that the completion doctrine, when applied to a symmetric monoidal category, yields a symmetric monoidal category. More precisely, we show that the KZ-doctrine $\mathtt{Ind}_\omega(\_)$ lifts to a KZ-doctrine on any of the 2-categories of monoidal categories appearing in Table 1, which, from the technical point of view, is the main result of paper.

Finally, we discuss how this result generalizes the algebraic approach to the

[1] We mention that, for any infinite cardinal $\aleph$ (respectively with no cardinality restrictions), there exists a similar KZ-doctrine of completion by colimits taken over filtered categories (or equivalently over chains) of cardinality $\aleph$ (respectively of arbitrary cardinality). The details can be found, e.g., in [32].

		small		locally small	
		monoidal	strict monoidal	monoidal	strict monoidal
STRICT	non symmetric	$\underline{\underline{\mathsf{MonCat}}}$	$\underline{\underline{\mathsf{sMonCat}}}$	$\underline{\underline{\mathsf{MonCAT}}}$	$\underline{\underline{\mathsf{sMonCAT}}}$
STRICT	symmetric	$\underline{\underline{\mathsf{SMonCat}}}$	$\underline{\underline{\mathsf{SsMonCat}}}$	$\underline{\underline{\mathsf{SMonCAT}}}$	$\underline{\underline{\mathsf{SsMonCAT}}}$
STRICT	strictly symmetric	$\underline{\underline{\mathsf{sSMonCat}}}$	$\underline{\underline{\mathsf{sSsMonCat}}}$	$\underline{\underline{\mathsf{sSMonCAT}}}$	$\underline{\underline{\mathsf{sSsMonCAT}}}$
STRONG	non symmetric	$\underline{\underline{\mathsf{MonCat}}}^{\star}$	$\underline{\underline{\mathsf{sMonCat}}}^{\star}$	$\underline{\underline{\mathsf{MonCAT}}}^{\star}$	$\underline{\underline{\mathsf{sMonCAT}}}^{\star}$
STRONG	symmetric	$\underline{\underline{\mathsf{SMonCat}}}^{\star}$	$\underline{\underline{\mathsf{SsMonCat}}}^{\star}$	$\underline{\underline{\mathsf{SMonCAT}}}^{\star}$	$\underline{\underline{\mathsf{SsMonCAT}}}^{\star}$
STRONG	strictly symmetric	$\underline{\underline{\mathsf{sSMonCat}}}^{\star}$	$\underline{\underline{\mathsf{sSsMonCat}}}^{\star}$	$\underline{\underline{\mathsf{sSMonCAT}}}^{\star}$	$\underline{\underline{\mathsf{sSsMonCAT}}}^{\star}$
MONOIDAL	non symmetric	$\underline{\underline{\mathsf{MonCat}}}^{\star\star}$	$\underline{\underline{\mathsf{sMonCat}}}^{\star\star}$	$\underline{\underline{\mathsf{MonCAT}}}^{\star\star}$	$\underline{\underline{\mathsf{sMonCAT}}}^{\star\star}$
MONOIDAL	symmetric	$\underline{\underline{\mathsf{SMonCat}}}^{\star\star}$	$\underline{\underline{\mathsf{SsMonCat}}}^{\star\star}$	$\underline{\underline{\mathsf{SMonCAT}}}^{\star\star}$	$\underline{\underline{\mathsf{SsMonCAT}}}^{\star\star}$
MONOIDAL	strictly symmetric	$\underline{\underline{\mathsf{sSMonCat}}}^{\star\star}$	$\underline{\underline{\mathsf{sSsMonCat}}}^{\star\star}$	$\underline{\underline{\mathsf{sSMonCAT}}}^{\star\star}$	$\underline{\underline{\mathsf{sSsMonCAT}}}^{\star\star}$

Legenda: The data in the definition of monoidal categories and functors (see Section 5 for the relevant definitions) give rise to many combinations according to whether the monoidality and the symmetry are strict or not and so on. To fix notation, we propose the nomenclature above. The idea is that, since we consider the categories with *strict* monoidal functors as the 'normal' categories, we explicitly indicate with simple and double superscripted $\star$'s the categories with, respectively, *strong* monoidal functors and simply *monoidal* functors. This is indicated by the leftmost column in the table. Clearly, the categories of symmetric monoidal categories consists always of *symmetric* monoidal functors. Moreover, sS means *strictly symmetric* while sMon means *monoidal strict*. We distinguish between categories of locally small and of small categories by using uppercase letters in the first case. Of course, there is an analogous table for the categories above considered as one-dimensional categories. As usual, we use a single underline in order to distinguish the two situations.

Table 1: A nomenclature for categories of monoidal categories

process semantics of Petri nets given in [30, 5] to the case in which infinite processes and composition operations on them are considered. In particular, the infinite processes of a Petri net can in this way be given an algebraic presentation which combines the *essentially algebraic* presentation of monoidal categories with the *monadic* presentation of their completion in terms of KZ-doctrines. However, the correspondence between infinite net processes obtained via the cocompletion doctrine and the algebraic theory of net processes is not as 'precise' as one would like. In fact, in addition to the symmetric monoidality, the categories of processes of Petri nets satisfy further axioms which are, in general, not preserved by $\mathtt{Ind}_\omega(\_)$. Therefore, although the arrows of the cocomplete category correspond precisely to infinite computations, they do not enjoy the global structural properties of finite net processes, i.e., $\mathtt{Ind}_\omega(\_)$ does not restrict to an endofunctor on the category of categories of net processes. It is still an open problem whether a more satisfactory solution to this problem can be found.

Concerning the organization of this paper, in Section 2 we briefly recall that the category $\underline{\mathsf{Set}}^{\underline{\mathsf{C}}^{\mathrm{op}}}$ of all presheaves on $\underline{\mathsf{C}}$ may be considered as the 'free' cocompletion of $\underline{\mathsf{C}}$ under all small colimits. This suggests immediately a strategy for identifying the cocompletion of $\underline{\mathsf{C}}$ for countable filtered colimits, i.e., to look for (a suitable representation of) an appropriate subcategory of $\underline{\mathsf{Set}}^{\underline{\mathsf{C}}^{\mathrm{op}}}$. In Section 3 we recall that countable filtered cocompletion and ω-chain cocompletion are equivalent notions. Then Section 4 gives the functors $\mathtt{Ind}_\omega(\_)$ bulding on the theory of KZ-doctrines, whose basic definition and results are summarized in Appendix A. Getting back to our original motivation, namely the process semantics of Petri nets, in Section 5 we study the extension of this inductive completion functor to monoidal categories, and in Section 6 we bring back these results to Petri nets.

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1 Motivations from Net Theory

In order to appreciate the discussion in this section, it is not necessary to know in detail what Petri nets are. Indeed, the relevant facts are, as already stressed in the introduction, that we have a category $\underline{\mathsf{C}}$ whose objects represent the states and whose arrows represent the finite processes of a computational device, say a net N. To make the situation more interesting, we assume that, in addition, there is a notion of parallel composition of transitions, expressed by the fact that $\underline{\mathsf{C}}$ is a *monoidal category.*

Building on such a formalization of its processes, it looks conceptually very simple to describe the infinite computations of N. In fact, since arrows in the category $\underline{\mathsf{C}}$ are *finite* processes, and since we understand infinite computations as 'limits' of countable sequences of finite processes, we can think of them as sequences of arrows in $\underline{\mathsf{C}}$, i.e., $\underline{\mathsf{C}}$-valued, ω-shaped diagrams

$$c_0 \xrightarrow{f_0} c_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} c_n \xrightarrow{f_n} c_{n+1} \xrightarrow{f_{n+1}} \cdots$$

which are exactly the functors from the partial ordered category $\underline{\omega} = \{0 < 1 < 2 < 3 < \cdots\}$ to $\underline{C}$.

However, this is only part of the story, actually the easiest. First of all, we are not interested in a set-theoretic treatment of infinite processes, but rather in their categorical description. In other terms, we aim at extending our category $\underline{C}$ to a larger category whose objects represent states and whose arrows represent the processes of N, including the infinite ones. Secondly, but not less importantly, we want to preserve for infinite computations the view already available for categories of finite net computations as models of (essentially) algebraic theories.

Changing the viewpoint, one may look at an ω-diagram F in $\underline{C}$ as a 'formal state', rather than as a computation, namely the state reached by (the computation represented by) F. Then, a tentative solution which immediately arises is provided by $\underline{C}^{\underline{\omega}}$, the category of functors from $\underline{\omega}$ to $\underline{C}$ and natural transformations. The tensor product $\otimes$ on $\underline{C}$ is easily lifted to $\underline{C}^{\underline{\omega}}$ by defining

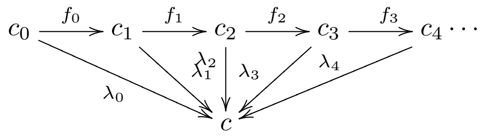
$$\begin{array}{ccc}
\underline{C}^{\underline{\omega}} \times \underline{C}^{\underline{\omega}} & \xrightarrow{\tilde{\otimes}} & \underline{C}^{\underline{\omega}} \\
(F, F') & \longmapsto & \otimes \circ \langle F, F' \rangle \\
{\scriptstyle (\sigma,\sigma')}\downarrow & & \downarrow{\scriptstyle \sigma \tilde{\otimes} \sigma'} \\
(G, G') & \longmapsto & \otimes \circ \langle G, G' \rangle
\end{array}$$

where $\langle \_, \_ \rangle$ is the pairing of functors induced by the product $\underline{C} \times \underline{C}$ and $\tilde{\otimes}$ acts on the natural transformations σ and σ' componentwise. The tensor $\tilde{\otimes}$ is exemplified in the diagram below.

$$\begin{array}{ccccccc}
c_0 \otimes c'_0 & \xrightarrow{f_0 \otimes f'_0} & c_1 \otimes c'_1 & \xrightarrow{f_1 \otimes f'_1} & c_2 \otimes c'_2 & \xrightarrow{f_2 \otimes f'_2} & c_3 \otimes c'_3 \cdots \\
{\scriptstyle \sigma_0 \otimes \sigma'_0}\downarrow & & {\scriptstyle \sigma_1 \otimes \sigma'_1}\downarrow & & {\scriptstyle \sigma_2 \otimes \sigma'_2}\downarrow & & {\scriptstyle \sigma_3 \otimes \sigma'_3}\downarrow \\
d_0 \otimes d'_0 & \xrightarrow[g_0 \otimes g'_0]{} & d_1 \otimes d'_1 & \xrightarrow[g_1 \otimes g'_1]{} & d_2 \otimes d'_2 & \xrightarrow[g_2 \otimes g'_2]{} & d_3 \otimes d'_3 \cdots
\end{array}$$

However, it is easily realized that the functor category $\underline{C}^{\underline{\omega}}$ is quite removed from the category we are looking for. For example, for any non-identity arrow f in $\underline{C}$, there are infinitely many $F \in \underline{C}^{\underline{\omega}}$ such that $F(j < j+1) = f$, for some $j \in \omega$, while for any $i \neq j$, $F(i < i+1)$ is an identity arrow. Although in our intended interpretation all these functors clearly represent the same computation, viz. f, they are distinct in $\underline{C}^{\underline{\omega}}$ and, even worse, they are not necessarily isomorphic to each other, which is the very least one would desire. Of course, a way out of this problem could be to construct a suitable quotient of $\underline{C}^{\underline{\omega}}$, or, more in the spirit of category theory, to make some appropriate arrows be isomorphisms; otherwise said, the notion of morphism for the category we search for is not at all self-evident.

Another conceptual approach to the issue of infinite computations which lies fully in the categorical framework is to exploit the notion of colimit. Suppose that we can 'complete' $\underline{C}$ by adding suitable objects and arrows so that we can ensure that every ω-diagram in the completed category has a colimit. Then, in particular, for every sequence $c_0 \xrightarrow{f_0} c_1 \xrightarrow{f_1} \cdots$ of processes of N, we have a *unique* (up to isomorphism) object c and a cocone



where, by definition, $f_i; \lambda_{i+1} = \lambda_i$ for any $i \in \omega$, with $\_;\_$ denoting the sequential composition of processes, i.e., the composition in $\underline{\mathsf{C}}$. Then, it follows immediately that the arrow $\lambda_0 \colon c_0 \to c$ represents the infinite computation $c_0 \xrightarrow{f_0} c_1 \xrightarrow{f_1} \cdots$.

In the following, we shall give (some different representations of) the *'free'* (in a lax sense to be explained later) completion of $\underline{\mathsf{C}}$ by ω-chains. Moreover, Section 5 will clarify how the two, seemingly different, approaches discussed in this section can be reconciled. Although we shall not achieve a representation of the category of infinite computations of N (or equivalently a free cocompletion of $\underline{\mathsf{C}}$) where 'just the needed points' and 'nothing else' is added, nevertheless, all the desired infinite computations will be represented faithfully, and all the computations which are 'intuitively' the same will have isomorphic colimits in the completed category. Also, $\underline{\mathsf{C}}$ will be embedded (fully and faithfully) by means of a *strict* monoidal functor in it.

Remark. Concerning foundational issues, we assume as usual the existence of a fixed *universe* U of *small* sets upon which *small* and *locally small* categories are built [24]. A category is *small* if the collection of all its arrows form a small set, i.e., it belongs to U; it is *locally small* if the collection of arrows between any two objects of the category is a small set.

Notation. In the following, $\underline{\mathsf{Set}}$, $\underline{\mathsf{Cat}}$ and $\underline{\mathsf{CAT}}$ are, respectively, the category of small sets and functions, the category of small categories and functors and the category of locally small categories and functors. Concerning notation, we shall use a double underlying to denote a 2-category. Thus, $\underline{\underline{\mathsf{Cat}}}$ and $\underline{\underline{\mathsf{CAT}}}$ are the 2-categories corresponding to $\underline{\mathsf{Cat}}$ and $\underline{\mathsf{CAT}}$. Apart from the *large* categories above, and unless differently specified, in the following $\underline{\mathsf{C}}$ stands for a generic locally small category. We denote indifferently by juxtaposition (from right to left) and by $\_ \circ \_$ the composition of functors, while the composition of arrows is always written as $\_ \circ \_$, except in the categories of net processes where, in order to emphasize the fact that it represents sequentialization, we write composition as $\_;\_$ and we use the (left to right) diagrammatic order. We tend to avoid parentheses around the arguments of functors. Finally, we shall preferably denote homsets in $\underline{\mathsf{C}}$ by $\mathsf{Hom}_{\underline{\mathsf{C}}}(a, b)$. However, since this notation can easily become heavy, we shall occasionally write $\underline{\mathsf{C}}[a, b]$.

2 Presheaf Categories as Free Cocompletions

Given a locally small $\underline{\mathsf{C}}$, a *presheaf* on $\underline{\mathsf{C}}$ is a contravariant functor $P \colon \underline{\mathsf{C}}^{\mathrm{op}} \to \underline{\mathsf{Set}}$. The (not necessarily locally small) category $\underline{\mathsf{Set}}^{\underline{\mathsf{C}}^{\mathrm{op}}}$ is the category of all presheaves on $\underline{\mathsf{C}}$. We remind the reader that $\underline{\mathsf{C}}$ is embedded fully and faithfully in $\underline{\mathsf{Set}}^{\underline{\mathsf{C}}^{\mathrm{op}}}$ via the *Yoneda embedding* Y defined as follow. To any $c \in \underline{\mathsf{C}}$ we associate the presheaf $\mathsf{Y}(c) = \mathsf{Hom}_{\underline{\mathsf{C}}}(\_, c)$, often denoted by h_c. This is the presheaf which associates to $d \in \underline{\mathsf{C}}$ the set $\mathsf{Hom}_{\underline{\mathsf{C}}}(d, c)$ and to $f\colon d' \to d$ in $\underline{\mathsf{C}}$ the function $(\_ \circ f)\colon \mathsf{Hom}_{\underline{\mathsf{C}}}(d, c) \to \mathsf{Hom}_{\underline{\mathsf{C}}}(d', c)$, as in the diagram below.

$$\begin{array}{ccc}
\underline{\mathsf{C}}^{\mathrm{op}} & \xrightarrow{\mathsf{Y}(c)} & \underline{\mathsf{Set}} \\
d & \longmapsto & \mathsf{Hom}_{\underline{\mathsf{C}}}(d, c) \\
{\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle (_\circ f)} \\
d' & \longmapsto & \mathsf{Hom}_{\underline{\mathsf{C}}}(d', c)
\end{array}$$

Now, Y can be extended to the arrows of $\underline{\mathsf{C}}$ by mapping $f\colon c \to c'$ to the (constant) natural transformation $(f \circ \_)\colon \mathsf{Y}(c) \stackrel{.}{\to} \mathsf{Y}(c')$. It is very easy to see that this definition makes Y into a (covariant) functor from $\underline{\mathsf{C}}$ to $\underline{\mathsf{Set}}^{\underline{\mathsf{C}}^{\mathrm{op}}}$.

$$\begin{array}{ccc}
\underline{\mathsf{C}} & \xrightarrow{\mathsf{Y}} & \underline{\mathsf{Set}}^{\underline{\mathsf{C}}^{\mathrm{op}}} \\
c & \longmapsto & \mathsf{Hom}_{\underline{\mathsf{C}}}(_, c) = \mathsf{Y}(c) \\
{\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle (f\circ _)} \\
c' & \longmapsto & \mathsf{Hom}_{\underline{\mathsf{C}}}(_, c') = \mathsf{Y}(c')
\end{array}$$

Functors of the form $\mathsf{Y}(c)$, i.e., those set-valued contravariant functors on $\underline{\mathsf{C}}$ isomorphic to $\mathsf{Y}(c)$ for some $c \in \underline{\mathsf{C}}$, are very important for at least two reasons. Firstly, they represent faithfully the category $\underline{\mathsf{C}}$, and secondly, if $\underline{\mathsf{C}}$ is small, they *generate* via small colimits all the others presheaves in $\underline{\mathsf{Set}}^{\underline{\mathsf{C}}^{\mathrm{op}}}$. They are called *representable functors*.

Lemma 2.1 (*Yoneda's Lemma*)
For any $P \in \underline{\mathsf{Set}}^{\underline{\mathsf{C}}^{\mathrm{op}}}$ we have that $\mathsf{Hom}_{\underline{\mathsf{Set}}^{\underline{\mathsf{C}}^{\mathrm{op}}}}(\mathsf{Y}(c), P) \cong P(c)$ via the natural isomorphism θ which sends $\sigma\colon \mathsf{Y}(c) \stackrel{.}{\to} P$ to $\sigma_c(id_c)$.

Corollary 2.2 (*Yoneda's Embedding*)
The functor $\mathsf{Y}\colon \underline{\mathsf{C}} \to \underline{\mathsf{Set}}^{\underline{\mathsf{C}}^{\mathrm{op}}}$ is full and faithful. Thus, Y determines an equivalence between $\underline{\mathsf{C}}$ and its replete image in $\underline{\mathsf{Set}}^{\underline{\mathsf{C}}^{\mathrm{op}}}$, i.e., between $\underline{\mathsf{C}}$ and the full subcategory of $\underline{\mathsf{Set}}^{\underline{\mathsf{C}}^{\mathrm{op}}}$ consisting of the representable functors.

Proof. Immediate: $\underline{\mathsf{Set}}^{\underline{\mathsf{C}}^{\mathrm{op}}}[\mathsf{Y}(c), \mathsf{Y}(c')] \cong \mathsf{Y}(c')(c) = \underline{\mathsf{C}}[c, c']$. ✓

There is also a contravariant version of Yoneda's embedding $\mathsf{Y}'\colon \underline{\mathsf{C}}^{op} \to \underline{\mathsf{Set}}^{\underline{\mathsf{C}}}$ defined as follows:

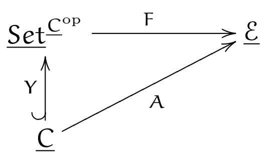
$$\begin{array}{ccc}
\underline{\mathsf{C}}^{op} & \xrightarrow{\mathsf{Y}'} & \underline{\mathsf{Set}}^{\underline{\mathsf{C}}} \\
c & \longmapsto & \mathsf{Hom}_{\underline{\mathsf{C}}}(c, _) = \mathsf{Y}'(c) \\
{\scriptstyle f}\big\downarrow & & \big\uparrow{\scriptstyle (_\circ f)} \\
c' & \longmapsto & \mathsf{Hom}_{\underline{\mathsf{C}}}(c', _) = \mathsf{Y}'(c')
\end{array}$$

Y' is dual to Y; in particular there is a version of Yoneda's Lemma which says that, for each $c \in \underline{\mathsf{C}}$ and for each $P \in \underline{\mathsf{Set}}^{\underline{\mathsf{C}}}$, one has

$$\underline{\mathsf{Set}}^{\underline{\mathsf{C}}}[\mathsf{Hom}_{\underline{\mathsf{C}}}(c, \_), P] \cong P(c).$$

It is worthwhile to recall that Y preserves limits and Y' preserves colimits, i.e., for any $\mathsf{F}\colon \mathsf{J} \to \underline{\mathsf{C}}$, if F has a limit, one has $\mathsf{Y}(\varprojlim \mathsf{F}) \cong \varprojlim(\mathsf{YF})$, and if F has a colimit one has $\mathsf{Y}'(\varinjlim \mathsf{F}) \cong \varprojlim(\mathsf{Y}'\mathsf{F})$.

Recall from general results in category theory that functor categories are as cocomplete (and complete) as their target categories, the colimits being computed *'pointwise'*. Thus, $\underline{\mathsf{Set}}^{\underline{\mathsf{C}}^{\mathrm{op}}}$ is cocomplete, and so we can consider the colimit of any small diagram of presheaves. Moreover, it can be shown that every presheaf on $\underline{\mathsf{C}}$ is a colimit of representables in a canonical way. This result implies quite directly that $\underline{\mathsf{Set}}^{\underline{\mathsf{C}}^{\mathrm{op}}}$ can be considered as the cocompletion of $\underline{\mathsf{C}}$ by all small colimits (and limits). This claim is substantiated by the fact that $\underline{\mathsf{Set}}^{\underline{\mathsf{C}}^{\mathrm{op}}}$ enjoys a weak form of 'universality', namely that for any cocomplete category $\underline{\mathcal{E}}$ and any functor $\mathsf{A}\colon \underline{\mathsf{C}} \to \underline{\mathcal{E}}$, there is a functor $\mathsf{F}\colon \underline{\mathsf{Set}}^{\underline{\mathsf{C}}^{\mathrm{op}}} \to \underline{\mathcal{E}}$ which preserves all small colimits and such that the following diagram commutes.



Moreover, F is, up to isomorphism, the unique functor which enjoys this property. The proofs of these facts can be found in [26].

The understanding of $\underline{\mathsf{Set}}^{\underline{\mathsf{C}}^{\mathrm{op}}}$ as the "free' completion of of a category $\underline{\mathsf{C}}$ by arbitrary small colimits, together with the fact that $\underline{\mathsf{C}}$ is embedded in $\underline{\mathsf{Set}}^{\underline{\mathsf{C}}^{\mathrm{op}}}$ via the Yoneda's embedding, gives a strong hint on what the completion of $\underline{\mathsf{C}}$ by ω-chains should be: an appropriate *full* subcategory of $\underline{\mathsf{Set}}^{\underline{\mathsf{C}}^{\mathrm{op}}}$. Formally, we have the following definition.

Definition 2.3 (*ω-Ind-Representable Functors*)
Given a locally small category $\underline{\mathsf{C}}$, let $\widehat{\underline{\mathsf{C}}}_\omega$ denote the full subcategory of $\underline{\mathsf{Set}}^{\underline{\mathsf{C}}^{\mathrm{op}}}$ which contains the representables h_c, $c \in \underline{\mathsf{C}}$, and $\varinjlim(\mathsf{Y} \circ \mathsf{F})$ for any functor $\mathsf{F}\colon \underline{\omega} \to \underline{\mathsf{C}}$, Y being the Yoneda embedding. The presheaves in $\widehat{\underline{\mathsf{C}}}_\omega$ are called ω-ind-representable (ind for inductively).

It can be shown (see e.g. [32], and later sections in this paper) that $\widehat{\underline{\mathsf{C}}}_\omega$ is closed under colimits of ω-chains and that it is *'universal'* (in the weak sense above) among the ω-chain cocomplete categories which extends $\underline{\mathsf{C}}$. Thus, $\widehat{\underline{\mathsf{C}}}_\omega$ provides a quick answer to the issue of ω-chain cocompletion of categories.

However, although ω-ind-representable presheaves are a highly elegant categorical notion, they are still rather far from our intuition motivated in Section 1. Moreover, some extra machinery is needed in order to express the construction of $\widehat{\underline{\mathsf{C}}}_\omega$ functorially on $\underline{\underline{\mathsf{CAT}}}$. Finally, in the case of monoidal categories, it is not easy to extend the monoidal structure of $\underline{\mathsf{C}}$ to $\widehat{\underline{\mathsf{C}}}_\omega$ in a direct way. In the following we shall give a representation of (a category equivalent to) $\widehat{\underline{\mathsf{C}}}_\omega$ that is better suited to our purposes.

3 ω-Chains versus ω-Filters

Although chains are quite simple structures, they are rather uncomfortable to manage, at least in our context, since, as categories, they are very strongly restricted. This situation, however, is common to other fields of mathematics, like partial order theory and topology, where one uses the—equivalent in many contexts—notion of directed set [27, 16, 33]. Thus, as a first step, we abandon ω-chains and we broaden the class of index categories we use for the colimits, but, as we shall see, this will not change the kind of cocompleteness.

Chains versus Directed Partial Orders

Definition 3.1
A non-empty subset D of a partial order P is directed if any pair of elements in D has an upper bound in D. Equivalently, D is directed if it contains an upper bound of any of its finite subsets. A chain is a non-empty partial order which is totally ordered.

It is well known that, for a partial order, having a least upper bound for each of its directed subsets of cardinality ω, and having a least upper bound for each of its chains of cardinality ω are equivalent notions. This result can be extended to the corresponding notion of cocompleteness in categories. For D a directed set, a functor $F\colon D \to \underline{\mathsf{C}}$ is called *ω-directed* if $|D| \leq \omega$, where $|\_|$ gives the cardinality of sets. Similarly, we shall call functors $F\colon \underline{\omega} \to \underline{\mathsf{C}}$ ω-chain functors, or simply *ω-chains*.

Definition 3.2
A category $\underline{\mathsf{C}}$ is ω-directed cocomplete if it admits colimits of all ω-directed functors. It is ω-chain cocomplete if it has colimits of all ω-chain functors.

Proposition 3.3
$\underline{\mathsf{C}}$ is ω-directed cocomplete if and only if it is ω-chain cocomplete.

Proof. One implication is trivial, since an ω-chain is a countable directed set. Let us show that when $\underline{\mathsf{C}}$ admits colimits of all ω-chains it has colimits of all ω-directed functors. Let $F\colon D \to \underline{\mathsf{C}}$ be a ω-directed functor. If D is finite then D has a greatest element, say d, and, therefore, F has a colimit, namely $F(d)$ with the obvious limit cocone. Otherwise, the proof is based on the following lemma [27].

Lemma. *Let D be an infinite directed set. Then, there exists a transfinite sequence $\{D_\alpha\}_\alpha$ of directed subsets of D, with $\alpha < |D|$, such that*

i) For any α

- *If α is finite so is D_α;*
- *If α is infinite, then $|D_\alpha| = |\alpha|$ (and therefore $|D_\alpha| < |D|$).*

ii) For any ordinals $\alpha < \beta < |D|$, $D_\alpha \subset D_\beta$.

iii) $D = \bigcup_\alpha D_\alpha$.

Therefore, we can find a countable sequence of *finite* directed subsets $\{D_i\}_{i<\omega}$ such that, $D_i \subset D_{i+1}$, for any $i < \omega$, and $D = \bigcup_{i\in\omega} D_i$. Let $in_i\colon D_i \to D$ denote the injection of D_i in D. For any $i < \omega$, let $F_i\colon D_i \overset{in_i}{\hookrightarrow} D \xrightarrow{F} \underline{\mathsf{C}}$ be the restriction of F to D_i, and let $\sigma_i\colon F_i \xrightarrow{\cdot} c_i$ be a colimit for F_i. Now, observe that for any $i \leq j$, since $D_i \subseteq D_j$, we have $(\sigma_j)_i\colon F_i \xrightarrow{\cdot} c_j$, where $(\sigma_j)_i$ is the restriction of σ_j to D_i. Then, by definition of colimit, there exists a unique induced arrow $f_{i,j}\colon c_i \to c_j$ such that $f_{i,j} \circ (\sigma_i)_d = (\sigma_j)_d$ for any $d \in D_i$.

It follows, again from the universal property of colimits, that the following definition defines a functor $G\colon \underline{\omega} \to \underline{\mathsf{C}}$.

$$G(i) = c_i \quad \text{for any } i \in \underline{\omega}; \qquad G(i \leq j) = f_{i,j}.$$

Since G is an ω-chain we can consider $\varinjlim G$ in $\underline{\mathsf{C}}$. Let $\lambda\colon G \xrightarrow{\cdot} c$ be the limit cocone. Then c, together with the cocone $F \xrightarrow{\cdot} c$ whose componenent at $d \in D$ is $\lambda_i \circ (\sigma_i)_d$, for any i such that $d \in D_i$, is $\varinjlim_D F$. ✓

Filtered Categories and Cofinal Functors

In this subsection we broaden further the kind of index categories over which colimits are considered. In particular, we recall the basic facts about *filtered categories* and *cofinal functors*.

Definition 3.4 (*Filtered Categories*)
A category J is filtered if it is not empty and

i) for all $j, j' \in \mathsf{J}$ there exists k and $u\colon j \to k$, $v\colon j' \to k$, i.e., $\begin{smallmatrix} j & \overset{u}{\searrow} & \\ & & k \\ j' & \underset{v}{\nearrow} & \end{smallmatrix}$.

ii) for all $i \overset{u}{\underset{v}{\rightrightarrows}} j$ in J, there exists $w\colon j \to k$ such that $w \circ u = w \circ v$, i.e., $i \overset{u}{\underset{v}{\rightrightarrows}} j \xrightarrow{w} k$ is commutative.

A functor $\mathsf{F}\colon \mathsf{J} \to \underline{\mathsf{C}}$ is filtered if J is filtered.

A good point is that colimits in $\underline{\mathsf{Set}}$ indexed by filtered categories are easily characterized.

Proposition 3.5
Let $\mathsf{F}\colon \mathsf{J} \to \underline{\mathsf{Set}}$ be filtered and suppose that J is small. Consider the set $\coprod_{j\in\mathsf{J}} \mathsf{F}(j)$ and the binary relation $\mathcal{R}$ on it defined as follows

$$in_i(x)\ \mathcal{R}\ in_j(y) \quad \Leftrightarrow \quad \exists k \in \mathsf{J} \text{ and } \begin{smallmatrix} i & \overset{u}{\searrow} & \\ & & k \\ j & \underset{v}{\nearrow} & \end{smallmatrix} \text{ such that } \mathsf{F}(u)(x) = \mathsf{F}(v)(y).$$

Then, (i) $\mathcal{R}$ is an equivalence relation; and (ii) $\left(\coprod_{j\in\mathsf{J}} \mathsf{F}(j)\right) \big/ \mathcal{R} \cong \varinjlim_{\mathsf{J}} \mathsf{F}$.

Cofinal subcategories are the categorical generalization of the set-theoretic notion of cofinal chains. Intuitively, a subcategory I of J is cofinal in J if the colimit of any J-indexed diagram coincides with the colimit of the same diagram restricted to I. Of course, there is no conceptual need to limit oneself to subcategories, and that is why one introduces *cofinal functors.*

Definition 3.6 (*Cofinal Functors*)
A functor $\phi\colon \mathsf{I} \to \mathsf{J}$ is cofinal if for any functor $\mathsf{F}\colon \mathsf{J} \to \underline{C}$

$$\varinjlim_{\mathsf{I}}(\mathsf{F}\circ\phi) \text{ exists} \quad \Rightarrow \quad \varinjlim_{\mathsf{J}}\mathsf{F} \text{ exists} \quad \text{and} \quad \varinjlim_{\mathsf{I}}(\mathsf{F}\circ\phi) \cong \varinjlim_{\mathsf{J}}\mathsf{F},$$

the isomorphism being via the canonical comparison map $\varinjlim_{\mathsf{I}}(\mathsf{F}\circ\phi) \to \varinjlim_{\mathsf{J}}\mathsf{F}$ induced by the colimit.
A subcategory I of J is cofinal if the inclusion functor is cofinal.

Of course the name is inherited from the corresponding notion in set theory and the 'co' prefix has nothing to do with duality in categories. For this reason, MacLane [25, chap. IX] and others use the term *'final'* to name the concept. However, once the reader has been warned about this mismatch, we prefer to keep using the classical terminology. The following key lemma gives a characterization of cofinal functors between filtered categories.

Lemma 3.7
For a functor $\phi\colon \mathsf{I} \to \mathsf{J}$ the following properties can be stated:

$\mathbf{F_1}$: *for any $j \in \mathsf{J}$, there exists $i \in \mathsf{I}$ such that $\mathsf{Hom}_{\mathsf{J}}(j, \phi(i)) \neq \varnothing$.*

$\mathbf{F_2}$: *for any $i \in \mathsf{I}$ and for any $j \overset{f}{\underset{g}{\rightrightarrows}} \phi(i)$ in J, there exists $h\colon i \to k$ in I such that $\phi(h)\circ f = \phi(h)\circ g$.*

Then, we have the following facts:

i) if ϕ is cofinal, then $\mathbf{F_1}$ holds;

ii) if I is filtered, then ϕ is cofinal if and only if $\mathbf{F_1}$ and $\mathbf{F_2}$ hold and, in this case, J is also filtered;

iii) if J is filtered and ϕ is full and faithful, then ϕ is cofinal if and only if $\mathbf{F_1}$ holds, and, in this case, I is also filtered.

The next proposition shows that requiring the existence of filtered colimits is equivalent to requiring the existence of directed colimits. Given a category J, by the *cardinality* of J, in symbols $|\mathsf{J}|$, we mean, as usual, the cardinality of the underlying set of arrows of J. If J is filtered and $|\mathsf{J}| \leq \omega$, we say that J is *ω-filtered.* We say that a functor $\mathsf{F}\colon \mathsf{J} \to \underline{C}$ is *ω-filtered* if J is ω-filtered. A category $\underline{C}$ is *ω-filtered cocomplete*, or countably filtered cocomplete, if it admits colimits of all ω-filtered functors.

PROPOSITION 3.8
A category $\underline{\mathsf{C}}$ is ω-filtered cocomplete if and only if it is ω-directed cocomplete, if and only if it is ω-chain cocomplete.

Proof. The second double implication is Proposition 3.3 and one direction of the first one is trivial. The other direction follows immediately by the following lemma [13].

> LEMMA. *Let J be small and filtered. Then, there exists a directed set D and $\phi\colon D \to \mathsf{J}$ which is cofinal. Moreover, if $|\mathsf{J}|$ is infinite, then $|\mathsf{J}| = |D|$.*

✓

4 A KZ-Doctrine for the ω-Ind Completion

As we already mentioned at the end of Section 2, the cocompletion by ω-chains of $\underline{\mathsf{C}}$ obtained directly from the category of preasheaves on $\underline{\mathsf{C}}$ is not well suited for a functorial formalization, which is indeed a desirable property. Moreover, in many occasions a more concrete description of the objects of $\widehat{\underline{\mathsf{C}}}_\omega$ may be useful. In particular, we think of something very close to the description of infinite processes we have sketched in Section 1. Close to this issue, there is the fact that one would often like a more algebraic description of colimits in terms of pseudo monads (see also the discussion at the beginning of Appendix A). In this section we study a KZ-doctrine for the ω-filtered cocompletion. In other words, we study alternative representations for the ω-ind-representable presheaves.

ω-Ind Objects

The following definition of ω-ind-object follows the same simple idea about representation of 'formal state' we discussed in Section 1.

DEFINITION 4.1 (*ω-Ind-Objects*)
A functor $X\colon \mathsf{J} \to \underline{\mathsf{C}}$ is an ω-ind-object, or simply an ind-object, if J is ω-filtered. We shall identify ω-ind-objects $X\colon \mathsf{I} \to \underline{\mathsf{C}}$ and $Y\colon \mathsf{J} \to \underline{\mathsf{C}}$ if there exists a cofinal $\phi\colon \mathsf{I} \to \mathsf{J}$ whose object component is an isomorphism and such that $Y \circ \phi = X$.

Thus, ω-ind-objects are nothing but countable filtered diagrams in $\underline{\mathsf{C}}$. We can think of ω-ind-objects as 'syntactic' representations of ω-ind-representable functors. In particular, we shall say that the ω-ind-object $X\colon \mathsf{I} \to \underline{\mathsf{C}}$ represents the ω-ind-representable presheaf

$$\mathsf{L}(X) = \varinjlim(\mathsf{I} \xrightarrow{X} \underline{\mathsf{C}} \xrightarrow{\mathsf{Y}} \widehat{\underline{\mathsf{C}}}_\omega) \cong \varinjlim_{i\in\mathsf{I}} \mathsf{Hom}_{\underline{\mathsf{C}}}(\_, X(i)) \cong \varinjlim_{i\in\mathsf{I}} \mathsf{h}_{X(i)}.$$

Observe that, since $\widehat{\underline{\mathsf{C}}}_\omega$ is ω-chain cocomplete, Proposition 3.8 guarantees the existence of $\mathsf{L}(X)$. Moreover, it is worth noticing that the equalities imposed by Definition 4.1 are perfectly harmless because they identify objects which represent isomorphic presheaves.

In order to simplify notation, we shall often use the so-called *indexed* notation for ω-ind-objects. We write $(X_i)_{i\in\mathsf{I}}$ for $X\colon \mathsf{I} \to \underline{\mathsf{C}}$ with $X(i) = X_i$. Admittedly this notation is rather poor but it will not be misleading, and therefore it will be acceptable, provided one never forgets that we are not handling sequences or chains of objects, but filtered diagrams in $\underline{\mathsf{C}}$. Of course, given an ω-ind-object $(X_i)_{i\in \mathsf{I}}$, we reserve the right to use X also in every context in which a functor is expected.

As already observed in Section 1, the key point is the definition of morphisms for ω-ind-objects. The right notion should be such that it identifies, i.e., it makes isomorphic, ω-ind-objects which intuitively should be the same. Moreover, it has to make the category of ω-ind-objects filtered cocomplete. Clearly, the theory exposed in Section 2 allows us to identify such notion of morphism immediately.

Definition 4.2
The category $\mathtt{Ind}_\omega(\underline{\mathsf{C}})$ *is the category whose objects are the* ω*-ind-objects of* $\underline{\mathsf{C}}$*, and whose homsets are defined by*

$$\mathsf{Hom}_{\mathtt{Ind}_\omega(\underline{\mathsf{C}})}(X,Y) = \mathsf{Hom}_{\widehat{\underline{\mathsf{C}}}_\omega}(\mathsf{L}(X),\mathsf{L}(Y)).$$

This makes L into a full and faithful functor from $\mathtt{Ind}_\omega(\underline{\mathsf{C}})$ to $\widehat{\underline{\mathsf{C}}}_\omega$. However, observe that it is far from being injective on the objects. Nevertheless, we have that $\mathtt{Ind}_\omega(\underline{\mathsf{C}})$ and $\widehat{\underline{\mathsf{C}}}_\omega$ are *(weakly)* equivalent.

Proposition 4.3
$\mathtt{Ind}_\omega(\underline{\mathsf{C}}) \dot{\cong} \widehat{\underline{\mathsf{C}}}_\omega$.

Proof. Exploiting Proposition 3.8 and Proposition 3.3, it is immediate to see that L is a weak equivalence, i.e., a full and faithful functor whose replete image is the whole target category, i.e., such that every object P in $\widehat{\underline{\mathsf{C}}}_\omega$ is isomorphic to some $\mathsf{L}(X)$ for X in $\mathtt{Ind}_\omega(C)$. Freyd and Scedrov [8] show that the hypothesis that every weak equivalence is a strict (classical) one is equivalent to the axiom of choice. Since in our context we have largely used such an axiom, we can also assume that L is an equivalence. ✓

Proposition 4.4
The category $\mathtt{Ind}_\omega(\underline{\mathsf{C}})$ *is locally small.*

Proof. Given the ω-ind-objects $(X_i)_{i\in\mathsf{I}}$ and $(Y_j)_{j\in\mathsf{J}}$, we have:

$$\begin{aligned}
\mathtt{Ind}_\omega(\underline{\mathsf{C}})[X,Y] &= \widehat{\underline{\mathsf{C}}}_\omega[\mathsf{L}(X),\mathsf{L}(Y)] \\
&= \underline{\mathsf{Set}}^{\mathsf{C}^{\mathrm{op}}}[\mathsf{L}(X),\mathsf{L}(Y)] \\
&\cong \underline{\mathsf{Set}}^{\mathsf{C}^{\mathrm{op}}}[\varinjlim_{\mathsf{I}} \mathtt{h}_{X_i}, \varinjlim_{\mathsf{J}} \mathtt{h}_{Y_j}] \\
&\cong \varprojlim_{\mathsf{I}} \underline{\mathsf{Set}}^{\mathsf{C}^{\mathrm{op}}}[\mathtt{h}_{X_i}, \varinjlim_{\mathsf{J}} \mathtt{h}_{Y_j}] \text{ (since } \mathsf{Y}' \text{ preserves colimits)} \\
&\cong \varprojlim_{\mathsf{I}} \left(\varinjlim_{\mathsf{J}} \mathtt{h}_{Y_j}\right) X_i \text{ (by the Yoneda's lemma)} \\
&\cong \varprojlim_{\mathsf{I}} \varinjlim_{\mathsf{J}} \mathsf{Hom}_{\underline{\mathsf{C}}}(X_i,Y_j).
\end{aligned}$$

Thus, since by hypothesis each $\mathsf{Hom}_{\underline{\mathsf{C}}}(X_i,Y_j)$ is a small set, as a $\varprojlim \varinjlim$ construction in $\underline{\mathsf{Set}}$ indexed by small categories, $\mathsf{Hom}_{\mathtt{Ind}_\omega(\underline{\mathsf{C}})}(X,Y)$ is a small set. ✓

PROPOSITION 4.5
If $\underline{\mathsf{C}}$ is small, then so is $\mathtt{Ind}_\omega(\underline{\mathsf{C}})$.

Proof. There is 'only' a small set of ω-filtered diagrams in $\underline{\mathsf{C}}$ and, therefore, the objects of $\mathtt{Ind}_\omega(\underline{\mathsf{C}})$ form a small set. Regarding morphisms, by Proposition 4.4, the collection of the morphisms of $\mathtt{Ind}_\omega(\underline{\mathsf{C}})$ are a family of small sets indexed by a small set, and therefore a small set. ✓

PROPOSITION 4.6
Consider the ω-ind-objects $(X_i)_{i\in\mathsf{I}}$ and $(Y_j)_{j\in\mathsf{J}}$ and suppose that there exists a cofinal $\phi\colon\mathsf{I}\to\mathsf{J}$ such that $Y\circ\phi = X$. Then, $(X_i)_{i\in\mathsf{I}}\cong(Y_j)_{j\in\mathsf{J}}$ in $\mathtt{Ind}_\omega(\underline{\mathsf{C}})$.

Proof. Since ϕ is cofinal $\mathsf{L}(X)=\mathsf{L}(Y\circ\phi)\cong\mathsf{L}(Y)$. ✓

Next, we give a more explicit representation of morphisms of ω-ind-objects. Recalling a computation we have done in Proposition 4.4, we have that

$$\mathtt{Ind}_\omega(\underline{\mathsf{C}})[(X_i)_{i\in\mathsf{I}},(Y_j)_{j\in\mathsf{J}}] = \underline{\mathsf{Set}}^{\underline{\mathsf{C}}^{\mathrm{op}}}[\mathsf{L}(X),\mathsf{L}(Y)] = \varprojlim_{\mathsf{I}}\varinjlim_{\mathsf{J}}\mathsf{Hom}_{\underline{\mathsf{C}}}(X_i,Y_j).$$

From Proposition 3.5, we know how to compute filtered colimits in $\underline{\mathsf{Set}}$. An element $x\in\varinjlim_{\mathsf{J}}\mathsf{Hom}_{\underline{\mathsf{C}}}(X_i,Y_j)$ is an equivalence class of arrows $[f]_\sim$ each representative of which is an arrow $f\colon X_i\to Y_j$ of $\underline{\mathsf{C}}$ for some $j\in\mathsf{J}$ and where

$$\Big(f\colon X_i\to Y_{j'}\Big)\sim\Big(g\colon X_i\to Y_{j''}\Big)\quad\Leftrightarrow\quad\exists\ \begin{matrix} j' \overset{u}{\searrow} \\ \quad k \\ j'' \underset{v}{\nearrow}\end{matrix}\ \text{in } \mathsf{J} \text{ st. } Y(u)\circ f = Y(v)\circ g.$$

Concerning limits, their calculus in $\underline{\mathsf{Set}}$ is much simpler than that of colimits.

PROPOSITION 4.7
Let I be a small category and $\mathsf{F}\colon\mathsf{I}\to\underline{\mathsf{C}}$ a functor. By $\langle f\rangle$ we denote a function $f\colon Obj(\mathsf{I})\to\bigcup_{i\in Obj(\mathsf{I})}\mathsf{F}(i)$ such that, for all $i\in Obj(\mathsf{I})$, it is $f(i)\in\mathsf{F}(i)$. Then,

$$\varprojlim_{\mathsf{I}}\mathsf{F} = \Big\{\langle f\rangle\ \Big|\ \mathsf{F}(h)(f(i)) = f(j),\ \forall h\colon i\to j \text{ in } \mathsf{I}\Big\}.$$

Getting back to our problem, the elements x in $\varprojlim_{\mathsf{I}}\varinjlim_{\mathsf{J}}\mathsf{Hom}_{\underline{\mathsf{C}}}(X_i,Y_j)$ are therefore a *collection of equivalence classes* $[f_i]_\sim$ indexed by the objects of I which are *compatible* in the precise sense that for any $h\colon i\to i'$ in I we have

$$[f_i]_\sim = [f_{i'}\circ X(h)]_\sim.$$

We shall denote this kind of families of equivalence classes with a notation similar to the one used for ω-ind-objects, namely, $([f_i])_{i\in\mathsf{I}}\colon(X_i)_{i\in\mathsf{I}}\to(Y_j)_{j\in\mathsf{J}}$, where f_i is an arrow from X_i to some Y_j. The square brackets remind us that each component is an equivalence class and the index i means that f_i is a representative for the i-th class. We almost always avoid explicit mention of $\sim$. However it should be taken into account that $\sim$, and thus the elements of $[f_i]_\sim$, of course depends on the actual J, while the compatibility of the various components depends also on I.

The composition of ω-ind-morphisms can be defined explicitly through the canonical function induced by the limit. However, its description in terms of families of equivalence classes is simple: given $([f_i])_{i\in\mathsf{I}}: (X_i)_{i\in\mathsf{I}} \to (Y_j)_{j\in\mathsf{J}}$ and $([g_j])_{j\in\mathsf{J}}: (Y_j)_{j\in\mathsf{J}} \to (Z_k)_{k\in\mathsf{K}}$, their composition is the I-indexed family whose i-th component is $[g \circ f]_{\sim'}$ for $(f: X_i \to Y_j) \in [f_i]_\sim$ and $(g: Y_j \to Z_k) \in [g_j]_{\sim'}$, with the equivalence $\sim$ being relative to J, and the equivalence $\sim'$ to K. In other words, the i-th class of the composition is obtained by considering the class (wrt. K) of the composition of one representative of the i-th component of $([f_i])_{i\in\mathsf{I}}$ and one representative of the j_i-th component of $([g_j])_{j\in\mathsf{J}}$, where j_i is determined by the chosen representative of f_i. Of course, it can be shown that this is well-defined, i.e., that the definition does not depend on the choice of f and g above and that it gives an ω-ind-morphism from $(X_i)_{i\in\mathsf{I}}$ to $(Z_k)_{k\in\mathsf{K}}$.

Remark. An alternative description of ω-ind-morphisms can be found via a *category of fractions* construction [9]. The interest of this approach, introduced in [35] (see [32] for a detailed survey), resides in the fact that it explains the cocompletion construction 'just' by making invertible a class of arrows in a universal way. Moreover, such a class of arrows is as simple as possible and, therefore, the approach gives insights on the subject by making intuitively clear how the arrows chosen similarly to Section 1 should be enriched in order to get cocompleteness.

When $\underline{\mathsf{C}}$ is a poset P, there is the following connection of $\mathrm{Ind}_\omega(\underline{\mathsf{C}})$ with the theory of complete posets.

Proposition 4.8

Let P be a small poset. Then $\mathrm{Ind}_\omega(P)$ is equivalent to the completion of P by countable ideals viewed as a category.

Proof. Recall that an ideal in P is a directed subset $I \subseteq P$ which is *downward closed*, i.e., such that $i \in I$ and $j \leq i$ implies $j \in I$. Now, observe that $\mathsf{Hom}_{\mathrm{Ind}_\omega(\mathrm{P})}(X,Y) = \varprojlim \varinjlim \mathsf{Hom}_P(X_i, Y_j)$ must be either a singleton or the empty set, since each $\mathsf{Hom}_P(X_i, Y_j)$ is such. Then $\mathrm{Ind}_\omega(\mathrm{P})$ is a preorder. Moreover, an ideal I of P is naturally an ω-ind-object, namely the inclusion $I \hookrightarrow P$. Conversely, an ω-ind-object $X: \mathsf{I} \to P$ can be thought as an ideal just by taking the *'downward' closure* of its image in P. Since X is cofinal in the ω-ind-object corresponding to such a closure, this defines an equivalence. ✓

A general treatment of the completion of posets in a categorical framework, namely via monads, has been given in [28].

$\mathrm{Ind}_\omega(\_)$ as a 2-endofunctor on $\underline{\underline{\mathsf{CAT}}}$

We have already seen that if $\underline{\mathsf{C}}$ is locally small, then so is $\mathrm{Ind}_\omega(\underline{\mathsf{C}})$. Therefore, it looks plausible that $\mathrm{Ind}_\omega(\_)$ is the object part of an endofunctor on $\underline{\mathsf{CAT}}$. In this subsection we show that this is the case. In particular, we show that $\mathrm{Ind}_\omega(\_)$ can be extended to the 2-cells of $\underline{\underline{\mathsf{CAT}}}$ obtaining in this way a 2-functor.

Let $\mathsf{F}: \underline{\mathsf{C}} \to \underline{\mathsf{D}}$ be a functor in $\underline{\underline{\mathsf{CAT}}}$. We define a functor $\mathrm{Ind}_\omega(\mathsf{F}): \mathrm{Ind}_\omega(\underline{\mathsf{C}}) \to \mathrm{Ind}_\omega(\underline{\mathsf{D}})$. Concerning the objects, the definition is evident: we map X to the composition of X with F.

$$\mathsf{I} \xrightarrow{X} \underline{\mathsf{C}} \xrightarrow{\mathsf{F}} \underline{\mathsf{D}}.$$

For the morphisms the situation is slightly more difficult. However, in terms of the representation of ω-ind-morphisms by families of equivalence classes, we have the following obvious definition, whose functorial nature is evident.

$$([f_i])_{i\in\mathsf{I}}\colon (X_i)_{i\in\mathsf{I}} \to (Y_j)_{j\in\mathsf{J}} \mapsto \ ([\mathsf{F}f_i])_{i\in\mathsf{I}}\colon (\mathsf{F}X_i)_{i\in\mathsf{I}} \to (\mathsf{F}Y_j)_{j\in\mathsf{J}}.$$

Observe that, of course, $[\mathsf{F}f_i]_\sim$ is not necessarily the image of $[f_i]_\sim$, since, intuitively, $\underline{\mathsf{D}}$ could have 'more' morphisms.

Consider now $\mathsf{F}, \mathsf{G}\colon \underline{\mathsf{C}} \to \underline{\mathsf{D}}$. Given a natural transformation $\alpha\colon \mathsf{F} \dot{\to} \mathsf{G}$ there is then an obvious candidate for $\mathtt{Ind}_\omega(\alpha)\colon \mathtt{Ind}_\omega(\mathsf{F}) \dot{\to} \mathtt{Ind}_\omega(\mathsf{G})$, namely the family $\{\alpha_X\}_{X\in\mathtt{Ind}_\omega(\underline{\mathsf{C}})}$ where $\alpha_X\colon (\mathsf{F}X_i)_{i\in\mathsf{I}} \to (\mathsf{G}X_i)_{i\in\mathsf{I}}$ is the arrow of $\mathtt{Ind}_\omega(\underline{\mathsf{D}})$ whose i-th component is $[\alpha_{X_i}]_\sim$. In other words, $\mathtt{Ind}_\omega(\alpha)$ is determined by (taking the equivalence classes of the component arrows of) $\alpha_X\colon \mathsf{F}X \dot{\to} \mathsf{G}X$. Observe that the condition $[\alpha_{X_i}]_\sim = [\alpha_{X_j} \circ \mathsf{F}X(h)]_\sim$ for any $h\colon i \to j$ in I comes directly from the naturality in $\underline{\underline{\mathsf{CAT}}}$ of α.

It is now very easy to check that $\{\alpha_X\}_{X\in\mathtt{Ind}_\omega(\underline{\mathsf{C}})}$ is a natural transformation and that this definition makes $\mathtt{Ind}_\omega(\_)$ into a 2-functor.

Proposition 4.9
$\mathtt{Ind}_\omega(\_)\colon \underline{\underline{\mathsf{CAT}}} \to \underline{\underline{\mathsf{CAT}}}$ *is a 2-functor.*

Constant ω-Ind-Objects: the 2-natural unit y

In this section we see that the Yoneda embedding $\mathsf{Y}\colon \underline{\mathsf{C}} \to \widehat{\underline{\mathsf{C}}}_\omega$ has a corresponding embedding $y\colon \underline{\mathsf{C}} \to \mathtt{Ind}_\omega(\underline{\mathsf{C}})$. This shall provide us with a 2-natural transformation for the KZ-doctrine we are building.

The category $\underline{1}$ consisting of a unique element and its identity arrow, i.e., the terminal object in $\underline{\mathsf{CAT}}$, is a filtered category. For any $c \in \underline{\mathsf{C}}$ we denote by $\underline{c}$ the ω-ind-object $\underline{c}\colon \underline{1} \to \underline{\mathsf{C}}$ which picks up c. These kind of ω-ind-objects are called *constant ω-ind-objects* and provide a full and faithful image of $\underline{\mathsf{C}}$ in $\mathtt{Ind}_\omega(\underline{\mathsf{C}})$ via the functor y defined below.

$$\begin{array}{ccc} \underline{\mathsf{C}} & \xrightarrow{\ y\ } & \mathtt{Ind}_\omega(\underline{\mathsf{C}}) \\ c & \longmapsto & \underline{c} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ d & \longmapsto & \underline{d} \end{array}$$

Observe that, by definition, $y(f)$ is $[f]_\sim$. However, since the index category for $\underline{d}$ is $\underline{1}$, in this case $\sim$ is trivial, i.e., $[f]_\sim$ consists of the unique element f.

Thus, y plays the role which Y plays in the case of presheaves. Of course there are many objects in $\mathtt{Ind}_\omega(\underline{\mathsf{C}})$ which can represent $\underline{\mathsf{C}}$ and, consequently, many possible embeddings y. (For instance, in Section 5 we shall use another y.) If we consider a constant functor $\underline{\underline{c}}\colon \mathsf{I} \to \underline{\mathsf{C}}$, which always takes the value c, we have that $\mathsf{L}(\underline{\underline{c}}) = \varinjlim_{\mathsf{I}} \mathsf{h}_c = \mathsf{h}_c$, i.e., $\underline{c}$ and $\underline{\underline{c}}$ are isomorphic in $\mathtt{Ind}_\omega(\underline{\mathsf{C}})$. The same happens if we consider a finite index category I and a functor $X\colon \mathsf{I} \to \underline{\mathsf{C}}$

which sends the greatest element of I to c. The ω-ind-objects X such that $X \cong \underline{c}$ for some $c \in \underline{C}$, or equivalently such that $\mathsf{L}(X) \cong \mathtt{h}_c$, are called *essentially constant ω-ind-objects*. Of course, y defines an equivalence between $\underline{C}$ and the full subcategory of the essentially constant ω-ind-objects in $\mathtt{Ind}_\omega(\underline{C})$.

A first connection with KZ-doctrines is the following proposition, where the reader will recognize the similarity with the definition of algebra for a KZ-doctrine (see Appendix A, Definition A.3) .

Proposition 4.10
A locally small category $\underline{C}$ is ω-filtered cocomplete if and only if $y\colon \underline{C} \to \mathtt{Ind}_\omega(\underline{C})$ has a left adjoint.

Proof. $\underline{C}$ has ω-filtered colimits if and only if for every $(X_i)_{i\in\mathsf{I}}$ in $\mathtt{Ind}_\omega(\underline{C})$ and c in $\underline{C}$ there is a natural isomorphism $\mathsf{Hom}_{\underline{C}}(\varinjlim X, c) \cong \underline{C}^{\mathsf{I}}[X, \Delta_c]$, where $\Delta_c\colon \mathsf{I} \to \underline{C}$ is the constant functor which selects c. By definition of $\underline{c}$, it is immediate to see that such cocones $X \stackrel{\cdot}{\to} \Delta_c$ are in one-to-one correspondence with ω-ind-morphisms $X \to \underline{c}$. It follows that $\underline{C}$ has all ω-filtered colimits if and only if there is a natural isomorphism

$$\mathsf{Hom}_{\underline{C}}(\varinjlim X, c) \cong \mathsf{Hom}_{\mathtt{Ind}_\omega(\underline{C})}(X, y(c)),$$

which is the isomorphism for the adjointness $\varinjlim \dashv y$. Observe that, since y is full and faithful, the adjunction is a reflection. ✓

Now, we have to see that the family $\{y_{\underline{C}}\}_{\underline{C}\in\underline{\underline{\underline{CAT}}}}$ is a 2-natural transformation $Id \stackrel{\cdot}{\to} \mathtt{Ind}_\omega(\_)$. The task is fairly easy: naturality is immediate, while for any $\alpha\colon \mathsf{F} \to \mathsf{G}$ in $\underline{\underline{\underline{CAT}}}$, the equation for 2-naturality is $\mathtt{Ind}_\omega(\alpha) y_{\underline{C}} = y_{\underline{D}}\alpha$, i.e.,

$$\underline{C} \xrightarrow{y_{\underline{C}}} \mathtt{Ind}_\omega(\underline{C}) \underset{\mathtt{Ind}_\omega(\mathsf{G})}{\overset{\mathtt{Ind}_\omega(\mathsf{F})}{\Downarrow \mathtt{Ind}_\omega(\alpha)}} \mathtt{Ind}_\omega(\underline{D}) = \underline{C} \underset{\mathsf{G}}{\overset{\mathsf{F}}{\Downarrow \alpha}} \underline{D} \xrightarrow{y_{\underline{D}}} \mathtt{Ind}_\omega(\underline{D})$$

Now, the X-th component of $\mathtt{Ind}_\omega(\alpha)$ is $([\alpha_{X_i}]_\sim)_{i\in\mathsf{I}}\colon (\mathsf{F}X_i)_{i\in\mathsf{I}} \to (\mathsf{G}X_i)_{i\in\mathsf{I}}$, and therefore the c-th component of $\mathtt{Ind}_\omega(\alpha)y_{\underline{C}}$ is $[\alpha_c]_\sim$. On the other hand, the c-th component of $y_{\underline{D}}\alpha$ is $y_{\underline{D}}(\alpha_c)$ which is again $[\alpha_c]_\sim$. Thus, we can conclude this subsection with the following proposition.

Proposition 4.11
$y\colon Id \stackrel{\cdot}{\to} \mathtt{Ind}_\omega(\_)$ is a 2-natural transformation.

Filtered Colimits in $\mathtt{Ind}_\omega(\underline{C})$: the 2-natural multiplication

In this subsection we show that $\mathtt{Ind}_\omega(\underline{C})$ is ω-filtered cocomplete. This is not surprising, since we already know that $\widehat{\underline{C}}_\omega$ is the 'free' cocompletion of $\underline{C}$ by ω-filtered colimits and we have shown that $\mathtt{Ind}_\omega(\underline{C})$ and $\widehat{\underline{C}}_\omega$ are equivalent. However, we shall see that the calculus of colimits in $\mathtt{Ind}_\omega(\underline{C})$ may be expressed 'uniformly' in $\underline{C}$, i.e., that it gives rise to a 2-natural transformation in $\underline{\underline{\underline{CAT}}}$.

Consider a filtered diagram T of ω-ind-objects, i.e., an ω-ind-object in the category $\mathtt{Ind}^2_\omega(\underline{\mathsf{C}}) = \mathtt{Ind}_\omega(\mathtt{Ind}_\omega(\underline{\mathsf{C}}))$. Suppose that $T(i) = (X_{i,j})_{j \in \mathsf{J}_i}$, and define the functor $U\colon \mathsf{K} \to \underline{\mathsf{C}}$ as follows:

- the objects of K are the pairs (i, j) where $i \in \mathsf{I}$ and $j \in \mathsf{J}_i$;
- the arrows of K are pairs $(\alpha, f)\colon (i,j) \to (h,k)$ where $\alpha\colon i \to h$ in I and $f\colon X_{i,j} \to X_{h,k}$ is a representative of the j-th component of $T(\alpha)\colon T(i) \to T(h)$, i.e., $(X_{i,j})_{j\in\mathsf{J}_i} \to (X_{h,k})_{k \in \mathsf{J}_h}$;

the composition in K being obviously given by $(\beta, g) \circ (\alpha, f) = (\beta \circ \alpha, g \circ f)$. Now, U is defined by

$$
\begin{array}{ccc}
\mathsf{K} & \xrightarrow{\quad U \quad} & \underline{\mathsf{C}} \\
(i,j) & \longmapsto & X_{i,j} \\
{\scriptstyle (\alpha,f)}\downarrow & & \downarrow{\scriptstyle f} \\
(h,k) & \longmapsto & X_{h,k}
\end{array}
$$

which, of course, gives a functor. Moreover, we have the following.

Lemma 4.12
K is filtered.

Proof. Easy, exploiting the fact that I and J_i are filtered. Concerning the cardinality of K, it is enough to recall that a countable union of countable sets is countable. ✓

Proposition 4.13
$U\colon \mathsf{K} \to \underline{\mathsf{C}}$ is the colimit of $T\colon \mathsf{I} \to \mathtt{Ind}_\omega(\underline{\mathsf{C}})$ in $\mathtt{Ind}_\omega(\underline{\mathsf{C}})$.

Proof. For any $i \in \mathsf{I}$ we can consider the functor

$$
\begin{array}{ccc}
T(i) & \xrightarrow{\quad u_i \quad} & \mathsf{K} \\
j & \longmapsto & (i,j) \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle (id,f)} \\
j' & \longmapsto & (i,j')
\end{array}
$$

Of course, we have $U \circ u_i = T(i)\colon \mathsf{J}_i \to \underline{\mathsf{C}}$, and therefore u_i induces a morphism $\lambda_i\colon \varinjlim(\mathsf{Y} \circ T(i)) \to \varinjlim(\mathsf{Y} \circ U)$, i.e., an ω-ind-morphism $\lambda_i\colon T(i) \to U$.

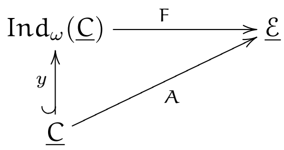
It is easy to see that the λ_i's form a cocone with vertex U. First of all, observe that, by definition, $(\lambda_i)_j$ the j-th component of λ_i is the class of the identity of $X_{i,j}$. Then, $(\lambda_i)_j$ contains any $f\colon X_{i,j} \to X_{h,j'}$ such that $(i,j) \xrightarrow{(\alpha,f)} (h,j')$ is in K. It is now immediate to conclude that for any $\alpha\colon i \to h$ in I we must have $\lambda_h \circ T(\alpha) = \lambda_i$. Consider now another cocone $\{\tau_i\}$, $\tau_i\colon T(i) \to Y$. Explicitly, we have $\tau_i\colon (X_{i,j})_{j\in\mathsf{J}_i} \to (Y_j)_{j\in\mathsf{J}}$. Then, by collecting together these arrows we have $\bar\tau\colon (X_{i,j})_{i\in\mathsf{I}, j \in \mathsf{J}_i} \to (Y_j)_{j \in \mathsf{J}}$, which, thanks to the naturality of the τ's, is easily shown to be an ω-ind-morphism $\bar\tau\colon U \to Y$. Of course we have $\bar\tau \circ \lambda_i = \tau_i$ for any $i \in \mathsf{I}$, and that $\bar\tau$ is the unique ω-ind-morphism $U \to Y$ which enjoys this property. ✓

Thus, we have the following.

Proposition 4.14
$\mathsf{Ind}_\omega(\underline{\mathsf{C}})$ is ω-filtered cocomplete.

Our claim that y plays the role which Y plays in the case of presheaves can now be fully justified by the following proposition which states the pseudo universal property enjoyed by $\mathsf{Ind}_\omega(\underline{\mathsf{C}})$.

Proposition 4.15
Let $\underline{\mathsf{C}}$ be a locally small category. For any ω-filtered cocomplete category $\underline{\mathcal{E}}$ and any functor $\mathsf{A}\colon \underline{\mathsf{C}} \to \underline{\mathcal{E}}$, there is a functor $\mathsf{F}\colon \mathsf{Ind}_\omega(\underline{\mathsf{C}}) \to \underline{\mathcal{E}}$ which preserves the ω-filtered colimits and such that the following diagram commutes.



Moreover, F is unique up to isomorphism.

Proof. Consider F which sends the ω-ind-object $(X_i)_{i\in\mathsf{I}}$ to $\varinjlim_{\mathsf{I}}(A \circ X)$ in $\underline{\mathcal{E}}$ and whose behaviour on the morphisms is induced by the universal property of colimits. Since without loss of generality we may assume that $\mathsf{F}(\underline{c}) = \varinjlim y(\mathsf{A}(c)) = \mathsf{A}(c)$, it follows immediately that the diagram commutes. Moreover, by exploiting the explicit definition of ω-filtered colimits in $\mathsf{Ind}_\omega(\underline{\mathsf{C}})$ given in Proposition 4.13, it is easy to check directly that F preserves them.

Now, suppose that $\mathsf{K}\colon \mathsf{Ind}_\omega(C) \to \underline{\mathcal{E}}$ renders the diagram commutative and preserves ω-filtered colimits. Then, for any $(X_i)_{i\in\mathsf{I}}$ in $\mathsf{Ind}_\omega(\underline{\mathsf{C}})$, we have:

$$\begin{array}{rcl} \mathsf{F}(X) & \cong & \varinjlim \mathsf{A}(X_i) \\ & \cong & \varinjlim \mathsf{K}(y(X_i)) \cong \mathsf{K}(\varinjlim(y(X_i))) \cong \mathsf{K}(X), \end{array}$$

where the last equality follows from $(X_i)_{i\in\mathsf{I}} = \varinjlim_{\mathsf{I}}(y(X_i))$ in $\mathsf{Ind}_\omega(\underline{\mathsf{C}})$, as the reader can check directly. Thus, we have $\mathsf{K} \cong \mathsf{F}$. ✓

Our next step is to remark that the construction given above is functorial. More precisely, since an ω-filtered diagram in $\mathsf{Ind}_\omega(\underline{\mathsf{C}})$ is an object of $\mathsf{Ind}^2_\omega(\underline{\mathsf{C}})$, the colimit construction defines a function $m_{\underline{\mathsf{C}}}$ from the objects of $\mathsf{Ind}^2_\omega(\underline{\mathsf{C}})$ to the objects of $\mathsf{Ind}_\omega(\underline{\mathsf{C}})$. It follows by the very definition of colimits that $m_{\underline{\mathsf{C}}}$ can be extended canonically to a left adjoint functor $\mathsf{Ind}^2_\omega(\underline{\mathsf{C}}) \to \mathsf{Ind}_\omega(\underline{\mathsf{C}})$. Moreover, by Proposition 4.10, the right adjoint to $m_{\underline{\mathsf{C}}}$ is $y_{\mathsf{Ind}_\omega(\underline{\mathsf{C}})}$. In other words, the proof of the functoriality of $m_{\underline{\mathsf{C}}}$ has been implicitly given in Proposition 4.13. Nevertheless, in the following we shall make explicit the definition of $m_{\underline{\mathsf{C}}}$ on the morphisms of $\mathsf{Ind}^2_\omega(\underline{\mathsf{C}})$. Consider $T\colon \mathsf{I} \to \mathsf{Ind}_\omega(\underline{\mathsf{C}})$ and $T'\colon \mathsf{I}' \to \mathsf{Ind}_\omega(\underline{\mathsf{C}})$ in $\mathsf{Ind}^2_\omega(\underline{\mathsf{C}})$. Suppose that $T(i) = (X_{i,j})_{j\in\mathsf{J}_i}$ and $T'(i') = (Y^{i'}_j)_{j\in\mathsf{J}'_{i'}}$. Thus, in the indexed notation, we write $((X_{i,j})_{j\in\mathsf{J}_i})_{i\in\mathsf{I}}$ for T and $((Y_{i,j})_{j\in\mathsf{J}'_i})_{i\in\mathsf{I}'}$ for T'. Consider now a morphism $\alpha\colon T \to T'$ in $\mathsf{Ind}^2_\omega(\underline{\mathsf{C}})$. By definition $\alpha = ([\alpha_i])_{i\in\mathsf{I}}$, is a compatible family of equivalence classes (wrt. I') of ω-ind-morphisms $\alpha_i = ([\alpha_{i,j}])_{j\in\mathsf{J}_i}\colon T(i) \to T'(i')$ (the equivalence being now wrt. $\mathsf{J}'_{i'}$) in $\mathtt{Ind}_\omega(\underline{\mathsf{C}})$, which in the indexed notation can be written as

$$\Big(\Big[([\alpha_{i,j}])_{j\in\mathsf{J}_i}\Big]\Big)_{i\in\mathsf{I}}.$$

Then, as it appears neatly in the proof of Proposition 4.13, the collection $\bar{\alpha} = ([\alpha_{i,j}])_{(i,j)\in\mathsf{K}}$ of all the equivalence classes (wrt. K') of representatives $\alpha_{i,j}$ of the j-th class of some representative α_i of the i-th component of α, is an ω-ind-morphism from $m_{\underline{\mathsf{C}}}(T) = U\colon \mathsf{K} \to \underline{\mathsf{C}}$ to $m_{\underline{\mathsf{C}}}(T') = U'\colon \mathsf{K}' \to \underline{\mathsf{C}}$, where U and U' are the colimits of T and T' determined as earlier in this section. We shall take $m_{\underline{\mathsf{C}}}(\alpha)$ to be $\bar{\alpha}$ and conclude as follows.

Proposition 4.16
$m_{\underline{\mathsf{C}}}\colon \mathtt{Ind}^2_\omega(\underline{\mathsf{C}}) \to \mathtt{Ind}_\omega(\underline{\mathsf{C}})$ is a functor which is left adjoint to $y_{\mathtt{Ind}_\omega(\underline{\mathsf{C}})}$.

Let us now show that the collection of functors $m_{\underline{\mathsf{C}}}\colon \mathtt{Ind}^2_\omega(\underline{\mathsf{C}}) \to \mathtt{Ind}_\omega(\underline{\mathsf{C}})$ is a 2-natural transformation (see [18] for the basic concepts of the theory of 2-categories) $\mathtt{Ind}^2_\omega(\_) \stackrel{\cdot}{\to} \mathtt{Ind}_\omega(\_)$. First, we have to show that

$$\begin{array}{ccc} \mathtt{Ind}^2_\omega(\underline{\mathsf{C}}) & \xrightarrow{m_{\underline{\mathsf{C}}}} & \mathtt{Ind}_\omega(\underline{\mathsf{C}}) \\ {\scriptstyle \mathtt{Ind}^2_\omega(\mathsf{F})}\downarrow & & \downarrow{\scriptstyle \mathtt{Ind}_\omega(\mathsf{F})} \\ \mathtt{Ind}^2_\omega(\underline{\mathsf{C}}) & \xrightarrow[m_{\underline{\mathsf{D}}}]{} & \mathtt{Ind}_\omega(\underline{\mathsf{D}}) \end{array}$$

commutes.

Let $T = ((X_{i,j})_{j\in\mathsf{J}_i})_{i\in\mathsf{I}}$ be in $\mathtt{Ind}^2_\omega(\underline{\mathsf{C}})$. Then, $\mathtt{Ind}^2_\omega(\mathsf{F})(T) = \mathtt{Ind}_\omega(\mathsf{F}) \circ T = \Big(\mathtt{Ind}_\omega(\mathsf{F})\Big((X_{i,j})_{j\in\mathsf{J}_i}\Big)\Big)_{i\in\mathsf{I}}$, which is $\Big(\mathsf{F}\circ(X_{i,j})_{j\in\mathsf{J}_i}\Big)_{i\in\mathsf{I}}$, i.e., $((\mathsf{F}X_{i,j})_{j\in\mathsf{J}_i})_{i\in\mathsf{I}}$. Then, $m_{\underline{\mathsf{D}}}\Big(\mathtt{Ind}^2_\omega(\mathsf{F})(T)\Big) = (\mathsf{F}X_{i,j})_{(i,j)\in\mathsf{K}'}$, where K' is built in $\mathtt{Ind}_\omega(\underline{\mathsf{D}})$ for the diagram $\mathtt{Ind}_\omega(\mathsf{F})\circ T$. On the other hand, $m_{\underline{\mathsf{C}}}\Big(((X_{i,j})_{j\in\mathsf{J}_i})_{i\in\mathsf{I}}\Big) = (X_{i,j})_{(i,j)\in\mathsf{K}}$, and therefore $\mathtt{Ind}_\omega(\mathsf{F})(m_{\underline{\mathsf{C}}}(T)) = \mathsf{F}\circ m_{\underline{\mathsf{C}}}(T) = (\mathsf{F}X_{i,j})_{(i,j)\in\mathsf{K}}$, where K is built in $\mathtt{Ind}_\omega(\underline{\mathsf{C}})$ for the diagram T. Observe that K and K' do not need to be isomorphic. More precisely, the objects in K and K' coincide, being the pairs (i,j) for $i \in \mathsf{I}$ and $j \in \mathsf{J}_i$. However, the morphisms of K' are pairs $(\alpha, f)\colon (i,j) \to (i',j')$ for $\alpha\colon i \to i'$ in I and $f\colon \mathsf{F}X_{i,j} \to \mathsf{F}X_{i',j'}$ in $\underline{\mathsf{D}}$ a representative of the j-th class of $\mathsf{F}T(\alpha)$, while in the morphisms of K the component f is a morphism $f\colon X_{i,j} \to X_{i',j'}$ in $\underline{\mathsf{C}}$, representative of the j-th class of $T(\alpha)$. Of course, these do not need to be the same, since, as observed earlier, $[\mathsf{F}f]_\sim$ is not necessarily the image through F of $[f]_\sim$. However, observe that, if ϕ is the functor defined by

$$\begin{array}{ccc} \mathsf{K} & \xrightarrow{\phi} & \mathsf{K}' \\ (i,j) & \longmapsto & (i,j) \\ {\scriptstyle (\alpha,f)}\downarrow & & \downarrow{\scriptstyle (\alpha,\mathsf{F}f)} \\ (i',j') & \longmapsto & (i',j') \end{array}$$

which is clearly well defined, we have $m_{\underline{D}}\Big(\mathrm{Ind}^2_\omega(\mathsf{F})(T)\Big)\phi = \mathrm{Ind}_\omega(\mathsf{F})(m_{\underline{C}}(T))$. Moreover, exploiting Lemma 3.7, point *(ii)*, it is easy to see that ϕ is cofinal. Then, by Definition 4.1, we conclude that $m_{\underline{D}}\Big(\mathrm{Ind}^2_\omega(\mathsf{F})(T)\Big) = \mathrm{Ind}_\omega(\mathsf{F})(m_{\underline{C}}(T))$, since we identify such objects in $\mathrm{Ind}_\omega(\underline{D})$.

Consider now a morphism α from T to T' in $\mathrm{Ind}^2_\omega(\underline{C})$. Then, it may be shown that $m_{\underline{D}}\Big(\mathrm{Ind}^2_\omega(\mathsf{F})(\alpha)\Big)$ and $\mathrm{Ind}_\omega(\mathsf{F})(m_{\underline{C}}(\alpha))$ coincide.

Remark. We would like to stress that the result above proves the naturality of the identification of ω-ind-objects imposed in Definition 4.1. In fact, it is important to notice that the proof given above does not rely on the fact that $\phi \colon \mathsf{K} \to \mathsf{K}'$ is the identity on the objects, but just on the fact that it is an isomorphism. Thus, the proof above applies to all the ω-ind-objects identified via the cofinal ϕ.

Next, we have to show that m is 2-natural, i.e., that for any $\alpha \colon \mathsf{F} \stackrel{\cdot}{\to} \mathsf{G}$ we have $\mathrm{Ind}_\omega(\alpha) m_{\underline{C}} = m_{\underline{D}} \mathrm{Ind}^2_\omega(\alpha)$. Consider an object $T = ((X_{i,j})_{j\in \mathsf{J}_i})_{i\in \mathsf{I}}$ of $\mathrm{Ind}^2_\omega(\underline{C})$, and let $U \colon \mathsf{K} \to \underline{C}$ be $m_{\underline{C}}(T)$. Then, the component at T of $\mathrm{Ind}_\omega(\alpha) * m_{\underline{C}}$ is $(\mathrm{Ind}_\omega(\alpha) * m_{\underline{C}})_T = \mathrm{Ind}_\omega(\alpha)_U = ([\alpha_{X_{i,j}}])_{(i,j)\in\mathsf{K}} \colon (\mathsf{F}X_{i,j})_{(i,j)\in\mathsf{K}} \to (\mathsf{G}X_{i,j})_{(i,j)\in\mathsf{K}}$.

On the other hand, $\mathrm{Ind}^2_\omega(\alpha)_T = ([\mathrm{Ind}_\omega(\alpha)_{T(i)}])_{i\in\mathsf{I}} = ([([\alpha_{X_{i,j}}])_{j\in\mathsf{J}_i}])_{i\in\mathsf{I}}$, and thus $m_{\underline{D}}(\mathrm{Ind}^2_\omega(\alpha)_T) = ([\alpha_{X_{i,j}}])_{(i,j)\in\mathsf{K}'}$. But this is again the situation we met before and, thus, we can conclude that the two morphisms coincide. Since the same holds for each T in $\mathrm{Ind}^2_\omega(\underline{C})$, it follows that $\mathrm{Ind}_\omega(\alpha) * m_{\underline{C}} = m_{\underline{D}} * \mathrm{Ind}^2_\omega(\alpha)$. Then, we have shown the following.

Proposition 4.17
$m \colon \mathrm{Ind}^2_\omega(\_) \stackrel{\cdot}{\to} \mathrm{Ind}_\omega(\_)$ is a 2-natural transformation.

Remark. It is worth noticing the primary role played in establishing the naturality of m by the fact that our index categories are filtered. There is no obvious way to achieve the same result working with chains or directed sets.

We complete this subsection by stating the following relevant fact.

Proposition 4.18
Let $\underline{C}$ and $\underline{D}$ be locally small categories. Then, for any $\mathsf{F} \colon \underline{C} \to \underline{D}$, the functor $\mathrm{Ind}_\omega(\mathsf{F})$ preserves ω-filtered colimits.

Proof. We have proved that $m_{\underline{C}}$ is 2-natural. In particular, the equation of naturality is $m_{\underline{D}} \circ \mathrm{Ind}^2_\omega(\mathsf{F}) = \mathrm{Ind}_\omega(\mathsf{F}) \circ m_{\underline{C}}$. However, since $m_{\underline{C}}$ is an explicit choice of colimits and the colimits of a given diagram are isomorphic, this can be read as $\mathrm{Ind}_\omega(\mathsf{F}) \circ \varinjlim_{\underline{C}} \cong \varinjlim_{\underline{D}} \circ \mathrm{Ind}^2_\omega(\mathsf{F})$, i.e., $\mathrm{Ind}_\omega(\mathsf{F})$ preserves ω-filtered colimits. ✓

The ω-Ind KZ-Doctrine

In this subsection we sum up the results by showing that the data $\mathrm{Ind}_\omega(\_)$, y, and m determine a KZ-doctrine on $\underline{\underline{\mathrm{CAT}}}$. We shall refer to Appendix A for the needed basic notions of the theory of KZ-doctrines as the need arises.

Recall that, by Proposition 4.16, we have a reflection $m_{\underline{C}} \dashv y_{\mathrm{Ind}_\omega(\underline{C})}$. Let $\eta_{\underline{C}} \colon id_{\mathrm{Ind}^2_\omega(\underline{C})} \stackrel{\cdot}{\to} y_{\mathrm{Ind}_\omega(\underline{C})} \circ m_{\underline{C}}$ be the unit of this adjunction. Then, we take

$\lambda_{\underline{C}}$ to be $\eta_{\underline{C}} * \mathtt{Ind}_\omega(y_{\underline{C}})$. For general reasons, we know that $\eta_{\underline{C}}$ gives the limit cocones. Let us give it explicitly.

Given $T = ((X_{i,j})_{j\in \mathsf{J}_i})_{i\in \mathsf{I}}$, we have $m_{\underline{C}}(T) = (X_{i,j})_{(i,j)\in \mathsf{K}}$ and therefore we have $y_{\mathtt{Ind}_\omega(\underline{C})}(m_{\underline{C}}(T)) = ((X_{i,j})_{(i,j)\in \mathsf{K}})_{\underline{1}}$, where we use the notation $(T)_{\underline{1}}$ for the singleton diagram of value T. Thus, η_T is an I-indexed family of ω-ind-morphisms α_i in $\mathtt{Ind}_\omega(\underline{C})$, where α_i is the 'inclusion' of $T(i)$ in the colimit of T, which is the J_i-indexed family of the equivalence classes (wrt. K) of the identities of $X_{i,j}$. In other words, we have $\eta_T = ([([id_{X_{i,j}}])_{j\in \mathsf{J}_i}])_{i\in \mathsf{I}} \colon ((X_{i,j})_{j\in \mathsf{J}_i})_{i\in \mathsf{I}} \to ((X_{i,j})_{(i,j)\in \mathsf{K}})_{\underline{1}}$. Thus, we can conclude the triangular identities, which in the particular case of a reflection take the form

$$\eta_{\underline{C}} * y_{\mathtt{Ind}_\omega(\underline{C})} = \mathbf{1} \qquad \text{and} \qquad m_{\underline{C}} * \eta_{\underline{C}} = \mathbf{1}.$$

Let us verify the KZ-doctrine axioms in Definition A.1.

$\mathbf{T_0}$: $m_{\underline{C}} \circ y_{\mathtt{Ind}_\omega(\underline{C})} = id$ and $m_{\underline{C}} \circ \mathtt{Ind}_\omega(y_{\underline{C}}) = id$.

$m_{\underline{C}}\Big(y_{\mathtt{Ind}_\omega(\underline{C})}\Big((X_i)_{i\in \mathsf{I}}\Big)\Big) = m_{\underline{C}}\Big(((X_i)_{i\in \mathsf{I}})_{\underline{1}}\Big) = (X_i)_{(*,i)\in \mathsf{K}}$. Once again, K determined from $\underline{1}$ and I is *not* isomorphic to I. In fact, its objects are pairs $(*, i)$ and its morphisms are pairs $(*, i) \stackrel{(id,f)}{\longrightarrow} (*, i')$, where f is a representative of the i-th class wrt. I of the identity on $(X_i)_{i\in \mathsf{I}}$. Thus, although every $f\colon i \to i'$ in I corresponds to (id, f) in K, the converse is not true. However, the embedding $\phi\colon \mathsf{I} \to \mathsf{K}$, which sends i to $(*, i)$ and f to (id, f), is easily shown to be *cofinal*. Thus, the last formula is equal to $(X_i)_{i\in \mathsf{I}}$. The same formal steps prove that $m_{\underline{C}} \circ y\mathtt{Ind}_\omega(\underline{C})$ is the identity also on the morphisms.

$m_{\underline{C}}\Big(\mathtt{Ind}_\omega(y_{\underline{C}})((X_i)_{i\in \mathsf{I}})\Big) = m_{\underline{C}}\Big(((X_i)_{\underline{1}})_{i\in \mathsf{I}}\Big) = (X_i)_{(i,*)\in \mathsf{K}}$. This time the objects of K are pairs $(i, *)$ for $i \in \mathsf{I}$ and the morphisms are pairs (α, k) where $\alpha\colon i \to i'$ is in I and k is a representative of the unique equivalence class wrt. $\underline{1}$ of $X(\alpha)$. However, because of the particular form of $\underline{1}$ there is a unique representative in that class. Therefore, in this case, K is isomorphic to I. Thus, the last formula is equal to $(X_i)_{i\in \mathsf{I}}$. The same argument can be used for the morphisms of $\mathtt{Ind}_\omega(\underline{C})$ to show $m_{\underline{C}} \circ \mathtt{Ind}_\omega(y_{\underline{C}}) = id$, as required.

Observe now that $\mathtt{Ind}_\omega(y_{\underline{C}})\colon \mathtt{Ind}_\omega(\underline{C}) \to \mathtt{Ind}^2_\omega(\underline{C})$, and thus

$$\lambda_{\underline{C}} = \eta_{\underline{C}} * \mathtt{Ind}_\omega(y_{\underline{C}})\colon \mathtt{Ind}_\omega(y_{\underline{C}}) \stackrel{\cdot}{\to} y_{\mathtt{Ind}^2_\omega(\underline{C})} \circ m_{\underline{C}} \circ \mathtt{Ind}_\omega(y_{\mathtt{Ind}_\omega(\underline{C})}) = y_{\mathtt{Ind}^2_\omega(\underline{C})},$$

as required.

Let us proceed to show that the remaining KZ-doctrine axioms hold in our context.

$\mathbf{T_1}$: $\lambda_{\underline{C}} * y_{\underline{C}} = \mathbf{1}$.

The lefthand side of the equation actually is $\eta_{\underline{C}} * \mathtt{Ind}_\omega(y_{\underline{C}}) * y_{\underline{C}}$, which by naturality of y is $\eta_{\underline{C}} * y_{\mathtt{Ind}_\omega(\underline{C})} * y_{\underline{C}}$. But the last two elements of this composition are one of the triangular identities for the adjunction, and therefore the formula above is an identity 2-cell.

$\mathbf{T_2}$: $m_{\underline{C}} * \lambda_{\underline{C}} = \mathbf{1}$.

The lefthand side of the equation is $m_{\underline{C}} * \eta_{\underline{C}} * \mathtt{Ind}_\omega(y_{\underline{C}})$, and using the other triangular identity involved we again can show that it equals $\mathbf{1}$.

$\mathbf{T_3}$: $m_{\underline{C}} * \mathtt{Ind}_\omega(m_{\underline{C}}) * \lambda_{\mathtt{Ind}^2_\omega(\underline{C})} = \mathbf{1}$.
Consider $T = ((X_{i,j})_{j\in \mathsf{J}_i})_{\in \mathsf{I}}$. We have $\mathtt{Ind}_\omega(y_{\mathtt{Ind}_\omega(\underline{C})})(T) = (((X_{i,j})_{j\in\mathsf{J}_i})_{\underline{1}})_{i\in\mathsf{I}}$. Thus, the unit $\eta_{\mathtt{Ind}_\omega(\underline{C})}$ at this object is the I-indexed family α of equivalence classes $[\alpha_i]$ whose representatives are ω-ind-morphisms

$$\alpha_i\colon ((X_{i,j})_{j\in\mathsf{J}_i})_{\underline{1}} \to ((X_{i,j})_{j\in\mathsf{J}_i})_{(i,*)\in\mathsf{K}_0},$$

where K_0 is built by $m_{\mathtt{Ind}_\omega(\underline{C})}$ for I and $\underline{1}$. Each α_i has a unique component which, by definition, is the equivalence class (wrt. K) of the identity ω-ind-morphism of $(X_{i,j})_{j\in\mathsf{J}_i}$. Then $\mathtt{Ind}_\omega(m_{\underline{C}})(\alpha)$ is the I-indexed family β of equivalence classes $[\beta_i]$ whose representatives are ω-ind-morphisms

$$\beta_i\colon (X_{i,j})_{(*,j)\in\mathsf{H}_i} \to ((X_{i,j})_{j\in\mathsf{J}_i})_{(i,*,j)\in\mathsf{L}_i},$$

where H_i is built by $m_{\underline{C}}$ for $\underline{1}$, and J_i and L_i corresponds to K_0 and J_i. Observe that each component of β_i is the equivalence class (wrt. L_i) of the identity of $X_{i,j}$ in $\underline{C}$. Finally, we must compute $m_{\underline{C}}(\beta)$. We have the index categories K built from I and the H_i's and K' built from $\underline{1}$ and the L_i's. Thus, $m_{\underline{C}}(\beta)$ is a $(i,*,j) \in \mathsf{K}$-indexed family γ whose components are equivalence classes (wrt. K') of the identity arrow of $X_{i,j}$. Now let K'' be the index category for $m_{\underline{C}}(T)$. Of course, K, K' and K'' all have isomorphic sets of objects and, as usual, it is not difficult to show that there exist cofinal functors $\phi\colon \mathsf{K}' \to \mathsf{K}$ and $\phi'\colon \mathsf{K}'' \to \mathsf{K}$. It follows that the component at T of $m_{\underline{C}} * \mathtt{Ind}_\omega(m_{\underline{C}}) * \eta_{Ind\underline{C}} * \mathtt{Ind}_\omega(y_{\mathtt{Ind}^2_\omega(\underline{C})}) = m_{\underline{C}} * \mathtt{Ind}_\omega(m_{\underline{C}}) * \lambda_{\mathtt{Ind}^2_\omega(\underline{C})}$ is the identity of $m_{\underline{C}}(T)$, i.e., $m_{\underline{C}} * \mathtt{Ind}_\omega(m_{\underline{C}}) * \lambda_{\mathtt{Ind}^2_\omega(\underline{C})} = \mathbf{1}$, as required.

Thus, we have proved the following.

Proposition 4.19
$\left(\mathtt{Ind}_\omega(\_), y, m, \{\lambda_{\underline{C}}\}_{\underline{C}\in\underline{\underline{\underline{CAT}}}}\right)$ *is a KZ-doctrine on* $\underline{\underline{\underline{CAT}}}$.

Moreover, by Proposition 4.5, we also have the following result concerning $\underline{\underline{\mathtt{Cat}}}$, the category of the small categories.

Proposition 4.20
$\left(\mathtt{Ind}_\omega(\_), y, m, \{\lambda_{\underline{C}}\}_{\underline{C}\in\underline{\underline{\mathtt{Cat}}}}\right)$ *is a KZ-doctrine on* $\underline{\underline{\mathtt{Cat}}}$.

We turn our attention to the category $\underline{\underline{\mathtt{Ind}_\omega\text{-}\mathtt{Alg}}}$ of $\mathtt{Ind}_\omega(\_)$-algebras. By Definition A.3, an algebra is a category $\underline{A}$ together with a functor $\mathfrak{a}\colon \mathtt{Ind}_\omega(\underline{A}) \to \underline{A}$ which is a reflection left adjoint for $y_{\underline{A}}\colon \underline{A} \to \mathtt{Ind}_\omega(\underline{A})$. By Proposition 4.10 we conclude immediately that the algebras are exactly the locally small ω-filtered cocomplete categories with a *choice* $\mathfrak{a}$ of colimits. Recall that, by the general theory of KZ-doctrines, the same category $\underline{A}$ gives rise to different algebras only via isomorphic $\mathfrak{a}$'s, i.e., via different choices of colimits in $\underline{A}$.

Let us consider the $\mathtt{Ind}_\omega(\_)$-homomorphisms. From the theory in Section A, we know that $\mathsf{F}\colon \underline{A} \to \underline{B}$ is a morphism of the algebras $(\underline{A}, \mathfrak{a})$ and $(\underline{B}, \mathfrak{b})$ if and only if the 2-cell $\phi = b * \mathtt{Ind}_\omega(\mathsf{F}) * \mathtt{Ind}_\omega(\mathfrak{a}) * \lambda_{\underline{A}}$ is invertible.

$$\begin{array}{ccc} \mathtt{Ind}_\omega(\underline{A}) & \xrightarrow{\mathtt{Ind}_\omega(\mathsf{F})} & \mathtt{Ind}_\omega(\underline{B}) \\ {\scriptstyle \mathfrak{a}}\downarrow & \overset{\phi}{\swarrow} & \downarrow{\scriptstyle \mathfrak{b}} \\ \underline{A} & \xrightarrow[\mathsf{F}]{} & \underline{B} \end{array}$$

Consider $A = (A_i)_{i \in \mathsf{I}}$ in $\mathtt{Ind}_\omega(\underline{A})$. Then, $\mathfrak{a}(A)$ is (a choice of) the colimit $\varinjlim_{\underline{A}} A$, and $\mathsf{F}(\mathfrak{a}(A))$ is $\mathsf{F}(\varinjlim_{\underline{A}} A)$. On the other hand, $\mathtt{Ind}_\omega(\mathsf{F})(A) = (\mathsf{F}A_i)_{i \in \mathsf{I}}$ is the translation through F of the diagram A in $\underline{B}$ and $\mathfrak{b}(\mathtt{Ind}_\omega(\mathsf{F})(A))$ is (a choice for) its colimit $\varinjlim_{\underline{B}} \mathsf{F}A$. In our context we have

$$\phi = b * \mathtt{Ind}_\omega(\mathsf{F}) * \mathtt{Ind}_\omega(\mathfrak{a}) * \eta_{\underline{A}} * \mathtt{Ind}_\omega(y_{\underline{A}}).$$

Moreover, the unit of the reflection $\mathfrak{a} \dashv y_{\underline{A}}$ is given by $\mathtt{Ind}_\omega(\mathfrak{a}) * \eta_{\underline{A}} * \mathtt{Ind}_\omega(y_{\underline{A}})$. Notice that the i-th component of $\eta = (\eta_{\underline{A}} * \mathtt{Ind}_\omega(y_{\underline{A}}))_A \colon ((A_i)_{\underline{1}})_{i \in \mathsf{I}} \to ((A_i)_{i \in \mathsf{I}})_{\underline{1}}$ is the class of $\eta_i \colon (A_i)_{\underline{1}} \to (A_i)_{i \in \mathsf{I}}$ whose unique component is the class of the identity of A_i. Then, $\mathtt{Ind}_\omega(\mathfrak{a})(\eta) = ([\mathfrak{a}(\eta_i)])_{i \in \mathsf{I}} \colon (A_i)_{i \in \mathsf{I}} \to (\mathfrak{a}((A_i)_{i \in \mathsf{I}}))_{\underline{1}}$ is the limit cocone for A. Therefore, by applying $\mathtt{Ind}_\omega(\mathsf{F})$ to $\mathtt{Ind}_\omega(\mathfrak{a})(\eta)$, we get $([\mathsf{F}\mathfrak{a}(\eta_i)])_{i \in \mathsf{I}} \colon (\mathsf{F}A_i)_{i \in \mathsf{I}} \to (\mathsf{F}\mathfrak{a}((A_i)_{i \in \mathsf{I}}))_{\underline{1}}$, which is the translation in $\underline{B}$ of the cocone, and finally, by applying $\mathfrak{b}$, we get $\mathfrak{b}(([\mathsf{F}\mathfrak{a}(\eta_i)])_{i \in \mathsf{I}}) \colon \mathfrak{b}((\mathsf{F}A_i)_{i \in \mathsf{I}}) \to (\mathsf{F}\mathfrak{a}((A_i)_{i \in \mathsf{I}}))$ which is the canonical comparison map $\varinjlim_{\underline{B}}(\mathsf{F}A) \to \mathsf{F}(\varinjlim_{\underline{A}} A)$. Then, we have that ϕ is invertible if and only if (by definition) F preserves ω-filtered colimits (up to isomorphism). Therefore, we can conclude with the following proposition.

Proposition 4.21
The 2-category of $\mathtt{Ind}_\omega(\_)$-algebras on $\underline{\underline{\mathsf{CAT}}}$ ($\underline{\underline{\mathsf{Cat}}}$) is the 2-category of the ω-filtered cocomplete locally small (small) categories with a choice of colimits and of the functors which preserve them up to isomorphism.
In equivalent terms, $\underline{\underline{\mathsf{Ind}_\omega}}$-$\underline{\underline{\mathsf{Alg}}}$ on $\underline{\underline{\mathsf{CAT}}}$ ($\underline{\underline{\mathsf{Cat}}}$) is the category $\underline{\underline{\omega}}$-$\underline{\underline{\mathsf{CAT}}}$ ($\underline{\underline{\omega}}$-$\underline{\underline{\mathsf{Cat}}}$) of the ω-chain cocomplete locally small (small) categories with a choice of colimits and ω-cocontinuous functors. It follows from Proposition A.13 that $\mathtt{Ind}_\omega(\_)$ determines a KZ-adjunction (see Definition A.11) from $\underline{\underline{\mathsf{CAT}}}$ ($\underline{\underline{\mathsf{Cat}}}$) to $\underline{\underline{\omega}}$-$\underline{\underline{\mathsf{CAT}}}$ ($\underline{\underline{\omega}}$-$\underline{\underline{\mathsf{Cat}}}$).

5 ω-Ind Completion of Monoidal Categories

In this section we show that the ω-filtered cocompletion of a monoidal category is a monoidal category in a canonical way. Moreover, the KZ-doctrine $(\mathtt{Ind}_\omega(\_), y, m, \lambda)$ lifts to KZ-doctrines on any of the 2-categories in Table 1, giving in this way their 'free' cocompletion.

We shall illustrate two equivalent approaches, the second being actually a variation of the first. As usual, we state definitions and results preferably for $\mathtt{Ind}_\omega(\underline{C})$, although everything which follows can be rephrased for $\mathtt{Ind}_\omega(\underline{C})$.

We recall that a *monoidal category* [7, 25] is a structure $(\underline{\mathsf{C}}, \otimes, e, \alpha, \lambda, \rho)$, where $\underline{\mathsf{C}}$ is a category, $\otimes: \underline{\mathsf{C}} \times \underline{\mathsf{C}} \to \underline{\mathsf{C}}$ is a functor, $\alpha: {\_}_1 \otimes ({\_}_2 \otimes {\_}_3) \xrightarrow{\sim} ({\_}_1 \otimes {\_}_2) \otimes {\_}_3$ is 'the *associativity*' natural isomorphism,[2] $\lambda: e \otimes {\_}_1 \xrightarrow{\sim} {\_}_1$ is 'the *left unit*' natural isomorphism and $\rho: {\_}_1 \otimes e \xrightarrow{\sim} {\_}_1$ is 'the *right unit*' natural isomorphism, e is an object in $\underline{\mathsf{C}}$, subject to the following Kelly-MacLane *coherence axioms* [23, 17]:

$$\begin{aligned} (\alpha_{x,y,z} \otimes id_k) \circ \alpha_{x,y\otimes z,k} \circ (id_x \otimes \alpha_{y,z,k}) &= \alpha_{x\otimes y,z,k} \circ \alpha_{x,y,z\otimes k}; \\ id_x \otimes \lambda_y \circ \alpha_{x,e,y} &= \rho_x \otimes id_y. \end{aligned} \tag{1}$$

A monoidal category is *strict* if α, λ and ρ are the identity natural transformation, i.e., if $\otimes$ is strictly monoidal. It is *symmetric* if it is given a *symmetry* natural isomorphism $\gamma: {\_}_1 \otimes {\_}_2 \xrightarrow{\sim} {\_}_2 \otimes {\_}_1$ satisfying the following axioms.

$$\begin{aligned} (\gamma_{x,z} \otimes id_y) \circ \alpha_{x,z,y} \circ (id_x \otimes \gamma_{y,z}) &= \alpha_{z,x,y} \circ \gamma_{x\otimes y,z} \circ \alpha_{x,y,z}; \\ \gamma_{y,x} \circ \gamma_{x,y} &= id_{x\otimes y}; \\ \rho_x \circ \gamma_{e,x} &= \lambda_x. \end{aligned} \tag{2}$$

When γ is the identity, $\underline{\mathsf{C}}$ is said *strictly symmetric.*

Given $\underline{\mathsf{C}} = (\underline{\mathsf{C}}, \otimes, e, \alpha, \lambda, \rho, \gamma)$ and $(\underline{\mathsf{D}}, \otimes', e', \alpha', \lambda', \rho', \gamma')$, a *monoidal functor* from $\underline{\mathsf{C}}$ to $\underline{\mathsf{D}}$ is a triple $(\mathsf{F}, \varphi^0, \varphi)$, where $\mathsf{F}: \underline{\mathsf{C}} \to \underline{\mathsf{D}}$ is a functor, $\varphi^0: e' \to \mathsf{F}(e)$ is an arrow in $\underline{\mathsf{D}}$, and $\varphi: \mathsf{F}({\_}_1) \otimes' \mathsf{F}({\_}_2) \xrightarrow{\cdot} \mathsf{F}({\_}_1 \otimes {\_}_2)$ is a natural transformation, required to satisfy

$$\begin{aligned} \mathsf{F}\alpha_{x,y,z} \circ \varphi_{x,y\otimes z} \circ (id_{\mathsf{F}x} \otimes' \varphi_{y,z}) &= \varphi_{x\otimes y,z} \circ (\varphi_{x\otimes y} \otimes' id_{Fz}) \circ \alpha'_{\mathsf{F}x,\mathsf{F}y,\mathsf{F}z}; \\ \mathsf{F}\lambda_x \circ \varphi_{e,x} \circ (\varphi^0 \otimes' id_{\mathsf{F}x}) &= \lambda'_{\mathsf{F}x} \\ \mathsf{F}\rho_x \circ \varphi_{x,e} \circ (id_{\mathsf{F}x} \otimes' \varphi^0) &= \rho'_{\mathsf{F}x}. \end{aligned} \tag{3}$$

Moreover, $(\mathsf{F}, \varphi^0, \varphi)$ is *symmetric* if

$$\mathsf{F}\gamma_{x,y} \circ \varphi_{x,y} = \varphi_{y,x} \circ \gamma'_{\mathsf{F}x,\mathsf{F}y}. \tag{4}$$

If φ^0 and φ are isomorphisms, then $(\mathsf{F}, \varphi^0, \varphi)$ is a *strong* monoidal functor, if they are the identity, then F is a *strict* monoidal functor. The combination of these data give the one-dimensional versions of the categories in Table 1.

A *monoidal transformation* between the functors $(\mathsf{F}, \varphi^0, \varphi)$ and $(\mathsf{F}', \varphi'^0, \varphi')$ is a natural transformation $\sigma: \mathsf{F} \xrightarrow{\cdot} \mathsf{F}'$ such that

$$\begin{aligned} \sigma_{x\otimes y} \circ \varphi_{x,y} &= \varphi'_{x,y} \circ (\sigma_x \otimes' \sigma_y) \\ \sigma_e \circ \varphi^0 &= \varphi'^0 \end{aligned} \tag{5}$$

By combining in a sensible way the data above, we get the 2-categories listed in Table 1 in page 3.

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[2]We use the symbols ${\_}_n$ for $n \in \omega$ as placeholders.

Cocompletion of Monoidal Categories: First Solution

The first issue is to extend $\otimes$ to a functor $\hat{\otimes}\colon \mathtt{Ind}_\omega(\underline{\mathsf{C}}) \times \mathtt{Ind}_\omega(\underline{\mathsf{C}}) \to \mathtt{Ind}_\omega(\underline{\mathsf{C}})$. Observe that by composing $\otimes$ with $y_{\underline{\mathsf{C}}}$ we get a functor $y_{\underline{\mathsf{C}}} \circ \otimes \colon \underline{\mathsf{C}} \times \underline{\mathsf{C}} \to \mathtt{Ind}_\omega(\underline{\mathsf{C}})$. Therefore, by the universality of $\mathtt{Ind}_\omega(\_)$, we get a functor

$$\otimes'\colon \mathtt{Ind}_\omega(\underline{\mathsf{C}} \times \underline{\mathsf{C}}) \to \mathtt{Ind}_\omega(\underline{\mathsf{C}})$$

which is the unique-up-to-isomorphism free extension of $\otimes$ to the ω-ind-objects. It is easy to realize that a possible choice for $\otimes'$ is exactly $\mathtt{Ind}_\omega(\otimes)$.

$$\begin{array}{ccc}
\mathtt{Ind}_\omega(\underline{\mathsf{C}} \times \underline{\mathsf{C}}) & \xrightarrow{\mathtt{Ind}_\omega(\otimes)} & \mathtt{Ind}_\omega(\underline{\mathsf{C}}) \\
{\scriptstyle y_{\underline{\mathsf{C}} \times \underline{\mathsf{C}}}} \uparrow & & \uparrow {\scriptstyle y_{\underline{\mathsf{C}}}} \\
\underline{\mathsf{C}} \times \underline{\mathsf{C}} & \xrightarrow[\otimes]{} & \underline{\mathsf{C}}
\end{array}$$

Thus, we need a canonical way of relating $\mathtt{Ind}_\omega(\underline{\mathsf{C}} \times \underline{\mathsf{C}})$ and $\mathtt{Ind}_\omega(\underline{\mathsf{C}}) \times \mathtt{Ind}_\omega(\underline{\mathsf{C}})$. We observe that $\mathtt{Ind}_\omega(\underline{\mathsf{C}} \times \underline{\mathsf{D}}) \stackrel{\bullet}{\cong} \mathtt{Ind}_\omega(\underline{\mathsf{C}}) \times \mathtt{Ind}_\omega(\underline{\mathsf{D}})$, although they are not at all isomorphic. Consider the mapping ∇ defined below

$$\begin{array}{ccc}
\mathtt{Ind}_\omega(\underline{\mathsf{C}} \times \underline{\mathsf{D}}) & \xrightarrow{\quad\nabla\quad} & \mathtt{Ind}_\omega(\underline{\mathsf{C}}) \times \mathtt{Ind}_\omega(\underline{\mathsf{D}}) \\
\mathsf{I} \xrightarrow{X} \underline{\mathsf{C}} \times \underline{\mathsf{D}} & \longmapsto & (\mathsf{I} \xrightarrow{\pi_0 X} \underline{\mathsf{C}}, \mathsf{I} \xrightarrow{\pi_1 X} \underline{\mathsf{D}}) \\
{\scriptstyle ([f_i])_{i \in \mathsf{I}}} \downarrow & & {\scriptstyle ([fst(f_i)])_{i \in \mathsf{I}}} \downarrow \qquad \downarrow {\scriptstyle ([snd(f_i)])_{i \in \mathsf{I}}} \\
\mathsf{J} \xrightarrow{Y} \underline{\mathsf{C}} \times \underline{\mathsf{D}} & \longmapsto & (\mathsf{J} \xrightarrow{\pi_0 Y} \underline{\mathsf{C}}, \mathsf{J} \xrightarrow{\pi_1 Y} \underline{\mathsf{D}})
\end{array}$$

where $fst\langle f, g\rangle = f$, $snd\langle f, g\rangle = g$ and π_i are the projections associated to the cartesian product.

Given $X\colon \mathsf{I} \to \underline{\mathsf{C}} \times \underline{\mathsf{D}}$, suppose $X(i) = (c_i, d_i)$. Then, the identity of X is $([(id_{c_i}, id_{d_i})])_{i \in \mathsf{I}}$, and therefore $\nabla(id_X)$ is the pair $(([id_{c_i}])_{i \in \mathsf{I}}, ([id_{d_i}])_{i \in \mathsf{I}})$ which is $(id_{\pi_0 X}, id_{\pi_1 X})$. Moreover, since $fst(g \circ f) = fst(g) \circ fst(f)$ and $snd(g \circ f) = snd(g) \circ snd(f)$, it is immediate to show that the definition above respects compositions. Thus, ∇ is a functor.

For a quasi-inverse of ∇, we consider the following Δ.

$$\begin{array}{ccc}
\mathtt{Ind}_\omega(\underline{\mathsf{C}}) \times \mathtt{Ind}_\omega(\underline{\mathsf{D}}) & \xrightarrow{\quad\Delta\quad} & \mathtt{Ind}_\omega(\underline{\mathsf{C}} \times \underline{\mathsf{D}}) \\
(\mathsf{I} \xrightarrow{X} \underline{\mathsf{C}}, \mathsf{J} \xrightarrow{Y} \underline{\mathsf{D}}) & \longmapsto & \mathsf{I} \times \mathsf{J} \xrightarrow{X \times Y} \underline{\mathsf{C}} \times \underline{\mathsf{D}} \\
{\scriptstyle ([f_i])_{i \in \mathsf{I}}} \downarrow \qquad \downarrow {\scriptstyle ([g_j])_{j \in \mathsf{J}}} & & \downarrow {\scriptstyle ([f_i \times g_j])_{i \in \mathsf{I}, j \in \mathsf{J}}} \\
(\mathsf{I}' \xrightarrow{X'} \underline{\mathsf{C}}, \mathsf{J}' \xrightarrow{Y'} \underline{\mathsf{D}}) & \longmapsto & \mathsf{I}' \times \mathsf{J}' \xrightarrow{X' \times Y'} \underline{\mathsf{C}} \times \underline{\mathsf{D}}
\end{array}$$

Also in this case it is immediate to see that Δ is a functor. In fact, the image of the identity of the pair X and Y is the $\mathsf{I} \times \mathsf{J}$-indexed family whose component (i, j)

is the class of the $id_{X_i} \times id_{Y_j}$ which is $id_{X_i \times Y_j}$, the identity of $(X \times Y)_{(i,j)}$. Thus, Δ respects the identities. Moreover, since $(f \circ f') \times (g \circ g') = (f \times g) \circ (f' \times g')$, it follows that Δ is a functor.

Now, given $X = ((c_i, d_i))_{i \in \mathsf{I}}$, we have $\Delta\nabla(X) = \pi_0 X \times \pi_1 X \colon \mathsf{I} \times \mathsf{I} \to \underline{\mathsf{C}} \times \underline{\mathsf{D}}$. Observe that $\phi_X \colon \mathsf{I} \to \mathsf{I} \times \mathsf{I}$ which sends i to (i, i) is clearly cofinal. Moreover, the following diagram commutes.

$$\begin{array}{ccc} \mathsf{I} & \xrightarrow{\phi_X} & \mathsf{I} \times \mathsf{I} \\ & \searrow^{X} & \downarrow{\scriptstyle \pi_0 X \times \pi_1 X} \\ & & \underline{\mathsf{C}} \times \underline{\mathsf{D}} \end{array}$$

It follows that X and $\pi_0 X \times \pi_1 X$ are isomorphic in $\mathtt{Ind}_\omega(\underline{\mathsf{C}} \times \underline{\mathsf{D}})$ via the canonical morphism $\bar{\phi}_X$ induced from ϕ_X by colimit, i.e., via the injection $\mathsf{L}(X) = \varinjlim \mathsf{Y}X$ of $\mathtt{Ind}_\omega(\underline{\mathsf{C}} \times \underline{\mathsf{D}})$ in the category of presheaves over $\underline{\mathsf{C}} \times \underline{\mathsf{D}}$. Since ϕ_X enjoys a universal property, it is clear that the family $\{\bar{\phi}_X\}_{X \in \mathtt{Ind}_\omega(\underline{\mathsf{C}} \times \underline{\mathsf{D}})}$ gives a natural transformation $Id \stackrel{\cdot}{\to} \Delta\nabla$.

On the other hand, given the pair $((X_i)_{i \in \mathsf{I}}, (Y_j)_{j \in \mathsf{J}})$ in $\mathtt{Ind}_\omega(\underline{\mathsf{C}}) \times \mathtt{Ind}_\omega(\underline{\mathsf{D}})$, we have $\nabla\Delta((X, Y)) = (\pi_0(X \times Y), \pi_1(X \times Y))$, where $\pi_0(X \times Y) \colon \mathsf{I} \times \mathsf{J} \to \underline{\mathsf{C}}$ and $\pi_1(X \times Y) \colon \mathsf{I} \times \mathsf{J} \to \underline{\mathsf{D}}$. Of course, $\mathsf{I} \times \mathsf{J}$ is cofinal both in I and in J, via the functors

$$\begin{array}{ccc} \mathsf{I} \times \mathsf{J} & \xrightarrow{\psi^0_{(X,Y)}} & \mathsf{I} \\ (i,j) & \longmapsto & i \\ {\scriptstyle (f,g)}\downarrow & & \downarrow{\scriptstyle f} \\ (i',j') & \longmapsto & i' \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathsf{I} \times \mathsf{J} & \xrightarrow{\psi^1_{(X,Y)}} & \mathsf{J} \\ (i,j) & \longmapsto & j \\ {\scriptstyle (f,g)}\downarrow & & \downarrow{\scriptstyle g} \\ (i',j') & \longmapsto & j' \end{array}$$

Moreover, the following diagrams commute.

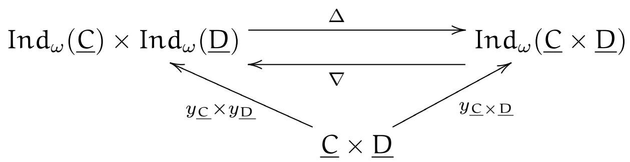
$$\begin{array}{ccc} \mathsf{I} \times \mathsf{J} & \xrightarrow{\psi^0_{(X,Y)}} & \mathsf{I} \\ & \searrow^{\pi_0(X \times Y)} & \downarrow{\scriptstyle X} \\ & & \underline{\mathsf{C}} \end{array} \qquad \begin{array}{ccc} \mathsf{I} \times \mathsf{J} & \xrightarrow{\psi 1_{(X,Y)}} & \mathsf{J} \\ & \searrow^{\pi_1(X \times Y)} & \downarrow{\scriptstyle Y} \\ & & \underline{\mathsf{D}} \end{array}$$

and so there exist two invertible ω-ind-morphisms $\bar{\psi}^0_{(X,Y)} \colon \pi_0(X \times Y) \to X$ and $\bar{\psi}^1_{(X,Y)} \colon \pi_1(X \times Y) \to Y$ induced by the universal property of colimits from $\psi^0_{(X,Y)}$ and $\psi^1_{(X,Y)}$. For general reasons, it follows that we have a natural transformation $\bar{\psi} \colon \nabla\Delta \stackrel{\cdot}{\to} Id$, where $\bar{\psi}_{(X,Y)} = (\bar{\psi}^0_{(X,Y)}, \bar{\psi}^1_{(X,Y)})$. In other words we have the following.

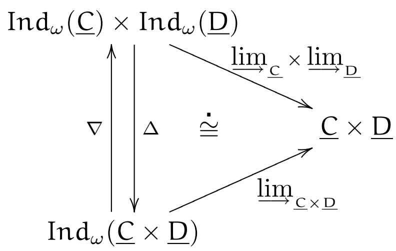
Proposition 5.1

$Id \cong \Delta\nabla$ via $\bar{\phi}$ and $\nabla\Delta \cong Id$ via $\bar{\psi}$. Therefore, $\mathtt{Ind}_\omega(\underline{\mathsf{C}} \times \underline{\mathsf{D}}) \stackrel{\cdot}{\cong} \mathtt{Ind}_\omega(\underline{\mathsf{C}}) \times \mathtt{Ind}_\omega(\underline{\mathsf{D}})$.

It is clear from the definition that Δ and ∇ are such that

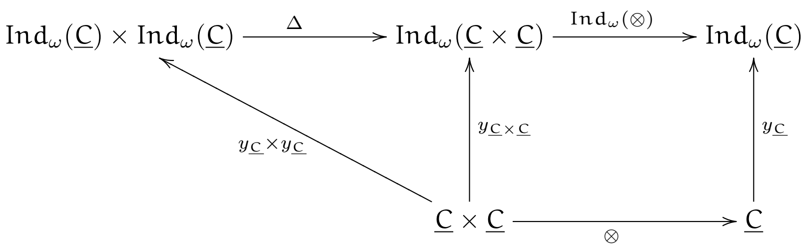


and, if $\underline{\mathsf{C}}$ and $\underline{\mathsf{D}}$ are ω-filtered cocomplete, then



i.e., $\varinjlim_{\underline{\mathsf{C}}\times\underline{\mathsf{D}}}(\mathsf{F}) = (\varinjlim_{\underline{\mathsf{C}}} \pi_0\mathsf{F}, \varinjlim_{\underline{\mathsf{D}}} \pi_1\mathsf{F})$ and $(\varinjlim_{\underline{\mathsf{C}}} \mathsf{F}, \varinjlim_{\underline{\mathsf{D}}} \mathsf{G}) = \varinjlim_{\underline{\mathsf{C}}\times\underline{\mathsf{D}}} \mathsf{F}\times\mathsf{G}$.

So we are allowed to define



Observe that this diagram commutes, which means that the tensors of $\underline{\mathsf{C}}$ and $\mathtt{Ind}_\omega(\underline{\mathsf{C}})$ coincide on the (essentially) constant ω-ind-objects. We shall see that actually the entire monoidal structure of $\underline{\mathsf{C}}$, and not merely the tensor, is preserved in $\mathtt{Ind}_\omega(C)$. In the following we shall denote $\mathtt{Ind}_\omega(\otimes)\circ\Delta$ by $\hat{\otimes}$. In terms of indexed representation of ω-ind-object we can then write

$$\begin{array}{ccccc} (X_i)_{i\in\mathsf{I}} & & (Y_j)_{j\in\mathsf{J}} & & (X_i\otimes Y_j)_{(i,j)\in\mathsf{I}\times\mathsf{J}} \\ {\scriptstyle ([f_i])_{i\in\mathsf{I}}}\downarrow & \hat{\otimes} & \downarrow{\scriptstyle ([g_j])_{j\in\mathsf{J}}} & = & \downarrow{\scriptstyle ([f_i\otimes g_j])_{(i,j)\in\mathsf{I}\times\mathsf{J}}} \\ (X'_i)_{i\in\mathsf{I}'} & & (Y'_j)_{j\in\mathsf{J}'} & & (X'_i\otimes Y'_j)_{(i,j)\in\mathsf{I}'\times\mathsf{J}'} \end{array}$$

To make explicit the remaining monoidal structure we have to identify the unit for $\hat{\otimes}$, to lift the coherence natural isomorphisms α, λ, ρ and γ to $\mathtt{Ind}_\omega(\underline{\mathsf{C}})$, and to prove that the axioms are satisfied. This task is fairly easy now. Concerning the unit, of course we take $\hat{e} = y_{\underline{\mathsf{C}}}(e) = \underline{e}$.

$\hat{\alpha}\colon {}_{-1} \mathbin{\hat{\otimes}} ({}_{-2} \mathbin{\hat{\otimes}} {}_{-3}) \xrightarrow{\sim} ({}_{-1} \mathbin{\hat{\otimes}} {}_{-2}) \mathbin{\hat{\otimes}} {}_{-3}$. For $X = (X_i)_{i \in \mathsf{I}}$, $Y = (Y_j)_{j \in \mathsf{J}}$, $Z = (Z_k)_{k \in \mathsf{K}}$ in $\mathtt{Ind}_\omega(\underline{\mathsf{C}})$, let H be $\mathsf{I} \times \mathsf{J} \times \mathsf{K}$ and define $\hat{\alpha}_{X,Y,Z}$ as follows

$$([\alpha_{X_i,Y_j,Z_k}])_{(i,j,k) \in \mathsf{H}}\colon (X_i \otimes (Y_j \otimes Z_k))_{(i,j,k) \in \mathsf{H}} \to ((X_i \otimes Y_j) \otimes Z_k)_{(i,j,k) \in \mathsf{H}}.$$

Observe that $\hat{\alpha}$ can be described as $\mathtt{Ind}_\omega(\alpha) * \Delta * (Id_{\mathtt{Ind}_\omega(\underline{\mathsf{C}})} \times \Delta)$. It follows that $\hat{\alpha}$, since it is the image of a natural isomorphism through a 2-functor, is a natural isomorphism.
$\hat{\lambda}\colon \hat{e} \mathbin{\hat{\otimes}} {}_{-1} \xrightarrow{\sim} {}_{-1}$. For $X = (X_i)_{i \in \mathsf{I}}$ in $\mathtt{Ind}_\omega(\underline{\mathsf{C}})$, the component at X of $\hat{\lambda}$ is

$$\hat{\lambda}_X = ([\lambda_{X_i}])_{i \in \mathsf{I}}\colon (e \otimes X_i)_{i \in \mathsf{I}} \to (X_i)_{i \in \mathsf{I}}.$$

This time $\hat{\lambda}$ can be written as $\mathtt{Ind}_\omega(\lambda) * \Delta(\hat{e}, \_)$, which implies that it is a natural isomorphism.
$\hat{\rho}\colon {}_{-1} \mathbin{\hat{\otimes}} \hat{e} \xrightarrow{\sim} {}_{-1}$. Given $X = (X_i)_{i \in \mathsf{I}}$ in $\mathtt{Ind}_\omega(\underline{\mathsf{C}})$, we define

$$\hat{\rho}_X = ([\rho_{X_i}])_{i \in \mathsf{I}}\colon (X_i \otimes e)_{i \in \mathsf{I}} \to (X_i)_{i \in \mathsf{I}}.$$

Observe that $\hat{\rho}$ is $\mathtt{Ind}_\omega(\rho) * \Delta(\_, \hat{e})$, and thus a natural isomorphism.
$\hat{\gamma}\colon {}_{-1} \mathbin{\hat{\otimes}} {}_{-2} \xrightarrow{\sim} {}_{-2} \mathbin{\hat{\otimes}} {}_{-1}$. For $X = (X_i)_{i \in \mathsf{I}}$ and $Y = (X_j)_{j \in \mathsf{J}}$, we define

$$\hat{\gamma}_{X,Y} = ([\gamma_{X_i,Y_j}])_{(i,j) \in \mathsf{I} \times \mathsf{J}}\colon (X_i \otimes Y_j)_{(i,j) \in \mathsf{I} \times \mathsf{J}} \to (Y_j \otimes X_i)_{(j,i) \in \mathsf{J} \times \mathsf{I}},$$

which again is $\mathtt{Ind}_\omega(\gamma) * \Delta$, and thus a natural isomorphism.

Now it is really simple to check that these definitions enjoy the coherence axioms (1) and (2). Thus, we have the following.

Proposition 5.2
For any monoidal category $(\underline{\mathsf{C}}, \otimes, e, \alpha, \lambda, \rho)$ the filtered cocomplete category $(\mathtt{Ind}_\omega(\underline{\mathsf{C}}), \hat{\otimes}, \hat{e}, \hat{\alpha}, \hat{\lambda}, \hat{\rho})$ is a monoidal category. Moreover, if $(\underline{\mathsf{C}}, \otimes, e, \alpha, \lambda, \rho, \gamma)$ is symmetric, then $(\mathtt{Ind}_\omega(\underline{\mathsf{C}}), \hat{\otimes}, \hat{e}, \hat{\alpha}, \hat{\lambda}, \hat{\rho}, \hat{\gamma})$ is a symmetric monoidal category. Finally, if $\underline{\mathsf{C}}$ is monoidal strict, then so is $\mathtt{Ind}_\omega(\underline{\mathsf{C}})$; if $\underline{\mathsf{C}}$ is strictly symmetric so is $\mathtt{Ind}_\omega(\underline{\mathsf{C}})$.

Proof. Concerning the cases where $\underline{\mathsf{C}}$ is strict monoidal or strictly symmetric, observe that the structure transformations $\hat{\alpha}$, $\hat{\lambda}$, $\hat{\rho}$ and $\hat{\gamma}$ are identities when the corresponding transformations of $\underline{\mathsf{C}}$ are so. ✓

As anticipated above, the embedding $y_{\underline{\mathsf{C}}}$ preserves the monoidal structure of $\underline{\mathsf{C}}$. Therefore, $\mathtt{Ind}_\omega(\underline{\mathsf{C}})$ can be considered the 'free' cocomplete monoidal category on $\underline{\mathsf{C}}$.

Proposition 5.3
The subcategory $y_{\underline{\mathsf{C}}}(\underline{\mathsf{C}})$ of $\mathtt{Ind}_\omega(\underline{\mathsf{C}})$ is isomorphic to $\underline{\mathsf{C}}$ in the monoidal sense, i.e., $y_{\underline{\mathsf{C}}}$ is a strict monoidal functor.

Proof. Of course, $y_{\underline{C}}(c \otimes d) = y_{\underline{C}}(c) \hat{\otimes} y_{\underline{C}}(d)$, since we identify $\underline{1}$ and $\underline{1} \times \underline{1}$. For the rest, observe that

$$\begin{aligned} y_{\underline{C}}(e) &= \hat{e}; \\ y_{\underline{C}}(\alpha_{x,y,z}) &= \hat{\alpha}_{y_{\underline{C}}(x), y_{\underline{C}}(y), y_{\underline{C}}(z)}; \\ y_{\underline{C}}(\lambda_x) &= \hat{\lambda}_{y_{\underline{C}}(x)}; \\ y_{\underline{C}}(\rho_x) &= \hat{\rho}_{y_{\underline{C}}(x)}; \\ y_{\underline{C}}(\gamma_{x,y}) &= \hat{\gamma}_{y_{\underline{C}}(x), y_{\underline{C}}(y)}; \end{aligned}$$

which is enough to conclude the desired result. ✓

We conclude this subsection studying the behaviour of $\mathtt{Ind}_\omega(\_)$ on monoidal functors and monoidal transformations. Let $(\mathsf{F}, \varphi^0, \varphi)$ be a monoidal functor between the monoidal categories $(\underline{C}, \otimes, e, \alpha, \lambda, \rho, \gamma)$ and $(\underline{D}, \otimes', e', \alpha', \lambda', \rho', \gamma')$. Consider $(\mathtt{Ind}_\omega(\mathsf{F}), y_{\underline{C}}(\varphi^0), \mathtt{Ind}_\omega(\varphi) * \Delta)$, i.e., a functor $\mathtt{Ind}_\omega(\mathsf{F})\colon \mathtt{Ind}_\omega(\underline{C}) \to \mathtt{Ind}_\omega(\underline{D})$, a morphism $y_{\underline{C}}(\varphi^0)\colon \hat{e}' \to \mathtt{Ind}_\omega(\mathsf{F})(\hat{e})$ and a natural transformation from $\mathtt{Ind}_\omega(\mathsf{F})(\_{}_1) \hat{\otimes}' \mathtt{Ind}_\omega(\mathsf{F})(\_{}_2) \to \mathtt{Ind}_\omega(\mathsf{F})(\_{}_1 \hat{\otimes} \_{}_2)$, whose component at the ω-ind-objects $X\colon \mathsf{I} \to \underline{C}$ and $Y\colon \mathsf{J} \to \underline{C}$ is

$$([\varphi_{X_i,Y_j}])_{(i,j)\in \mathsf{I}\times\mathsf{J}}\colon (\mathsf{F}(X_i) \otimes' \mathsf{F}(Y_j))_{(i,j)\in\mathsf{I}\times\mathsf{J}} \to (\mathsf{F}(X_i \otimes Y_j))_{(i,j)\in\mathsf{I}\times\mathsf{J}}.$$

It is just a matter of a few calculations to verify that the axioms (3) hold for $(\mathtt{Ind}_\omega(\mathsf{F}), y_{\underline{C}}(\varphi^0), \mathtt{Ind}_\omega(\varphi) * \Delta)$. Moreover, if $(\mathsf{F}, \varphi^0, \varphi)$ is symmetric, then (4) also holds, i.e., $(\mathtt{Ind}_\omega(\mathsf{F}), y_{\underline{C}}(\varphi^0), \mathtt{Ind}_\omega(\varphi) * \Delta)$ is symmetric. Clearly, strongness and strictness are also preserved. Let $\sigma\colon (\mathsf{F}, \varphi^0, \varphi) \overset{\cdot}{\to} (\mathsf{F}', \varphi'^0, \varphi')$ be a monoidal transformation. Recall that the component of $\mathtt{Ind}_\omega(\sigma)$ at the ω-ind-objects $X = (X_i)_{i\in\mathsf{I}}$ is $([\sigma_{X_i}])_{i\in\mathsf{I}}\colon (\mathsf{F}X_i)_{i\in\mathsf{I}} \to (\mathsf{F}'X_i)_{i\in\mathsf{I}}$. Therefore, it follows easily that, when σ satisfies (5), $\mathtt{Ind}_\omega(\sigma)$ is a monoidal transformation from $(\mathtt{Ind}_\omega(\mathsf{F}), y_{\underline{C}}(\varphi^0), \mathtt{Ind}_\omega(\varphi) * \Delta)$ to $(\mathtt{Ind}_\omega(\mathsf{F}'), y_{\underline{C}}(\varphi'^0), \mathtt{Ind}_\omega(\varphi') * \Delta)$. Therefore, we can state the following proposition.

Proposition 5.4
The KZ-doctrine $\mathtt{Ind}_\omega(\_)$ *on* $\underline{\underline{\underline{CAT}}}$ *lifts to KZ-doctrines on* $\underline{\underline{B}}$*, for any* $\underline{\underline{B}}$ *appearing in Table 1.*

Proof. Concerning the categories of locally small categories, the result follows immediately from the previous considerations about monoidal functors and transformations and from Proposition 5.2. In the cases where $\underline{\underline{B}}$ is a category of small categories, it follows from the above and Proposition 4.20. ✓

For each $\underline{\underline{B}}$ appearing in Table 1, let $\underline{\underline{\omega}}$-$\underline{\underline{B}}$ be the category consisting of the ω-chain cocomplete categories in $\underline{\underline{B}}$ with a choice of colimits and of the functors in $\underline{\underline{B}}$ which preserve ω-chain colimits up to isomorphism. Then, by Proposition A.13, we have the following.

Proposition 5.5
For any $\underline{\underline{B}}$ *in Table 1,* $\mathtt{Ind}_\omega(\_)$ *determines a KZ-adjunction from* $\underline{\underline{B}}$ *to* $\underline{\underline{\omega}}$-$\underline{\underline{B}}$*.*

The extension of the monoidal structure of $\underline{C}$ to $\mathrm{Ind}_\omega(\underline{C})$ given in the previous subsection may look rather far from our intended interpretation motivated in Section 1. For instance, the tensor of two ω-chains is (represented by) a two-dimensional structure $\omega \times \omega$. However, this is just a comfortable representation for the tensor. One could consider another representation taking for example the 'diagonal' cofinal chain in $\omega \times \omega$, which is isomorphic as ω-ind-object to the 'whole square', and which corresponds to the motivating diagram shown in Section 1. In this subsection, we study an alternative description of the monoidal structure of $\mathrm{Ind}_\omega(\underline{C})$ which is more intuitive and better suited for our intended applications to Petri nets.

Let $\underline{C}^\omega$ be the full subcategory of $\mathrm{Ind}_\omega(\underline{C})$ consisting of the ω-chains, i.e., the ω-ind-objects indexed by $\underline{\omega}$.[3] Then we have the following.

PROPOSITION 5.6
$\underline{C}^\omega \overset{\cdot}{\cong} \mathrm{Ind}_\omega(\underline{C})$.

Proof. The inclusion functor $\underline{C}^\omega \hookrightarrow \mathrm{Ind}_\omega(\underline{C})$ is by definition full and faithfull. We show that its replete image is $\mathrm{Ind}_\omega(\underline{C})$. Then, by exploting the result of [8] as in Proposition 4.3, we have the desired result.

Let X be an ω-ind-object, i.e., a countable filtered diagram in $\underline{C}$. We have to show that it is isomorphic in $\mathrm{Ind}_\omega(\underline{C})$ to an ω-chain. By applying the lemma stated in the proof of Proposition 3.8, and thanks to Proposition 4.6, we may assume that X is indexed over a countable directed set D. Then, working as in Proposition 3.3, we find a countable sequence of finite directed subsets $\{D_i\}_{i\in\omega}$ such that, $D_i \subset D_{i+1}$, for any $i \in \omega$, and $D = \bigcup_{i\in\omega} D_i$, from which we can extract the corresponding sequence of greatest elements $\{c_i\}_{i\in\omega}$. Now, define the functors $\phi\colon \underline{\omega} \to D$ and $Y\colon \underline{\omega} \to \underline{C}$ as follows:

$$\phi(i) = c_i \qquad \text{and} \qquad \phi(i < i+1) = c_i < c_{i+1};$$
$$Y(i) = X(c_i) \qquad \text{and} \qquad Y(i < i+1) = X(c_i < c_{i+1}).$$

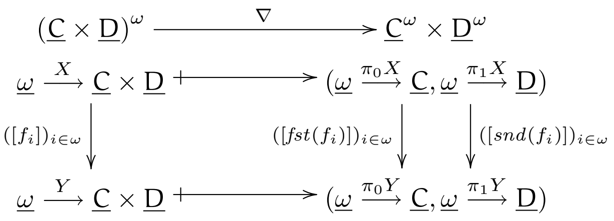
Clearly, by Lemma 3.7 *(iii)*, we have that ϕ is cofinal, and since $Y \circ \phi = X$, by Proposition 4.6, we conclude that X and Y are isomorphic in $\mathrm{Ind}_\omega(\underline{C})$. Since Y is an ω-chain, this concludes the proof. ✓

Observe that, as immediate consequence of the proposition above, we have that $\underline{C}^\omega$ is ω-chain (filtered) cocomplete. Of course, working with $\underline{C}^\omega$, we have to redefine y. We shall consider the obvious choice $\bar{y}(c) = \underline{\underline{c}} = \underline{\omega} \xrightarrow{c} \underline{C}$, the constant chain. Of course, we still have that $\bar{y}\colon \underline{C} \to \underline{C}^\omega$ is full and faithful. Moreover, $\mathsf{L} \circ \bar{y} = \mathsf{L} \circ y$.

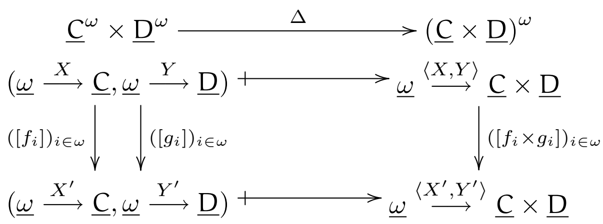
Restricting our attention to $\underline{\omega}$ makes possible expressing the commutativity of $(\_)^\omega$ and $\_ \times \_$ by an isomorphism.

[3] As in the case discussed in the note 1, the results which follow can be restated for any cardinal $\aleph$ and for the corresponding subcategory $\underline{C}^\aleph$ of $\aleph$-chains in $\underline{\mathrm{Set}}^{\underline{C}^{\mathrm{op}}}$.

Proposition 5.7
There is an isomorphism $(\underline{\mathsf{C}} \times \underline{\mathsf{D}})^\omega \cong \underline{\mathsf{C}}^\omega \times \underline{\mathsf{D}}^\omega$*, defined by* Δ *and* ∇ *below.*

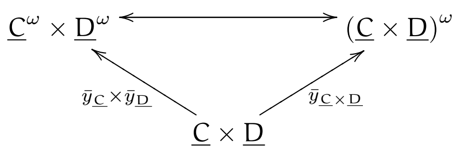


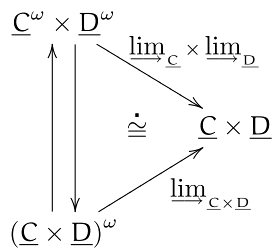
fst and snd being as in the previous subsection.



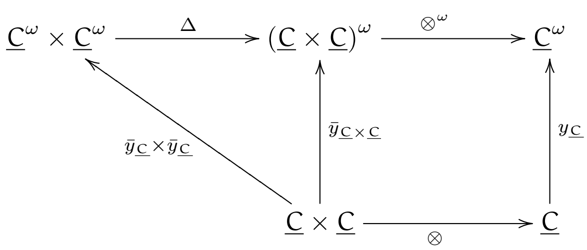
$\langle \_, \_ \rangle$ *being the pairing of functors.*

As in the previous subsection, we have the following commutative diagrams.





the second diagram existing when $\underline{\mathsf{C}}$ and $\underline{\mathsf{D}}$ are ω-chain cocomplete, and thus we can define



In the following, $\otimes^\omega \circ \Delta$ will be denoted by $\tilde{\otimes}$. Writing the tensor in terms of the indexed representation of ω-ind-object and morphisms makes clear the correspondence of this approach with the discussion in Section 1.

$$\begin{array}{ccccc} (X_i)_{i\in\omega} & & (Y_i)_{i\in\omega} & & (X_i\otimes Y_i)_{i\in\omega} \\ {\scriptstyle ([f_i])_{i\in\omega}}\downarrow & \tilde{\otimes} & \downarrow{\scriptstyle ([g_i])_{i\in\omega}} & = & \downarrow{\scriptstyle ([f_i\otimes g_i])_{i\in\omega}} \\ (X'_i)_{i\in\omega} & & (Y'_i)_{i\in\omega} & & (X'_i\otimes Y'_i)_{i\in\omega} \end{array}$$

So, given the symmetric monoidal category $(\underline{C}, \otimes, e, \alpha, \lambda, \rho, \gamma)$, the monoidal structure on $\underline{C}^\omega$ is $(\underline{C}^\omega, \tilde{\otimes}, \tilde{e}, \tilde{\alpha}, \tilde{\lambda}, \tilde{\rho}, \tilde{\gamma})$, where

- $\tilde{e} = \bar{y}_{\underline{C}}(e) = \underline{e}$;
- $\tilde{\alpha}_{X,Y,Z} = ([\alpha_{X_i,Y_i,Z_i}])_{i\in\omega}$;
- $\tilde{\lambda}_X = ([\lambda_{X_i}])_{i\in\omega}$;
- $\tilde{\rho}_X = ([\rho_{X_i}])_{i\in\omega}$;
- $\tilde{\gamma}_{X,Y} = ([\gamma_{X_i,Y_i}])_{i\in\omega}$.

Showing that these data form a symmetric monoidal category is a routine task. Therefore, we can summarize the results in the following propositions.

Proposition 5.8
For any monoidal category $(\underline{C}, \otimes, e, \alpha, \lambda, \rho)$ the ω-filtered cocomplete category $(\underline{C}^\omega, \tilde{\otimes}, \tilde{e}, \tilde{\alpha}, \tilde{\lambda}, \tilde{\rho})$ is a monoidal category. Moreover, if $(\underline{C}, \otimes, e, \alpha, \lambda, \rho)$ is symmetric, then $(\underline{C}^\omega, \tilde{\otimes}, \tilde{e}, \tilde{\alpha}, \tilde{\lambda}, \tilde{\rho}, \tilde{\otimes})$ is a symmetric monoidal category. Finally, if $\underline{C}$ is monoidal strict, then so is $\underline{C}^\omega$; if $\underline{C}$ is strictly simmetric so is $\underline{C}^\omega$.

Proposition 5.9
The subcategory $\bar{y}_{\underline{C}}(\underline{C})$ of $\underline{C}^\omega$ is isomorphic to $\underline{C}$ in the monoidal sense, i.e., $\bar{y}_{\underline{C}}$ is a strict monoidal functor.

Proposition 5.10
$\underline{C}^\omega$ is, up to equivalence, the free ω-chain cocomplete monoidal category on $\underline{C}$.

Proof. Immediate from Proposition A.12 and Proposition 5.6. ✓

6 Applications to Petri Nets

The previous sections have shown how we can build the (pseudo) free ω-filtered cocomplete category $\mathsf{Ind}_\omega(\underline{C})$ over a given $\underline{C}$. In particular, in Section 5 we have proved that the construction lifts to a KZ-doctrine on $\underline{\underline{\mathsf{sSsMonCat}}}$, respectively on $\underline{\underline{\mathsf{SsMonCat}}}$, giving in this way the completion of strictly symmetric strict monoidal categories, respectively symmetric strict monoidal categories. This section, which matches in style Section 1, explains how these facts bring us close again to Petri nets.

Following [30, 5], a *Petri net* is a structure $N = (\partial^0_N, \partial^1_N : T_N \to S^\oplus_N)$, where T_N is a set of *transitions*, S_N is a set of *places*, $S^\oplus_N$ is the free commutative monoid on S_N, i.e., the monoid of finite multisets on S_N, in this context usually called (finite) *markings* of N, and ∂^0_N and ∂^1_N are *functions* which associate to any transition respectively a source and a target marking. For $t \in T_N$, we write $t: u \to v$ if $\partial^0_N(t) = u$ and $\partial^1_N(t) = v$. A *morphism* of PT nets from N to N' consists of a pair of functions $\langle f, g\rangle$, where $f: T_N \to T_{N'}$ is a *function* and $g: S^\oplus_N \to S^\oplus_{N'}$ is a *monoid homomorphism* such that $\langle f, g\rangle$ respects source and target, i.e.,

$$\partial^0_{N'} \circ f = g \circ \partial^0_N \quad \text{and} \quad \partial^1_{N'} \circ f = g \circ \partial^1_N.$$

These data, with the obvious componentwise composition of morphisms, define the category $\underline{\mathsf{Petri}}$ of PT nets.

For N a net and $w \in S^\oplus_N$, the set $\mathcal{R}[N, w]$ of markings of N *reachable* from w is defined inductively by the rules

$$w \in \mathcal{R}[N, w], \qquad\qquad \frac{u \oplus u' \in \mathcal{R}[N, u] \quad \text{and} \quad t: u \to v \in T_N}{v \oplus v' \in \mathcal{R}[N, u]}.$$

We say that a place $a \in S_N$ is *bounded* if for all $w \in S^\oplus_N$ there exists $n \in \omega$ such that for all $u \in \mathcal{R}[N, w]$ with $u(a) \leq n$. Otherwise, we say that a is *unbounded*.

Definition 6.1 (*Process Nets and Processes*)
A process (or occurrence) net is a net Θ such that

i) for all $t \in T_\Theta$, $\partial^0_\Theta(t)$ and $\partial^1_\Theta(t)$ are sets (as opposed to multisets);

ii) for all pairs $t_0 \neq t_1 \in T_\Theta$ and $i = 0, 1$, $\partial^i_\Theta(t_0) \cap \partial^i_\Theta(t_1) = \varnothing$;

iii) the 'flow' relation $\prec$ on $T_\Theta \cup S_\Theta$ obtained as transitive closure of

$$\big\{(a, t) \bigm| a \in \partial^0_\Theta(t)\big\} \cup \big\{(t, a) \bigm| a \in \partial^1_\Theta(t)\big\}$$

is irreflexive, i.e., Θ is acyclic;

iv) $T_\Theta \cup S_\Theta$ is countable and for all $x \in T_\Theta \cup S_\Theta$, the set $\{y \mid y \prec x\}$ is finite.

We shall denote by $\min(\Theta)$ and $\max(\Theta)$ the sets of places of Θ which are, respectively, minimal and maximal with respect to $\prec$.

Given $N \in \underline{\mathsf{Petri}}$, a process of N is a morphism $\pi: \Theta \to N$, where Θ is a process net, and π is a net morphism which maps places to places (as opposed to morphisms which map places to markings) and such that the set $\pi^{-1}(a) \cap \min(\Theta)$ is finite for all the bounded $a \in S_N$.
A process is finite if the underlying process net is such.

For the purpose of defining processes at the right level of abstraction, we need to make some identifications. Of course, we shall consider as identical process nets which are isomorphic and, consequently, we shall make no distinction between two processes $\pi: \Theta \to N$ and $\pi': \Theta' \to N$ for which there exists an isomorphism $\varphi: \Theta \to \Theta'$ such that $\pi' \circ \varphi = \pi$. Observe that the constraint on π

is relevant, since we certainly want process morphisms to map a single component of the process net to a single component of N. Otherwise said, process are nothing but labellings of Θ, which in turn is essentially a partial ordering of transitions, with an appropriate element of N.

Remark. We recall that an ordinal number is ...

Definition 6.2 (*f-indexed orderings*)
Given a countable set A together with a set B and a function $f\colon A \to B$, an f-indexed ordering of A is a family $\{\ell_b \mid b \in B\}$ of bijections $\ell_b\colon f^{-1}(b) \to \eta$, η being a countable ordinal smaller than $\omega \cdot \omega$.

Informally, an f-indexed ordering of A is a family of total orderings, one for each of the partitions of A induced by f.

Definition 6.3 (*Concatenable Processes*)
A concatenable process of N is a triple $CP = (\pi, \ell, L)$ where

- *$\pi\colon \Theta \to N$ is a process of N;*
- *ℓ is a π-indexed ordering of* $\min(\Theta)$*;*
- *L is a π-indexed ordering of* $\max(\Theta)$*.*

Two concatenable processes CP and CP' are isomorphic if their underlying processes are isomorphic via an isomorphism φ which respects the ordering, i.e., such that $\ell'_{\pi'(\varphi(a))}(\varphi(a)) = \ell_{\pi(a)}(a)$ and $L'_{\pi'(\varphi(b))}(\varphi(b)) = L_{\pi(b)}(b)$ for all $a \in \min(\Theta)$ and $b \in \max(\Theta)$. As in the case of processes, we identify isomorphic concatenable processes.
A concatenable process is finite if the underlying process is such.

In order to define an operation of concatenation of concatenable processes, we associate a source and a target to a concatenable process $CP = (\pi\colon \Theta \to N, \ell, L)$ by considering the formal sums $u = \bigoplus_{a \in S_N} \eta_a \cdot a$, where η_a is the codomain of ℓ_a, respectively of L_a, if $a \in \pi(\min(\Theta))$, respectively $a \in \pi(\max(\Theta))$, and η_a is 0 otherwise. It is easy to verify that S_N^ω, the set of source and targets of concatenable processes of N, is the set of the formal sums $\bigoplus_{a \in S_N} \eta_a \cdot a$ such that $\eta_a \in \omega{\cdot}\omega$ and $\eta_a \in \omega$ if and only if a is bounded in N. It follows immediately that S_N^ω is a monoid with unit the sum $\bigoplus_{a \in S_N} 0 \cdot a$ under the operation

$$\Big(\bigoplus_{a \in S_N} \eta_a \cdot a\Big) \oplus \Big(\bigoplus_{a \in S_N} \eta'_a \cdot a\Big) = \bigoplus_{a \in S_N} (\eta_a + \eta'_a) \cdot a,$$

where $\_ + \_$ is the usual sum of ordinals. Given their resemblance to markings, in the following we shall call the elements of S_N^ω *generalized markings* of N. The concatenation of $(\pi_0\colon \Theta_0 \to N, \ell^0, L^0)\colon u \to v$ and $(\pi_1\colon \Theta_1 \to N, \ell^1, L^1)\colon v \to w$ is the concatenable process $(\pi\colon \Theta \to N, \ell, L)\colon u \to w$ defined as follows (see also Figure 2), where, in order to simplify notations, we assume that S_{Θ_0} and S_{Θ_1} are disjoint.

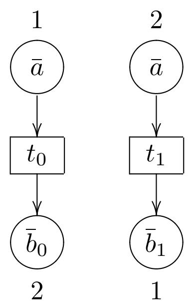


Figure 1: A net N and one of its concatenable processes $\alpha\colon a \to \omega \cdot b$

- Let A be the set of pairs (x, y) such that $x \in \max(\Theta)$, $y \in \min(\Theta)$, $\pi_0(y) = a = \pi_1(x)$ and $L_a^0(x) = \ell_a^1(y)$. By the definitions of concatenable processes and of their sources and targets, an element of $\max(\Theta_0)$ belongs exactly to one pair of A, and of course the same happens to $\min(\Theta_1)$. Consider $S_0 = S_{\Theta_0} \setminus \max(\Theta_0)$ and $S_1 = S_{\Theta_1} \setminus \min(\Theta_1)$. Then, let $in_0\colon S_{\Theta_0} \to S_0 \cup A$ be the function which is the identity on $x \in S_0$ and maps $x \in \max(\Theta_1)$ to the corresponding pair in A. Define $in_1\colon S_{\Theta_1} \to S_1 \cup A$ analogously. Then,

$$\Theta = (\partial^0, \partial^1\colon T_{\Theta_0} + T_{\Theta_1} \to (S_0 \cup S_1 \cup A)^{\oplus}),$$

  where

  - $\partial^0 = in_0^{\oplus} \circ \partial^0_{\Theta_0} + in_1^{\oplus} \circ \partial^0_{\Theta_1}$;
  - $\partial^1 = in_0^{\oplus} \circ \partial^1_{\Theta_0} + in_1^{\oplus} \circ \partial^1_{\Theta_1}$;

- Suppose $\pi_i = \langle f_i, g_i \rangle$, for $i = 0, 1$ and consider the function $g(x) = g_i(x)$ if $x \in S_i$ and $g((x, y)) = g_0(x) = g_1(y)$ otherwise. Then $\pi = \langle f_0 + f_1, g \rangle$.

- $\ell_a(x) = \ell_a^0(x)$ if $x \in \min(\Theta_0)$ and $\ell_a((x, y)) = \ell_a^0(x)$ if $(x, y) \in \min(\Theta)$.

- $L_a(x) = L_a^1(x)$ if $x \in \max(\Theta_1)$ and $L_a((x, y)) = L_a^1(y)$ if $(x, y) \in \max(\Theta)$.

Proposition 6.4
Under the above defined operation of sequential composition, the concatenable processes of N form a category $\mathcal{IP}[N]$ with objects the elements of S_N^{ω} and

with identities those processes consisting only of places, which therefore are both minimal and maximal, and such that $\ell = L$.

Concatenable processes admit also a tensor operation $\otimes$ such that, given $CP_0 = (\pi_0: \Theta_0 \to N, \ell^0, L^0): u \to v$ and $CP_1 = (\pi_1: \Theta_1 \to N, \ell^1, L^1): u' \to v'$, $CP_0 \otimes CP_1$ is the concatenable process $(\pi: \Theta \to N, \ell, L): u \oplus u' \to v \oplus v'$ given below.

- $\Theta = (\partial^0_{\Theta_0} + \partial^0_{\Theta_1}, \partial^1_{\Theta_0} + \partial^1_{\Theta_1}: T_{\Theta_0} + T_{\Theta_1} \to (S_{\Theta_0} + S_{\Theta_1})^{\oplus})$,
- $\pi = \pi_0 + \pi_1$;
- $\ell_a(in_0(x)) = \ell^0_a(x)$ and $\ell_a(in_1(y)) = u(a) + \ell^1_a(y)$.
- $L_a(in_0(x)) = L^0_a(x)$ and $L_a(in_1(y)) = v(a) + L^1_a(y)$.

It is easy to verify that $\otimes$ is a functor $\otimes: \mathcal{IP}[N] \times \mathcal{IP}[N] \to \mathcal{IP}[N]$. The concatenable processes consisting only of places are the analogous of the symmetries. In particular, for any $u = \bigoplus_{a \in S_N} \eta_a \cdot a$ and $v = \bigoplus_{a \in S_N} \eta'_a \cdot a$ in $\S^{\omega}_N$, the concatenable process which consists of $|\eta_a + \eta'_a|$ places mapped by π to the corresponding places of N and such that $L_a(x) = v(a) + \ell_a(x)$ and $\ell_a(x) = L_a(x) - u(b_i)$ corresponds to the component at (u, v) of the symmetry isomorphism. Moreover, the category $\mathcal{IP}[N]$ enjoys the axioms (2) for $\_;\_$, $\_ \otimes \_$ and γ as given above. Therefore, since the product is strictly associative and the unit is strict, $\mathcal{IP}[N]$ is a symmetric strict monoidal category.

Proposition 6.5
Under the above defined tensor product $\mathcal{IP}[N]$ is a symmetric strict monoidal category whose symmetry isomorphism is the family $\{\bar{\gamma}(u,v)\}_{u,v \in S^{\otimes}_N}$.

The transitions t of N are faithfully represented in the obvious way by concatenable processes with a unique transition which is in the post-set of any minimal place and in the pre-set of any maximal place, minimal and maximal places being in one-to-one correspondence, respectively, with $\partial^0_N(t)$ and $\partial^1_N(t)$ (see also Figure 3).

Definition 6.6 (*Finite Concatenable Processes [5]*)
Let $\mathcal{CP}[N]$ be the full subcategory of $\mathcal{IP}[N]$ consisting of the finite concatenable processes.

The arrows of $\mathcal{P}[N]$ have a nice computational interpretation in terms of a slight refinement of the classical notion of process consisting of a suitable layer of labels to the minimal and to the maximal places of process nets in order to distinguish among different istances of a place in a process of N. The equivalence of the following definition of $\mathcal{P}[N]$ with the original one in [5] has been proved in [**?**].

Definition 6.7
The category $\mathcal{P}[N]$ is the monoidal quotient (see Appendix **??***) of $\mathcal{F}(N)$, the free symmetric strict monoidal category generated by N, modulo the axioms*

$$\begin{array}{rcll} \gamma_{a,b} & = & id_{a\oplus b} & \text{if } a,b \in S_N \text{ and } a \neq b, \\ t;(id \otimes \gamma_{a,a} \otimes id) & = & t & \text{if } t \in T_N \text{ and } a \in S_N, \\ (id \otimes \gamma_{a,a} \otimes id);t & = & t & \text{if } t \in T_N \text{ and } a \in S_N, \end{array}$$

where γ is the symmetry isomorphism of $\mathcal{F}(N)$.

Proposition 6.8
$\mathcal{CP}[N]$ and $\mathcal{P}[N]$ are isomorphic.

Proof. See [5]. ✓

Proposition 6.9
$\mathcal{P}[N]$ embeds fully and faithfully in $\mathcal{IP}[N]$ preserving the monoidal structure.

Proof. **To be written.** ✓

Proposition 6.10
$\mathcal{IP}[N]$ is cocomplete and it is an algebra for the completion

Proof. **To be written.** ✓

Proposition 6.11
$\mathcal{IP}[N] \cong \mathtt{Ind}_\omega(\mathcal{P}[N])$.

Proof. **To be written.** ✓

Among other interesting categorical formalizations of net computations [30, 5] (see [32] for a survey), we would like to recall two relevant monoidal constructions of processes, namely:

i) $\mathcal{T}[N]$, which gives the free *strictly* symmetric strict monoidal category on N, corresponding to the notion of *commutative processes* of N [3]; we shall call $\underline{\mathtt{CatPetri}}$ the category of such monoidal categories and symmetric strict monoidal functors;

ii) $\mathcal{P}[N]$, which gives the symmetric strict monoidal category obtained by quotienting the free symmetric strict monoidal category on N whose monoid of objects is $S^\oplus$ via the axioms

$$\begin{array}{rcll} \gamma_{a,b} & = & id_{a\otimes b} & \text{if } a,b \in S \text{ and } a \neq b, \\ t;(id \otimes \gamma_{a,a} \otimes id) & = & t & \text{if } t \in T \text{ and } a \in S, \\ (id \otimes \gamma_{a,a} \otimes id);t & = & t & \text{if } t \in T \text{ and } a \in S, \end{array}$$

corresponding to the *concatenable processes* of N, which are a slight refinement of the standard notion of process [10]; we shall refer to the category of small symmetric monoidal categories with the properties above and symmetric strict monoidal functors as $\underline{\mathtt{CatProc}}$.

Since we already know that the monoidal structure of such categories is preserved by the ω-ind-completion process, the remaining question is whether the additional structure is preserved by the ω-ind-completion or, in other words, if $\mathtt{Ind}_\omega(\_)$ lifts to KZ-monads on $\underline{\mathtt{CatPetri}}$ and on $\underline{\mathtt{CatProc}}$. Clearly, this would be the best possible result from our point of view, since it would allow a full application of the theory of the cocompletion of monoidal categories to the case of Petri nets, thus giving a full account of infinite behaviours of nets. Unfortunately, this is not the case. More precisely, only the objects of $\underline{\mathtt{CatPetri}}$ are rather close to keep their structure under the cocompletion construction. In fact, we know from Proposition 5.2 that $\mathtt{Ind}_\omega(\mathcal{T}[N])$ is a strictly symmetric strict monoidal category. However, $\mathtt{Ind}_\omega(\mathcal{T}[N])$ does not belong to $\underline{\mathtt{CatPetri}}$ since its monoid of objects is *not* free. The situation is worse for $\underline{\mathtt{CatProc}}$. Observe, in fact, that when $\underline{\mathsf{C}}$ is not strictly symmetric, then the primary requirement about the monoid of objects being commutative fails immediately. In fact, in this case, the tensor of diagrams in general will not be commutative because of the arrows.

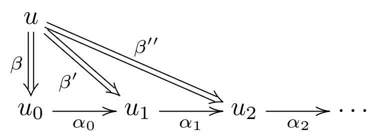
Thus, $\mathtt{Ind}_\omega(\_)$ does *not* restrict to an endofunctor on the categories we are mainly concerned with. A possible way out of this problem, which is currently under investigation, consists of looking for an alternative presentation of the cocompletion doctrine, i.e., for a doctrine whose functor is isomorphic to $\mathtt{Ind}_\omega(\_)$ but better suited for the case of Petri nets. For the time being, however, we present the following considerations about the relationships between Petri nets and the cocompletion of their categories of processes which aim at showing that, at the level of a single net, $\mathtt{Ind}_\omega(\_)$ behaves as expected, giving a faithful description of infinite processes. We shall focus on the concatenable processes $\mathcal{P}[N]$, although, as in Section 1, all the following comes from rather general properties.

First of all, we need to show that $\mathtt{Ind}_\omega(\mathcal{P}[N])$ can be considered as the category of infinite concatenable processes of N. Consider a net $N \in \underline{\mathtt{Petri}}$ and an ω-chain

$$u_0 \xrightarrow{\alpha_0} u_1 \xrightarrow{\alpha_1} u_2 \cdots u_n \xrightarrow{\alpha_n} u_{n+1} \cdots$$

i.e., an ω-ind-object U in $\mathcal{P}[N]^\omega$. We look at this chain as a limit point for an infinite computation (and not as the infinite computation itself!), i.e., as a sort of generalized *infinite marking* represented by the computation which produces it from the finite markings. Observe that the adjective 'generalized' is appropriate, since, in general, the infinite marking above depends on the transitions which appear in the chain, not just on their sources and targets. For instance, if we consider a net with two transitions $t, t' : a \to a$, then the chains consisting respectively of a sequence of t and a sequence of t' represent different infinite markings.

In order to substantiate the intuition about morphisms, let us start with the following case. Let $\underline{u}$ be the standard representative of u in $\mathtt{Ind}_\omega(\mathcal{P}[N])$, i.e., $\underline{u} = y(u)$ the diagram with value u^I on the singleton filtered category $\underline{1}$. Given the particular shapes of $\underline{1}$ and $\underline{\omega}$, an arrow from $\underline{u}$ to U in $\mathtt{Ind}_\omega(\mathcal{P}[N])$, is an equivalence class $[\beta]\colon \underline{u} \to U$ of arrows $\beta\colon u \to u_n$ in $\mathcal{P}[N]$, where $(\beta\colon u \to u_n) \sim (\beta'\colon u \to u_k)$ with $n \leq k$ if and only if $\beta; \alpha_{n+1}; \cdots; \alpha_k = \beta'$.



Now, recalling the characterization of arrows in $\mathcal{P}[N]$ as concatenable processes, we conclude that an arrow from $\underline{u}$ to U in $\mathtt{Ind}_\omega(\mathcal{P}[N])$ is an ω-chain of concatenable processes embedded into each other, i.e., it represents a unique infinite process. Before getting to the generality of arrows between ω-ind-objects, it is worthwhile to point out the following particular case. Observe that each ω-ind-object is the limit in $\mathtt{Ind}_\omega(\underline{C})$ of its component constant ω-ind-objects. (To see this just apply the definition of colimit in $\mathtt{Ind}_\omega(\underline{C})$ given in Proposition 4.13.) Then, it follows immediately from the discussion above that, for any $n \in \omega$, the component at n of the limit cocone for U, say $\lambda_n\colon u_n \to U$, contains the set

$$\{\alpha_n,\, \alpha_n; \alpha_{n+1},\, \alpha_n; \alpha_{n+1}; \alpha_{n+2}, \ldots\}$$

as a cofinal subset. Then, $\lambda_0\colon u_0 \to U$ represents the limit of the sequence of processes α_i, as expected.

Consider now the ω-ind-objects $U = u_0 \xrightarrow{\alpha_0} u_1 \xrightarrow{\alpha_1} \cdots$ and $V = v_0 \xrightarrow{\beta_0} v_1 \xrightarrow{\beta_1} \cdots$ and an arrow $([\sigma_i])_{i\in\omega}\colon U \to V$. As explained above, each component $[\sigma_i]$ represents an infinite process leaving from u_i, i.e., leaving from the i-th approximation of the generalized marking U. Now, the 'compatibility' condition on the components of $([\sigma_i])_{i\in\omega}$ means that for any $n \leq k$ and for any $\sigma_i\colon u_n \to v_{n'}$ and $\sigma_k\colon u_k \to v_{k'}$, representatives of, respectively, the i-th and the k-th component of the ω-ind-morphism, assuming without loss of generality $n' \leq k'$, we must have $\alpha_n; \cdots; \alpha_{k-1}; \sigma_k = \sigma_n; \beta_{n'}; \cdots; \beta_{k'-1}$. It follows that the infinite processes (corresponding to) $[\sigma_i]$ form a sequence of embedded processes which leave from better and better approximations of U. Then, this chain admits a limit process which is the infinite process corresponding to $([\sigma_i])_{i\in\omega}$. In other words, morphisms from generalized infinite markings are defined via continuity from 'finite' approximation morphisms. The same, of course, happens for the composition of infinite processes which, therefore, are concatenable. A similar description in term of continuity may be given for the parallel composition of infinite processes.

Of course, the previous informal discussion could be easily translated into a formal proof of the fact that $\mathcal{P}[N]^\omega$ captures the usual intuitive notion of infinite processes, thus yielding a smooth extension of the algebraic theory of Petri nets of [30, 5] to an axiomatization in terms of monoidal categories of the infinite causal behaviour of N. For the purpose of this paper, however, we simply claim that the following definition is completely adequate.

Definition 6.12
$\mathcal{P}[N]^\omega$ is, up to equivalence, the (symmetric strict monoidal) category of infinite concatenable processes of N.

Finally, since by Proposition 5.6 we know that $\underline{\mathsf{C}}^\omega$ and $\mathtt{Ind}_\omega(\underline{\mathsf{C}})$ are equivalent categories, we have the following proposition.

Proposition 6.13
$\mathtt{Ind}_\omega(\mathcal{P}[N])$ is, up to equivalence, the (symmetric strict monoidal) category of the infinite concatenable processes of N.

Conclusions and Further Work

Of course, besides the problems with the semantics of nets we have noticed in the previous section, there are many other applications to be investigated, and we plan to explore many of these in the near future. For example, a mainstream in the research on infinite computations focuses on *topology*—more precisely on metric spaces (see [4] and references therein). Roughly, the approach consists of defining a suitable distance between finite computations and applying to the resulting metric space a standard *Cauchy completion*, thus yielding a complete metric space where the infinite computations are the cluster points. One of the most valuable aspects of this approach is that, by choosing appropriate metrics, it is possible to factor out those infinite computations which do not enjoy certain properties, in particular *fairness* properties [6]. It is indeed a very interesting question whether these results can be recovered in the categorical framework building on the seminal paper [22].

Moreover since by now there are several categorical approaches to the semantics of computing systems in which objects represent states and arrows computations, this also yields a general method to construct and manipulate infinite computations of those systems. A notable example is given by Meseguer's concurrent rewriting systems [29]. This issue deserves to be fully investigated in future.

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A KZ-Doctrines and Pseudo-Monads

Free constructions are defined up to isomorphism, which reflects the fact that the stress is on the essential structure to be added, irrespective of the actual representation chosen for such a structure. Correspondingly, free constructions, as left adjoints to forgetful functors, give rise to *monads* [25], i.e., to algebraic constructions. In the case of completion of categories by colimits, however, as discussed in Section 2, the freeness condition is verified only up to isomorphism of the functors involved. Thus, the free object is identified up to equivalence of categories, in the precise sense that the (infinitely many) categories which enjoy the universal property are 'only' equivalent—as opposed to isomorphic—to each other. Correspondingly, the cocompletion functors do not give rise to an adjunction or, equivalently, to a monad (however see [19]). We would like to stress that, since the notion of colimit is defined only up to isomorphism, it is *not* a strictly algebraic operation, and therefore it would not be reasonable to expect a stronger form of universality.

Situations like this arise often in the everyday practice in mathematics, and a lot of work has been done in order to formalize them in category theory, e.g. [1, 20, 11, 12, 34, 2, 35], where equality is replaced by equivalence of morphisms or, even weaker, by the mere existence of a 2-cell between two morphisms. Needless to say, it is very often the case that this 2-cells have to be related by coherence isomorphisms themselves. So these constructions make sense in 3-categories, like $\underline{\underline{\mathsf{Cat}}}$. There are many natural examples of situations where the *pseudo* version (i.e. up to coherent isomorphisms) or the *lax* version (i.e. up to coherent 2-cells) of algebraic laws seems to be the natural requirement. Perhaps the most evident example is the case of monoidal categories [7] where the standard notion of *monoidal functor* is not required to commute with the monoidal structure 'on the nose', but only up to isomorphisms or up to coherent 2-cells.

Of course another such example is, in our opinion, the cocompletion construction we are interested in. Thus, instead of trying to 'fish out' some peculiar representatives in order to make the colimit notion behave strictly algebraically, we prefer to adopt a viewpoint also taken by other authors [19, 20, 35] who recognize its 2-categorical *'lax'* nature and formalize it as a pseudo-adjunction, or equivalently as a pseudo-monad. However, the problem with this approach is that the needed coherence conditions may look quite overwhelming sometimes. For instance, Zöberlein's *2-doctrines* [35], i.e., 2-functors on 2-categories with unit and multiplication natural only up to isomorphisms for which the laws for monads, algebras and homomorphisms hold up to isomorphism, must be provided with 19 coherence axioms. Fortunately enough, in the nice case of *coquasi-idempotent doctrines* [35], which are what is needed for the cocompletion construction, most of them disappear. In the following we shall recall the basics of *KZ-doctrines* or *KZ-monads* [20, 34] (KZ standing for Kock-Zöberlein), which are a simpler representation of the cited notion. In particular, the most relevant feature of KZ-doctrines is that all we need about coherent isomorphisms of 1-cells is contained in a single piece of information, namely a family of 2-cells.

<u>Remark</u>. In the following we shall be dealing with 2-categories. For the basic defini-

tions the reader is referred to [18]. As a matter of notation, we shall denote by $\_ * \_$ the horizontal composition and by $\_ \bullet \_$ the vertical composition of 2-cells, while we stick to the classical $\_ \circ \_$ for the horizontal composition of 1-cells. Identity 1-cells are written as id_C, or simply id, while for the identity 2-cell of a 1-cell f we use f itself, since confusion is never possible. Moreover, when the 1-cell involved is not relevant, we write $\mathbf{1}$ to indicate a generic identity 2-cell.

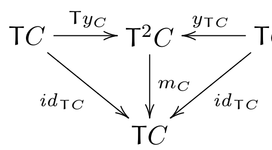
Definition A.1

A KZ-doctrine on a 2-category $\underline{\underline{C}}$ *is a tuple* $(\mathsf{T}, y, m, \lambda)$*, where*

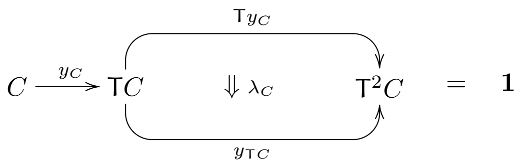
- $\mathsf{T}\colon \underline{\underline{C}} \to \underline{\underline{C}}$ *is a 2-endofunctor;*
- $y\colon Id \stackrel{\cdot}{\to} \mathsf{T}$ *and* $m\colon \mathsf{T}^2 \stackrel{\cdot}{\to} \mathsf{T}$ *are 2-natural transformations;*
- λ *is a family of 2-cells* $\{\lambda_C\colon \mathsf{T}y_C \Rightarrow y_{\mathsf{T}C}\colon \mathsf{T}C \to \mathsf{T}^2C\}_{C\in\underline{\underline{C}}}$ *indexed by the objects of* $\underline{\underline{C}}$*;*

satisfying the following axioms[4]

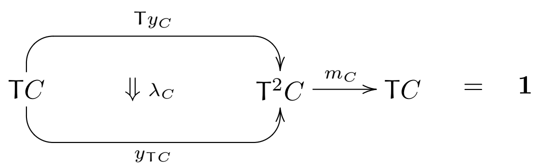
$\mathbf{T_0}$: $m_C \circ \mathsf{T}y_C = m_C \circ y_{\mathsf{T}C} = id_{\mathsf{T}C}$*;*



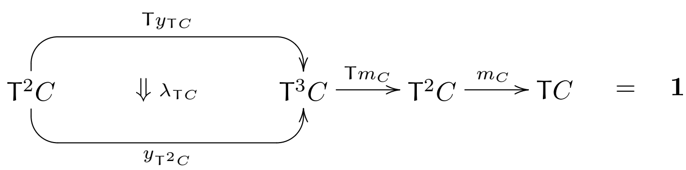
$\mathbf{T_1}$: $\lambda_C * y_C = \mathbf{1}$*;*



$\mathbf{T_2}$: $m_C * \lambda_C = \mathbf{1}$*;*



$\mathbf{T_3}$: $m_C * \mathsf{T}m_C * \lambda_{\mathsf{T}C} = \mathbf{1}$.



[4]It can be shown that the equations are only between 2-cells with the same source and target, i.e., that they are well given.

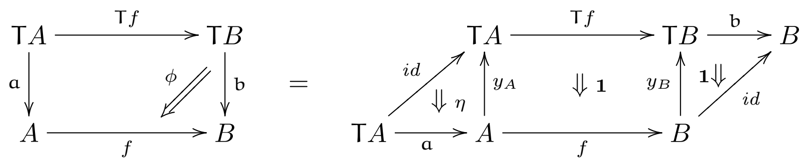
Thus, T, y and m play the role of the *functor*, *unit* and *multiplication* of an ordinary 2-monad [2]. In particular, y and m are *actual* (not pseudo) 2-natural transformations. As anticipated, the only additional 2-dimensional information around is λ and every coherence isomorphism is obtained from it. Axiom $\mathbf{T_0}$ corresponds to the *unit law* of monads, that therefore holds strictly also in KZ-doctrines. Axioms $\mathbf{T_1}$, $\mathbf{T_2}$ and $\mathbf{T_3}$ express the coherence of λ with the unit and the multiplication. Observe that there is no explicit mention of a pseudo form of the *multiplication law*. However, we shall see later that this is indeed the case and, for any $C \in \underline{\underline{C}}$, there exists an isomorphism $\mu_C\colon m_C \circ \mathsf{T}m_C \Rightarrow m_C \circ m_{\mathsf{T}C}\colon \mathsf{T}^3C \to \mathsf{T}C$.

Proposition A.2
For any $C \in \underline{\underline{C}}$ we have a reflection $m_C \dashv y_{\mathsf{T}C}\colon \mathsf{T}^2C \rightharpoonup \mathsf{T}C$, the unit of the adjunction being $\mathsf{T}m_C * \lambda_{\mathsf{T}C}\colon id_{\mathsf{T}^2C} \Rightarrow y_{\mathsf{T}C} \circ m_C$.

Definition A.3 (T-*Algebras*)
An algebra for T is an object $A \in \underline{\underline{C}}$ together with a structure map $\mathfrak{a}\colon \mathsf{T}A \to A$ which is a reflection left adjoint for $y_A\colon A \to \mathsf{T}A$.

Thus, *structures* are *adjoints* to *units* [20]. Observe that, since $\mathfrak{a} \dashv y_A$ is a reflection, we have $\mathfrak{a} \circ y_A = id$. Therefore, as in the case of the KZ-doctrine itself, the *unit law* for the structure of an algebra holds strictly. Since we have $m_C \dashv y_{\mathsf{T}C}$, for any $C \in \underline{\underline{C}}$ there is a 'free' algebra on C, namely $(\mathsf{T}C, m_C)$.

Given T-algebras $(A, \mathfrak{a})$ and $(B, \mathfrak{b})$, consider a morphism $f\colon A \to B$ in $\underline{\underline{C}}$. By naturality of y, we have that $\mathsf{T}f \circ y_A = y_B \circ f$. Thus, we can consider the identity $\mathbf{1}\colon \mathsf{T}f \circ y_A \Rightarrow y_B \circ f$ and its mate (see e.g. [18]) $\phi\colon \mathfrak{b} \circ \mathsf{T}f \Rightarrow f \circ \mathfrak{a}$ under the adjunctions $\mathfrak{a} \dashv y_A$ and $\mathfrak{b} \dashv y_B$ wrt. f and $\mathsf{T}f$, i.e.,



We shall refer to ϕ as the *canonical 2-cell* associated to f.

Definition A.4 (T-*homomorphisms*)
A T-homomorphism f from the T-algebra $(A, \mathfrak{a})$ to the T-algebra $(B, \mathfrak{b})$ is a morphism $f\colon A \to B$ whose canonical 2-cell is invertible.

Since the calculus of mates preserves composition, given the algebras $(A, \mathfrak{a})$ and $(B, \mathfrak{b})$, we have that if ϕ_f and ϕ_g are the canonical 2-cells of $f\colon A \to B$ and $g\colon B \to C$, then the canonical 2-cell $\phi_{g \circ f}$ associated to $g \circ f$ is $\phi_g * \phi_f$. Moreover, a simple shot of pasting shows that the canonical 2-cell associated to id_A is the identity 2-cell. Therefore, we have the following.

Proposition A.5
T-algebras and T-homomorphisms form a category $\underline{\mathsf{T}\text{-}\mathsf{Alg}}$ which is lifted to a 2-category $\underline{\underline{\mathsf{T}\text{-}\mathsf{Alg}}}$ by enriching it with all the 2-cells in $\underline{\underline{\mathsf{C}}}$.

It follows immediately from the definitions that the forgetful functor

$$\begin{array}{ccc} \underline{\underline{\mathsf{T}\text{-}\mathsf{Alg}}} & \xrightarrow{\ \mathsf{u}\ } & \underline{\underline{\mathsf{C}}} \\ (A,\mathfrak{a}) & \longmapsto & A \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ (B,\mathfrak{b}) & \longmapsto & B \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \underline{\underline{\mathsf{T}\text{-}\mathsf{Alg}}} & \xrightarrow{\ \mathsf{u}\ } & \underline{\underline{\mathsf{C}}} \\ \xrightarrow{f} & & \xrightarrow{f} \\ {\scriptstyle \alpha}\Downarrow & \longmapsto & \Downarrow{\scriptstyle \alpha} \\ \xrightarrow[g]{} & & \xrightarrow[g]{} \end{array}$$

is *faithful* and *locally fully faithful*, i.e., $\underline{\underline{\mathsf{T}\text{-}\mathsf{Alg}}}[f,g] = \underline{\underline{\mathsf{C}}}[f,g]$.

Next, we state two important cases in which one can conclude that a morphism is a T-homomorphism.

Proposition A.6
Let $(A,\mathfrak{a})$, $(B,\mathfrak{b})$ be T-algebras. If $f\colon A \to B$ is invertible, then f is a T-homomorphism. Moreover, if f is a left adjoint, then f is a T-homomorphism.

T-algebras are characterized by structure maps which are adjoint to a given morphism. Therefore, the structure on a given algebra is unique up to isomorphisms. Moreover, fixed a structure map the unit of the adjunction is uniquely determined via λ.

Proposition A.7
Let $(A,\mathfrak{a})$ be an algebra and suppose that η is the unit of $\mathfrak{a} \dashv y_A$. Then, $\eta = \mathsf{T}\mathfrak{a} * \lambda_A$.

Since the canonical 2-cells of morphisms $f\colon A \to B$ are mates of an identity 2-cell under adjunctions whose units can be expressed through λ and whose counits are identities, the following result is very natural.

Proposition A.8
Let $(A,\mathfrak{a})$ and $(B,\mathfrak{b})$ be algebras and consider $f\colon A \to B$. Then, the canonical 2-cell associated to f is $\phi_f = \mathfrak{b} * \mathsf{T}f * \mathsf{T}a * \lambda_A$.

Another relevant property of the multiplication m is the following.

Proposition A.9
For any $C \in \underline{\underline{\mathsf{C}}}$ there is a coreflection $\mathsf{T}y_C \dashv m_C$.

We show now, as promised, that the associativity law for m holds up to a canonical isomorphism. In fact, since $m_C\colon \mathsf{T}^2C \to \mathsf{T}C$ is left adjoint to $y_{\mathsf{T}C}$, by

Proposition A.6, it is a T-homomorphism with canonical 2-cell $\mu_C: m_C \circ \mathsf{T}m_C \Rightarrow m_C \circ m_{\mathsf{T}C}$, i.e.,

$$\begin{array}{ccc} \mathsf{T}^3C & \xrightarrow{\mathsf{T}m_C} & \mathsf{T}^2C \\ {\scriptstyle m_{\mathsf{T}C}}\downarrow & \overset{\mu_C}{\swarrow\!\!\swarrow} & \downarrow{\scriptstyle m_C} \\ \mathsf{T}^2C & \xrightarrow[m_C]{} & \mathsf{T}C \end{array}$$

We complete this appendix about KZ-doctrines by recalling that a KZ-doctrines give rise to a particular kind pseudo-adjunctions. We first recall the notion of *pseudo natural transformation.*

Definition A.10
Let $\mathsf{F}: \underline{\underline{C}} \to \underline{\underline{D}}$ *and* $\mathsf{G}: \underline{\underline{C}} \to \underline{\underline{D}}$ *be 2-functors. A pseudo natural transformation* $\sigma: \mathsf{F} \dot{\to} \mathsf{G}$ *is a family* $\{\sigma_C: \mathsf{F}C \to \mathsf{G}C\}_{C \in \underline{\underline{C}}}$ *of 1-cells in* $\underline{\underline{D}}$ *together with a family* $\{\sigma_f: \sigma_{C'} \circ \mathsf{F}f \to \mathsf{G}f \circ \sigma_C \mid f: C \to C' \text{ in } \underline{\underline{C}}\}$ *of invertible 2-cells in* $\underline{\underline{D}}$*, such that*

- *for each* $\mu: f \Rightarrow f': C \to C'$ *we have* $(\mathsf{G}\mu * \sigma_C) \centerdot \sigma_f = \sigma_{f'} \centerdot (\sigma_{C'} * \mathsf{F}\mu)$*;*
- $\sigma_{id_C} = \mathbf{1}_{\sigma_C}$*;*
- $\sigma_{g \circ f}$ *coincides with the pasting of the following 2-cells.*

$$\begin{array}{ccc} \mathsf{F}C & \xrightarrow{\sigma_C} & \mathsf{G}C \\ {\scriptstyle \mathsf{F}f}\downarrow & \overset{\sigma_f}{\nearrow\!\!\nearrow} & \downarrow{\scriptstyle \mathsf{G}f} \\ \mathsf{F}C' & \xrightarrow[\sigma_{C'}]{} & \mathsf{G}C' \\ {\scriptstyle \mathsf{F}g}\downarrow & \overset{\sigma_g}{\nearrow\!\!\nearrow} & \downarrow{\scriptstyle \mathsf{G}g} \\ \mathsf{F}C'' & \xrightarrow[\sigma_{C''}]{} & \mathsf{G}C'' \end{array}$$

Definition A.11 (*KZ-adjunction*)
A KZ-adjunction is a tuple $(\mathsf{F}, G, \eta, \varepsilon)$ *where* $\mathsf{F}: \underline{\underline{C}} \to \underline{\underline{D}}$ *and* $\mathsf{G}: \underline{\underline{D}} \to \underline{\underline{C}}$ *are 2-functors,* $\eta: Id_{\underline{\underline{C}}} \dot{\to} \mathsf{GF}$ *is a 2-natural transformation and* $\varepsilon: \mathsf{FG} \dot{\to} Id_{\underline{\underline{D}}}$ *is a pseudo natural transformation such that*

$$\begin{array}{rcl} \varepsilon\mathsf{F} \centerdot \mathsf{F}\eta & = & \mathbf{1}; \\ \mathsf{G}\varepsilon \centerdot \eta\mathsf{G} & = & \mathbf{1}; \end{array} \qquad \text{and} \qquad \begin{array}{rcl} \varepsilon_\varepsilon * \mathsf{F}\eta\mathsf{G} & = & \mathbf{1}_\varepsilon; \\ \mathsf{G}\varepsilon_{\mathsf{F}f} & = & \mathbf{1}. \end{array}$$

A KZ-adjunction determines a family of equivalences between the hom-categories $\mathsf{Hom}_{\underline{\underline{D}}}(\mathsf{F}C, D) \overset{\cdot}{\cong} \mathsf{Hom}_{\underline{\underline{C}}}(C, \mathsf{G}D)$ which also take a particular form. For any C in $\underline{\underline{C}}$ and D in $\underline{\underline{D}}$, let $\mathcal{H}$ and $\mathcal{K}$ be the functors defined below.

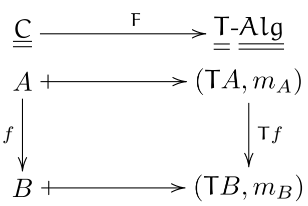
$$\begin{array}{ccc} \mathsf{Hom}_{\underline{\underline{D}}}(\mathsf{F}C, D) & \xrightarrow{\mathcal{H}} & \mathsf{Hom}_{\underline{\underline{C}}}(C, \mathsf{G}D) \\ f & \longmapsto & (\mathsf{G}f \circ \eta_C) \\ {\scriptstyle\alpha}\Downarrow & & \Vert{\scriptstyle \mathsf{G}\alpha * \eta_C} \\ g & \longmapsto & (\mathsf{G}g \circ \eta_C) \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \mathsf{Hom}_{\underline{\underline{C}}}(C, \mathsf{G}D) & \xrightarrow{\mathcal{K}} & \mathsf{Hom}_{\underline{\underline{D}}}(\mathsf{F}C, D) \\ f & \longmapsto & (\varepsilon_D \circ \mathsf{F}f) \\ {\scriptstyle\alpha}\Downarrow & & \Vert{\scriptstyle \varepsilon_D * \mathsf{F}\alpha} \\ g & \longmapsto & (\varepsilon_D \circ \mathsf{F}g) \end{array}$$

Then, we have that $\mathcal{H} \circ \mathcal{K} = Id$ and that the family $\{\psi_f = \epsilon_f * \mathsf{F}\eta_C\}_{f \in \mathsf{Hom}_{\underline{\underline{D}}}(\mathsf{F}C, D)}$ gives a natural isomorphism $\psi: \mathcal{K} \circ \mathcal{H} \stackrel{.}{\to} Id$. Moreover, (the collection of all the) $\mathcal{H}$ is *natural* in C and D while $\mathcal{K}$ is natural in C and natural up to isomorphism in D. This characterization of KZ-adjunctions in terms of equivalences of hom-categories gives us the following proposition, which makes explicit the kind of universality enjoyed by $\mathsf{F}C$.

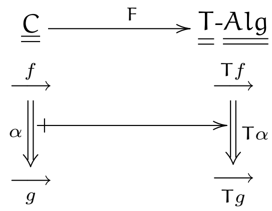
Proposition A.12
For any $f: C \to \mathsf{G}D$ in $\underline{\underline{C}}$, the morphism $\varepsilon_D \circ \mathsf{F}f: \mathsf{F}C \to D$ of $\underline{\underline{D}}$ enjoys the following universal property (wrt. η_C): for any $g: \mathsf{F}C \to D$ and $\alpha: f \Rightarrow g \circ \eta_C$, there exists a unique $\beta: \varepsilon_D \circ \mathsf{F}f \Rightarrow g$ such that $\mathsf{G}\beta * \eta_C = \alpha$.
Moreover, if C' is equivalent to $\mathsf{F}C$ in $\underline{\underline{D}}$, i.e., such that there exist $h: \mathsf{F}C \to C'$ and $k: C' \to \mathsf{F}C$ together invertible 2-cells $k \circ h \Rightarrow id_{\mathsf{F}C}$ and $h \circ k \Rightarrow id_{C'}$, then $\varepsilon_D \circ \mathsf{F}f \circ k$ enjoys the same universal property (wrt. $\eta_C \circ h$) of $\varepsilon_D \circ \mathsf{F}f$.

The next proposition makes explicit the link between KZ-doctrines and KZ-adjunctions.

Proposition A.13
Let $(\mathsf{T}, y, m, \lambda)$ be a KZ-doctrine and consider the 2-functor



and



Let $\varepsilon_{(A,\mathfrak{a})}$ be $\mathfrak{a}: \mathsf{FU}(A, \mathfrak{a}) = (\mathsf{T}A, m_A) \to (A, \mathfrak{a})$, where U is the forgetful functor $\underline{\underline{\mathsf{T}}}\text{-}\underline{\underline{\mathsf{Alg}}} \to \underline{\underline{C}}$, and for each T-homomorphism f let ε_f be ϕ_f, the canonical 2-cell associated to f.
Then $(\mathsf{F}, \mathsf{U}, y, \varepsilon)$ is a KZ-adjunction from $\underline{\underline{C}}$ to $\underline{\underline{\mathsf{T}}}\text{-}\underline{\underline{\mathsf{Alg}}}$.

The reader is suggested to restate Proposition A.12 for the KZ-adjunction obtained from a KZ-doctrine and to compare the result with Proposition 4.15.

In the paper we see how the notions concerning KZ-doctrines are perfectly suited to describe the cocompletion construction. Of course, this is not surprising since they arose from Kock's work on completion of categories [19].

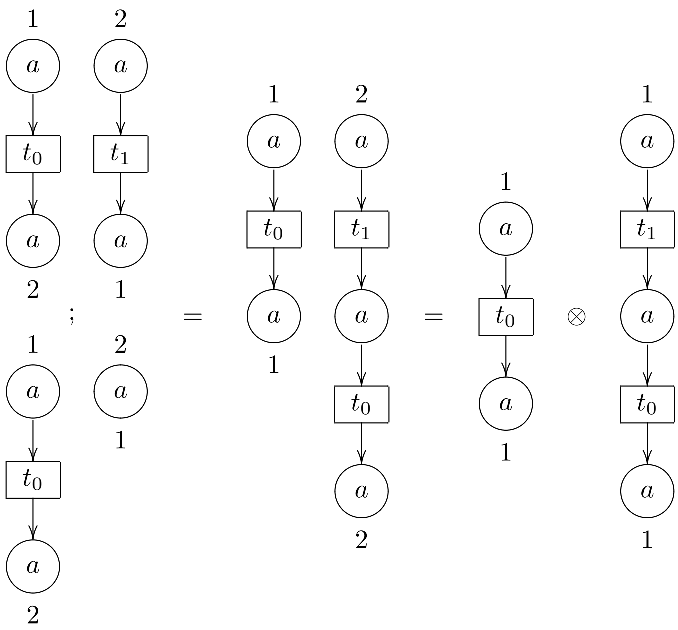


Figure 2: An example of the algebra of concatenable processes

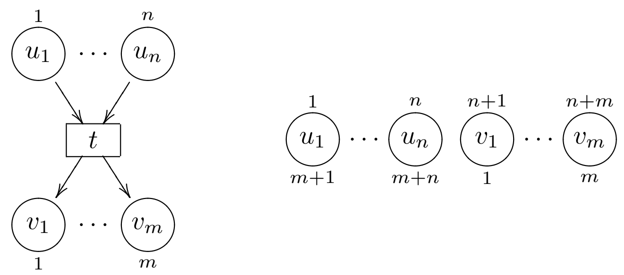


Figure 3: A transitions $t_{u,v}\colon u \to v$ and the symmetry $\gamma(u, v)$ in $\mathcal{IP}[N]$