A mia madre Liana
ed a Rosanna

The moving power of mathematical invention
is not reasoning but imagination.
Augustus De Morgan

Foreword

I still wonder whether the title of this thesis is the right one or whether it should have been somehow different. I admit I like it. However, the first idea that came to my mind and that, in my opinion, described properly the thesis has been something very similar to the following:

> *"On the semantics and on the structure of place/transition Petri nets in their status of nicely intuitive formalization of the notion of causality and influential instance of the concurrency paradigm, and on the relationships which they bear to other models for concurrency, investigated by means of processes, unfoldings, infinite computations, some algebra, some category theory, ..."*

The reader will probably agree that a similar title would not fit nicely in whatever page layout. Since this was certainly my opinion, I followed the aesthetic motivation and I decided to shorten it drastically, arriving gradually to the actual title. However, I must admit that probably my choice has not been nice to the *"other models for concurrency"* which disappeared completely from the title. (Not from the thesis, though!) Well, I shall make justice here by saying that, despite the title, I consider the part of the thesis dealing with "other models" as relevant as the others.

A similar remark applies indeed to categories: although category theory is not explicitly mentioned in the title, it plays a considerable role in the formal development to follow. This is because it provides a formal framework in which certain interesting questions can be asked naturally and (sometimes) answered.

In order for the reader to get in tune with the author's choice of dedicating (large part of) his doctoral thesis to the issue of categorical semantics for Petri nets, I indicate below the two "postulates" upon which such a choice is based.

i) Petri net are interesting from the point of view of noninterleaving concurrency, since, informally speaking, they are flexible enough to model all the sensibile cause/effect interactions which may occur between a set of computing agents;

ii) Category Theory is useful both "in the small" to look for appropriate axiomatic descriptions of algebraic structures, e.g., such as the processes of net, and "in the large" to establish formal relationships between different structures, e.g., how to translate uniformly from Petri nets to transition systems.

In other words, the first point above means that a convincing causal semantics for Petri nets is likely to yield convincing causal semantics for a large class of other models. The second point, instead, implies that the categorical paradigm possesses a good ability of abstracting away from undesired details, while keeping consistency of the desired ones. I would consider it an excellent outcome if this thesis could convince a skeptic reader at least of the second postulate.

Acknowledgements

It is customary to start the acknowledgment list of a thesis by saying sometime like "without my supervisor this thesis could not have been written". Well, I certainly will not escape this tradition. Actually, I dare say that never ever as in my case this is indeed true. In fact, the topics treated and the tools exploited here were at the beginning so hostile to me, that I had to be pushed rather strongly and repeatedly in order to get through with them. I heartily give full credit for this to my supervisor Ugo Montanari. But of course there is more than that. I met him when I was a third year undergraduate student: since then he supported almost all my professional activities and he taught me, directly or indirectly, almost everything I know about computer science.

In the autumn 91, I spent three months in California working in strict contact with José Meseguer at the SRI International. First of all, I like to remember the nice time I had there and to thank Narciso Martí-Oliet and José Meseguer for their friendship. But clearly, that stay also represented a very good professional opportunity for me. In particular, I learnt from José what it means to write a paper and to put it in a good shape. (I hope that the reader will not raise strong objections!)

From March to August 92 I was at DAIMI in Aarhus. The relevance of that stay for my education is witnessed by the fact that the works I produced there jointly with Mogens Nielsen and Glynn Winskel have their place in this thesis as Chapter 2. I really like to mention that, thanks to the favorable environment that DAIMI provides and thanks to the friendship that Mogens and Glynn showed to me, being in particular always lavish with helpful suggestions, I think I really made the best out of my stay in Denmark. Probably above all the rest, that period raised my confidence in my working skills, so marking an improvement in my approach to the research activity. In Aarhus I also happened to meet more than once Madhavan Mukund, with whom I shared a flat for five months, P.S. Thiagarajan and Jeremy Gunawardena. I have benefited a lot from several discussions with them.

During the entire PhD programme I have of course taken advantage by the high standard of the Computer Science Department in Pisa, which provided several good courses and seminars. In particular, I had good advices from PierPaolo Degano and Roberto Gorrieri. Moreover, I have enjoyed many interesting discussions with Andrea Corradini, Roberto Di Meglio, Gianluigi Ferrari, Simone Martini and Luca Roversi. I would also like to thank my room-mates Fabio Gadducci, Corrado Priami, Gioia Ristori and Laura Semini for the company. (Paola Quaglia is *intentionally* not thanked.) In particular, I cannot forget that Corrado and Fabio heroically tolerated the smoke of my cigarettes.

During the years 90–92 I have taken part in the project "Modelling Distributed Concurrency" at Hewlett-Packard Laboratories, Pisa Science Centre. I thank the project managers Wulf Rehder, Monica Frontini and Lorenzo Coslovi. I mostly regret that my colleagues and good friends Luca Aceto, Cosimo Laneve, Martina Marré and Daniel Yankelevich working at that project have all left at the beginning of 93. Doubtless, had I had their suggestions, this thesis would have been better.

During the last months several people have spent some time with me chatting about work and other issues. In particular, I warmly thank Gerard Boudol, Ilaria Castellani, Matthew Hennessy, Furio Honsell, David Murphy, Nino Salibra and Colin Stirling. In connection with Chapter 3, I had helpful long discussions with Andrea Corradini and Pino Rosolini. Also Bart Jacobs, Anders Kock and Eugenio Moggi gave me good advices on this topic.

I am pleased to thank Narciso Martí-Oliet and Axel Poigné, the reviewers of this thesis, for their nice reports and for the helpful suggestions they gave. Unofficially, also Andrea Corradini and Fabio Gadducci have acted as reviewers spotting many typos in the text. Of course, all the remaining mistakes are mine.

Switching to the most personal side of the story, I want to thank Rosanna, who shared with me the last ten years. And now, it comes the turn of my parents, grandparents and friends. Well, I'd better switch to Italian now... Ringrazio i miei genitori Liana e Antonio per il loro continuo supporto morale. Non dimentico i miei nonni Maria Cristina e Aldo. Constatare la loro soddisfazione nel vedermi progredire negli studi, dopo una partenza ben poco promettente, è stato un ulteriore stimolo per me. Voglio anche ricordare i miei amici Sara Tesini e Pietro Falaschi che negli ultimi tempi mi hanno dimostrato un insospettato affetto. Infine, nonostante la sua pazzia galoppante che ha causato qualche problema, non posso fare a meno di riconoscere che il mio vecchio maestro Giorgio Rugieri ha giocato un ruolo rilevante nella mia formazione.

This thesis has been written using TEX, LATEX and XY-pic.

Contents

On the Semantics of Petri Nets: Processes, Unfoldings and Infinite Computations. An Overview

This thesis is concerned with *Petri nets* [109] and related models of concurrency. Petri nets are unanimously considered among the most representative *models for concurrency*, since they are a fairly simple and natural model of *concurrent* and *distributed* computation. Notwithstanding their "naturality"—perhaps because of that—Petri nets are, in our opinion, still far from being completely understood.

The focus here is on the semantics and on the structure of Petri nets. To address the issue, we exploit standard categorical tools which on the one hand help in *axiomatizing* the structure of net computations as *monoidal categories*, via left adjoint functors corresponding to free constructions, and, on the other hand, provide a nice framework to formalize the intuitive connections between nets and other models of concurrency, via *adjunctions* which express *translations* between models.

Process Semantics for Petri Nets

In recent works, Degano, Meseguer and Montanari [97, 16] have shown that the semantics of Petri nets can be understood in terms of *symmetric monoidal categories*—where objects are states, arrows processes, and the tensor product and the arrow composition model respectively the operations of parallel and sequential composition of processes. This yields an axiomatization of the causal behaviour of nets as an *essentially algebraic theory* whose models are monoidal categories.

More precisely, [16] introduces the *concatenable processes* of a Petri net N, a slight refinement of Goltz and Resig's non-sequential processes of N [36] on which an operation of sequential composition can be defined, and shows that they can be characterized abstractly as the arrows of a symmetric strict monoidal category $\mathcal{P}[N]$. However, [16] provides only a partial axiomatization of the non-sequential behaviour of N, since the construction of the category $\mathcal{P}[N]$ is based on a concrete, seemingly ad hoc chosen, underlying category of symmetries. We present here a *completely abstract*, *purely algebraic* description of the category of concatenable processes of N.

The construction of the concatenable processes of N is unsatisfactory in another respect: it is *not functorial*. In other words, given a morphism between two nets, which can be safely thought of as a *simulation*, it may not be possible to identify a corresponding monoidal functor between the respective categories of computations. This situation, besides showing that perhaps our undestanding of the structure of Petri nets is still incomplete, prevents us to identify the *category* (of the categories) *of net behaviours*, i.e., to axiomatize the behaviour of Petri nets "in the large".

We present an analysis of the functoriality issue and a possible solution based on the new notion of *strong concatenable processes* of N, a refinement of concatenable processes still rather close to the standard notion of non-sequential process. We show that, similarly to the concatenable processes, the strong concatenable processes of N can be axiomatized as the arrows of a symmetric strict monoidal category $\mathcal{Q}[N]$, and that, differently from $\mathcal{P}[\_]$, $\mathcal{Q}[\_]$ is a functor. The key feature of $\mathcal{Q}[\_]$ is that it associates to a net N a monoidal category whose objects form a free, *non-commutative* monoid. The reason for renouncing to the commutativity of such monoids is a strong *negative* new result which shows that $\mathcal{Q}[\_]$ is a reasonable proposal: under very mild assumptions, *no* mapping from nets to symmetric strict monoidal categories with commutative monoids of objects can be extended to a functor. Clearly, the functoriality of $\mathcal{Q}[\_]$ provides a category of symmetric monoidal categories which is our attempt to identify the category of net computations.

Unfolding Semantics for Petri Nets

A seminal approach to net semantics is the Nielsen, Plotkin and Winskel's unfolding [106]. This approach explains the behaviour of nets through a chain of *coreflections*

$$\underline{\mathsf{Safe}} \hookleftarrow \underline{\mathsf{Occ}} \hookleftarrow \underline{\mathsf{PES}} \hookleftarrow \underline{\mathsf{Dom}},$$

where $\underline{\mathsf{Safe}}$, $\underline{\mathsf{Occ}}$, $\underline{\mathsf{PES}}$ and $\underline{\mathsf{Dom}}$ are, respectively, the categories of *1-safe* nets, *occurrence* nets, *prime event structures* and *dI-domains*. Roughly speaking, the *unfolding semantics* consists, as the name indicates, in "unfolding" a net to simple denotational structures in which the identity of every event in its computations is unambiguous. The relevance of these constructions resides in the fact that they provide 1-safe Petri nets with an abstract semantics where *causality* is taken in full account. In addition, the unfolding a 1-safe net N to an occurrence net has the great merit of collecting together all the processes of N as a *whole*, so accounting at the same time for concurrency and nondeterminism.

We show how the unfolding semantics of 1-safe nets can be extended to the full category of Petri nets, by presenting a chain of *adjunctions*

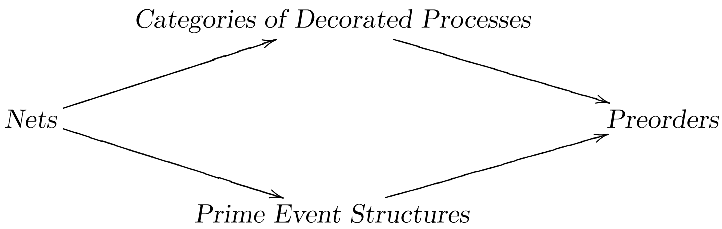
$$\underline{\mathsf{PTNets}} \leftharpoonup \underline{\mathsf{DecOcc}} \leftharpoonup \underline{\mathsf{Occ}},$$

PTNets and DecOcc being, respectively, the category of Petri nets and a category of appropriately *decorated occurrence* nets.

This work has already appeared in [98, 99].

Process versus Unfolding Semantics for Petri Nets

In order to reconcile the process and the unfolding semantics for Petri nets, we introduce a new symmetric strict monoidal category $\mathcal{DP}[N]$ whose arrows represent a new notion of process, here called *decorated concatenable processes.* Decorated concatenable processes give a *process-oriented* account of the unfolding construction, in the precise sense that, for a net N, the preorder category of decorated concatenable processes and the partial order of finite configurations of the associated event structure are equivalent. In other words, the following diagram commutes up to isomorphism.



It is to be stressed that our concern here is at the level of a *single* net, which means that the correspondence we establish is not functorial. Nevertheless, we think that this is an interesting result, since $\mathcal{DP}[\_]$ provides a natural and unified account of the *algebraic/category-theoretic*, the *process-oriented* and the *unfolding/denotational* views of net semantics.

This work appears also as [100, 101].

Models Related to Petri Nets

The case graph of a Petri net, and thus its behaviour, can be understood also as a *transition system* with some added structure which makes it possible to identify a relation of concurrency between transitions, see e.g. [105, 146], where this is stated in categorical terms. This brings to the foreground several models which have recently appeared in literature and which are based on the idea of extending transition systems to a noninterleaving model, see e.g. [129, 2, 132, 115, 146], and calls for an investigation of the relationships between nets and such models.

We present here a broad study of *transition systems with independence* achieved by formally relating them with several other models by means of *reflections* and *coreflections.* Table A summarizes the categories of models we consider and Figure A, where arrows represent coreflections and "backward" arrows represent reflections, the cube of relationships we prove.

Beh./Int./Lin.	Hoare languages	HL
Beh./Int./Bran.	Synchronization Trees	ST
Beh./Nonint./Lin.	deterministic Labelled Event Structures	dLES
Beh./Nonint./Bran.	Labelled Event Structures	LES
Sys./Int./Lin.	deterministic Transition Systems	dTS
Sys./Int./Bran.	Transition Systems	TS
Sys./Nonint./Lin.	deterministic Transition Systems with Independence	dTSI
Sys./Nonint./Bran.	Transition System with Independence	TSI

Table A: The models

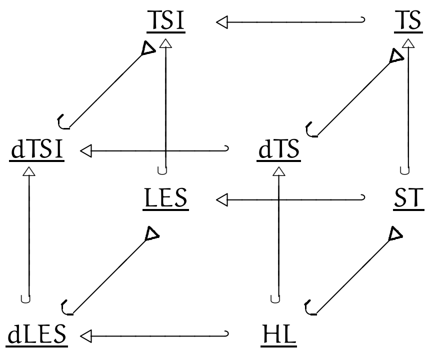


Figure A: The cube of relationships

It is interesting to observe that each of the models chosen is a representative of one of the eight classes of models obtained by combining the following three parameters, which are among the most relevant parameters with regard to which models for concurrency can be classified:

Behaviour or System models;
Interleaving or Noninterleaving models;
Linear time or Branching time models.

When modelling a system, of course, a choice concerning these parameters corresponds to choosing the level of abstraction of the resulting semantics. Then, once the behaviour of a system—in particular of a Petri net—has been specified in terms of a transition system with independence, it is possible to translate it by means of the adjoint functors in Figure A to another semantics at the desired level of abstraction.

Apart from transition systems with independence, which are new, and deterministic labelled event structures, each of the other models is a canonical and "universally" accepted representative of its class. Unfortunately, for the class of behavioural, linear-time, noninterleaving models there does not, at present, seem to be an obvious choice of a corresponding canonical model. However, although not canonical, the choice of deterministic labelled event structures is certainly fair and, more important, it is *not* at all compelled. In order to show this, and for the sake of completeness, we investigate the relationships between this model and two of the most-studied models in the class: the pomsets of Pratt [114], and the traces of Mazurkiewicz [94]. In particular, we introduce a category $\underline{\mathsf{dSL}}$ of deterministic languages of pomsets and a category $\underline{\mathsf{GTL}}$ of generalized trace languages in which the independence relation is context-dependent. These categories are proved equivalent to $\underline{\mathsf{dLES}}$. In other words, we present the chain of equivalences

$$\underline{\mathsf{dSL}} \cong \underline{\mathsf{dLES}} \cong \underline{\mathsf{GTL}}$$

which, besides identifying models which can replace $\underline{\mathsf{dLES}}$ in Figure A, also introduces new interesting *deterministic behavioural models* for concurrency and formalizes their mutual relationships.

This work has already appeared in [124, 125, 126].

Infinite Computations of Petri Nets

When modeling perpetual systems, describing finite processes is not enough: we need to consider also *infinite behaviours*. Actually, infinite computations of Petri nets have occasionally been considered [9], mainly in connection with acceptors of ω-languages [138, 12, 49]. These approaches, of course, focused just on sequential computations and treat nets simply as generalized automata. Our interest, instead, resides on *processes*, i.e., on structures able to describe computations more intensionally, taking into account causality. More precisely, we aim to define an *algebra* of net computations which includes *infinite processes* as well. To the best of our knowledge, this issue is still completely unexplored.

In order to fulfill our programme, we first address the general issue of *completion of categories* by colimits of arbitrary chains. Since chain cocompleteness coincides with the completeness by colimits taken over filtered index categories and for technical reasons filtered colimits are also needed, we present the theory of cocompletion of categories by such kind of colimits. More precisely, for $\underline{\underline{\mathsf{CAT}}}$ the 2-category of locally small categories, for any infinite cardinal $\aleph$, we define a *Kock-Zöberlain (KZ-)doctrine* [76, 149] $\mathtt{Ind}(\_)_\aleph \colon \underline{\underline{\mathsf{CAT}}} \to \underline{\underline{\mathsf{CAT}}}$ which associates to each locally small category its completion by $\aleph$-filtered colimits, or its $\aleph$-*ind*-completion (ind standing for *ind*uctive), and such that the $\aleph$-filtered cocomplete categories with functors preserving $\aleph$-filtered colimits are exactly the *algebras* for the doctrine. Although related results have already appeared in several different forms in the literature, e.g. [71, 43, 76, 149, 58], the presentation here is a rather complete survey which integrates the best features of the existing approaches and explores the application of these ideas to computer science.

		small		locally small	
		monoidal	strict monoidal	monoidal	strict monoidal
STRICT	non symmetric	$\underline{\underline{\mathsf{MonCat}}}$	$\underline{\underline{\mathsf{sMonCat}}}$	$\underline{\underline{\mathsf{MonCAT}}}$	$\underline{\underline{\mathsf{sMonCAT}}}$
	symmetric	$\underline{\underline{\mathsf{SMonCat}}}$	$\underline{\underline{\mathsf{SsMonCat}}}$	$\underline{\underline{\mathsf{SMonCAT}}}$	$\underline{\underline{\mathsf{SsMonCAT}}}$
	strictly symmetric	$\underline{\underline{\mathsf{sSMonCat}}}$	$\underline{\underline{\mathsf{sSsMonCat}}}$	$\underline{\underline{\mathsf{sSMonCAT}}}$	$\underline{\underline{\mathsf{sSsMonCAT}}}$
STRONG	non symmetric	$\underline{\underline{\mathsf{MonCat}}}^{\star}$	$\underline{\underline{\mathsf{sMonCat}}}^{\star}$	$\underline{\underline{\mathsf{MonCAT}}}^{\star}$	$\underline{\underline{\mathsf{sMonCAT}}}^{\star}$
	symmetric	$\underline{\underline{\mathsf{SMonCat}}}^{\star}$	$\underline{\underline{\mathsf{SsMonCat}}}^{\star}$	$\underline{\underline{\mathsf{SMonCAT}}}^{\star}$	$\underline{\underline{\mathsf{SsMonCAT}}}^{\star}$
	strictly symmetric	$\underline{\underline{\mathsf{sSMonCat}}}^{\star}$	$\underline{\underline{\mathsf{sSsMonCat}}}^{\star}$	$\underline{\underline{\mathsf{sSMonCAT}}}^{\star}$	$\underline{\underline{\mathsf{sSsMonCAT}}}^{\star}$
MONOIDAL	non symmetric	$\underline{\underline{\mathsf{MonCat}}}^{\star\star}$	$\underline{\underline{\mathsf{sMonCat}}}^{\star\star}$	$\underline{\underline{\mathsf{MonCAT}}}^{\star\star}$	$\underline{\underline{\mathsf{sMonCAT}}}^{\star\star}$
	symmetric	$\underline{\underline{\mathsf{SMonCat}}}^{\star\star}$	$\underline{\underline{\mathsf{SsMonCat}}}^{\star\star}$	$\underline{\underline{\mathsf{SMonCAT}}}^{\star\star}$	$\underline{\underline{\mathsf{SsMonCAT}}}^{\star\star}$
	strictly symmetric	$\underline{\underline{\mathsf{sSMonCat}}}^{\star\star}$	$\underline{\underline{\mathsf{sSsMonCat}}}^{\star\star}$	$\underline{\underline{\mathsf{sSMonCAT}}}^{\star\star}$	$\underline{\underline{\mathsf{sSsMonCAT}}}^{\star\star}$

Legenda: The data in the definition of monoidal categories and functors (see Section A.2 for the relevant definitions) give rise to many combinations according to whether the monoidality and the symmetry are strict or not and so on. To fix notation, we propose the nomenclature above. The idea is that, since we consider the categories with *strict* monoidal functors as the "normal" categories, we explicitly indicate with simple and double superscripted $\star$'s the categories with, respectively, *strong* monoidal functors and simply *monoidal* functors. This is indicated by the leftmost column in the table. Clearly, the categories of symmetric monoidal categories consists always of *symmetric* monoidal functors. Moreover, sS means *strictly symmetric* while sMon means *monoidal strict*. We distinguish between categories of locally small and of small categories by using uppercase letters in the first case. Of course, there is an analogous table for the categories above considered as one-dimensional categories. We use a single underline in order to distinguish the two situations.

Table B: A nomenclature for categories of monoidal categories

Then, we show that the cocompletion doctrine, when applied to a symmetric monoidal category, yields a symmetric monoidal category. More precisely, we show that the KZ-doctrine $\mathtt{Ind}(\_)_{\aleph}$ lifts to a KZ-doctrine on any of the 2-categories of monoidal categories appearing in Table B, which, from the technical point of view, is main result of Chapter 3.

We discuss how this result generalizes the algebraic approach to the process semantics of Petri nets to the case in which infinite processes and composition operations on them are considered. In particular, the infinite processes of a Petri net can in this way be given an algebraic presentation which combines the *essentially algebraic* presentation of monoidal categories with the *monadic* presentation of their completion in terms of KZ-doctrines.

We should like to remark mention that, since in the last years many computing systems have been given a semantics through the medium of category theory, the general pattern being to look at objects as representing states and at arrows as representing computations, the theory of cocompletion of categories yields a general method to construct and manipulate infinite computations of those systems. The main purpose of Chapter 3 is to substantiate this claim by studying in detail the case of Petri nets.

This work appears also as [123].

Put up in a place
where it's easy to see,
the cryptic admonishment T. T. T.
When you feel how depressingly
slowly you climb,
it's well to remember that
Things Take Time.
Piet Hein, Grooks

"Fermi!" disse il Grande Bastardo.
"Stiamo inventando troppe cose in una volta."
Stefano Benni, La compagnia dei Celestini

Chapter 1

Processes and Unfoldings

ABSTRACT. The sematics of Petri nets has been investigated in several different ways. Apart from the classical "token game", we can model the behaviour of Petri nets via non-sequential processes, via algebraic approaches, which view Petri nets as essentially algebraic theories whose models are monoidal categories, and, in the case of safe nets, via unfolding constructions, which provide formal relationships between nets and domains.

In this chapter we extend Winskel's result to PT nets and we show that the unfolding semantics can be reconciled with the process-oriented and the algebraic points of view. In our formal development a relevant role is played by a category of occurrence nets appropriately decorated to take into account the history of tokens. The structure of decorated occurrence nets at the same time provides natural unfoldings for PT nets and suggests a new notion of processes, the decorated processes, which induce on Petri nets the same semantics as that of the unfolding. In addition, the decorated processes of a net form a symmetric monoidal category which yield an algebraic explanation of net behaviours.

In addition, we propose solutions to some open problems in the algebraic/categorical theory of net processes.

I would rather discover one cause
that gain the kingdom of Persia.

Democritus

I problemi sono universali.
Le soluzioni sono individuali.
Just do it.

Nike's commercial

This chapter is based on joint work with José Meseguer and Ugo Montanari [98, 99, 100, 101] and on [121, 122].

Introduction

Petri nets, introduced by C.A. Petri in [109] (see also [110, 119, 120, 30]), are a widely used model of concurrency. This model is attractive from a theoretical point of view because of its simplicity and because of its intrinsically concurrent nature, and has often been used as a semantic basis on which to interpret concurrent languages (see for example [140, 108, 139, 15]).

For *Place/Transition (PT) nets*, having a satisfactory semantics—one that does justice to their truly concurrent nature, yet is abstract enough—remains in our view an unresolved problem. Certainly, several different semantics have been proposed in the literature. Most of them can be coarsely classified as *process-oriented* semantics, *unfolding* semantics or *algebraic* semantics, though the latter class is not as clearly delimited and not as widely diffused as the former two. Of course, such classes are not at all disjoint, as this chapter aims to support. We further discuss these approaches below.

At the most basic operational level we have of course the "token game", the computational mechanisms semantics of Petri nets. The development of theory Petri nets, focusing on the noninterleaving aspects of concurrency, brought to the foreground various notions of process, e.g. [111, 36, 9, 97, 16]. Generally speaking, Petri net processes—whose standard version is given by the Goltz-Reisig *non-sequential processes* [36]—are structures needed to account for the *causal relationships* which rule the occurrences of events in computations. Thus, ideally, processes are simply computations in which the explicit information about such causal connections is added. More precisely, since it is a well-established idea that, as far as the theory of computation is concerned, causality can be faithfully described by means of partial orderings—though "heretic" ideas appear sometimes—abstractly, the processes of a net N are ordered sets whose elements are labelled by transitions of N. In concrete, in order to describe exactly what multisets of transitions are processes of and what are not, one defines a process of N to be a map $\pi\colon \Theta \rightarrow N$ which maps transitions to transitions and places to places respecting the "bipartite graph structure" of nets, where Θ is a *finite deterministic occurrence net*, i.e., roughly speaking, a finite deterministic 1-safe acyclic net whose "flow relation" induces a partial ordering of its elements, the minimal and maximal elements of which are places. Of course, the role of π is to "label" the places and the transitions of Θ with places and transitions of N in a way compatible with the structure of N.

The main criticism raised against process models is that they do not provide a semantics for a net as a *whole*, but specify only the meaning of single, deterministic computations, while the accurate description of the fine interplay between concurrency and nondeterminism is one of the most valuable features of nets.

Other semantic investigations have capitalized on the *algebraic structure* of PT nets, first noticed by Reisig [119] and later exploited by Winskel to identify a sensible notion of *morphism* between nets [141, 144]. The clear advantage of these approaches resides in the fact that they tend to clarify both the structure of the single PT net, so giving insights about their essential properties, and the global structure of the class of all nets. Providing, for example, useful net combinators associated to standard categorical constructions such as product and coproduct, which can be used to give a simple account of corresponding compositional operations at the level of a concurrent programming language, such as various forms of parallel and non-deterministic composition [143, 144, 97].

An original interpretation of the algebraic structure of PT nets has been proposed in [97], where the theory of *monoidal categories* is exploited to the purpose. Unlike the preceding approaches, [97] yields an algebraic theory of Petri nets in which notions such as firing sequence, case graph, relationships between net descriptions at different abstraction levels, duality and invariants find adequate algebraic/categorical formulations. However, generally speaking, the algebraic approaches are often too concrete, and a more abstract semantics—one allowing greater semantic identifications between nets—would be sometimes preferable.

Roughly speaking the *unfolding semantics* consists, as the name indicates, in "unfolding" a net to simple denotational structures in which the identity of every event in its computations is unambiguous. More precisely, very attractive formulation for such a semantics would be an *adjoint functor* assigning an abstract denotation to each PT net and preserving certain compositional properties in the assignment. This is exactly what Winskel has done for the subcategory of safe nets [143]. In that work—which builds on the previous work [106]—the denotation of a safe net is a *Scott domain* [128, 133], and Winskel shows that there exists a coreflection—a particularly nice form of adjunction—between the category $\underline{\mathsf{Dom}}$ of (coherent) *finitary prime algebraic domains* and the category $\underline{\mathsf{Safe}}$ of *safe Petri nets*. This construction is completely satisfactory: from the intuitive point of view it gives the "truly concurrent" semantics of safe nets in the most universally accepted type of model, while from the formal point of view the existence of an adjunction guarantees its "naturality". Winskel's coreflection factorizes through the chain of coreflections

$$\underline{\mathsf{Safe}} \underset{\hookleftarrow}{\xrightarrow{\;\mathcal{U}[\_]\;}} \underline{\mathsf{Occ}} \underset{\mathcal{N}[\_]}{\overset{\mathcal{E}[\_]}{\rightleftarrows}} \underline{\mathsf{PES}} \underset{\mathcal{P}r[\_]}{\overset{\mathcal{L}[\_]}{\rightleftarrows}} \underline{\mathsf{Dom}}$$

where $\underline{\mathsf{PES}}$ is the category of *prime event structures* (with binary conflict relation), which is equivalent to $\underline{\mathsf{Dom}}$, $\underline{\mathsf{Occ}}$ is the category of *occurrence nets* [143] and $\hookleftarrow$ is the inclusion functor.

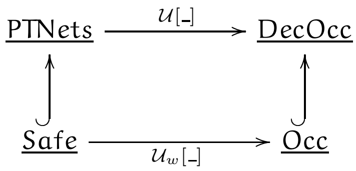
Recently, various attempts have been made to extend this chain or, more generally, to identify a suitable semantic domain for PT nets. Among them, we recall [115], where, in order to obtain a model "mathematically more attractive than Petri nets", a *geometric* model of concurrency based on n-categories as models of higher dimensional automata is introduced, but it is not clear whether the modelling power obtained is greater than that of ordinary PT nets; [50], in which the authors give semantics to PT nets in terms of generalized trace languages and discuss how using their work it could perhaps be possible to obtain a concept of unfolding for PT nets; and [23], where the unfolding of Petri nets is given in term of a branching process. However, the nets considered in [23] are not really PT nets because their transitions are restricted to have pre and post-sets where all places have no multiplicities. A yet more recent approach is [51], where the unfolding is explained in terms of a notion of local event structure. Finally, we would like to cite [105, 46, 107].

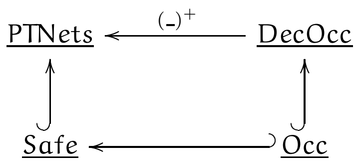
A large part of this chapter is devoted to present an extension of Winskel's approach from safe nets to the category of PT nets. We define the *unfoldings* of PT nets and relate them by an *adjunction* to occurrence nets and therefore—exploiting the already existing adjunctions—to prime event structures and finitary prime algebraic domains. The adjunctions so obtained are extensions of the corresponding Winskel's coreflections.

The category $\underline{\mathsf{PTNets}}$ which we consider for the unfolding functor is quite general. Objects are PT nets in which markings may be infinite and transitions are allowed to have infinite pre- and post-sets, but, as usual, with finite multiplicities. The only technical restriction we impose, with respect to the natural extension to nets with infinite markings of the general formulation in [97], is the usual condition that transitions must have non-empty pre-sets. Actually, the objects of $\underline{\mathsf{PTNets}}$ strictly include those of the categories considered in [143, 144]. Although a technical restriction applies to the morphisms—they are required to map places belonging to the initial marking or to the post-set of the same transition to disjoint multisets—they are still quite general. In particular, the category $\underline{\mathsf{PTNets}}$ has *initial* and *terminal* objects, and has *products* and *coproducts* which faithfully model, respectively, the operations of parallel and non-deterministic composition of nets as in [144] and in [97]. It is worth remarking that, while coproducts do *not* exist in the categories of generally marked, non-safe PT nets considered in the above cited works, they do in $\underline{\mathsf{PTNets}}$. This quite interesting fact is due to the aforesaid restriction we impose on the arrows of $\underline{\mathsf{PTNets}}$.

Concerning the presentation of these results, in Section 1.5 we start the formal development regarding the unfolding semantics by defining the category $\underline{\mathsf{PTNets}}$. In the same section it is shown that it has products and coproducts. In Section 1.6 we introduce a new kind of nets, the *decorated occurrence nets*, which naturally represent the unfoldings of PT nets and can account for the multiplicities of places in transitions. They are occurrence nets in which places belonging to the post-set of the same transition are partitioned into *families*. Families are used to relate places corresponding in the unfolding to multiple instances of the same place in the original net. When all the families of a decorated occurrence net have cardinality one, we have (a net isomorphic to) an ordinary occurrence net. Therefore, $\underline{\mathsf{Occ}}$ is (isomorphic to) a full subcategory of DecOcc, the category of decorated occurrence nets. Products and coproducts of decorated occurrence nets are studied in the second part of Section 1.7.

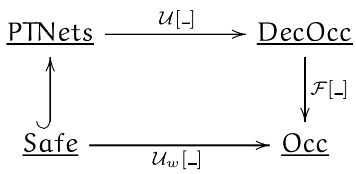
In Section 1.7, we show an adjunction $\langle (\_)^+, \mathcal{U}[\_] \rangle : \underline{\mathsf{DecOcc}} \rightharpoonup \underline{\mathsf{PTNets}}$ whose right adjoint $\mathcal{U}[\_]$ gives the unfoldings of PT nets. This adjunction restricts to Winskel's coreflection from Occ to Safe as illustrated by the following commutative diagrams:





i.e., the left and the right adjoint, when restricted respectively to Safe and Occ, coincide with the corresponding adjoints of Winskel's coreflection.

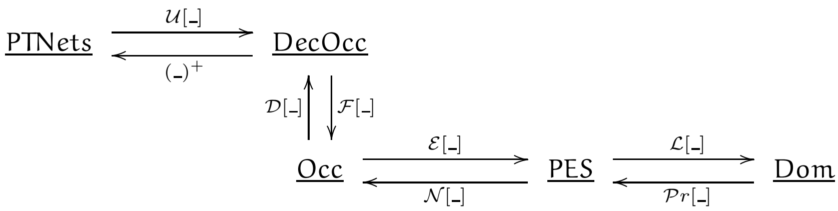
Then, in Section 1.8, we relate decorated occurrence nets to occurrence nets by means of an adjunction $\langle \mathcal{D}[\_], \mathcal{F}[\_] \rangle : \underline{\mathsf{Occ}} \rightharpoonup \underline{\mathsf{DecOcc}}$, where $\mathcal{F}[\_]$ is the *forgetful* functor which forgets about families. Moreover, the diagram



(1.1)

commutes.

Therefore, we get the desired adjunction between Dom and PTNets as the composition of the chain of adjunctions



It follows from the commutative diagram (1.1) that, when PTNets is restricted to Safe, all the right adjoints in the above chain coincide with the corresponding functors defined by Winskel. In this sense, this work generalizes the work of Winskel and gives an abstract, truly concurrent semantics for PT nets. Moreover, the existence of left adjoints guarantees the "naturality" of this generalization.

We have already mentioned that the three views of net semantics we are discussing are not mutually exclusive. In fact, a unification of the process-oriented and algebraic views has recently been proposed in [16] (see also [17]), by showing that the *commutative processes* [9] of a net N are isomorphic to the arrows of a strictly symmetric monoidal category $\mathcal{T}[N]$. Moreover, [16] introduced the *concatenable processes* of N to account, as the name indicates, for the issue of process concatenation. Let us briefly reconsider the ideas which led to their definition.

Given the definition of process discussed above, one can assign the natural *source* and *target* states to a process $\pi\colon \Theta \to N$ by considering the multisets of places of N which are the image via π of, respectively, the minimal and maximal (wrt. to the ordering identified by Θ) places of Θ. Now, the simple minded attempt to concatenate a process $\pi_1\colon \Theta_1 \to N$ with source u to a process $\pi_0\colon \Theta_0 \to N$ with target u by merging the maximal places of Θ_0 with the minimal places of Θ_1 in a way which preserves the labellings breaks down immediately. In fact, if more than one place of u is labelled by a single place of N, there are many ways to put in one-to-one correspondence the maximal places of Θ_0 and the minimal places of Θ_1 respecting the labels, i.e., there are many possible concatenations of π_0 and π_1, each of which gives a possibly different process of N. In other words, process concatenations has to do with *merging tokens* rather than *merging places*, as the above argument shows clearly.

Therefore, any attempt to deal with process concatenation must disambiguate the *identity* of each token in a process. This is exactly the idea of *concatenable processes*, which are simply Goltz-Reisig processes in which the minimal and maximal places carrying the same label are linearly ordered. This yields immediately an operation of concatenation, since the ambiguity about token indentities is broken using the additional information given by the orderings. Moreover, the existence of concatenation brings us easily to the definition of a category of concatenable processes. It turns out that such category is a *symmetric monoidal category* in which the tensor product represents faithfully the parallel composition of processes [16]. The relevance of this result resides in the fact that it describes processes of Petri nets as *essentially algebraic theories* (whose models are given by symmetric monoidal categories), which indeed is a remarkable property.

Naturally linked to the fact that they are algebraic structures, concatenable processes can also be described in abstract terms. In [16] the authors give such an abstract description by providing for each net N a symmetric monoidal category $\mathcal{P}[N]$ whose arrows are in one-to-one correspondence with the concatenable processes of N. Since this category is obtained as a "free" (in a weak sense to explained later) construction, this yields an explanation of Petri net processes as a *term algebra* by means of which one can easily "compute" with them. In particular, the distributivity of tensor product and arrow composition in monoidal categories is shown to capture the basic facts about net computations, so providing a *model of computation* for Petri nets.

However, strictly speaking, the category $\mathcal{P}[N]$ is only *partially* axiomatized in [16], since it is built on a concrete category of symmetries Sym_N which is constructed in an ad hoc way. After recalling in Section 1.1 the basic facts about the algebraic approach to Petri nets as given in [97, 16, 17], in Section 1.2 we show that also Sym_N can be characterized abstractly, thus yielding a *purely algebraic* and *completely abstract* characterization of the category of concatenable processes of N. Namely, we shall see that $\mathcal{P}[N]$ is the *free symmetric strict monoidal* category on the net N modulo two simple additional axioms. We remark that a similar conjecture has been proposed in [37].

In spite of accounting for algebraic and process-oriented aspects in a simple and unified way, the approach of [16] is still somehow unsatisfactory, since it is *not functorial*, a property which would be greatly recommendable indeed, since it would guarantee (to a certain extent) a "good" quality to the semantics induced by $\mathcal{P}[\_]$. More strongly, given a morphism between two nets, which can be safely thought of as a *simulation*, it may be not possible to identify a corresponding monoidal functor between the respective categories of computations. This situation, besides showing that perhaps our understanding of the algebraic structure of Petri nets is still incomplete, has also other drawbacks, the most relevant of which is probably that it prevents us to identify the *category* (of the categories) *of net behaviours*, i.e., to axiomatize the behaviour of Petri nets "in the large".

In Section 1.3, we present an analysis of the issue of functoriality of the process semantics of nets founded on symmetric monoidal categories, and a possible solution based on the new notion of *strong concatenable processes* of N, introduced in Section 1.4. These are a slight refinement of concatenable processes which are still rather close to the standard notion of process: namely, they are Goltz-Reisig processes whose minimal and maximal places are *linearly* ordered. In the paper we show that, similarly to the concatenable processes, the strong concatenable processes of N can be axiomatized as a free construction on N, by building on N an abstract symmetric monoidal category $\mathcal{Q}[N]$ and by proving that the arrows of $\mathcal{Q}[N]$ are isomorphic to the strong concatenable processes of N.

The key feature of $\mathcal{Q}[\_]$ is that, differently from $\mathcal{P}[\_]$, it associates to net N a symmetric monoidal category whose objects form a free, *non-commutative* monoid. The reason for renouncing to commutativity, a choice that at a first glance may seem odd, is explained in Section 1.3, where the following negative result is proved:

> *under very reasonable assumptions, no mapping from nets to symmetric monoidal categories whose monoids of objects are commutative can be extended to a functor, since there exists a morphism of nets which does not have a corresponding symmetric monoidal functor between the appropriate categories.*

Thus, abandoning the commutativity of the monoids of objects seems to be a price which is necessary to pay in order to obtain a functorial semantics of nets. Then, bringing such condition to the net level, instead of taking multisets of places as sources and targets of computations, we consider *strings* of places, a choice which leads us directly to strong concatenable processes. Correspondingly, a transition of N will be represented by many arrows in $\mathcal{Q}[N]$, one for each different "linearization" of its pre-set and its post-set. However, such arrows will be "linked" to each other by a "naturality" condition, in the precise sense that, when collected together, they form a natural transformation between appropriate functors. Such naturality axiom is the second relevant feature of $\mathcal{Q}[\_]$ and it is actually the key to keep the computational interpretation of the new category $\mathcal{Q}[N]$ surprisingly close to the category $\mathcal{P}[N]$ of concatenable processes.

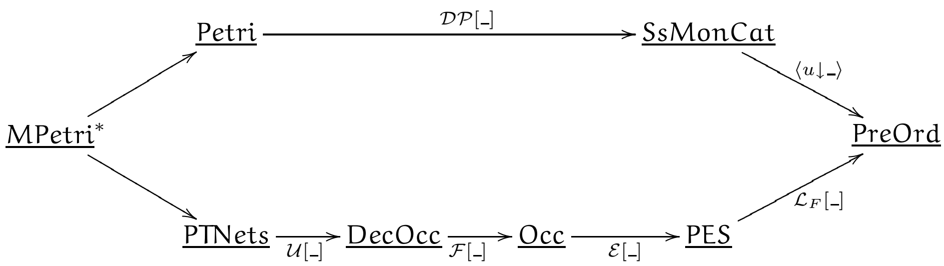
Clearly, the functoriality of $\mathcal{Q}[\_]$ allows us to identify a category $\underline{\mathsf{SSMC}}^{\otimes}$, together with a "forgetful" functor from it to the category of Petri nets, which represents our proposed axiomatization of net computations in categorical terms. Although we are aware that this contribution constitutes simply a first attempt towards the aim, we honestly think that the results illustrated here help to deepen the understanding of the subject. We remark that the refinement of concatenable processes represented by strong concatenable processes is similar and comparable to the one which brought from Goltz-Reisig processes to them. Of course, the passage here is more "hazardous" on the intuitive ground, since it brings us to model Petri nets, which after all are just multiset rewriting systems, using strings. However, it is important to remind the negative result in Section 1.3, which makes strong concatenable processes interesting, showing that, in a sense, they are the least refinement of Goltz-Reisig processes which yield an operation of sequential composition and a functorial treatment.

Getting back to the relationships between the various kind of semantics for Petri nets, concerning process and unfolding semantics, in the case of safe nets the question is easily answered by exploiting the existence of a coreflection of $\underline{\mathsf{Occ}}$ into $\underline{\mathsf{Safe}}$, which directly implies the existence of an isomorphism between the processes of N and the deterministic finite subnets of $\mathcal{U}[N]$, i.e., the finite configurations of $\mathcal{EU}[N]$. (More details about such correspondence will be given in Section 1.9.) Thus, in this case, the process and unfolding semantics coincide, although it should not be forgotten that the latter has the great merit of collecting together all the processes of N as a *whole*, so accounting at the same time for concurrency and nondeterminism.

In Section 1.9, we study the relationships between the algebraic paradigm, the process semantics described above and the unfolding semantics for PT nets given in Sections 1.5–1.8. We find that, in the context of general PT nets, the latter two notions do not coincide. In particular, the unfolding of a net N contains information strictly more concrete than the collection of the processes of N. However, we show that the difference between the two semantics can be axiomatized quite simply. In particular, we introduce a new notion of process, whose definition is suggested by the idea of families in decorated occurrence nets, and which are therefore called *decorated processes*, and we show that they capture the unfolding semantics, in the precise sense that there is a one-to-one translation between decorated processes of N and finite configurations of $\mathcal{EFU}[N]$. Then, following the approach of [16], we axiomatize the notion of decorated (concatenable) process in terms of monoidal categories. More precisely, we define an abstract symmetric monoidal category $\mathcal{DP}[N]$ and we show that its arrows represent *decorated concatenable processes.*

The natural environment for the development of a theory of net processes based on monoidal categories is, as illustrated in [16], a category Petri of *unmarked* nets, i.e., nets without initial markings, whose transitions have *finite* pre- and post-sets. Although there seem to be no formal reasons preventing one from extending the theory to nets with infinite markings, such an extension would be at least technically rather involved. Therefore, our choice here is to follow [16] and define $\mathcal{DP}[\_]$ on the category Petri of unmarked nets with finite markings introduced in [97].

On the other hand, since the existence of *left adjoint* functors for $\mathcal{F}[\_]$ and $\mathcal{E}[\_]$ requires infinite markings, and since the unfolding of a net is considered with respect to an initial marking, PTNets and all the categories of nets considered in [98] (and in related works) are categories of *marked* nets whose transitions have possibly *infinite* pre- and post-sets. Moreover, because of technical reasons, the transitions are forced to have *nonempty* pre-sets. In order to solve such discrepancy, we simply restrict our attention to a category MPetri* of marked nets whose transitions have nonempty pre-sets and finite pre- and post-sets. Therefore, summing up, our result is that the following diagram commutes up to isomorphism for each net N with initial marking u in MPetri*



where the two leftmost arrows represent respectively the insertion of N in Petri obtained by forgetting the initial marking and the injection of N in PTNets, $\langle u{\downarrow}\_\rangle$ indicates the construction of the (comma) category of the objects under u in $\mathcal{DP}[N]$, SsMonCat is the category of the symmetric strict monoidal categories, PreOrd is the category of preorders and $\mathcal{L}_F$ returns the finite configurations of prime event structu+res ordered by inclusion. In this sense, we claim that $\mathcal{DP}[\_]$ is, at the same time, the algebraic, the process-oriented and the category-theoretic counterpart of the unfolding construction.

It is to be stressed that our concern here is *not* "in the large", i.e., on the structure of the category of nets, but at the level of a *single* net, which means that the diagram above is defined only at the object level, i.e., the correspondence we establish is *not* functorial. Of course, this is due to the fact that $\mathcal{DP}[\_]$, exactly as $\mathcal{P}[\_]$, is not a functor. Nevertheless, we think that this is an interesting result, since it provides a natural and unified account of the *algebraic*, the *process-oriented* and the *denotational* views of net semantics. We remark that a similar approach has been followed in [107] in the case of elementary net systems—a particular class of safe nets without self-looping transitions—for unfoldings and non-sequential processes.

Finally, we conclude this chapter by briefly discussing some natural variations on the unfolding theme in order to justify further the construction.

<u>Notation</u>. Given a category $\underline{\mathsf{C}}$, we denote the composition of arrows in $\underline{\mathsf{C}}$ by the usual $\_ \circ \_$ in the usual right to left order. The identity of $c \in \underline{\mathsf{C}}$ is written as id_c. However, we make the following exception. When dealing with a category in which arrows are meant to represent computations, in order to stress this computational meaning, we write arrow composition from left to right, i.e., in the diagramatic order, and we denote it by $\_;\_$. Moreover, when no ambiguity arises, id_c is simply written as c. We assume that tensor product binds more strictly than arrow composition, i.e., $f \otimes g; h \otimes k$ stands for $(f \otimes g);(h \otimes k)$. The reader is referred to the Appendix for the categorical concepts used. A thorough introduction to category theory can be acquired from any of the textbooks [90, 1, 103, 127, 26, 92].

<u>Remark</u>. Concerning foundational issues, following [89], we assume as usual the existence of a fixed *universe* $\mathfrak{U}$ [55, 136, 43] of *small* sets upon which categories are built (see also Appendix A.1). However, since the explicit distinction between "small" and "large" objects plays a significant role only in Chapter 3, in this and in the following chapter we shall avoid any further reference to small sets. Of course, in order to make the categories we shall define in Chapter 1 and Chapter 2 agree with the definition given in Appendix A.1, it is enough to read "set" as "small set" where appropriate.

1.1 Petri Nets and their Computations

In this section we briefly recall some of the basic definitions about Petri nets [109, 110, 120] (for a comprehensive introduction to the theory, see [119]). In particular, we remind their algebraic description as introduced in [97] and their processes [111, 30, 36, 9, 16].

It is easy to notice that a graph can be thought of as a functor from the category $\bullet \rightrightarrows \bullet$ to $\underline{\mathsf{Set}}$, the category of sets and functions, whose object component selects a set of arcs and a set of nodes and whose arrow component picks up the source and target fuctions. Needless to say, a morphism of graphs is then a natural transformation between the corresponding functors. Thus we have $\underline{\mathsf{Graph}}$, the category of graphs, automatically defined as a category of functors [92] (see also Appendix A.1).

Definition 1.1.1 *(Graphs)*
A graph is a structure $G = (\partial^0_G, \partial^1_G \colon A_G \to N_G)$*, where* A_G *is a set of arcs,* N_G *is a set of nodes, and* ∂^0_G *and* ∂^1_G *are functions assigning to each arc, respectively, a source and a target node.*

A morphism of graphs from G_0 *to* G_1 *consists of a pair of functions* $\langle f, g\rangle$*, where* $f \colon A_{G_0} \to A_{G_1}$ *and* $g \colon N_{G_0} \to N_{G_1}$ *are such that the following diagrams commute.*

$$\begin{array}{ccc} A_G & \xrightarrow{\partial^0_G} & N_G \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ A_{G'} & \xrightarrow[\partial^0_{G'}]{} & N_{G'} \end{array} \quad \text{and} \quad \begin{array}{ccc} A_G & \xrightarrow{\partial^1_G} & N_G \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ A_{G'} & \xrightarrow[\partial^1_{G'}]{} & N_{G'} \end{array}$$

This gives the category $\underline{\mathsf{Graph}}$ *of graphs.*

Of course, there is no particular reason to restrict oneself to $\underline{\mathsf{Set}}$; one could consider graphs over any category without loosing the structural properties of graphs. Yet more interesting is to consider graphs with some *algebraic structure* on nodes and arcs. This leads directly to monads [90, 22] (see also Appendix A.1), and in particular monads on $\underline{\mathsf{Graph}}$. Since the kind of "algebraic graphs" one would like to capture do not necessarily have the same algebraic structure on the arcs and on the nodes, one arrives to the following general pattern. Given two monads (T, η, μ) and (T', η', μ') together with a morphism of monads $\sigma \colon T \stackrel{\cdot}{\to} T'$, consider the mapping

$$A \underset{\partial^1}{\overset{\partial^0}{\rightrightarrows}} N \quad \rightsquigarrow \quad TA \xrightarrow{\sigma_A} T'A \underset{T'\partial^1}{\overset{T'\partial^0}{\rightrightarrows}} T'N\,.$$

It is not difficult to show that this gives a monad on $\underline{\mathsf{Graph}}$, whose unit and multiplication are, respectively, $\langle \eta, \eta' \rangle$ and $\langle \mu, \mu' \rangle$. The main interest is then on the categories of free algebras for such monads. For example, the basic category of Petri nets considered in [97] is an instance of this pattern obtained by choosing the identity monad for T and the "commutative monoids" monad for T', and other useful instances are also discussed in [97]. In this sense Petri nets are monoids. Let us give the relevant definitions.

Given a set S, and a function μ from S to the set of natural numbers ω, we write $[\![\mu]\!]$ to indicate the support of μ that is the subset of S consisting of those elements s such that $\mu(s) > 0$. Moreover, we denote by $S^\oplus$ the set of *finite multisets* of S, i.e., the set of all functions from S to ω with finite support. Of course, any function $g \colon S_0 \to S_1$ can be "freely" extended to a function $g^\oplus \colon S_0^\oplus \to S_1^\oplus$ defined by

$$g^\oplus(\mu)(s') = \sum_{s \in g^{-1}(s')} \mu(s).$$

This definition makes $(\_)^{\oplus}$ an endofunctor on $\underline{\mathsf{Set}}$. Consider now a finite multiset of finite multisets $\phi\colon S^{\oplus} \to \omega$. It can be considered a formal "linear combination" of multisets, and thus identified with the multiset μ such that

$$\mu(s) = \sum_{\nu \in S^{\oplus}} \phi(\nu)\nu(s).$$

It is then easy to see that $(\_)^{\oplus}$ is a *commutative monad* [72, 73, 74, 75] on $\underline{\mathsf{Set}}$ whose multiplication is the operation of linear combination of multisets above and whose unit maps $s \in S$ to the function which yields 1 on s and zero elsewhere. Clearly, the $(\_)^{\oplus}$-algebras are the commutative monoids and the $(\_)^{\oplus}$-homomorphisms are monoid homomorphisms.

We shall represent a finite multiset $\mu \in S^{\oplus}$ as a formal sum $\bigoplus_{s\in S} \mu(s) \cdot s$. Moreover, we shall often denote $\mu \in S^{\oplus}$ by $\bigoplus_{i\in I} n_i s_i$ where $\{s_i \mid i \in I\} = [\![\mu]\!]$ and $n_i = \mu(s_i)$, i.e., as a sum whose summands are all nonzero. For instance, the multiset which contains the unique element s with multiplicity one is written as $1 \cdot s$, or simply s. In this setting, the multiplication of finite multisets is written as

$$\bigoplus_{\mu \in S^{\oplus}} n_\mu \cdot \mu = \bigoplus_{\mu \in S^{\oplus}} n_\mu \cdot \left(\bigoplus_{s \in S} \mu(s) \cdot s \right) = \bigoplus_{s\in S} \Big(\sum_{\mu \in S^{\oplus}} n_\mu \mu(s) \Big) \cdot s,$$

while the monoid homomorphism condition for a function $g\colon S_0^{\oplus} \to S_1^{\oplus}$ is

$$g(\mu) = \bigoplus_{s \in S_0} \mu(s) \cdot g(1 \cdot s).$$

Finally, given $S' \subseteq S$, we will write $\bigoplus S'$ for $\bigoplus_{s\in S'} 1 \cdot s = \bigoplus_{s \in S'} s$.

Definition 1.1.2 *(Petri Nets)*
A Place/Transition Petri (PT) net is a structure $N = (\partial_N^0, \partial_N^1 \colon T_N \to S_N^{\oplus})$, where T_N is a set of transitions, S is a set of places, and ∂_N^0 and ∂_N^1 are functions.

A morphism of PT nets from N_0 to N_1 consists of a pair of functions $\langle f, g\rangle$, where $f\colon T_{N_0} \to T_{N_1}$ is a function and $g\colon S_{N_0}^{\oplus} \to S_{N_1}^{\oplus}$ is a monoid homomorphism such that $\langle f, g\rangle$ respects source and target, i.e., they make the two rectangles obtained by choosing the upper or lower arrows in the parallel pairs of the diagram below commute.

$$\begin{array}{ccc}
T_{N_0} & \overset{\partial^0_{N_0}}{\underset{\partial^1_{N_0}}{\rightrightarrows}} & S^{\oplus}_{N_0} \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\
T_{N_1} & \overset{\partial^0_{N_1}}{\underset{\partial^1_{N_1}}{\rightrightarrows}} & S^{\oplus}_{N_1}
\end{array}$$

This gives the category $\underline{\mathsf{Petri}}$ of PT nets.

This describes a Petri net precisely as a graph whose set of nodes is a free commutative monoid, i.e., the set of *finite multisets* on a given set of *places*. Source and target of an arc, here called *transition*, are meant to represent, respectively, the *marking* which enables the transition, i.e., the states which allow the transition to fire, and the marking produced by the firing of the transition.

Observe that we are only considering finite markings. Although this is clearly a restriction, it does not have serious drawbacks on the pratical ground of system modelization and verification. Moreover, it usually does not have strong consequences also from the theoretical point of view, since, to the best of our knowledge, not many notions in the theory require the existence of infinite markings. However, we shall encounter two of those notions in which infinite markings are needed, one in Section 1.5 and one in Chapter 3, since the existence of a left adjoint for the unfolding functor requires infinite markings and, of course, so do infinite computations.

<u>Notation</u>. To simplify notation, we assume the standard constraint that $T_N \cap S_N = \varnothing$—which of course can always be achieved by an appropriate renaming. Moreover, we shall sometimes use a single letter to denote a morphism $\langle f, g\rangle$. In these cases, the type of the argument will identify which component we are referring to. Observe further that by the very definition of free algebras, an $(\_)^{\oplus}$-homomorphism $g\colon S_{N_0}^{\oplus} \to S_{N_1}^{\oplus}$, which constitutes the place component of a morphism $\langle f, g\rangle$, is completely defined by its behaviour on S_{N_0}, the generators of the free algebra $S_{N_0}^{\oplus}$. Therefore, we will often define morphisms between nets by giving their transition components and a map $g\colon S_{N_0} \to S_{N_1}^{\oplus}$ for their place components: it is implicit that they have to be thought of as lifted to the corresponding $(\_)^{\oplus}$-homomorphisms.

Another point which is worth raising concerns the observation that the morphisms in <u>Petri</u> are total, while in the literature nets (and many other models) are often provided with *partial* morphisms (see e.g. [144, 146]). Since the intuition about morphisms in categories of models of computation is that they represent "simulations," partial morphisms model situations in which some computational step may be simulated vacuosly. Partial morphisms may be recovered in this approach simply by considering a sligthly refined monad for the algebra of transitions, namely the *lifting* monad.

A *pointed set* is a pair (S, s) where S is a set and $s \in S$ is a chosen element of S: the pointed element. Morphisms of pointed sets are functions that preserve the pointed elements. Therefore, pointed set morphisms provide a convenient way to treat partial functions between sets as total functions. Of course, this yields a monad $(\_)_0$ on <u>Set</u> whose functor part adds a pointed element to a set, whose multiplication forgets it and whose unit is the inclusion of S in $S + \{*\}$.

We will regard $S^{\oplus}$ also as a pointed set whose pointed element is the empty multiset, i.e., the function which always yields zero, that, in the following, we denote by 0. Of course, this is nothing but defining a morphism of monads $\sigma\colon (\_)_0 \stackrel{.}{\to} (\_)^{\oplus}$

where $\sigma_S\colon (S,*) \to S^{\oplus}$ sends $*$ to 0 and s to $1 \cdot s$. Thus, following the general pattern above we have the following definition.

Definition 1.1.3 *(Pointed Petri Nets)*
A pointed (PT) net is a structure $N = (\partial_N^0, \partial_N^1\colon (T_N, 0) \to S_N^{\oplus})$, *where* $(T_N, 0)$ *is a pointed set of transitions,* S *is a set of places, and* ∂_N^0 *and* ∂_N^1 *are morphisms of pointed sets.*

A morphism of pointed nets from N_0 *to* N_1 *consists of a pair of functions* $\langle f, g\rangle$, *where* $f\colon T_{N_0} \to T_{N_1}$ *is a morphism of pointed sets and* $g\colon S_{N_0}^{\oplus} \to S_{N_1}^{\oplus}$ *is a monoid homomorphism such that* $\langle f, g\rangle$ *respects source and target, i.e.,* $g \circ \partial_{N_0}^0 = \partial_{N_1}^0 \circ f$ *and* $g \circ \partial_{N_0}^1 = \partial_{N_1}^1 \circ f$. *In other words, a pointed net morphism is a morphism of the underlying PT nets which, in addition, preserves the pointed element.*

This gives the category $\underline{\mathsf{Petri}}_0$ *of pointed nets.*

The next issue we need to treat concerns the *initial marking*, i.e., the initial state, of a net. It is rather common to consider the kind of nets we defined above closer to *system schemes* than to *systems*, since they lack an initial state from which to start computing and, of course, different initial markings can give rise to very different behaviours for the same net. Although this distinction is clearly agreeable, we shall not put much emphasis on it, since in the categorical framework this is not always necessary. We shall for instance define processes and computations of unmarked nets, so obtaining the collection of the computations for any possible inital marking, the point being that it is always possible to recover all the relevant information about the behaviour for a given marking via canonical constructions such as comma categories [90] (see also Appendix A.1).

Definition 1.1.4 *(Marked Petri Nets)*
A marked (pointed) PT net is a pair (N, u_N), *where* N *is a (pointed) PT net and* $u_N \in S_N^{\oplus}$ *is the initial marking.*

A morphism of marked (pointed) PT nets from N_0 *to* N_1 *consists of a (pointed) PT net morphism* $\langle f, g\rangle\colon N_0 \to N_1$ *which preserves the initial marking, i.e., such that* $g(u_{N_0}) = u_{N_1}$.

This gives the category $\underline{\mathsf{MPetri}}$ *of marked PT nets and the category* $\underline{\mathsf{MPetri}}_0$ *of marked pointed nets.*

Transitions are the basic units of computation in a PT net. A transition t with $\partial_N^0(t) = u$ and $\partial_N^1(t) = v$—usually written $t\colon u \to v$—performs a computation *consuming* the tokens in u and *producing* the tokens in v. A finite number of transitions can be composed in parallel to form a *step*, which, therefore, is a finite multiset of transitions. We write $u[\alpha\rangle v$ to denote a step α with source u and target v. The set $S[N]$ of steps of N is generated by the rules:

$$\frac{t\colon u \to v \text{ in } N \text{ and } w \text{ in } S^{\oplus}}{(u \oplus w)[t\rangle(v \oplus w) \text{ in } S[N]} \qquad \frac{u[\alpha\rangle v \text{ and } u'[\beta\rangle v' \text{ in } S[N]}{(u \oplus u')[\alpha \oplus \beta\rangle(v \oplus v') \text{ in } S[N]}.$$

Notably, this construction can be defined using the general pattern given above by considering also for the transitions the commutative monoid monad.

Definition 1.1.5 *(Petri Commutative Monoids)*
A Petri commutative monoid is a structure $M = (\partial^0_M, \partial^1_M\colon (T_M, +, 0) \to S^{\oplus}_M)$, *where* $(T_M, +, 0)$ *is a commutative monoid,* S *is a set, and* ∂^0_M *and* ∂^1_M *are monoid homomorphisms.*

A morphism of Petri commutative monoids from M_0 *to* M_1 *consists of a pair of functions* $\langle f, g\rangle$, *where* $f\colon (T_{N_0}, +, 0) \to (T_{N_1}, +, 0)$ *and* $g\colon S^{\oplus}_{N_0} \to S^{\oplus}_{N_1}$ *are monoid homomorphisms which respect source and target.*

This gives the category $\underline{\mathsf{CMonPetri}}$ *of Petri commutative monoids.*

Then, we have the following.

Proposition 1.1.6 *(Steps are Petri Commutative Monoids)*
Given a net N *in* $\underline{\mathsf{Petri}}$, *consider the Petri commutative monoid*

$$\mathcal{S}[N] = ((\partial^0_N)^{\oplus}, (\partial^1_N)^{\oplus}\colon T^{\oplus}_N \to S^{\oplus}).$$

There is a transition $\alpha\colon u \to v$ *in* $\mathcal{S}[N]$ *if and only if* $u[\alpha\rangle v$ *belongs to* $S[N]$.

Moreover, $\mathcal{S}[N]$ *is the universal Petri commutative monoid on* N. *Thus,* $\mathcal{S}[\_]$ *extends to a functor which is left adjoint to the forgetful functor* $\underline{\mathsf{CMonPetri}} \to \underline{\mathsf{Petri}}$.

Proof. Immediate. ✓

Observe that there are obvious left adjoint functors also for the forgetful functors $\underline{\mathsf{CMonPetri}} \to \underline{\mathsf{Petri}}_0$ and $\underline{\mathsf{Petri}}_0 \to \underline{\mathsf{Petri}}$ and their composition is the functor $\mathcal{S}[\_]$ given in the previous proposition.

$$\underline{\mathsf{Petri}} \;\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}\; \underline{\mathsf{Petri}}_0 \;\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}\; \underline{\mathsf{CMonPetri}}$$

A finite number of steps of N can be sequentially composed thus yielding a *step sequence*. The set of step sequences, denoted $SS[N]$, is given by the rules:

$$\frac{u \text{ in } S^{\oplus}}{u[\varnothing\rangle u} \qquad \frac{u_0[\alpha_0\rangle \ldots [\alpha_{n-1}\rangle u_n \text{ in } SS[N] \text{ and } u_n[\alpha_n\rangle u_{n+1} \text{ in } S[N]}{u_0[\alpha_0\rangle \cdots [\alpha_{n-1}\rangle[\alpha_n\rangle u_{n+1} \text{ in } SS[N]}.$$

Following the same idea which led to Petri commutative monoids, one could try to define a structure which captures step sequences by providing Petri commutative monoids with a sequentialization operation which makes them be categories. However, such a structure would be richer than needed, since, besides sequences of steps, we would also have around steps built of sequences. The need to relate such "induced" arrows with step sequences brings directly to monoidal categories.

Definition 1.1.7 *(Petri Categories)*
A Petri category is a structure $C = (\partial^0_C, \partial^1_C\colon (T_C, +, 0) \rightarrow S^{\oplus}_C, \_;\_, id)$, *where* $(\partial^0_C, \partial^1_C\colon (T_C, +, 0) \rightarrow S^{\oplus}_C)$ *is a Petri commutative monoid,* $id\colon S_C \rightarrow T_C$ *is a function which associates to each* $u \in S$ *an arrow* $id(u)\colon u \rightarrow u$ *and* $\_;\_\colon T_C \times T_C \rightarrow T_C$ *is a partial function, called sequentialization, defined on the pairs* (α, β) *such that* $\partial^1_C(\alpha) = \partial^0_C(\beta)$. *In addition, they satisfy the axioms:*

i) $\partial^0_C(\alpha;\beta) = \partial^0_C(\alpha)$ *and* $\partial^1_C(\alpha;\beta) = \partial^1_C(\beta)$;

ii) $\alpha; id(\partial^1_C(\alpha)) = \alpha$ *and* $id(\partial^0_C(\alpha)); \alpha = \alpha$;

iii) $(\alpha;\beta);\gamma = \alpha;(\beta;\gamma)$;

iv) $(\alpha + \alpha');(\beta + \beta') = (\alpha;\beta) + (\alpha';\beta')$, *for* $\alpha\colon u \rightarrow v, \quad \alpha'\colon u' \rightarrow v'$
$\beta\colon v \rightarrow w, \quad \beta'\colon v' \rightarrow w'$.

Thus, in other words, a Petri category is a strict monoidal category [90] (see Appendix A.2 for further references) which is strictly symmetric and whose object set form a free monoid.

A morphism of Petri categories from C_0 *to* C_1 *is a morphism* $\langle f, g\rangle$ *of the underlying Petri commutative monoids which, in addition, preserves identities and sequential composition, i.e.,* $f(id(u)) = id(g(u))$ *and* $f(\alpha;\beta) = f(\alpha); f(\beta)$.

This gives the category <u>CatPetri</u> *of Petri categories.*

As before, there is a left adjoint $\mathcal{T}[\_]\colon \underline{\mathsf{Petri}} \rightarrow \underline{\mathsf{CatPetri}}$ to the forgetful functor $\underline{\mathsf{CatPetri}} \rightarrow \underline{\mathsf{Petri}}$. It can be easily defined as follows.

$$\frac{t\colon u \rightarrow v \text{ in } N}{t\colon u \rightarrow v \text{ in } \mathcal{T}[N]} \qquad \frac{u \text{ in } S^{\oplus}}{u\colon u \rightarrow u \text{ in } \mathcal{T}[N]}$$

$$\frac{\alpha\colon u \rightarrow v \text{ and } \beta\colon v \rightarrow w \text{ in } \mathcal{T}[N]}{\alpha;\beta\colon u \rightarrow w \text{ in } \mathcal{T}[N]} \qquad \frac{\alpha\colon u \rightarrow v \text{ and } \alpha'\colon u' \rightarrow v' \text{ in } \mathcal{T}[N]}{\alpha \oplus \alpha'\colon u \oplus u' \rightarrow v \oplus v' \text{ in } \mathcal{T}[N]}$$

subject to the equations which make it a strictly symmetric strict monoidal category, the empty marking 0 being the unit and the $u \in S^{\oplus}$ being the identities.

It is easy to realize that $\mathcal{T}[\_]$ factors through $\mathcal{S}[\_]$, and therefore we have the following chain of adjunctions.

$$\underline{\mathsf{Petri}} \underset{\longleftarrow}{\overset{\longrightarrow}{\bot}} \underline{\mathsf{Petri}}_0 \underset{\longleftarrow}{\overset{\longrightarrow}{\bot}} \underline{\mathsf{CMonPetri}} \underset{\longleftarrow}{\overset{\longrightarrow}{\bot}} \underline{\mathsf{CatPetri}}$$

The interesting fact about Petri categories is that the axioms of monoidal categories induce identifications also on step sequences, thus yielding a representation of net behaviour which is more abstract than step sequences. More precisely, the arrows of $\mathcal{T}[N]$, called *commutative processes* in [16], have been found in [9], even though following different motivations and through quite a different formalization, and characterized as the least quotient of step sequences which is more abstract than processes [36]. In order to be more precise on this we first need to introduce the classical notion of process of a Petri net.

Unlike step sequences, processes provide a causality oriented explanation of net behaviours, which is achieved by decorating sequences with explicit information about the *causal links* which ruled the firing of the transitions in the sequence. As usual, one assumes that such links may be expressed faithfully as a partial order of transitions, the ordering being considered a cause/effect relationship (however, see [44, 45, 116, 117, 112] for criticisms about this assumption). Thus, roughly speaking, a process of a net N consists of a partial order built on a multisubset of transitions of N. The formalization of this gives the following notion of process net.

Notation. In the following, we use the standard notation ${}^\bullet a$, for $a \in S_N$, to mean the *pre-set* of a, i.e., ${}^\bullet a = \{t \in T_N \mid a \in [\![\partial^1_N(t)]\!]\}$. Similarly, $a^\bullet = \{t \in T_N \mid a \in [\![\partial^0_N(t)]\!]\}$, the *post-set* of a. These notations are extended in the obvious way to the case of sets of places. Recall that the terminology pre- and post-set is used also for transitions to indicate, respectively, ${}^\bullet t = [\![\partial^0_N(t)]\!]$ and $t^\bullet = [\![\partial^1_N(t)]\!]$.

Definition 1.1.8 *(Process Nets)*
A process net is a net Θ such that

i) for all $t \in T_\Theta$
   (a) $\partial^0_\Theta(t) \neq 0$ and $\partial^1_\Theta(t) \neq 0$;
   (b) for all $a \in S_\Theta$ it is $\partial^0_\Theta(t)(a) \leq 1$ and $\partial^1_\Theta(t)(a) \leq 1$;

ii) for all $a \in S_\Theta$, it is $|{}^\bullet a| \leq 1$ and $|a^\bullet| \leq 1$, where $|\_|$ gives the cardinality of sets;

iii) $\prec$ is irreflexive, where $\prec$ is the transitive closure of the relation

$$\prec^1 = \{(a,t) \mid a \in S_\Theta,\ t \in T_\Theta,\ t \in a^\bullet\} \cup \{(t,a) \mid a \in S_\Theta,\ t \in T_\Theta,\ t \in {}^\bullet a\};$$

moreover, $\forall t \in T_\Theta,\ \{t' \in T_\Theta \mid t' \prec t\}$ is finite.

Let $\underline{\mathsf{Petri}}_*$ be the lluf[1] *subcategory of $\underline{\mathsf{Petri}}$ determined by the morphisms which*

[1]Following [26], a lluf subcategory of $\underline{\mathsf{A}}$ is a subcategory which has the same objects as $\underline{\mathsf{A}}$.

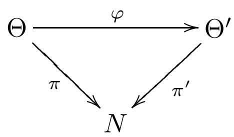
map places to places (as opposed to morphisms which map places to markings). Then, we define $\underline{\mathsf{ProcNets}}$ *to be the full subcategory of* $\underline{\mathsf{Petri}}_*$ *determined by process nets.*

Thus, in process nets every transition has non-empty pre- and post-set. Moreover, each place belongs at most to one pre-set and at most to one post-set. This makes of the "flow" relation $\prec$ be a pre-order. Thus, requiring it to be irreflexive, which is equivalent to requiring that the net be acyclic, identifies a partial order on the transitions. The constraint about the cardinality of the set of predecessors of a transition is then the fairly intuitive requirement that each transition be finitely caused. (See [143] for a discussion in terms of event structures of this issue.) The expert reader will have already noticed that the process nets above are the usual (deterministic) occurrence nets. However, we prefer to reserve this terminology for the kind of nets used in the context of the unfolding semantics in Section 1.7.

Definition 1.1.9 *(Processes)*
Given $N \in \underline{\mathsf{Petri}}$*, a process of* N *is a morphism* $\pi\colon \Theta \to N$ *in* $\underline{\mathsf{Petri}}_*$*, where* Θ *is a process net. We say that* $\pi\colon \Theta \to N$ *is finite if* Θ *is so.*

Let $\underline{\mathsf{Proc}}[N]$*, the category of processes of* N*, be the comma category* $\langle \underline{\mathsf{ProcNets}} {\downarrow} N \rangle$ *of the process nets over* N *in* $\underline{\mathsf{Petri}}_*$*. We shall denote by* $\underline{\mathsf{Proc}}_f[N]$ *the full subcategory of* $\underline{\mathsf{Proc}}[N]$ *consisting of the finite processes.*

We recall that the objects of $\underline{\mathsf{Proc}}[N]$ are morphisms $\pi\colon \Theta \to N$ in $\underline{\mathsf{Petri}}_*$, while its morphisms $\phi\colon \pi \to \pi'$ are morphisms $\varphi\colon \Theta \to \Theta'$ such that the following diagram commutes.



Thus, $\underline{\mathsf{Proc}}[N]$ may be considered as a generalization of the usual *"prefix ordering"* of computations. For the purpose of defining processes at the right abstraction level, we need to make some identifications of process nets. Of course, we shall consider as identical process nets which are isomorphic and, consequently, we shall make no distinction between two processes $\pi\colon \Theta \to N$ and $\pi'\colon \Theta' \to N$ such that there exists an isomorphism $\varphi\colon \Theta \to \Theta'$ such that $\pi' \circ \varphi = \pi$. Observe that the choice of $\underline{\mathsf{Petri}}_*$ for π is relevant, since, also in the case of pointed nets, we certainly want processes to be total and to map a single component of the process net to a single component of N. Otherwise said, processes are nothing but labellings of Θ with an appropriate element of N.

It is worth noticing that the usual definition includes only finite processes, although infinite ones have been considered for instance in [9]. We consider the broader definition above, since in Section 3.7 we shall be talking about infinite processes. However, in this section we only consider finite processes. Moreover, the definition of processes for nets is usually given with respect to some initial marking. In the following, we shall briefly see that, in this setting, this does not make any difference.

Definition 1.1.10 *(Marked Process Nets and Processes)*
A marked process net is a pair (Θ, u) *where* Θ *is a process net and* u *is the set of the minimal (wrt.* $\prec$*) elements of* Θ*, which are necessarily places. Let* $\underline{\mathsf{MProcNets}}$ *denote the full subcategory of* $\underline{\mathsf{MPetri}}_*$ *consisting of marked process nets ad morphisms,* $\underline{\mathsf{MPetri}}_*$ *being the lluf subcategory of* $\underline{\mathsf{MPetri}}$ *consisting of the morphisms which map places to places.*

The category $\underline{\mathsf{MProc}}[(N, u)]$ *of processes of a marked net* (N, u) *is the comma category* $\langle\underline{\mathsf{MProcNets}}{\downarrow}(N, u)\rangle$ *of the marked process nets over* (N, u) *in* $\underline{\mathsf{MPetri}}_*$*, i.e., a process of marked nets is obtained by considering processes which map the minimal elements of process nets to* u*, the initial marking of* N*. Similarly to the previous case,* $\underline{\mathsf{MProc}}_f[(N, u)]$ *will denote the category of finite processes of* N*.*

There is the following obvious link between $\underline{\mathsf{Proc}}[N]$ and $\underline{\mathsf{MProc}}[(N, u)]$.

Proposition 1.1.11
$\underline{\mathsf{MProc}}[(N, u)]$ *(respectively* $\underline{\mathsf{MProc}}_f[(N, u)]$*) is the full subcategory of* $\underline{\mathsf{Proc}}[N]$ *(respectively* $\underline{\mathsf{Proc}}_f[(N, u)]$*) consisting of those processes* $\pi\colon \Theta \to N$ *such that* $\pi(\min(\Theta)) = u$*, where* $\min(\Theta)$ *denotes the set of minimal elements of* Θ*.*

We can now get quickly back to the characterization of $\mathcal{T}[N]$ we referred to earlier. Observe that, by definition, processes are more concrete than step sequences, i.e., different processes may give rise to common step sequences, corresponding to the fact that different partial orders may have common linearizations. On the other hand, different step sequences may be sequentializations of the same process

Proposition 1.1.12 *(Step Sequences and Petri Categories)*
For N *in* $\underline{\mathsf{Petri}}$*, the arrows of* $\mathcal{T}[N]$*, i.e., the commutative processes of* N*, are obtained by quotienting the processes of* N *by the least equivalence such that if a step sequence is a linearization of two different processes, then they are equivalent.*

Proof. See [9, 16]. ✓

The categories of processes $\underline{\mathsf{Proc}}[N]$ are not completely satisfactory, at least for two relevant reasons. Firstly, when "categorizing" computational formalisms the focus is on representing states—respectively types, formulas—as objects and computations—respectively terms, proofs—as arrows, and this is because the focus is actually on computations and, after all, categories consist mainly of arrows. Therefore, to consider $\underline{\mathsf{Proc}}[N]$ as a satisfactory solution would be, at the very least, out of the current mainstream of research. Secondly, both the monoidal structure of Petri nets and the monoidal structure which intuitively should be enjoyed by the computations of a concurrent system under the operation of parallel composition are *not* reflected in $\underline{\mathsf{Proc}}[N]$. Clearly, $\mathcal{T}[N]$ does not suffer of these problems. However, the very observation that $\mathcal{T}[N]$ provides a description of the behaviour of N even more abstract than step sequences of course put it out of the game, since we are looking for a causal semantics for nets. It is easy to realize that the problem with $\mathcal{T}[N]$ concerning causality is due to the *strict* symmetry which confuses the "causal streams" of computations. Therefore, a possible way out of this problem is to look for a *non* strictly symmetric version of $\mathcal{T}[N]$, where the flow of causality is taken into account by the *symmetry isomorphism*.

Of course, another possible solution consists of looking for a version of $\underline{\mathsf{Proc}}[N]$ in which the processes are represented by arrows, sequential composition being the concatenation of processes. More strongly, inspired by the current trends in the development of the theory of computation, one would certainly like to describe the processes of a net N as an algebra whose operations model a minimal set of combinators on processes which capture the essence of concurrency. Clearly, in the present case, the core of such algebra must consist of the operations of *sequential* and *parallel* composition of processes. The problem which arises immediately is that non-sequential processes cannot be concatenated when multiplicities are present: in order to support such an operation one must *disambiguate* the identity of all the tokens in the multisets source and target of processes. In other words, one must recognize that process concatenation has to do with tokens rather than with places. This is the approach followed in [16] where the above sketched variation of the processes of N is modelled by means of a symmetric strict monoidal category $\mathcal{P}[N]$. Next, we briefly recall such a construction, which represents the meeting point of the two development lines indicated above.

A *symmetric strict monoidal category* (SSMC in the following) is a structure $(\underline{\mathsf{C}}, \otimes, e, \gamma)$, where $\underline{\mathsf{C}}$ is a category, e is an object of $\underline{\mathsf{C}}$, called the *unit* object, $\otimes\colon \underline{\mathsf{C}} \times \underline{\mathsf{C}} \to \underline{\mathsf{C}}$ is a functor, called the *tensor product*, subject to the following equations

$$\otimes \circ \langle \otimes \times 1_{\underline{\mathsf{C}}} \rangle = \otimes \circ \langle 1_{\underline{\mathsf{C}}} \times \otimes \rangle; \qquad (1.2)$$

$$\otimes \circ \langle \underline{e}, 1_{\underline{\mathsf{C}}} \rangle = 1_{\underline{\mathsf{C}}}; \qquad (1.3)$$

$$\otimes \circ \langle 1_{\underline{\mathsf{C}}}, \underline{e} \rangle = 1_{\underline{\mathsf{C}}}; \qquad (1.4)$$

where $\underline{e}\colon \underline{\mathsf{C}} \to \underline{\mathsf{C}}$ is the constant functor which associate e and id_e respectively to each object and each morphism of $\underline{\mathsf{C}}$, $\langle \_, \_ \rangle$ is the pairing of functor induced by the cartesian product, and γ is a natural transformation $\otimes \stackrel{\cdot}{\to} \otimes \circ \Delta$, where Δ is the endofunctor on $\underline{\mathsf{C}} \times \underline{\mathsf{C}}$ which "swaps" its arguments, subject to the following

Kelly-MacLane *coherence* axioms [87, 62, 67]:

$$(\gamma_{x,z} \otimes id_y) \circ (id_x \otimes \gamma_{y,z}) = \gamma_{x \otimes y, z}; \tag{1.5}$$
$$\gamma_{y,x} \circ \gamma_{x,y} = id_{x \otimes y}. \tag{1.6}$$

Of course, equation (1.2) states that the tensor is associative on both objects and arrows, while (1.3) and (1.4) state that e and id_e are, respectively, the unit object and the unit arrow for $\otimes$. Concerning the coherence axioms, axiom (1.6) says that $\gamma_{y,x}$ is the inverse of $\gamma_{x,y}$, while (1.5), the *real key* of symmetric monoidal categories, links the symmetry at composed objects to the symmetry at the components.[2]

Remark. Adapting the general definition of monoidal category to the special case of SSMC's, one finds that there is a further axiom to state, namely $\gamma_{e,x} = id_x$. Observe however that it follows from the others. In fact, by (1.3) we have that $e \otimes e = e$ and thus $\gamma_{e,x} = \gamma_{e \otimes e,x}$, which by (1.5) is equal to $(\gamma_{e,x} \otimes id_e) \circ (id_e \otimes \gamma_{e,x})$. Now, by (1.3) and (1.4) we have that $\gamma_{e,x} = \gamma_{e,x} \circ \gamma_{e,x}$ and thus, multiplying both terms by $\gamma_{x,e}$ and exploiting (1.6), we have $\gamma_{e,x} = id_{e \otimes x} = id_x$.

A *symmetry* in a symmetric monoidal category is any arrow obtained as composition and tensor of components of γ and identities. We shall write $Sym_{\underline{\mathsf{C}}}$ to denote the lluf subcategory of a symmetric monoidal category $\underline{\mathsf{C}}$ whose arrows are the symmetries of $\underline{\mathsf{C}}$. It is important to stress that, in our context, i.e., from the point of view of the semantics of concurrency, symmetries provide a precise and elegant way to account for *causality streams* in computations. This will be clear shortly.

A *symmetric strict monoidal functor* from $(\underline{\mathsf{C}}, \otimes, e, \gamma)$ to $(\underline{\mathsf{D}}, \otimes', e', \gamma')$, is a functor $\mathsf{F}\colon \underline{\mathsf{C}} \to \underline{\mathsf{D}}$ such that

$$\mathsf{F}(e) = e', \tag{1.7}$$
$$\mathsf{F}(x \otimes y) = \mathsf{F}(x) \otimes' \mathsf{F}(y), \tag{1.8}$$
$$\mathsf{F}(\gamma_{x,y}) = \gamma'_{\mathsf{F}x,\mathsf{F}y}. \tag{1.9}$$

Let $\underline{\mathsf{SsMonCat}}$ be the category of SSMC's and symmetric strict monoidal functors (see also Appendix A.2 and Table A.3). In the following, we shall be concerned with a particular kind of SSMC's, namely those whose objects form a *free commutative monoid*. Let $\underline{\mathsf{SSMC}}^{\oplus}$ be the full subcategory of $\underline{\mathsf{SsMonCat}}$ consisting of such categories. Remarkably, a very similar kind of categories have appeared as distinguished algebraic structures also in [88], where thery are called PROP's (for Product and Permutation categories) , and in [85]. The difference between the categories we use and PROP's is that the monoid of objects of the latter have a single generator, i.e., it is the monoid of natural numbers with addition.

[2]Strictly speaking the coherence axioms of SSMC's consists of both (1.5) and (1.6). However, by abuse of language, we shall often say "the coherence axiom" to refer to (1.5).

Let N be a PT net in Petri. In order to define $\mathcal{P}[N]$, we start by introducing the *vectors of permutations* (*vperms*) of N,[3] which will play the role of the symmetry isomorphism of the symmetric strict monoidal category $\mathcal{P}[N]$.

Remark. A *permutation* of n elements is an *automorphism* of the segment of the first n positive natural numbers, i.e., an isomorphism of $\{1, \ldots, n\}$ with itself. Permutations can be represented by a matrix-like notation: the permutation σ such that $\sigma(i) = \sigma_i$ is written as

$$\begin{pmatrix} 1 & 2 & \cdots & n \\ \sigma_1 & \sigma_2 & \cdots & \sigma_n \end{pmatrix}.$$

Sometimes we use a graphical variation of the notation above, according to which the permutation σ is depicted by drawing a line from i to $\sigma(i)$ (see for example Figure 1.5 in page 34).

The set $\Pi(n)$ of the $n!$ permutations of n elements is a group under the operation of composition of functions. The unit element of $\Pi(n)$ is the identity function on $\{1, \ldots, n\}$ and the inverse of σ is its inverse function σ^{-1}. The group $\Pi(n)$ is called the *symmetric group* on n elements, or the symmetric group of order $n!$. As a notation, when $\sigma \in \Pi(n)$, we write $|\sigma| = n$. Due to its triviality, the notion of permutation of zero elements is never considered. However, to simplify notation, we shall assume that the empty function $\varnothing\colon \varnothing \to \varnothing$ is the (unique) permutation of zero elements. Observe that $\Pi(0)$ and $\Pi(1)$ are isomorphic, consisting each of a single permutation; in other words, they are (abstractly) the same group.

We say that $\sigma \in \Pi(n)$ is a *transposition* if there exists $i < n$ such that $\sigma(i) = i+1$, $\sigma(i+1) = i$ and $\sigma(k) = k$ elsewhere. We shall denote such a σ simply as $\{i \to i+1, i+1 \to i\}$. Thus, transpositions are just *"swappings"* of adjacent elements. They are a relevant kind of permutations, since any permutation can be written as the composition of transpositions.

Definition 1.1.13 *(Vectors of Permutations)*
For $u \in S_N^{\oplus}$, a vperm $s\colon u \to u$ is a function which assigns to each $a \in S_N$ a permutation $s(a) \in \Pi(u(a))$. Given $u = n_1 a_1 \oplus \ldots \oplus n_k a_k$ in $S_N^{\oplus}$, we shall represent a vperm s on u as a vector of permutations, $\langle \sigma_{a_1}, \ldots, \sigma_{a_k} \rangle$, where $s(a_j) = \sigma_{a_j}$, whence their name.

Thus, the vperms on a given $u \in S_N^{\otimes}$ are a product of permutation groups. Moreover, they form a symmetric monoidal category under the operations defined below (see also Figure 1.1).

Definition 1.1.14 *(Operations on vperms)*
Given the vperms $s = \langle \sigma_{a_1}, \ldots, \sigma_{a_k} \rangle\colon u \to u$ and $s' = \langle \sigma'_{a_1}, \ldots, \sigma'_{a_k} \rangle\colon u \to u$ their sequential composition $s; s'\colon u \to u$ is the vperm $\langle \sigma_{a_1}; \sigma'_{a_1}, \ldots, \sigma_{a_k}; \sigma'_{a_k} \rangle$, where $\sigma; \sigma'$ is the composition of permutations, which we write in diagrammatic order.

[3] Vperms are called *symmetries* in [16]. Here, in order to avoid confusion with the general notion of symmetry in a symmetric monoidal category, we prefer to use another term.

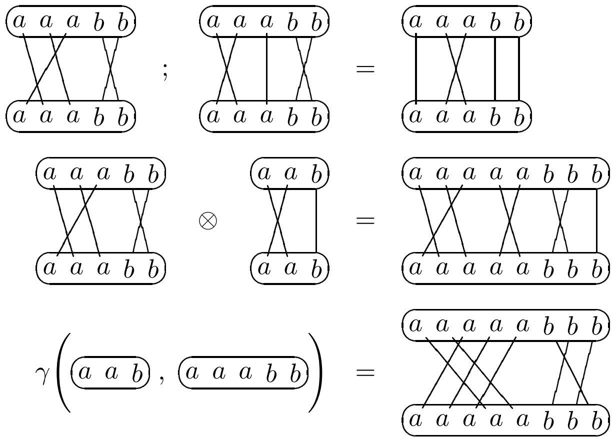


Figure 1.1: The monoidal structure of vperms

Given the vperms $s = \langle \sigma_{a_1}, \dots, \sigma_{a_k} \rangle \colon u \to u$ *and* $s' = \langle \sigma'_{a_1}, \dots, \sigma'_{a_k} \rangle \colon v \to v$ *(where possibly* $\sigma_{a_j} = \varnothing$ *for some* j*), their parallel composition* $s \otimes s' \colon u \oplus v \to u \oplus v$ *is the vperm*

$$\langle \sigma_{a_1} \otimes \sigma'_{a_1}, \dots, \sigma_{a_k} \otimes \sigma'_{a_k} \rangle,$$

where

$$(\sigma \otimes \sigma')(x) = \begin{cases} \sigma(x) & \textit{if } 0 < x \leq |\sigma| \\ \sigma'(x - |\sigma|) + |\sigma| & \textit{if } |\sigma| < x \leq |\sigma| + |\sigma'| \end{cases}$$

Let γ *be* $\{1 \to 2, 2 \to 1) \in \Pi(2)$ *and consider* $u_i = n_1^i a_1 \oplus \ldots \oplus n_k^i a_k$*,* $i = 1, 2$*, in* $S^{\oplus}$*, the interchange vperm* $\gamma(u_1, u_2)$ *is the vperm* $\langle \sigma_{a_1}, \dots, \sigma_{a_k} \rangle \colon u_1 \oplus u_2 \to u_1 \oplus u_2$ *where*

$$\sigma_{a_j}(x) = \begin{cases} x + n_j^2 & \textit{if } 0 < x \leq n_j^1 \\ x - n_j^1 & \textit{if } n_j^1 < x \leq n_j^1 + n_j^2 \end{cases}$$

It is now immediate to see that $\_;\_$ is associative. Moreover, for each $u \in S^{\oplus}$ the vperm $u = \langle id_{a_1}, \dots, id_{a_n} \rangle \colon u \to u$, where id_{a_j} the identity permutation, is of course an identity for sequential composition. Let us consider $\otimes$. For 0 the empty multiset on S, the (unique) vperm $s \colon 0 \to 0$ is clearly a unit for parallel composition. Moreover, it follows immediately from the definitions that $\otimes$ is strictly associative, but not commutative. Furthermore, $\otimes$ and $\_;\_$ satisfy the equations

$$(p \otimes p'); (q \otimes q') = (p; q) \otimes (p'; q') \quad \text{and} \quad id_u \otimes id_v = id_{u \oplus v}.$$

Proposition 1.1.15 *(Sym_S is symmetric strict monoidal)*
Given a net N, let Sym_N be the category whose objects are the elements of $S_N^{\oplus}$ and whose arrows are the vperms $s: u \to u$ for $u \in S^{\oplus}$ with the given identities and composition. By the pevious arguments above, this is a category.
Consider the mapping $\otimes: Sym_N \times Sym_N \to Sym_N$ defined as follows

$$\begin{array}{ccc} Sym_N \times Sym_N & \xrightarrow{\otimes} & Sym_N \\ (u, u') & \longmapsto & u \oplus u' \\ \Big\downarrow{\scriptstyle (s,s')} & & \Big\downarrow{\scriptstyle (s \otimes s')} \\ (u, u') & \longmapsto & u \oplus u' \end{array}$$

By the equations given above, $\otimes$ is a functor, and since it is strictly associative, Sym_N is a strict monoidal category.
Consider the permutation $\gamma = \{1 \to 2, 2 \to 1\}$ in $\Pi(2)$. It is easy to verify that, for any $s: u \to u$ and $s': v \to v$, we have the following equalities

$$\begin{aligned} \gamma(u,v); (s' \otimes s) &= (s \otimes s'); \gamma(u,v) \\ (\gamma(u,v) \otimes id_w); (id_v \otimes \gamma(u,w)) &= \gamma(u, v \oplus w) \\ \gamma(u,v); \gamma(v,u) &= id_{u \oplus v} \end{aligned}$$

the first of which expresses that the family $\gamma = \{\gamma(u,v)\}_{u,v \in Sym_N}$ is a natural transformation and the others correspond to axioms (1.5) and (1.6). It follows that Sym_N is a symmetric strict monoidal category with symmetry isomorphism γ.

Observe that, although Sym_N is in general not strictly symmetric, it keeps being so on the objects. More strongly, the objects form a free commutative monoid.

Definition 1.1.16 *(The category $\mathcal{P}[N]$)*
Let N be a net in Petri*. Then $\mathcal{P}[N]$ is the category which includes Sym_N as subcategory and has the additional arrows defined by the following inference rules:*

$$\frac{t: u \to v \text{ in } T_N}{t: u \to v \text{ in } \mathcal{P}[N]}$$

$$\frac{\alpha: u \to v \text{ and } \beta: u' \to v' \text{ in } \mathcal{P}[N]}{\alpha \otimes \beta: u \oplus u' \to v \oplus v' \text{ in } \mathcal{P}[N]} \qquad \frac{\alpha: u \to v \text{ and } \beta: v \to w \text{ in } \mathcal{P}[N]}{\alpha; \beta: u \to w \text{ in } \mathcal{P}[N]}$$

plus axioms expressing the fact that $\mathcal{P}[N]$ is a strict monoidal category:

$$\begin{array}{rcl} \alpha; id_v = \alpha = id_u; \alpha & \text{and} & (\alpha;\beta);\delta = \alpha;(\beta;\delta) \\ (\alpha \otimes \beta) \otimes \delta = \alpha \otimes (\beta \otimes \delta) & \text{and} & id_0 \otimes \alpha = \alpha = \alpha \otimes id_0 \\ (\alpha_0 \otimes \alpha_1); (\beta_0 \otimes \beta_1) = (\alpha_0; \beta_0) \otimes (\alpha_1; \beta_1) & \text{for} & \alpha_i: u_i \to v_i,\ \beta_i: v_i \to w_i \\ \gamma(u,u'); \beta \otimes \alpha = \alpha \otimes \beta; \gamma(v,v') & \text{for} & \alpha: u \to v,\ \beta: u' \to v' \end{array}$$

and the following axioms involving vperms and transitions

$$\begin{array}{ll} t; s = t & \text{where } t\colon u \to v \text{ in } T_N \text{ and } s\colon v \to v \text{ in } Sym_N, \\ s; t = t & \text{where } t\colon u \to v \text{ in } T_N \text{ and } s\colon u \to u \text{ in } Sym_N. \end{array} \qquad (\Psi)$$

Thus, $\mathcal{P}[N]$ is build on the category Sym_N by adding the transitions of N and freely closing with respect to sequential and parallel composition of arrows so that $\mathcal{P}[N]$ is made strict monoidal. The axiom involving vperms extends the naturality of γ to the newly added arrows. The other axioms γ must satisfy do not depend on the new transitions and so they follow directly from Proposition 1.1.15.

The intended interpretation of the data above is as follows. As usual, a single transition $t_0\colon u_0 \to v$ consumes the tokens in u_0 and produces those in v. Of course, given $t'_0\colon v \to w_0$, in the composition $t_0; t'_0$ we say that t'_0 causally depends on t_0. Consider now $t_1\colon u_1 \to v$ and $t'_1\colon v \to w_1$. Then, in accordance with the fact that $(t_0 \otimes t_1); (t'_0 \otimes t'_1) = (t_0; t'_0) \otimes (t_1; t'_1)$, we may stipulate that in the process $(t_0 \otimes t_1); (t'_0 \otimes t'_1)\colon u_0 \oplus u_1 \to w_0 \oplus w_1$ the transition t'_0 depends on t_0 and the transition t'_1 depends on t_1, while in $(t_0 \otimes t_1); (t'_1 \otimes t'_0)$ it is t_0 that causes t'_1 and t_1 that causes t'_0. Of course, both of those scenarios are possible since in $\mathcal{P}[N]$ we have that $(t'_0 \otimes t'_1) \neq (t'_1 \otimes t'_0)$. Now, since

$$(t_0 \otimes t_1); \gamma(v, v); (t'_0 \otimes t'_1) = (t_0 \otimes t_1); (t'_1 \otimes t'_0),$$

vperms may be viewed as formal operations that "exchange causes", by exchanging the tokens produced by parallel transitions. Observe that this interpretation is also well supported by the particular form that the interchange vperm takes on disjoint pairs u and v. Then, $\gamma(u, v)$ is the identity, corresponding to the fact that in this case no ambiguity is possible concerning what transition produced what token in $u \oplus v$ and, therefore, $(t_0 \otimes t_1); (t'_0 \otimes t'_1)$ and $(t_0 \otimes t_1); (t'_1 \otimes t'_0)$ have in this case to be considered as the same process. Now, the meaning of the "naturality" of γ is apparent. The same applies to the axiom $s; t; s' = t$ since exchanging two tokens consumed by or produced by a single t does not influence the causal behaviour.

As alredy mentioned, this nice interpretation of the arrows of $\mathcal{P}[N]$ may be pursued further by relating them to a slight refinement of the classical notion of finite process, namely the refinement consisting of adding a suitable layer of labels to the minimal and maximal places of finite process nets.

Definition 1.1.17 *(f-indexed orderings)*
Given sets A and B together with a function $f\colon A \to B$, an f-indexed ordering of A is a family $\{\ell_b \mid b \in B\}$ of bijections $\ell_b\colon f^{-1}(b) \to \{1, \ldots, |f^{-1}(b)|\}$, with $f^{-1}(b)$ being as usual the set $\{a \in A \mid f(a) = b\}$.

Therefore, an f-indexed ordering of A is a family of total orderings, one for each of the partitions of A induced by f. By abuse of language, we shall keep calling an

Figure 1.2: A net and one of its concatenable processes $\pi\colon 2a \to b \oplus c$

f-indexed ordering of $C \subseteq A$ any ordering obtained by considering the restriction of f to C. In the following, given a finite process net Θ, let $\min(\Theta)$ and $\max(\Theta)$ denote, respectively, its minimal and maximal elements, which must be places.

Definition 1.1.18 *(Concatenable Processes)*
A concatenable process of N is a triple $CP = (\pi, \ell, L)$ where

- *$\pi\colon \Theta \to N$ is a finite process of N;*
- *ℓ is a π-indexed ordering of $\min(\Theta)$;*
- *L is a π-indexed ordering of $\max(\Theta)$.*

Two concatenable processes CP and CP' are isomorphic if their underlying processes are isomorphic (in $\underline{\texttt{Proc}}_f[N]$) via an isomorphism φ which respects the ordering, i.e., such that $\ell'_{\pi'(\varphi(a))}(\varphi(a)) = \ell_{\pi(a)}(a)$ and $L'_{\pi'(\varphi(b))}(\varphi(b)) = L_{\pi(b)}(b)$ for all $a \in \min(\Theta)$ and $b \in \max(\Theta)$. As in the case of processes, we shall identify isomorphic concatenable processes.

Concatenable processes can be represented by drawing the underlying process nets and labelling their elements according to π, ℓ and L. When $|\pi^{-1}(a)| = 1$ for some place a, we omit the trivial labelling. Figure 1.2 shows a simple example. We use the standard graphical representation of nets in which circles are places, boxes are transitions, and sources and targets are directed arcs whose weights represent multiplicities, unitary weights being omitted.

As the reader will have already guessed, it is clearly possible to define an operation of concatenation of concatenable processes, whence their name. We can associate a source and a target in $S_N^{\oplus}$ to any concatenable process CP, namely by taking the image through π of, respectively, $\min(\Theta)$ and $\max(\Theta)$, where Θ is the underlying process net of CP. Then, the concatenation of $(\pi_0\colon \Theta_0 \to N, \ell^0, L^0)\colon u \to v$ and

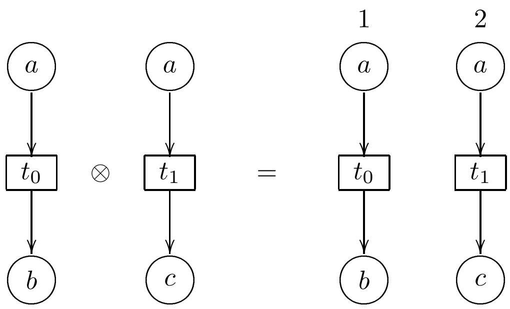


Figure 1.3: The process of Figure 1.2 as tensor of two simpler processes.

$(\pi_1: \Theta_1 \to N, \ell^1, L^1): v \to w$ is the concatenable process $(\pi: \Theta \to N, \ell, L): u \to w$ defined as follows.

- In order to simplify notations, suppose that S_{Θ_0} and S_{Θ_1} are disjoint. Let A be the set of pairs (x, y) such that $x \in \max(\Theta)$, $y \in \min(\Theta)$, $\pi_0(y) = a = \pi_1(x)$ and $L^0_a(x) = \ell^1_a(y)$. By the definitions of concatenable processes and of their sources and targets, an element of $\max(\Theta_0)$ belongs exactly to one pair of A, and of course the same happens to $\min(\Theta_1)$. Consider $S_0 = S_{\Theta_0} \setminus \max(\Theta_0)$ and $S_1 = S_{\Theta_1} \setminus \min(\Theta_1)$. Then, let $in_0: S_{\Theta_0} \to S_0 \cup A$ be the function which is the identity on $x \in S_0$ and maps $x \in \max(\Theta_1)$ to the corresponding pair in A. Define $in_1: S_{\Theta_1} \to S_1 \cup A$ analogously. Then,

$$\Theta = (\partial^0, \partial^1: T_{\Theta_0} + T_{\Theta_1} \to (S_0 \cup S_1 \cup A)^{\oplus}),$$

  where
  - $\partial^0 = in_0^{\oplus} \circ \partial^0_{\Theta_0} + in_1^{\oplus} \circ \partial^0_{\Theta_1}$;
  - $\partial^1 = in_0^{\oplus} \circ \partial^1_{\Theta_0} + in_1^{\oplus} \circ \partial^1_{\Theta_1}$;

- Suppose $\pi_i = \langle f_i, g_i \rangle$, for $i = 0, 1$ and consider the function $g(x) = g_i(x)$ if $x \in S_i$ and $g((x, y)) = g_0(x) = g_1(y)$ otherwise. Then $\pi = \langle f_0 + f_1, g \rangle$.

- $\ell_a(x) = \ell^0_a(x)$ if $x \in \min(\Theta_0)$ and $\ell_a((x, y)) = \ell^0_a(x)$ if $(x, y) \in \min(\Theta)$.

- $L_a(x) = L^1_a(x)$ if $x \in \max(\Theta_1)$ and $L_a((x, y)) = L^1_a(y)$ if $(x, y) \in \max(\Theta)$.

Under this operation of sequential composition, the concatenable processes of N form a category $\mathcal{CP}[N]$ with object the finite multisets on S_N and identities those processes consisting only of places, which therefore are both minimal and maximal, and such that $\ell = L$.

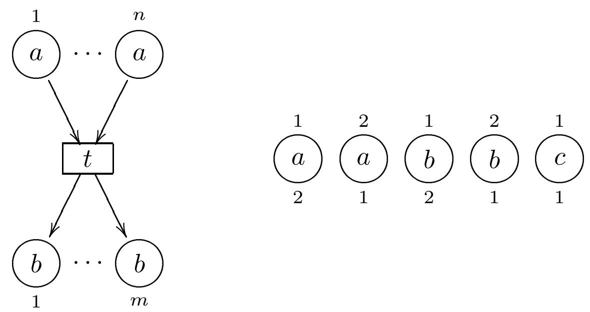


Figure 1.4: A transitions $t: na \to mb$ and the symmetry $\gamma(a \oplus b \oplus c, a \oplus b)$ in $\mathcal{P}[N]$

Concatenable processes, of course, admit also a tensor operation $\otimes$ such that, given $CP_0 = (\pi_0: \Theta_0 \to N, \ell^0, L^0): u \to v$ and $CP_1 = (\pi_1: \Theta_1 \to N, \ell^1, L^1): u' \to v'$, $CP_0 \otimes CP_1$ is the concatenable process $(\pi: \Theta \to N, \ell, L): u \oplus u' \to v \oplus v'$ given below.

- $\Theta = (\partial^0_{\Theta_0} + \partial^0_{\Theta_1}, \partial^1_{\Theta_0} + \partial^1_{\Theta_1}: T_{\Theta_0} + T_{\Theta_1} \to (S_{\Theta_0} + S_{\Theta_1})^{\oplus})$,
- $\pi = \pi_0 + \pi_1$;
- $\ell_a(in_0(x)) = \ell^0_a(x)$ and $\ell_a(in_1(y)) = |\pi_0^{-1}(a)| + \ell^1_a(y)$.
- $L_a(in_0(x)) = L^0_a(x)$ and $L_a(in_1(y)) = |\pi_0^{-1}(a)| + L^1_a(y)$.

The concatenable processes consisting only of places are the analogous of the vperms. In particular, for any $u = n_1 a_1 \oplus \cdots \oplus n_k a_k$ and $v = m_1 b_1 \oplus \cdots \oplus m_h b_h$, the concatenable process which consists of as many places as elements in the multiset $u \oplus v$, i.e., $\Sigma_{i=1}^k n_i + \Sigma_{i=1}^h m_i$, mapped by π to the corresponding places of N and such that $L_{a_i}(x) = v(a_i) + \ell_{a_i}(x)$ and $\ell_{b_i}(x) = L_{b_i}(x) - u(b_i)$ corresponds to the interchange vperm $\gamma(u, v)$ (see also Figure 1.4). Moreover, the category $\mathcal{CP}[N]$ enjoys the axioms (1.2)–(1.6) for $\_;\_$, $\_\otimes\_$ and γ as given above. Therefore, $\mathcal{CP}[N]$ is a SSMC. It is easy to see that the subcategory of symmetries of $\mathcal{CP}[N]$ is isomorphic to Sym_N. Finally, since the transitions t of N are faithfully represented in the obvious way by concatenable processes with a unique transition which is in the post-set of any minimal place and in the pre-set of any maximal place, minimal and maximal places being in one-to-one correspondence, respectively, with $\partial^0_N(t)$ and $\partial^1_N(t)$ (see also Figure 1.4), it is possible to show the following.

Proposition 1.1.19
$\mathcal{CP}[N]$ and $\mathcal{P}[N]$ are isomorphic.

Proof. See [16]. ✓

Furthermore, it can be shown—and it is indeed easy to get convinced—that concatenable processes are the least refinement of processes on which such a sequentialization may be given. In other words, they are the finest concretion of classical processes which may act as arrows in a category. Observe that this consideration, together with the fact that concatenable processes correspond with the arrows of $\mathcal{P}[N]$, links the two approaches we discussed earlier: making $\mathcal{T}[N]$ not strictly symmetric and making the objects of $\underline{\mathsf{Proc}}[N]$ be the arrows of some category.

As already remarked in the introduction, $\mathcal{P}[N]$ provides an axiomatization of the causal behaviour of N which, however, builds on the concrete choice of the category of vperms on S_N as the subcategory of symmetries of $\mathcal{P}[N]$. Clearly, it would be nice to be able to give an abstract characterization also of Sym_N. This is what we shall do in the next section.

1.2 Axiomatizing Concatenable Processes

In this section we show that the category of vperms Sym_N can be described in abstract terms, thus yielding a fully axiomatic characterization of concatenable processes. We start by showing that we can associate a free SSMC to each net N. Although this fact is not very surprising, our proof of it will give a "minimal" description of the free category on N which will be useful later on.

PROPOSITION 1.2.1
The forgetful functor $\mathcal{U}\colon \underline{\mathsf{SSMC}}^{\oplus} \to \underline{\mathsf{Petri}}$ *has a left adjoint* $\mathcal{F}\colon \underline{\mathsf{Petri}} \to \underline{\mathsf{SSMC}}^{\oplus}$.

Proof. Consider the category $\mathcal{F}(N)$ whose objects are the elements of $S_N^{\oplus}$ and whose arrows are generated by the inference rules

$$\frac{u \in S_N^{\oplus}}{id_u\colon u \to u \text{ in } \mathcal{F}(N)} \qquad \frac{a \text{ and } b \text{ in } S_N}{c_{a,b}\colon a \oplus b \to a \oplus b \text{ in } \mathcal{F}(N)} \qquad \frac{t\colon u \to v \text{ in } T_N}{t\colon u \to v \text{ in } \mathcal{F}(N)}$$

$$\frac{\alpha\colon u \to v \text{ and } \beta\colon u' \to v' \text{ in } \mathcal{F}(N)}{\alpha \otimes \beta\colon u \oplus u' \to v \oplus v' \text{ in } \mathcal{F}(N)} \qquad \frac{\alpha\colon u \to v \text{ and } \beta\colon v \to w \text{ in } \mathcal{F}(N)}{\alpha;\beta\colon u \to w \text{ in } \mathcal{F}(N)}$$

modulo the axioms expressing that $\mathcal{F}(N)$ is a strict monoidal category, namely,

$$\begin{array}{rcl} \alpha; id_v = \alpha = id_u; \alpha & \text{and} & (\alpha;\beta);\gamma = \alpha;(\beta;\gamma), \\ (\alpha \otimes \beta) \otimes \gamma = \alpha \otimes (\beta \otimes \gamma) & \text{and} & id_0 \otimes \alpha = \alpha = \alpha \otimes id_0, \\ id_u \otimes id_v = id_{u \oplus v} & \text{and} & (\alpha \otimes \alpha');(\beta \otimes \beta') = (\alpha;\beta) \otimes (\alpha';\beta'), \end{array} \tag{1.10}$$

the latter whenever the righthand term is defined, and the following axioms

$$c_{a,b}; c_{b,a} = id_{a \oplus b} \tag{1.11}$$

$$c_{u,u'};(\beta \otimes \alpha) = (\alpha \otimes \beta); c_{v,v'} \quad \text{for } \alpha\colon u \to v,\ \beta\colon u' \to v' \tag{1.12}$$

where $c_{u,v}$ for $u, v \in S_N^\oplus$ denote *any* term obtained from $c_{a,b}$ for $a, b \in S_N$ by applying recursively the following rules (compare with axiom (1.5)):

$$\begin{array}{rcl} c_{0,u} & = & id_u \quad = \quad c_{u,0} \\ c_{a\oplus u,v} & = & (id_a \otimes c_{u,v}); (c_{a,v} \otimes id_u) \\ c_{u,v\oplus a} & = & (c_{u,v} \otimes id_a); (id_v \otimes c_{u,a}) \end{array} \tag{1.13}$$

Observe that equation (1.12), in particular, equalizes all the terms obtained from (1.13) for fixed u and v. In fact, let $c_{u,v}$ and $c'_{u,v}$ be two such terms and take α and β to be, respectively, the identities of u and v. Now, since $id_u \otimes id_v = id_{u\oplus v} = id_v \otimes id_u$, from (1.12) we have that $c_{u,v} = c'_{u,v}$ in $\mathcal{F}(N)$. Then, we claim that the collection $\{c_{u,v}\}_{u,v\in S_N^\oplus}$ is a symmetry natural isomorphism which makes $\mathcal{F}(N)$ into a SSMC and that, in addition, $\mathcal{F}(N)$ is the free SSMC on N.

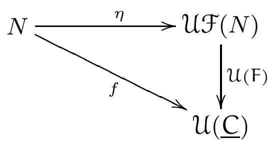
In order to show the first claim, observe that the naturality of c is expressed directly from axiom (1.12). Then, we need to check that for any u and v we have $c_{u,v}; c_{v,u} = id_{u\oplus v}$, which follows easily from (1.11) by induction on the least of the sizes of u and v.

<u>base cases</u>. If $u = 0$ or $v = 0$ then the thesis follows from the first of (1.13). If instead $|u| = |v| = 1$, then required equation is (1.11).

<u>inductive step</u>. Without loss of generality, assume that $u = a \oplus u'$. Then, by (1.13),

$$\begin{array}{rcl} c_{u,v}; c_{v,u} & = & (id_a \otimes c_{u',v}); (c_{a,v} \otimes id_{u'}); (c_{v,a} \otimes id_{u'}); (id_a \otimes c_{v,u'}) \\ & = & (id_a \otimes c_{u',v}); ((c_{a,v}; c_{v,a})) \otimes id_{u'}); (id_a \otimes c_{v,u'}) \\ & = & (id_a \otimes c_{u',v}); (id_a \otimes c_{v,u'}) \\ & = & id_a \otimes (c_{u',v}; c_{v,u'}) = id_a \otimes id_{u'\oplus v} = id_{u\oplus v}. \end{array}$$

Consider now the net $\mathcal{UF}(N)$ obtained from $\mathcal{F}(N)$ by forgetting about its categorical structure. More precisely, the markings of $\mathcal{UF}(N)$ are the markings of N and its transitions are the arrows $\mathcal{F}(N)$ with the given sources and targets. Consider then the Petri net morphism $\eta\colon N \to \mathcal{UF}(N)$ where η_p is the identity homomorphism and η_t is the obvious injection of T_N in $T_{\mathcal{UF}(N)}$. We show that η is universal, i.e., that for any SSMC $\underline{C}$ in $\underline{SSMC}^\oplus$ and for any Petri net morphism $f\colon N \to \mathcal{U}(\underline{C})$, there is a unique symmetric strict monoidal functor $\mathsf{F}\colon \mathcal{F}(N) \to \underline{C}$ which makes the following diagram commute.



Let $\underline{C} = (\underline{C}, \otimes, 0, \gamma)$ and $f\colon N \to \mathcal{U}(\underline{C})$ be as in the hypothesis above. Then, in order for the diagram to commute and in order for F to be a symmetric strict monoidal functor, its definition on the generators of $\mathcal{F}(N)$ is compelled to be:

$$\mathsf{F}(u) = f_p(u), \quad \mathsf{F}(t) = f_t(t), \quad \mathsf{F}(id_u) = id_{\mathsf{F}(u)}, \quad \mathsf{F}(c_{a,b}) = \gamma_{f_p(a),f_p(b)}.$$

Of course, also the extension of F to composition and tensor is uniquely determined, namely, it must be $\mathsf{F}(\alpha;\beta) = \mathsf{F}(\beta)\circ\mathsf{F}(\alpha)$ and $\mathsf{F}(\alpha\otimes\beta) = \mathsf{F}(\alpha)\otimes\mathsf{F}(\beta)$. To conclude the proof we only need to show that F is a well-defined symmetric strict monoidal functor, since, in this case, it is necessarily the unique which makes the diagram commute.

In order to show that F is well-defined, it is enough to see that it preserves the axioms which generates $\mathcal{F}(N)$. Since $\underline{\mathsf{C}}$ is a strict monoidal category and since $\mathsf{F}(id_u) = id_{\mathsf{F}(u)}$, axioms (1.10) are clearly preserved. Moreover, since $\underline{\mathsf{C}}$ is symmetric with symmetry isomorphism γ, we have that

$$\mathsf{F}(c_{a,b};c_{b,a}) = \gamma_{\mathsf{F}(b),\mathsf{F}(a)}\circ\gamma_{\mathsf{F}(a),\mathsf{F}(b)} = id_{\mathsf{F}(a)\oplus\mathsf{F}(b)} = id_{\mathsf{F}(a\oplus b)},$$

i.e., F respects axiom (1.11). Thus, the last missing steps are to show that F preserves axiom (1.12) and that it is a symmetric strict monoidal functor, which actually reduces to show that for each u and v we have $\mathsf{F}(c_{u,v}) = \gamma_{\mathsf{F}(u),\mathsf{F}(v)}$. We start by showing, by induction on the structure of the c's, that for any term $c_{u,v}$ generated from (1.13), we have $\mathsf{F}(c_{u,v}) = \gamma_{\mathsf{F}(u),\mathsf{F}(v)}$.

<u>base cases</u>. If $u = 0$ then $c_{u,v} = id_v$ and thus $\mathsf{F}(c_{u,v}) = id_{\mathsf{F}(v)}$. However, we have already shown in the remark in page 22 that $\gamma_{e,x} = id_x$ holds in any symmetric monoidal category. Thus, since $\mathsf{F}(u) = 0$, we have $\mathsf{F}(c_{u,v}) = \gamma_{\mathsf{F}(u),\mathsf{F}(v)}$ as required. A symmetric argument applies if $v = 0$. If instead $|u| = |v| = 1$, the claim is proved by appealing directly to the definition of F.

<u>inductive step</u>. Suppose that $u = a\oplus u'$. Then, exploting the induction hypothesis, $\mathsf{F}(c_{u,v}) = (\gamma_{\mathsf{F}(a),\mathsf{F}(v)}\otimes id_{\mathsf{F}(u')})\circ(id_{\mathsf{F}(a)}\otimes\gamma_{\mathsf{F}(u'),\mathsf{F}(v)})$ and thus, by the coherence axiom (1.5) of symmetric monoidal categories, we have $\mathsf{F}(c_{u,v}) = \gamma_{\mathsf{F}(a)\oplus\mathsf{F}(u'),\mathsf{F}(v)}$ which is $\gamma_{\mathsf{F}(a\oplus u'),\mathsf{F}(v)}$, i.e., $\gamma_{\mathsf{F}(u),\mathsf{F}(v)}$. If instead we have that $v = v'\oplus a$ and $c_{u,v}$ is generated by the last of (1.13), then the claim is proved similarly by using the inverse of (1.5), i.e., $\gamma_{x,y\otimes z} = (id_y\otimes\gamma_{x,z})\circ(\gamma_{x,y}\otimes id_z)$, which, of course, holds in any symmetric monoidal category.

Now, since $\mathsf{F}(c_{u,v}) = \gamma_{\mathsf{F}(u),\mathsf{F}(v)}$ and since γ is a natural transformation, it follows immediately that F preserves axiom (1.12).

This shows that $\mathcal{F}$ is left adjoint to $\mathcal{U}$, in symbols $\mathcal{F}\dashv\mathcal{U}$. ✓

Thus, establishing the adjunction $\underline{\mathsf{Petri}} \rightharpoonup \underline{\mathsf{SSMC}}^{\oplus}$, we have identified the free SSMC on N as a category generated, modulo some equations, from the net N viewed as a graph enriched with formal arrows id_u, which play the role of the identities, and $c_{a,b}$ for $a,b\in S_N$, which generate all the needed symmetries. In the following, we speak of the *free* SSMC on N to mean $\mathcal{F}(N)$ as constructed above.

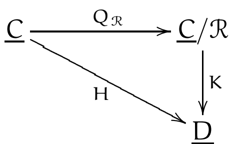
<u>REMARK</u>. Observe that the above choice is the only *sensible* one for the notion of free SSMC on N. In fact, the notion of free algebra, or free construction, makes sense *only* when one has a functor $\mathcal{F}$ which is *left adjoint* to a *forgetful functor.* or, in other words, when you have a *monadic* functor. Then, the free algebra on an object a is $\mathcal{F}(a)$. Now, it is not possible to have a forgetful functor from a category of SSMC's to $\underline{\mathsf{Petri}}$ unless one restricts oneself to $\underline{\mathsf{SSMC}}^{\oplus}$. *Ergo* the term "free" makes sense *only* in this case.

The following is the adaptation to SSMC's of the usual notion of quotient algebras characterized, as usual, by a universal property. It is worth noticing that the theory treats only the quotient of categories obtained by imposing equalities of arrows belonging to the same homset.

Proposition 1.2.2 *(Monoidal Quotient Categories)*
For a given SSMC $\underline{\mathsf{C}}$*, let* $\mathcal{R}$ *be a function which assigns to each pair of objects* a *and* b *of* $\underline{\mathsf{C}}$ *a binary relation* $\mathcal{R}_{a,b}$ *on the homset* $\underline{\mathsf{C}}(a,b)$*. Then, there exist a* SSMC $\underline{\mathsf{C}}/\mathcal{R}$ *and a symmetric strict monoidal functor* $\mathsf{Q}_{\mathcal{R}}\colon \underline{\mathsf{C}} \to \underline{\mathsf{C}}/\mathcal{R}$ *such that*

i) If $f\mathcal{R}_{a,b}f'$ *then* $\mathsf{Q}_{\mathcal{R}}(f) = \mathsf{Q}_{\mathcal{R}}(f')$*;*

ii) For each symmetric strict monoidal $\mathsf{H}\colon \underline{\mathsf{C}} \to \underline{\mathsf{D}}$ *such that* $\mathsf{H}(f) = \mathsf{H}(f')$ *whenever* $f\mathcal{R}_{a,b}f'$*, there exists a unique functor* $\mathsf{K}\colon \underline{\mathsf{C}}/\mathcal{R} \to \underline{\mathsf{D}}$*, which is necessarily symmetric strict monoidal, such that the following diagram commutes.*



Proof. Say that $\mathcal{R}$ is a *congruence* if $\mathcal{R}_{a,b}$ is an equivalence for each a and b and if $\mathcal{R}$ respects composition, i.e., whenever $f\mathcal{R}_{a,b}f'$ then, for all $h\colon a' \to a$ and $k\colon b \to b'$, we have $(k \circ f \circ h)\mathcal{R}_{a',b'}(k \circ f' \circ h)$. Clearly, if $\mathcal{R}$ is a congruence, the following definition is well-given: $\underline{\mathsf{C}}/\mathcal{R}$ is the category whose objects are those of $\underline{\mathsf{C}}$, whose homset $\underline{\mathsf{C}}/\mathcal{R}(a,b)$ is $\underline{\mathsf{C}}(a,b)/\mathcal{R}_{a,b}$, i.e., the quotient of the corresponding homset of $\underline{\mathsf{C}}$ modulo the appropriate component of $\mathcal{R}$, and whose composition of arrows is given by $[g]_{\mathcal{R}} \circ [f]_{\mathcal{R}} = [g \circ f]_{\mathcal{R}}$. In fact, since $\mathcal{R}_{a,b}$ is an equivalence $\underline{\mathsf{C}}/\mathcal{R}(a,b)$ is well-defined, and since $\mathcal{R}$ preserves the composition, so is the composition in $\underline{\mathsf{C}}/\mathcal{R}$.

Let $\underline{\mathsf{C}} = (\underline{\mathsf{C}}, \otimes, e, \gamma)$. Call $\mathcal{R}$ a $\otimes$-*congruence* if it is a congruence in the above sense and it respects tensor, i.e., if $f\mathcal{R}_{a,b}f'$ then, for all $h\colon a' \to b'$ and $k\colon a'' \to b''$, we have $(h \otimes f \otimes k)\mathcal{R}_{a' \otimes a \otimes a'', b' \otimes b \otimes b''}(h \otimes f' \otimes k)$. It is easy to check that, if $\mathcal{R}$ is a $\otimes$-congruence, then the definition $[f]_{\mathcal{R}} \otimes [g]_{\mathcal{R}} = [f \otimes g]_{\mathcal{R}}$ makes the quotient category $\underline{\mathsf{C}}/\mathcal{R}$ into a SSMC with symmetry isomorphism given by the natural transformation whose component at (u,v) is $[\gamma_{u,v}]_{\mathcal{R}}$ and unit object e.

Observe now that, given $\mathcal{R}$ as in the hypothesis, it always possible to find the least $\otimes$-congruence $\mathcal{R}'$ which includes (componentwise) $\mathcal{R}$. Then, take $\underline{\mathsf{C}}/\mathcal{R}$ to be $\underline{\mathsf{C}}/\mathcal{R}'$ and $\mathsf{Q}_{\mathcal{R}}$ to be the obvious projection of $\underline{\mathsf{C}}$ into $\underline{\mathsf{C}}/\mathcal{R}$. Clearly, $\mathsf{Q}_{\mathcal{R}}$ is a symmetric strict monoidal functor.

Now, let $\mathsf{H}\colon \underline{\mathsf{C}} \to \underline{\mathsf{D}}$ be a monoidal functor as in the hypothesis and consider the mapping of objects and arrows of $\underline{\mathsf{C}}/\mathcal{R}$ to, respectively, objects and arrows of $\underline{\mathsf{D}}$ given by $\mathsf{K}(a) = \mathsf{H}(a)$ and $\mathsf{K}([f]_{\mathcal{R}}) = \mathsf{H}(f)$. It follows from definition of functor that the family $\{\mathcal{S}_{a,b}\}_{a,b \in \underline{\mathsf{C}}}$, where $\mathcal{S}_{a,b}$ is the binary relation $\{(f,g) \mid \mathsf{H}(f) = \mathsf{H}(g)\}$ on $\underline{\mathsf{C}}(a,b)$, is a congruence. Moreover, since $\mathsf{H}(f \otimes g) = \mathsf{H}(f) \otimes \mathsf{H}(g)$, we have that $\{\mathcal{S}_{a,b}\}_{a,b \in \underline{\mathsf{C}}}$

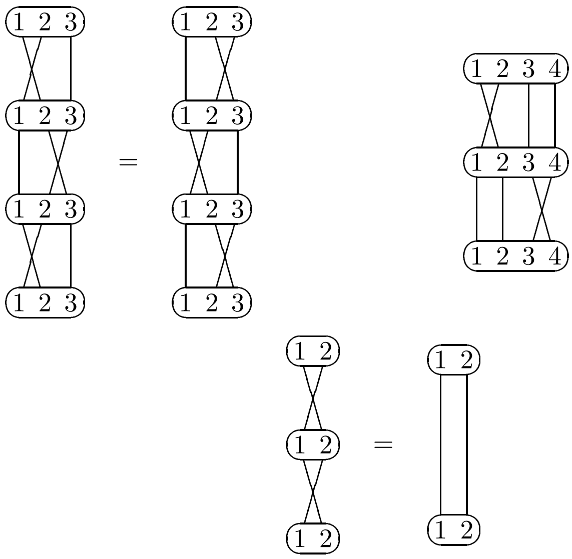


Figure 1.5: Some instances of the axioms of permutations

is a $\otimes$-congruence. Then, if H satisfies the condition in the hypothesis, i.e., if $\mathcal{R} \subseteq \mathcal{S}$, since $\mathcal{R}'$ is the least $\otimes$-congruence which contains $\mathcal{R}$, we have that $f\mathcal{R}'_{a,b}g$ implies $\mathsf{H}(f) = \mathsf{H}(g)$, i.e., K is well-defined. Moreover, since H is a functor, it follows that $\mathsf{K}([id_a]_{\mathcal{R}}) = id_{\mathsf{H}(a)} = id_{\mathsf{K}(a)}$ and $\mathsf{K}([g]_{\mathcal{R}} \circ [f]_{\mathcal{R}}) = \mathsf{H}(g) \circ \mathsf{H}(f) = \mathsf{K}([g]_{\mathcal{R}}) \circ \mathsf{K}([f]_{\mathcal{R}})$, i.e., K is a functor. In the same way, one shows that $\mathsf{K}([f]_{\mathcal{R}} \otimes [g]_{\mathcal{R}}) = \mathsf{K}([f]_{\mathcal{R}}) \otimes \mathsf{K}([g]_{\mathcal{R}})$. Then, since $\mathsf{K}([\gamma_{u,v}]_{\mathcal{R}}) = \mathsf{H}(\gamma_{u,v}) = \gamma'_{\mathsf{K}(u),\mathsf{K}(v)}$, where γ' is the symmetry isomorphism of $\underline{\mathsf{D}}$, one concludes that K is in $\underline{\mathsf{SsMonCat}}$.

Clearly, K renders commutative the diagram above and it is indeed the unique functor which enjoys such a property for the given H. ✓

Our next step is to show that $\mathcal{P}[N]$ is the quotient of $\mathcal{F}(N)$ modulo two simple additional axioms. In order to show this, we need the following lemma, originally proved in [104] (see also [11]).

LEMMA 1.2.3
The symmetric group $\Pi(n)$ is (isomorphic to) the group G freely generated from the set $\{\tau_i \mid 1 \le i < n\}$, modulo the equations (see also Figure 1.5)

$$\begin{aligned} \tau_i\tau_{i+1}\tau_i &= \tau_{i+1}\tau_i\tau_{i+1}; \\ \tau_i\tau_j &= \tau_j\tau_i \quad \text{if } |i-j| \ge 1; \\ \tau_i\tau_i &= e; \end{aligned} \tag{1.14}$$

where e is the unit element of the group.

Proof. The proof is by induction on n. First of all, observe that for $n = 0$ and $n = 1$ the set of generators is empty and the equations are vacuous. Hence, G is the free group on the empty set of generators, i.e., the group consisting only of the unit element, which is (isomorphic to) $\Pi(0)$ and $\Pi(1)$.

Suppose now that the thesis holds for $n \geq 1$ and let us prove it for $n+1$. It is immediately evident that the permutations of $n+1$ elements are generated by the n transpositions, i.e., by those pemutations which leave all the elements fixed but two adjacent ones, which are exchanged. Moreover, the transpositions satisfy axioms (1.14), as a quick look to Figure 1.5 shows. It follows that the order of G must be not smaller than the order of $\Pi(n+1)$, i.e., $|G| \geq (n+1)!$, where $|\_|$ gives the cardinality of sets. Moreover, there is a group homomorphism $h: G \to \Pi(n+1)$ which sends τ_i to the transposition $(i\ i+1)$, and since the transpositions generate $\Pi(n+1)$, we have that h is surjective. Thus, in order to conclude the proof, we only need to show that h injective, which clearly follows if we show that $|G| = (n+1)!$.

Let H be the subgroup of G generated by $\{\tau_1, \tau_2, \ldots, \tau_{n-1}\}$ and consider the $n+1$ cosets $H_1, \ldots, H_{n+1}$, where $H_i = H\tau_n \cdots \tau_i = \{x\tau_n \cdots \tau_i \mid x \in H\}$, $1 \leq i \leq n$, and $H_{n+1} = H$. Then, for $1 \leq i \leq n+1$ and $1 \leq j \leq n$, consider $H_i\tau_j$. The following cases are possible.

<u>$i > j+1$</u>. By the second of axioms (1.14), τ_j is permutable with each of $\tau_i, \ldots, \tau_n$ and, therefore,

$$\begin{aligned} H_i\tau_j &= H\tau_n \cdots \tau_i\tau_j \\ &= H\tau_j\tau_n \cdots \tau_i \\ &= H\tau_n \cdots \tau_i = H_i. \end{aligned}$$

<u>$i < j$</u>. Again by the second of (1.14), τ_j is permutable with each of $\tau_i, \ldots, \tau_{j-2}$ and, therefore,

$$\begin{aligned} H_i\tau_j &= H\tau_n \cdots \tau_i\tau_j \\ &= H\tau_n \cdots \tau_{j+1}\tau_j\tau_{j-1}\tau_j \cdots \tau_i \\ &= H\tau_n \cdots \tau_{j+1}\tau_{j-1}\tau_j\tau_{j-1} \cdots \tau_i && \text{by the first of (1.14)} \\ &= H\tau_{j-1}\tau_n \cdots \tau_{j+1}\tau_j\tau_{j-1} \cdots \tau_i && \text{by the second of (1.14)} \\ &= H\tau_n \cdots \tau_i = H_i. \end{aligned}$$

<u>$i = j$</u>. Then $H_j\tau_j = H\tau_n \cdots \tau_j\tau_j$. i.e., by the third of (1.14), $H\tau_n \cdots \tau_{j+1} = H_{j+1}$.

<u>$i = j+1$</u>. Then $H_{j+1}\tau_j = H\tau_n \cdots \tau_{j+1}\tau_j = H_j$.

In other words, for $1 \leq j \leq n$, the sets $H_1 \ldots H_{n+1}$ remain all unchanged by post-multiplication by τ_j, except that H_j and H_{j+1} which are exchanged with each other. Now, since each element of G is a product $\tau_{i_1} \cdots \tau_{i_k}$, it belongs to $H\tau_{i_1} \cdots \tau_{i_k}$, i.e., to one of the H_i. Hence, G is contained in the union of the H_i's. It follows immediately that, if H is finite, we have that $|G| \leq (n+1) \cdot |H|$. However, by induction hypothesis, H is (isomorphic to) $\Pi(n)$, and thus H is finite and $|H| = n!$. Therefore, $|G| \leq (n+1)!$, which concludes the proof. ✓

The lemma above is easily adapted to vperms as follows.

LEMMA 1.2.4

The arrows of Sym_N are generated via sequential composition by the vperms of the kind $id_u \otimes \gamma(a,a) \otimes id_v : u \oplus 2a \oplus v \to u \oplus 2a \oplus v$. Moreover, two such compositions yield the same vperm if and only if this can be shown by using the axioms (compare with (1.14) by distributing the terms on the two sides of the equality sign)

$$\begin{aligned}((id_{u\oplus a} \otimes \gamma(a,a) \otimes id_v)\,;\,(id_u \otimes \gamma(a,a) \otimes id_{a\oplus v}))^3 &= id_{u\oplus 3a\oplus v},\\ ((id_u \otimes \gamma(a,a) \otimes id_{2b\oplus v})\,;\,(id_{u\oplus 2a} \otimes \gamma(b,b) \otimes id_v))^2 &= id_{u\oplus 2a\oplus 2b\oplus v},\\ (id_u \otimes \gamma(a,a) \otimes id_v)^2 &= id_{u\oplus 2a\oplus v}.\end{aligned} \tag{1.15}$$

where f^n indicates the composition of f with itself n times.

Proof. Concerning the first claim, a vperm $p = \langle \sigma_{a_1}, \ldots, \sigma_{a_n} \rangle$ coincides with the tensor $\sigma_{a_1} \otimes \cdots \otimes \sigma_{a_n}$ which, exploiting the functoriality of $\otimes$, can be written as $(\sigma_{a_1} \otimes \cdots \otimes id_{u_n}); \cdots ; (id_{u_1} \otimes \cdots \otimes \sigma_{a_n})$. Now, since σ_{a_i} is a permutation, it is a composition of transpositions, and since the transposition $\tau_i : na \to na$ can be written as $id_{(i-1)a} \otimes \gamma(a,a) \otimes id_{(n-i-1)a}$ in Sym_N, we have $\sigma_{a_i} = (id_{u'_1} \otimes \gamma(a_i,a_i) \otimes id_{u''_1}); \cdots ; (id_{u'_k} \otimes \gamma(a_i,a_i) \otimes id_{u''_k})$. Therefore, the vperms $id_u \otimes \gamma(a,a) \otimes id_v$ generate via composition all the vperms of Sym_N.

Concerning the axiomatization, it is easy to verify that the equations (1.15) hold in Sym_N (compare with Figure 1.5). On the other hand, suppose that the sequences

$$\begin{aligned} p &= (id_{u_1} \otimes \gamma(a_1,a_1) \otimes id_{v_1}); \cdots ; (id_{u_n} \otimes \gamma(a_n,a_n) \otimes id_{v_n})\\ q &= (id_{u'_1} \otimes \gamma(b_1,b_1) \otimes id_{v'_1}); \cdots ; (id_{u'_m} \otimes \gamma(b_m,b_m) \otimes id_{v'_m}) \end{aligned}$$

evaluate to the same vperm $\sigma_{c_1} \otimes \cdots \otimes \sigma_{c_k}$. We have to show that p and q can be proved equal using axioms (1.15) . To this aim, observe first that every a_i appearing in p and every b_i appearing in q must be one of the c_i's. Moreover, observe that, by repeated applications of the second of (1.15), we can reorganize p and q in such a way that all the terms involving c_1—if any—are grouped together and immediately followed by all the terms involving c_2—if any—and so on. Let us denote by p' and q' the terms so obtained and let us focus on the sequences p'_i and q'_i of terms involving c_i respectively in p' and q'. The following cases are possible.

i) p'_i and q'_i are both empty. Then, there is nothing to show.

ii) Either p'_i or q'_i—without loss of generality say p'_i—is empty. Then, σ_{c_i} is the identity and since q'_i evaluates to it, by Lemma 1.2.3, q'_i can be proved equal to the identity permutation using axioms (1.14). Now notice that the axioms (1.14) coincide with the axioms (1.15) instantiated to c_i. Therefore, the proof that q'_i is the identity permutation can be mimicked to prove using instances of axioms (1.15) that q'_i is an identity in Sym_N. Then we can drop q'_i from q'.

iii) Both p'_i and q_i are nonempty. Then, they must both evaluate to σ_{c_i} and, therefore, they can be proved equal using axioms (1.14). Then exploting the observation in the previous case, p'_i and q'_i can be proved equal using axioms (1.15).

Thus, p and q are proved equal using axioms (1.15), which concludes the proof. ✓

We are now ready to show the promised characterization of $\mathcal{P}[N]$.

Proposition 1.2.5
$\mathcal{P}[N]$ is the monoidal quotient of the free SSMC *on N modulo the axioms*

$$\begin{array}{rcll} c_{a,b} & = & id_{a\oplus b} & \text{if } a,b\in S_N \text{ and } a\neq b, \qquad (1.16)\\ s;t;s' & = & t & \text{if } t\in T_N \text{ and } s,s' \text{ are symmetries.} \qquad (1.17)\end{array}$$

Proof. We show that $\mathcal{P}[N]$ enjoys the universal property of $\mathcal{F}(N)/\mathcal{R}$ stated in Proposition 1.2.2, where $\mathcal{R}$ is the congruence generated from equations (1.16) and (1.17). It follows then from general facts about universal constructions that $\mathcal{P}[N]$ is isomorphic to $\mathcal{F}(N)/\mathcal{R}$.

First of all observe that $\mathcal{P}[N]$ belongs to $\underline{\mathsf{SSMC}}^{\oplus}$. Therefore, corresponding to the Petri net *inclusion* morphism $N\to\mathcal{U}\mathcal{P}[N]$, there is a symmetric strict monoidal functor $\mathsf{Q}\colon\mathcal{F}(N)\to\mathcal{P}[N]$ which is the identity on the places and on the transitions of N, i.e., such that

$$\begin{array}{rcl} \mathsf{Q}(a) & = & a \quad \text{if } a\in S_N,\\ \mathsf{Q}(t) & = & t \quad \text{if } t\in T_N,\\ \mathsf{Q}(c_{a,b}) & = & \gamma(a,b) \quad \text{if } a,b\in S_N.\end{array}$$

Thus, since $\gamma(a,b)=id_{a\oplus b}$ if $a\neq b\in S_N$, we have that $\mathsf{Q}(c_{a,b})=\mathsf{Q}(id_{a\oplus b})$ for $a\neq b\in S_N$. Moreover, since Q is a symmetric monoidal functor, it sends symmetries to symmetries. Therefore, since (1.17) holds in $\mathcal{P}[N]$, if s and s' are symmetries and $t\in T_N$, we have that $\mathsf{Q}(s;t;s')=\bar{s};t;\bar{s}'=t=\mathsf{Q}(t)$. We shall show that Q is universal among the functors which equalize the pairs $(c_{a,b},id_{a\oplus b})$ with $a\neq b$ and the pairs $(s;t;s',t)$ with s and s' symmetries and $t\in T_N$.

We start by showing that Sym_N can be embedded in $\mathit{Sym}_{\mathcal{F}(N)}$, though not via a symmetric strict monoidal functor. Consider the mapping G of objects and arrows of Sym_N to, respectively, objects and arrows of $\mathit{Sym}_{\mathcal{F}(N)}$ which is the identity on the objects and such that

$$\begin{array}{rcl} \mathsf{G}(id_u\otimes\gamma(a,a)\otimes id_v) & = & id_u\otimes c_{a,a}\otimes id_v,\\ \mathsf{G}(p;q) & = & \mathsf{G}(p);\mathsf{G}(q)\\ \mathsf{G}(id_u) & = & id_u.\end{array}$$

It follows from Lemma 1.2.4 that the above definition defines G on all vperms. Thus, in order to see that G is a functor we only need to see that it is well-defined which, by exploiting Lemma 1.2.4, can be seen by showing that it respects axioms (1.15).

i) From (1.13) we have that $(id_a\otimes c_{a,a});(c_{a,a}\otimes id_a)=c_{a\oplus a,a}$ and then from (1.12) we have $c_{a\oplus a,a};(id_a\otimes c_{a,a})=(c_{a,a}\otimes id_a);c_{a\oplus a,a}$, which, again by (1.13), yields

$(id_a \otimes c_{a,a}); (c_{a,a} \otimes id_a); (id_a \otimes c_{a,a}) = (c_{a,a} \otimes id_a); (id_a \otimes c_{a,a}); (c_{a,a} \otimes id_a)$, which is $((id_a \otimes c_{a,a}); (c_{a,a} \otimes id_a))^3 = id_{3a}$. Then, tensoring both terms by id_u on the left and id_v on the right and using (1.10), we have the required equality $((id_{u\oplus a} \otimes c_{a,a} \otimes id_v); (id_u \otimes c_{a,a} \otimes id_{a\oplus v}))^3 = id_{u\oplus 3a\oplus v}$

ii) By two applications of the last of (1.10), we have

$$(c_{a,a} \otimes id_{2b}); (id_{2a} \otimes c_{b,b}) = (c_{a,a} \otimes c_{b,b}) = (id_{2a} \otimes c_{b,b}); (c_{a,a} \otimes id_{2b}).$$

Then, applying the tensor of id_u and id_v as before, the required equality follows.

iii) $(id_u \otimes c_{a,a} \otimes id_v); (id_u \otimes c_{a,a} \otimes id_v) = id_{u\oplus 2a\oplus v}$ follows immediately from (1.11).

Therefore, G is a functor. Of course, G is not symmetric strict monoidal, since $\mathsf{G}(\gamma(a,b)) = id_{a\oplus b} \neq c_{a,b}$, i.e., axiom (1.9) does not hold. However, G is monoidal in the sense that conditions (1.7) and (1.8) hold. The claim is immediate for (1.7), while $\mathsf{G}(s \otimes s') = \mathsf{G}((s \otimes id_v); (id_u \otimes s')) = (\mathsf{G}(s) \otimes id_v); (id_u \otimes \mathsf{G}(s')) = \mathsf{G}(s) \otimes \mathsf{G}(s')$.

Let $\underline{\mathsf{C}} = (\underline{\mathsf{C}}, \otimes, e, \gamma)$ be a SSMC and suppose that there exists a symmetric strict monoidal functor $\mathsf{H}\colon \mathcal{F}(N) \to \underline{\mathsf{C}}$ such that, for any pair $a \neq b \in S_N$ and for any symmetries s and s', $\mathsf{H}(c_{a,b}) = \mathsf{H}(id_{a\oplus b})$ and $\mathsf{H}(s; t; s') = \mathsf{H}(t)$. We have to show that there exists a unique $\mathsf{K}\colon \mathcal{P}[N] \to \underline{\mathsf{C}}$ such that $\mathsf{H} = \mathsf{KQ}$. We consider the following definition of K on objects and generators

$$\begin{array}{rcll} \mathsf{K}(u) & = & \mathsf{H}(u) & \text{if } u \in S_N^{\oplus}, \\ \mathsf{K}(s) & = & \mathsf{H}(\mathsf{G}(s)) & \text{if } s \text{ is a symmetry} \\ \mathsf{K}(t) & = & \mathsf{H}(t) & \text{if } t \in T_N, \end{array}$$

extendend to $\mathcal{P}[N]$ by $\mathsf{K}(\alpha;\beta) = \mathsf{K}(\alpha); \mathsf{K}(\beta)$ and $\mathsf{K}(\alpha \otimes \beta) = \mathsf{K}(\alpha) \otimes \mathsf{K}(\beta)$. First of all, we have to show that K is well-defined, i.e., that the equations which hold in $\mathcal{P}[N]$ are preserved by K. To this aim, recall that $\mathcal{P}[N]$, by definition, is the category generated on top of Sym_N from the transitions of N modulo the axioms of monoidality, i.e., (1.2)–(1.6) plus the functoriality of $\otimes$ and the naturality of the symmetry isomorphism, and axioms (Ψ). Now, since H is symmetric strict monoidal and since G is monoidal, it is immediate to see that the functoriality of $\otimes$ and the axioms (1.2)–(1.4) are preserved. The *key* to show that the same happens for the naturality of the symmetry, (1.5) and (1.6) is to show that $\mathsf{K}(\gamma(u,v)) = \gamma_{\mathsf{K}(u),\mathsf{K}(v)}$. In fact, once this fact is established, the aforesaid points follow from the fact that $\underline{\mathsf{C}}$ is a SSMC. Thus, we proceed to prove our claim by induction on the least of the sizes of u and v.

<u>base cases</u>. If $u = 0$ then $\gamma(u,v) = id_v$ and since $\mathsf{K}(0) = e$, we have that $\mathsf{K}(\gamma(u,v)) = id_{\mathsf{K}(v)} = \gamma_{e,\mathsf{K}(v)}$. Otherwise, if $v = 0$, a symmetric argument applies. If instead $|u| = |v| = 1$, we have the following two cases.

$u = v = a$. Then $\mathsf{K}(\gamma(a,a)) = \mathsf{H}(c_{a,a}) = \gamma_{\mathsf{K}(a),\mathsf{K}(a)}$, since H is symmetric.

$u \neq v$ Then $\mathsf{K}(\gamma(u,v)) = \mathsf{H}(id_{u\oplus v})$ which, by hypothesis on H, is $\gamma_{\mathsf{K}(u),\mathsf{K}(v)}$.

<u>inductive step</u>. Suppose that $u = a \oplus u'$. Then, by the coherence axiom (1.5), $\mathsf{K}(\gamma(u,v)) = (\mathsf{K}(\gamma(a,v)) \otimes id_{\mathsf{K}(u')}) \circ (id_{\mathsf{K}(a)} \otimes \mathsf{K}(\gamma(u',v)))$ and thus, exploting the induction hypothesis, $\mathsf{K}(\gamma(u,v)) = (\gamma_{\mathsf{K}(a),\mathsf{K}(v)} \otimes id_{\mathsf{K}(u')}) \circ (id_{\mathsf{K}(a)} \otimes \gamma_{\mathsf{K}(u'),\mathsf{K}(v)})$, which,

again by (1.5), is $\gamma_{\mathsf{K}(a)\otimes\mathsf{K}(u'),\mathsf{K}(v)}$, i.e., $\gamma_{\mathsf{K}(a\oplus u'),\mathsf{K}(v)}$. If instead we have that $v = v' \oplus a$ the induction is mantained similarly by using the inverse of (1.5).

Thus, the last step we have left is to prove that axioms (Ψ) are mantained. This is clear, since if s is a symmetry, then $\mathsf{G}(s)$ is also a symmetry. Therefore $\mathsf{K}(s;t) = \mathsf{H}(\mathsf{G}(s);t) = \mathsf{H}(\mathsf{G}(s);t;id) = \mathsf{H}(t) = \mathsf{K}(t)$ and, reasoning in the same way, $\mathsf{K}(t;s) = \mathsf{K}(t)$. Now, the fact that K is a functor follows from its own definition and from the fact that G is so. Observe that, by showing that the symmetry isomorphism of $\mathcal{P}[N]$ is preserved, we have actually shown that K is a symmetric strict monoidal functor.

Our next task is to show that $\mathsf{H} = \mathsf{KQ}$. This is of course the case for objects and transitions. Observe further that, since Q is symmetric strict monoidal, we have that $\mathsf{Q}(c_{u,v}) = \gamma(u,v)$, whence it follows that $\mathsf{KQ}(c_{u,v}) = \gamma_{\mathsf{H}(u),\mathsf{H}(v)} = \mathsf{H}(c_{u,v})$. Now, since the symmetries of $\mathcal{F}(n)$ are by definition generated via tensor and composition from c and from identities, we conclude that H and KQ coincide on $Sym_{\mathcal{F}(N)}$. Then, one proves that $\mathsf{H} = \mathsf{KQ}$ by proving, by easy induction on the structure of the terms, that each arrow of $\mathcal{F}(N)$ can be written as the composition of symmetries and arrows of the kind $id_u \otimes t \otimes id_v$, for $t \in T_N$.

Finally, concerning the uniqueness condition on K, observe that it must necessarily be $\mathsf{K}(id_u \otimes \gamma(a,a) \otimes id_v) = id_{\mathsf{H}(u)} \otimes \gamma_{\mathsf{H}(a),\mathsf{H}(a)} \otimes id_{\mathsf{H}(v)}$, which, by Lemma 1.2.4, defines K uniquely on Sym_N. Moreover, the behaviour of K on the arrows formed as composition and tensor of transitions is uniquely determined by H. Therefore, the proof is concluded. ✓

The next corollary gives an alternative form for axiom (1.17).

Corollary 1.2.6
Axiom (1.17) in Proposition 1.2.5 can be replaced by the axioms

$$\begin{aligned} t;(id_u \otimes c_{a,a} \otimes id_v) &= t \qquad \text{if } t \in T \text{ and } a \in S, \\ (id_u \otimes c_{a,a} \otimes id_v);t &= t \qquad \text{if } t \in T \text{ and } a \in S. \end{aligned} \tag{1.18}$$

Proof. Since $(id_u \otimes \gamma_{a,a} \otimes id_v)$ and all the identities are symmetries, axiom (1.17) implies the present ones. It is easy to see that, on the contrary, the axioms above, together with axiom (1.16) implies (1.17).

Let $s: u \to u$ by a symmetry of $\mathcal{F}(N)$ and suppose $s \neq id_u$. By repeated applications of (1.13), together the functoriality of $\otimes$, we obtain the following equality:

$$s = (id_{u_1} \otimes c_{a_1,b_1} \otimes id_{v_1}); \dots ; (id_{u_h} \otimes c_{a_h,b_h} \otimes id_{v_h})$$

for some $h \in \omega$. Moreover, by exploiting axiom (1.16), we can drop every term in which $a_i \neq b_i$. Thus we have

$$s = (id_{u_1} \otimes c_{a_1,a_1} \otimes id_{v_1}); \dots ; (id_{u_k} \otimes c_{a_k,a_k} \otimes id_{v_k}) \tag{1.19}$$

for some $k \leq h$. Then, by (1.19) and repeated applications of the axioms (1.18), one can prove $s;t;s' = t$. ✓

The next corollary sums up the purely algebraic characterization of the category of concatenable processes that we have proved in this note.

COROLLARY 1.2.7
The category $\mathcal{P}[N]$ of concatenable processes of N is the category whose objects are the elements of $S_N^{\oplus}$ and whose arrows are generated by the inference rules

$$\frac{u \in S_N^{\oplus}}{id_u\colon u \to u \text{ in } \mathcal{P}[N]} \qquad \frac{a \text{ in } S_N}{c_{a,a}\colon a \oplus a \to a \oplus a \text{ in } \mathcal{P}[N]} \qquad \frac{t\colon u \to v \text{ in } T_N}{t\colon u \to v \text{ in } \mathcal{P}[N]}$$

$$\frac{\alpha\colon u \to v \text{ and } \beta\colon u' \to v' \text{ in } \mathcal{P}[N]}{\alpha \otimes \beta\colon u \oplus u' \to v \oplus v' \text{ in } \mathcal{P}[N]} \qquad \frac{\alpha\colon u \to v \text{ and } \beta\colon v \to w \text{ in } \mathcal{P}[N]}{\alpha ; \beta\colon u \to w \text{ in } \mathcal{P}[N]}$$

modulo the axioms expressing that $\mathcal{P}[N]$ is a strict monoidal category, namely,

$$\begin{aligned} \alpha; id_v = \alpha = id_u; \alpha \quad &\text{and} \quad (\alpha;\beta);\gamma = \alpha;(\beta;\gamma), \\ (\alpha\otimes\beta)\otimes\gamma = \alpha\otimes(\beta\otimes\gamma) \quad &\text{and} \quad id_0 \otimes \alpha = \alpha = \alpha \otimes id_0, \\ id_u \otimes id_v = id_{u\oplus v} \quad &\text{and} \quad (\alpha\otimes\alpha');(\beta\otimes\beta') = (\alpha;\beta)\otimes(\alpha';\beta'), \end{aligned}$$

the latter whenever the righthand term is defined, and the following axioms

$$\begin{aligned} c_{a,a}; c_{a,a} &= id_{a\oplus b} \\ t; (id_u \otimes c_{a,a} \otimes id_v) &= t \quad \text{if } t \in T, \\ (id_u \otimes c_{a,a} \otimes id_v); t &= t \quad \text{if } t \in T, \\ c_{u,u'}; (\beta \otimes \alpha) &= (\alpha\otimes\beta); c_{v,v'} \quad \text{for } \alpha\colon u \to v,\ \beta\colon u' \to v' \end{aligned}$$

where $c_{u,v}$ for $u, v \in S_N^{\oplus}$ is obtained from $c_{a,a}$ by applying recursively the following rules:

$$\begin{aligned} c_{a,b} &= id_{a\oplus b} \quad \text{if } a = 0 \text{ or } b = 0 \text{ or } (a, b \in S_N \text{ and } a \neq b) \\ c_{a\oplus u,v} &= (id_a \otimes c_{u,v}); (c_{a,v} \otimes id_u) \\ c_{u,v\oplus a} &= (c_{u,v} \otimes id_a); (id_v \otimes c_{u,a}) \end{aligned}$$

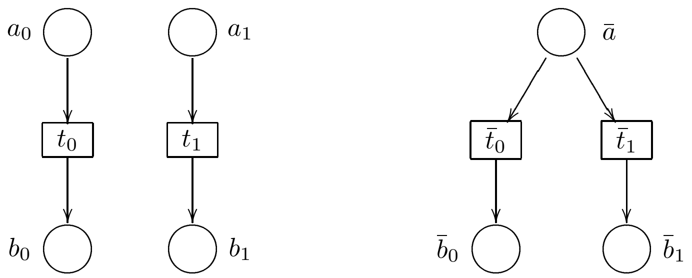
Proof. Easy from Proposition 1.2.1, Proposition 1.2.5 and Corollary 1.2.6. ✓

1.3 A Functorial Construction for Processes

Among the primary requirements usually imposed on constructions like $\mathcal{P}[\_]$ there is the *functoriality*. One of the main reasons which support the choice of a categorical treatment of semantics is the need of specifying further the structure of the systems under analysis by giving explicitly the morphisms or, in other words, by specifying how the given systems simulate each other. This, in turn, means to choose precisely what the relevant (behavioural) structure of the systems is. It is therefore clear that such morphisms should be preserved at the semantic level. In our specific case, the functoriality of $\mathcal{P}[\_]$ would mean that if N can be mapped to N' via a morphism $\langle f, g\rangle$, which by the very definition of net morphisms implies that step sequences of N can be simulated by step sequences of N', there must be a way, namely $\mathcal{P}[\langle f, g\rangle]$, to see the processes of N as processes of N'.

Unfortunately, this is not possible for $\mathcal{P}[\_]$. More precisely, although it may be possible to find a tricky way to extend $\mathcal{P}[\_]$ to net morphisms, it is definitely not possible to make it a monoidal functor, i.e., a functor which respects the monoidal structure of processes, which is certainly what is to be done in our case. The problem, as illustrated by the following example, is due to the particular shape of the symmetries of $\mathcal{P}[N]$ which, on the other hand, is exactly what makes $\mathcal{P}[N]$ capture quite precisely the notion of processes of N.

Example 1.3.1
Consider the nets N and $\bar{N}$ in the picture below. We have $S_N = \{a_0, a_1, b_0, b_1\}$ and T_N consisting of the two transitions $t_0\colon a_0 \to b_0$ and $t_1\colon a_1 \to b_1$, while $S_{\bar{N}} = \{\bar{a}, \bar{b}_0, \bar{b}_1\}$ and $T_{\bar{N}}$ contains $\bar{t}_0\colon \bar{a} \to \bar{b}_0$ and $\bar{t}_1\colon \bar{a} \to \bar{b}_1$.



Consider now the net morphism $\langle f, g\rangle$ where $f(t_i) = \bar{t}_i$, $g(a_i) = \bar{a}$ and $g(b_i) = \bar{b}_i$, for $i = 0, 1$. We claim that $\langle f, g\rangle$ cannot be extended to a monoidal functor $\mathcal{P}[\langle f, g\rangle]$ from $\mathcal{P}[N]$ to $\mathcal{P}[\bar{N}]$. Suppose in fact that F is such an extension. Then, it must be $F(t_0 \otimes t_1) = F(t_0) \otimes F(t_1) = \bar{t}_0 \otimes \bar{t}_1$. Moreover, since $t_0 \otimes t_1 = t_1 \otimes t_0$, we would have

$$\bar{t}_0 \otimes \bar{t}_1 = F(t_1 \otimes t_0) = \bar{t}_1 \otimes \bar{t}_0,$$

which is impossible since the leftmost and the rightmost terms in the formula above are different in $\mathcal{P}[\bar{N}]$.

It is easy to observe from the above discussion that as soon as one tries to impose axioms on $\mathcal{P}[N]$ which guarantee to get a functor, one annihilates all the symmetries and, therefore, destroys the ability of $\mathcal{P}[N]$ to cope with causality.

The problem can be explained *formally* by saying that the category Sym_N, sitting inside $\mathcal{P}[N]$, is *not free*, and that is why we cannot find an extension to $\mathcal{P}[N]$ of the morphism $\langle f, g\rangle: N \to \bar{N} \hookrightarrow \mathcal{P}[\bar{N}]$. In fact, Proposition 1.2.5 shows that Sym_N is the free category of symmetries on $S_N^{\oplus}$ modulo axiom (1.16), i.e., the SSMC which has $S_N^{\oplus}$ as objects and whose arrows are freely generated from a family of formal arrows $c_{u,v}$, for $u, v \in S_N^{\oplus}$ modulo the axioms which express that c is a natural transformation, the axioms (1.2)–(1.6) of SSMC's, and the axiom

$$c_{a,b} = id_{a\oplus b} \quad \text{if } a \neq b \text{ in } S_N.$$

Clearly, it is exactly this conditional axiom with a *negative premise* which prevents Sym_N from being the free model of the axioms (and us from getting any free models of them). To make things worse, the theory illustrated so far makes it clear that, in order for $\mathcal{P}[N]$ to have the interesting computational meaning it has, such an axiom is strictly needed.

There does not seem to be an easy and satisfactory solution to the functoriality problem for $\mathcal{P}[\_]$. A possible solution which comes naturally to the mind would consist of looking for a *non strict* monoidal functor, i.e., a functor F together with a natural transformation $\varphi: F(x_1) \otimes F(x_2) \stackrel{\cdot}{\to} F(x_1 \otimes x_2)$ which substitutes the equality required by strict functors. However, simple examples show that this idea does not bring anywhere, at least unless $\mathcal{P}[\_]$ is heavily modified also on the objects, since it is not possible to choose the components of φ "naturally".

However, we present here a solution to the problem based on a rather radical change of models for net behaviours. Namely, instead of considering SSMC's with commutative monoids of objects, we choose as semantic domain a category $\underline{\mathsf{SSMC}}^{\otimes}$ of SSMC's whose objects form free, *non-commutative* monoids. The reason for renouncing to commutativity is explained in the following subsection. The most interesting thing is, in our view, that the objects of $\underline{\mathsf{SSMC}}^{\otimes}$ still admit a very nice computational interpretation. We are aware that our approach is to be considered only a first step towards a satisfactory solution. With the same strenght, however, we believe that our results contribute to deepen the understanding of the subject.

A Negative Result about Functoriality

In this subsection we show that the problem illustrated in Example 1.3.1 is serious, actually deep enough to prevent any naive modification of $\mathcal{P}[\_]$ to be functorial.

PROPOSITION 1.3.2
Let $\mathcal{X}[\_]$ be a function which assigns to each net N a symmetric strict monoidal category whose monoid of objects is commutative and contains S_N, the places of N. Suppose further the group of symmetries at any object of $\mathcal{X}[N]$ is finite. Finally, suppose that there exists a net N with a place $a \in N$ such that, for each $n > 1$, we have that the symmetry of $\mathcal{X}[N]$ at (na, na) is not an identity.

Then, there exists a Petri net morphism $\langle f, g\rangle : N_0 \to N_1$ which cannot be extended to a symmetric strict monoidal functor from $\mathcal{X}[N_0]$ to $\mathcal{X}[N_1]$.

Proof. The key of the proof is the following observation about monoidal categories. Let $\underline{C}$ be a symmetric strict monoidal category with symmetry isomorphism c. Then, for all $a \in \underline{C}$ and for all $n \geq 1$, we have $(c_{a,(n-1)a})^n = id$, where, in order to simplify notations, throughout the proof we write na and $c^n_{x,y}$ to denote, respectively, the tensor product of n copies of a and sequential composition of n copies of $c_{x,y}$. To show that, consider for $i = 1, \ldots, n$ the functor F_i from $\underline{C}^n$, the cartesian product of n copies of $\underline{C}$, to $\underline{C}$ defined as follows.

$$
\begin{array}{ccc}
\underline{C}^n & \xrightarrow{\quad \mathsf{F}_i \quad} & \underline{C} \\[1ex]
(x_1, \ldots, x_n) & \longmapsto & x_i \cdots x_n \cdots x_{i+1} \\
\Big\downarrow {\scriptstyle (f_1, \ldots, f_n)} & & \Big\downarrow {\scriptstyle (f_i \cdots f_n f_1 \cdots f_{i+1})} \\
(y_1, \ldots, y_n) & \longmapsto & y_i \cdots y_n \cdots y_{i+1}
\end{array}
$$

Moreover, consider the natural transformations $\phi_i : \mathsf{F}_i \stackrel{\bullet}{\to} \mathsf{F}_{i+1}$, $i = 1, \ldots, n-1$ and $\phi_n : F_n \to F_1$ whose components at $x_1, \ldots, x_n$ are, respectively, $c_{x_i, x_{i+1} \cdots x_n x_1 \cdots x_{i-1}}$ and $c_{x_n, x_1 \cdots x_{n-1}}$. Finally, let ϕ be the sequential composition of $\phi_1, \ldots, \phi_n$. Then ϕ is a natural transformation $x_1 \cdots x_n \stackrel{\bullet}{\to} x_1 \cdots x_n$ built up only from components of c. From the Kelly-MacLane coherence theorem [87, 62] (see also Appendix A.2) we know that there is at most natural transformation consting only of identities and components of c, and since the identity of F_1 is such, we have that $\phi = id_{\mathsf{F}_1}$. Then, instantiating each variable with a, we obtain $(c_{a,(n-1)a})^n = id_{na}$, as required.

It may be worth to observe that the above property holds also for $n = 0$, provided we define $0a = e$ and $c^0_{x,y} = id$.

It is now easy to conclude the proof. Let N' be a net such that, for each n, we have $c'_{na,na} \neq id$, where c' is the symmetry natural isomorphism of $\mathcal{X}[N']$. Analogously, let N be a net with two distinct places a and b and with *no* transitions, and let c' be the symmetry natural isomorphism of $\mathcal{X}[N]$. Since the group of symmetries at ab is finite, so is its *cyclic* subgroup generated by $c_{a,b}$, i.e., there exists $k > 1$, the order of the subgroup, such that $(c_{a,b})^k = id$ and $(c_{a,b})^n \neq id$ for any $1 \leq n < k$.

Let p be any prime number greater than k. We claim that the Petri net morphism $\langle f, g\rangle : N \to N'$, where f is the (unique) function $\varnothing \to T_{N'}$ and g is the monoid homomorphism such that $g(b) = (p-1)a$ and g is the identity on the other places of N, cannot be extended to a symmetric strict monoidal functor $\mathsf{F} : \mathcal{X}[N] \to \mathcal{X}[N']$. In fact, from the first part of this proof, we know that $(c_{a,(p-1)a})^p = 1$. Moreover, by general result of group theory, the order of the cyclic subgroup generated by $c_{a,(p-1)a}$ must be a factor of p and then, in this case, 1 or p. In other words, either $c_{a,(p-1)a} = id$ or $(c_{a,(p-1)a})^n \neq id$ for all $1 \leq n < p$. If the second situation occurs, then we have $\mathsf{F}((c_{a,b})^k) = id$ and also $\mathsf{F}((c_{a,b})^k) = (c'_{\mathsf{F}(a),\mathsf{F}(b)})^k = (c'_{a,(p-1)a})^k \neq id$, i.e., F cannot exists. Thus, in order to conclude the proof, we only need to show that, in our hypothesis, $c'_{a,(p-1)a} \neq id$. To this aim, it is enough to observe that $c'_{a,(p-1)a} = id$ implies

$c'_{na,na} = id$ for $n = p-1$, which is against our hypothesis on N'. In fact, $c'_{ka,(p-1)a} = ac'_{(k-1)a,(p-1)a}; c'_{a,(p-1)a}na$, whence it follows directly that $c'_{(p-1)a,(p-1)a} = id$. ✓

The contents of the previous proposition may be restated in different terms by saying that in the *free* category of symmetries on a commutative monoid M there are infinite homsets. This means that dropping axiom (1.16) in the definition of $\mathcal{P}[N]$ causes an "explosion" of the structure of the symmetries. More precisely, if we omit axiom (1.16), we can find some object u such that the group of symmetries on u has infinite order. Of course, since symmetries represent causality, this makes the category so obtained completely useless for the kind of application we have in mind.

The hypothesis of Proposition 1.3.2 can be certainly weakened in several ways, clearly to the price of complicating the proof. However, we avoided such complications, since the conditions stated above are *already* weak enough if one wants to regard $\mathcal{X}[N]$ as a category of processes of N. In fact, since places represent the atomic bricks on which states are built, one needs to consider them in $\mathcal{X}[N]$, since symmetries regulate the "flow of causality", there will be $c_{na,na}$ different from the identity, and since in a computation we can have only finitely many "causality streams", there will not be categories with infinite groups of symmetries. Therefore, the given result means that there is no chance to have a functorial construction of the processes of N on the line of $\mathcal{P}[N]$ whose objects form a commutative monoid.

THE CATEGORY $\mathcal{Q}[N]$

This subsection introduces the symmetric strict monoidal category $\mathcal{Q}[N]$ which is meant to represent the processes of a Petri net N and which supports a functorial construction. This allows us to characterize the category of the categories of net behaviours, or, in other words, to axiomatize the behaviour of nets "in the large". In fact, although [97] and [16] clarify how the behaviour of a single net may be captured by a symmetric strict monoidal category, because of the missing functoriality of $\mathcal{P}[\_]$, nothing is said about what the semantic domain for Petri nets is.

Necessarily, there is a price to be payed. Here, the idea is to abandon the commutativity of the monoids of the objects. More precisely, we build the arrows of $\mathcal{Q}[N]$ starting from Sym^*_N, the *free* category of symmetries over the *set* S_N. This makes transitions to have many corresponding arrows in $\mathcal{Q}[N]$; however, all the arrows of $\mathcal{Q}[N]$ which differ only for instances of transitions will be equated by a "naturality" condition which, therefore, guarantees that $\mathcal{Q}[N]$ remains close to the category $\mathcal{P}[N]$ of concatenable processes. Namely, the arrows of $\mathcal{Q}[N]$ correspond to Goltz-Reisig processes in which the minimal and the maximal places are *totally* ordered.

Similarly to Sym_N, Sym^*_N serves a double purpose. From the categorical point of view it provides the symmetry isomorphism of a symmetric monoidal category, while from the semantics viewpoint it regulates the flow of causal dependency. It should be noticed, however, that here the point of view is strictly more concrete than in the case of Sym_N. In fact, generally speaking, a symmetry in $\mathcal{Q}[N]$ must be interpreted as a *"reorganization"* of the tokens in the global state of the net which, when reorganizing multiple instances of the same place, as a by-product, yields a exchange of causes exactly as Sym_N does for $\mathcal{P}[N]$.

NOTATION. In the following, we use $S^{\otimes}$ to indicate the set of (finite) strings on set S, more commonly denoted by S^*. In the same way, we use $\otimes$ to denote string concatenation, while 0 denotes the empty string. As usual, for $u \in S^{\otimes}$, we indicate by $|u|$ the lenght of u and by u_i the its i-th element.

The construction of $S^{\otimes}$, which under the operation of string concatenation is the *free monoid* on S, admits a corresponding monad $((\_)^{\otimes}, \eta, \mu)$ on **Set**. In this case $(\_)^{\otimes}$ is the functor which associates to each set S the monoid $S^{\otimes}$ and to each $f\colon S_0 \to S_1$ the monoid homomorphism $f^{\otimes}\colon S_0^{\otimes} \to S_1^{\otimes}$ such that $f^{\otimes}(u) = \bigotimes f(u_i)$, $\eta_S\colon S \to S^{\otimes}$ is the injection of S in $S^{\otimes}$ and $\mu_S\colon S^{\otimes^2} \to S^{\otimes}$ is the obvious monoid homomorphism which maps a string of elements of $S^{\otimes}$ to the concatenation of its component strings. Recall that the algebras for such a monad are the monoids and the homomorphisms are the monoids homomorphisms.

DEFINITION 1.3.3 *(Permutations)*
Let S be a set. The category Sym^*_S has for objects the strings $S^{\otimes}$ and an arrow $p\colon u \to v$ if and only if $p \in \Pi(|u|)$, i.e., p is a permutation of $|u|$ elements, and v is the string obtained from u by applying the permutation p, i.e., $v_{p(i)} = u_i$.
Arrows composition in Sym^*_S is obviously given by the product of permutations, i.e., their composition as functions, here and in the following denoted by $\_;\_$.

Graphically, we represent an arrow $p\colon u \to v$ in Sym^*_S by drawing a line between u_i and $v_{p(i)}$, as for example in Figure 1.6. Of course, it is possible to define a tensor product on Sym^*_S together with interchange permutations which make it be a symmetric monoidal category (see also Figure 1.6, where γ is once again the permutation $\{1 \to 2, 2 \to 1\}$).

DEFINITION 1.3.4 *(Operations on Permutations)*
Given the permutations $p\colon u \to v$ and $p'\colon u' \to v'$ in Sym^*_S their parallel composition $p \otimes p'\colon u \otimes u' \to v \otimes v'$ is the permutation such that

$$i \mapsto \begin{cases} p(i) & \text{if } 0 < i \le |u| \\ p'(i - |u|) + |u| & \text{if } |u| < i \le |u| + |u'| \end{cases}$$

Given $\sigma \in \Pi(m)$ and m strings u_i for $i = 1, \dots, m$ in $S^{\otimes}$, the interchange permutation $\sigma(u_1, \dots, u_m)$ is the permutation p such that

$$p(i) = i - \sum_{j=1}^{h-1} |u_j| + \sum_{\pi(j)<\pi(h)} |u_j| \quad \text{if } \sum_{j=1}^{h-1} |u_j| < i \le \sum_{j=1}^{h} |u_j|.$$

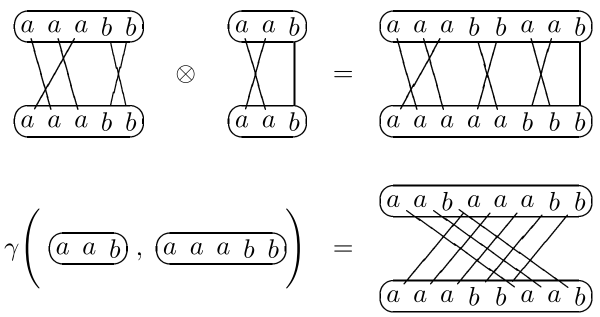


Figure 1.6: The monoidal structure of Sym_S^*

Clearly, $\otimes$ so defined is associative and furthermore a simple calculation shows that it satisfies the equations

$$(p \otimes p'); (q \otimes q') = (p; q) \otimes (p'; q') \quad \text{and} \quad id_u \otimes id_v = id_{u \otimes v}.$$

It follows easily that the mapping $\otimes : Sym_S^* \times Sym_S^* \rightarrow Sym_S^*$ defined by

$$\begin{array}{ccc} Sym_S^* \times Sym_S^* & \xrightarrow{\otimes} & Sym_S^* \\ (u, u') & \longmapsto & u \otimes v \\ {\scriptstyle (p,p')} \downarrow & & \downarrow {\scriptstyle (p \otimes p')} \\ (v, v') & \longmapsto & v \otimes v' \end{array}$$

is a functor which makes Sym_S^* be a strict monoidal category. Finally, the symmetric structure of Sym_S^* is made explicit by the interchange permutations.

PROPOSITION 1.3.5 *(Sym_S^* is symmetric strict monoidal)*
For any set S, the family $\gamma = \{\gamma(u, v)\}_{u,v \in Sym_S^*}$ provides the symmetry isomorphism which endowes Sym_S^* with a symmetric monoidal structure.

Proof. Recall that $\gamma(u, v)$ is the interchange permutation defined from the permutation $\gamma = \{1 \rightarrow 2, 2 \rightarrow 1\}$ in $\Pi(2)$. It is just a matter of few calculations to verify that, for any $p : u \rightarrow u'$ and $p' : v \rightarrow v'$, the equations which define a symmetry isomorphism i.e., the naturality and axioms (1.5) and (1.6), which in the current case reduce to

$$\begin{array}{rcl} \gamma(u, v); (p' \otimes p) & = & (p \otimes p'); \gamma(u', v') \\ (\gamma(u, v) \otimes w); (v \otimes \gamma(u, w)) & = & \gamma(u, v \otimes w) \\ \gamma(u, v); \gamma(v, u) & = & id_{u \otimes v} \end{array}$$

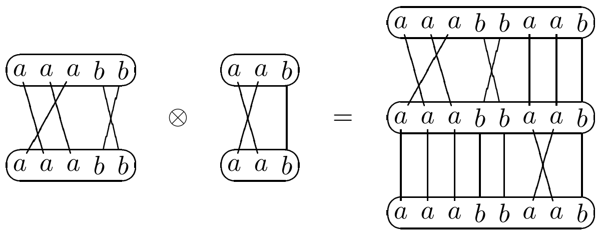


Figure 1.7: The parallel composition of permutations

hold. Observe that, in fact,

$$\gamma(u,v)(i) = \begin{cases} i + |v| & \text{if } 0 < i \leq |u| \\ i - |u| & \text{if } |u| < i \leq |u| + |v| \end{cases}$$

which shows the second equation. Moreover, it implies that $(\gamma(u,v); (p' \otimes p))(i)$ is $p(i) + |v|$ if $0 < i \leq |u|$ and is $p'(i - |u|)$ if otherwise $|u| < i \leq |u| + |v|$. On the other hand, we have that $((p \otimes p'); \gamma(u', v'))(i)$ is $p(i) + |v'| = p(i) + |v|$ if $0 < i \leq |u|$ and $p'(i - |u|) + |u| - |u| = p'(i - |u|)$ if $|u| < i \leq |u| + |v|$. Therefore, the first equation is shown. Concerning the last equation, we have that

$$(\gamma(u,v) \otimes w)(i) = \begin{cases} i + |v| & \text{if } 0 < i \leq |u| \\ i - |u| & \text{if } |u| < i \leq |u| + |v| \\ i & \text{if } |u| + |v| < i \leq |u| + |v| + |w| \end{cases}$$

and, since

$$(v \otimes \gamma(u,w))(i) = \begin{cases} i & \text{if } 0 < i \leq |v| \\ i + |w| & \text{if } |v| < i \leq |v| + |u| \\ i - |u| & \text{if } |v| + |u| < i \leq |v| + |u| + |w|, \end{cases}$$

we have the required equality. ✓

The previous proposition justifies the use of the name *symmetries* for the arrows of the groupoid category Sym^*_S. The key point about Sym^*_S is that it is a free construction.

Proposition 1.3.6
Let $(\underline{\mathsf{C}}, \otimes, e, \gamma)$ be a SSMC *and F be a function from the set S to the set of objects of $\underline{\mathsf{C}}$. Then, there exists a unique symmetric strict monoidal functor $\mathsf{F}\colon Sym^*_S \to \underline{\mathsf{C}}$ which extends F.*

Proof. To shorten notation, in the proof we write u to indicate id_u. There is of course a compelled choice for the behaviour of F on the objects: the monoidal extension of F, i.e., the mapping

$$\mathsf{F}(0) = e \quad \text{and} \quad \mathsf{F}(u \otimes v) = \mathsf{F}(u) \otimes \mathsf{F}(v) \text{ for } u, v \in S^{\otimes}.$$

Concerning the morphisms, we know by Lemma 1.2.3 that each arrow in Sym^*_S can be written as composition of transpositions. Moreover, observe that the transposition $\{i \to i+1, i+1 \to i\}: u \otimes a \otimes b \otimes v \to u \otimes b \otimes a \otimes v$, where u is a string of length $i-1$, coincides in Sym^*_S with the tensor of $\gamma(a,b): a \otimes b \to b \otimes a$ with appropriate identities, namely $(u \otimes \gamma(a,b) \otimes v)$. Thus, recalling also that $0 \otimes \gamma(a,b) = \gamma(a,b) = \gamma(a,b) \otimes 0$, the following definition defines F on all the arrows of Sym^*_S.

$$\begin{aligned} \mathsf{F}(u \otimes \gamma(a,b) \otimes v) &= \mathsf{F}(u) \otimes \gamma_{\mathsf{F}(a),\mathsf{F}(b)} \otimes \mathsf{F}(v) \quad a,b \in S,\ u,v \in S^{\otimes}; \\ \mathsf{F}(p;p') &= \mathsf{F}(p') \circ \mathsf{F}(p). \end{aligned} \tag{1.20}$$

Observe that both the equations (1.20) are forced by the definition of symmetric *strict* monoidal functor, i.e., axioms (1.7)–(1.9) in page 22. It follows that the extension of F to a strict monoidal functor, if it exists, is unique and must be given by (1.20). Then, in order to conclude the proof, we only need to show that F is well-defined and that it is a symmetric monoidal functor.

We first show that F is well-defined. To this aim, it is enough to show that the axioms (1.14) of Lemma 1.2.3 are preserved by F. In fact, this implies that applying the definition of F to two different factorizations of p actually yield the same result, i.e., it implies that F is well-defined. Concering axioms (1.14), the third one matches directly with the fact that the inverse of $\gamma_{\mathsf{F}(a),\mathsf{F}(b)}$ is $\gamma_{\mathsf{F}(b),\mathsf{F}(a)}$, while the second one follows easily from the fact that $\_ \otimes \_$ is a functor. In fact, in the hypothesis, we have $\tau_i = (u \otimes \gamma(a,b) \otimes v \otimes c \otimes d \otimes w)$ and $\tau_j = (u \otimes b \otimes a \otimes v \otimes \gamma(c,d) \otimes w)$. Thus, we have

$$\begin{aligned} \mathsf{F}(\tau_i;\tau_j) &= (\mathsf{F}(u) \otimes \mathsf{F}(b) \otimes \mathsf{F}(a) \otimes \mathsf{F}(v) \otimes \gamma_{\mathsf{F}(c),\mathsf{F}(d)} \otimes \mathsf{F}(w)) \circ \\ &\qquad (\mathsf{F}(u) \otimes \gamma_{\mathsf{F}(a),\mathsf{F}(b)} \otimes \mathsf{F}(v) \otimes \mathsf{F}(c) \otimes \mathsf{F}(d) \otimes \mathsf{F}(w)) \\ &= (\mathsf{F}(u) \otimes \gamma_{\mathsf{F}(a),\mathsf{F}(b)} \otimes \mathsf{F}(v) \otimes \gamma_{\mathsf{F}(c),\mathsf{F}(d)} \otimes \mathsf{F}(w)) \\ &= (\mathsf{F}(u) \otimes \gamma_{\mathsf{F}(a),\mathsf{F}(b)} \otimes \mathsf{F}(v) \otimes \mathsf{F}(d) \otimes \mathsf{F}(c) \otimes \mathsf{F}(w)) \circ \\ &\qquad (\mathsf{F}(u) \otimes \mathsf{F}(a) \otimes \mathsf{F}(b) \otimes \mathsf{F}(v) \otimes \gamma_{\mathsf{F}(c),\mathsf{F}(d)} \otimes \mathsf{F}(w)) \\ &= \mathsf{F}(\tau_j;\tau_i) \end{aligned}$$

Finally, concerning the third axiom, we have

$$\begin{aligned} \mathsf{F}(\tau_i;\tau_{i+1};\tau_i) &= (\mathsf{F}(u) \otimes \gamma_{\mathsf{F}(b),\mathsf{F}(c)} \otimes \mathsf{F}(a) \otimes \mathsf{F}(v)) \circ \\ &\qquad (\mathsf{F}(u) \otimes \mathsf{F}(b) \otimes \gamma_{\mathsf{F}(a),\mathsf{F}(c)} \otimes \mathsf{F}(v)) \circ \\ &\qquad\quad (\mathsf{F}(u) \otimes \gamma_{\mathsf{F}(a),\mathsf{F}(b)} \otimes \mathsf{F}(c) \otimes \mathsf{F}(v)) \\ &= (\mathsf{F}(u) \otimes \mathsf{F}(b) \otimes \gamma_{\mathsf{F}(a),\mathsf{F}(c)} \otimes \mathsf{F}(v)) \circ \\ &\qquad (\mathsf{F}(u) \otimes \gamma_{\mathsf{F}(a),\mathsf{F}(b)\otimes\mathsf{F}(c)} \otimes \mathsf{F}(v)) \\ &= (\mathsf{F}(u) \otimes \gamma_{\mathsf{F}(a),\mathsf{F}(c)\otimes\mathsf{F}(b)} \otimes \mathsf{F}(v)) \circ \\ &\qquad (\mathsf{F}(u) \otimes \mathsf{F}(a) \otimes \gamma_{\mathsf{F}(b),\mathsf{F}(c)} \otimes \mathsf{F}(v)) \\ &= (\mathsf{F}(u) \otimes \mathsf{F}(c) \otimes \gamma_{\mathsf{F}(a),\mathsf{F}(b)} \otimes \mathsf{F}(v)) \circ \\ &\qquad (\mathsf{F}(u) \otimes \gamma_{\mathsf{F}(a),\mathsf{F}(c)} \otimes \mathsf{F}(b) \otimes \mathsf{F}(v)) \circ \\ &\qquad\quad (\mathsf{F}(u) \otimes \mathsf{F}(a) \otimes \gamma_{\mathsf{F}(b),\mathsf{F}(c)} \otimes \mathsf{F}(v)) \\ &= \mathsf{F}(\tau_{i+1};\tau_i;\tau_{i+1}) \end{aligned}$$

where the third equation is by naturality of γ and the others follow from the coherence axiom for γ.

Let us prove now that F is a symmetric monoidal functor. Since $\underline{\mathsf{C}}$ is a symmetric strict monoidal category, we have that $\gamma_{e,x} = \gamma_{e\otimes e,x} = \gamma_{e,x} \otimes e \circ e \otimes \gamma_{e,x} = \gamma_{e,x} \circ \gamma_{e,x}$, and since $\gamma_{e,x}$ is invertible, it follows that $\gamma_{e,x} = id_x$. Of course, the same holds for every symmetric strict monoidal category. Therefore, since $\mathsf{F}(id_u) = \mathsf{F}(\gamma(0,u))$ and $\gamma_{e,\mathsf{F}(u)} = id_{\mathsf{F}(u)}$, we have that the $\mathsf{F}(id_u) = id_{\mathsf{F}(u)}$. This, together with the second of the equations (1.20), means that F is a functor.

Observe that for $p\colon u \to v$ and $p'\colon u' \to v'$ in Sym^*_S we have $p \otimes p' = (p \otimes u'); (v \otimes p')$ (see also Figure 1.7). Then, we have that

$$\mathsf{F}(p \otimes p') = \mathsf{F}(v \otimes p') \circ \mathsf{F}(p \otimes u') = (\mathsf{F}(v) \otimes \mathsf{F}(p')) \circ (\mathsf{F}(p) \otimes \mathsf{F}(u')) = \mathsf{F}(p) \otimes \mathsf{F}(p'),$$

i.e., F is a strict monoidal functor.

Finally, thanks to the coherence axiom for symmetries, i.e., axiom (1.5), we have that $\gamma(a, b \otimes c) = (\gamma(a,b) \otimes c); (b \otimes \gamma(a,c))$ and thus, by the aforesaid axiom and by the coherence of γ,

$$\begin{aligned}
\mathsf{F}(\gamma(a, b \otimes c)) &= \mathsf{F}((\gamma(a,b) \otimes c); (b \otimes \gamma(a,c))) \\
&= (F(b) \otimes \gamma_{F(a),F(c)}) \circ (\gamma_{F(a),F(b)} \otimes F(c)) \\
&= \gamma_{F(a),F(b)\otimes F(c)} = \gamma_{\mathsf{F}(a),\mathsf{F}(b\otimes c)}.
\end{aligned}$$

Now, by considering the inverse of (1.5), we have $\gamma(a \otimes b, c) = (a \otimes \gamma(b,c)); (\gamma(a,c) \otimes b)$ and $\gamma_{\mathsf{F}(a\otimes b),\mathsf{F}(c)} = (\gamma_{\mathsf{F}(a),\mathsf{F}(c)} \otimes \mathsf{F}(b)) \circ (\mathsf{F}(a) \otimes \gamma_{\mathsf{F}(b),\mathsf{F}(c)})$. Therefore, it follows easily by induction that $\mathsf{F}(\gamma(u,v)) = \gamma_{\mathsf{F}(u),\mathsf{F}(v)}$. Then, F maps each component of the symmetry natural isomorphism of Sym^*_S to the corresponding component of γ, i.e., F is a symmetric monoidal functor. ✓

In other words, the previous proposition proves that the mapping $S \mapsto Sym^*_S$ extends to a *left adjoint* functor $\underline{\mathsf{Set}} \rightharpoonup \underline{\mathsf{SsMonCat}}$. Equivalently, we can say that Sym^*_S is the free symmetric strict monoidal category on the set S.

<u>Remark</u>. It may be worth remarking that this adjunction does not exist if $\underline{\mathsf{SsMonCat}}$ is replaced by $\underline{\mathsf{SsMonCat}}^\star$ (see Appendix A.2) or, *a fortiori*, by $\underline{\mathsf{SsMonCat}}^{\star\star}$, since the last category does not admit a free category of symmetries on a set S. In fact, a mapping F from S to the objects of a symmetric monoidal category $(\underline{\mathsf{C}}, \otimes, e, \alpha, \lambda, \rho, \gamma)$ can be extended to a (non-strict) monoidal functor F from Sym^*_S to $\underline{\mathsf{C}}$ in two different way, namely,

$$\begin{aligned}
\mathsf{F}(0) &= e; & \mathsf{F}(u \otimes v) &= \mathsf{F}(u) \otimes \mathsf{F}(v); \\
\mathsf{F}(\gamma(u,v)) &= \gamma_{F(u),F(v)}; & \mathsf{F}(p \otimes p') &= \mathsf{F}(p) \otimes \mathsf{F}(p'),
\end{aligned}$$

with $\varphi^0 = id$ and $\varphi = id_{\mathsf{F}(x_1)\otimes\mathsf{F}(x_2)}$ and

$$\begin{aligned}
\mathsf{F}(0) &= e; & \mathsf{F}(u \otimes v) &= \mathsf{F}(v) \otimes \mathsf{F}(u); \\
\mathsf{F}(\gamma(u,v)) &= \gamma_{F(v),F(u)}; & \mathsf{F}(p \otimes p') &= \mathsf{F}(p') \otimes \mathsf{F}(p),
\end{aligned}$$

with $\varphi^0 = id$ and $\varphi = \gamma_{\mathsf{F}(x_1),\mathsf{F}(x_2)}$.

COROLLARY 1.3.7
Let $\underline{\mathsf{S}}$ be the symmetric strict monoidal category whose monoid of objects is $S^{\otimes}$ and whose arrows are freely generated from a family of arrows $c_{u,v}: u \otimes v \to v \otimes u$, for $u, v \in S^{\otimes}$, subject to the axioms (1.2)–(1.6) (with γ properly replaced by c) plus the axiom which expresses the naturality of $\{c_{u,v}\}_{u,v \in S^{\otimes}}$. Then $\underline{\mathsf{S}}$ and Sym^*_S are isomorphic.

Proof. By definition, $\underline{\mathsf{S}}$ is the free monoidal category on S. In fact, since the axioms which define $\underline{\mathsf{S}}$ hold all symmetric strict monoidal categories, it is immediate to verify that $\underline{\mathsf{S}}$ enjoys the universal property stated in Proposition 1.3.6. Then, exploiting in the usual way the uniqueness condition in this universal property, we have that the functors $\mathsf{F}: Sym^*_S \to \underline{\mathsf{S}}$ and $\mathsf{G}: \underline{\mathsf{S}} \to Sym^*_S$ which are identity on the objects and which map, respectively, $\gamma(u, v)$ to $c_{u,v}$ and $c_{u,v}$ to $\gamma(u, v)$ are each other's inverse. ✓

Now, we can define of $\mathcal{Q}[N]$. In the following, given a string $u \in S^{\otimes}$, let $\mathcal{M}(u)$ denote the multiset corresponding to u. In the following, given a net N we denote by Sym^*_N the category $Sym^*_{S_N}$.

DEFINITION 1.3.8 *(The category $\mathcal{Q}[N]$)*
Let N be a net in $\underline{\mathsf{Petri}}$. Then $\mathcal{Q}[N]$ is the category which includes Sym^*_N as subcategory and has the additional arrows defined by the following inference rules:

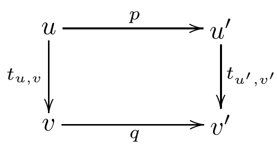
$$\frac{t: \mathcal{M}(u) \to \mathcal{M}(v) \text{ in } T_N}{t_{u,v}: u \to v \text{ in } \mathcal{Q}[N]}$$

$$\frac{\alpha: u \to v \text{ and } \beta: u' \to v' \text{ in } \mathcal{Q}[N]}{\alpha \otimes \beta: u \otimes u' \to v \otimes v' \text{ in } \mathcal{Q}[N]} \qquad \frac{\alpha: u \to v \text{ and } \beta: v \to w \text{ in } \mathcal{Q}[N]}{\alpha;\beta: u \to w \text{ in } \mathcal{Q}[N]}$$

plus the axioms expressing the fact that $\mathcal{Q}[N]$ is a symmetric strict monoidal category with symmetry isomorphism γ, and the following axiom involving transitions and symmetries.

$$p; t_{u',v'} = t_{u,v}; q \quad \text{where } p: u \to u' \text{ in } Sym^*_N \text{ and } q: v \to v' \text{ in } Sym^*_N. \qquad (\Phi)$$

It is worthwhile to notice that axiom (Φ) entails, as a particular case, the axioms (Ψ) of $\mathcal{P}[N]$. In fact, axiom (Φ) asserts that any diagram of the kind



commutes. Now, fixed $u = u'$ and $v = v'$, choosing $p = id$ and $q = id$ one obtains, respectively, the first and the second of axioms (Ψ).

Exploiting Corollary 1.3.7, it is easy to prove that the following is an alternative description of $\mathcal{Q}[N]$.

PROPOSITION 1.3.9
$\mathcal{Q}[N]$ is (isomorphic to) the category $\underline{\mathsf{C}}$ whose objects are the elements of $S_N^{\otimes}$ and whose arrows are generated by the inference rules

$$\frac{u \in S_N^{\otimes}}{id_u\colon u \to u \text{ in } \underline{\mathsf{C}}} \qquad \frac{u, v \text{ in } S_N^{\otimes}}{c_{u,v}\colon u \otimes v \to u \otimes v \text{ in } \underline{\mathsf{C}}} \qquad \frac{t\colon \mathcal{M}(u) \to \mathcal{M}(v) \text{ in } T_N}{t_{u,v}\colon u \to v \text{ in } \underline{\mathsf{C}}}$$

$$\frac{\alpha\colon u \to v \text{ and } \beta\colon u' \to v' \text{ in } \underline{\mathsf{C}}}{\alpha \otimes \beta\colon u \otimes u' \to v \otimes v' \text{ in } \underline{\mathsf{C}}} \qquad \frac{\alpha\colon u \to v \text{ and } \beta\colon v \to w \text{ in } \underline{\mathsf{C}}}{\alpha;\beta\colon u \to w \text{ in } \underline{\mathsf{C}}}$$

modulo the axioms expressing that $\underline{\mathsf{C}}$ is a strict monoidal category, namely,

$$\begin{array}{rcl} \alpha; id_v = \alpha = id_u; \alpha & \text{and} & (\alpha;\beta);\delta = \alpha;(\beta;\delta), \\ (\alpha \otimes \beta) \otimes \delta = \alpha \otimes (\beta \otimes \delta) & \text{and} & id_0 \otimes \alpha = \alpha = \alpha \otimes id_0, \\ id_u \otimes id_v = id_{u \otimes v} & \text{and} & (\alpha \otimes \alpha');(\beta \otimes \beta') = (\alpha;\beta) \otimes (\alpha';\beta'), \end{array} \tag{1.21}$$

the latter whenever the righthand term is defined, the following axioms expressing that $\underline{\mathsf{C}}$ is symmetric with symmetry natural isomorphism c

$$\begin{array}{rcl} c_{u,u'};(\beta \otimes \alpha) & = & (\alpha \otimes \beta); c_{v,v'} \quad \text{for } \alpha\colon u \to v,\ \beta\colon u' \to v', \\ c_{u,v \otimes w} & = & (c_{u,v} \otimes id_w);(id_v \otimes c_{u,w}), \\ c_{u,v}; c_{v,u} & = & id_{u \otimes v}, \end{array} \tag{1.22}$$

and the following axiom corresponding to axiom (Φ)

$$p; t_{u',v'}; q = t_{u,v} \quad \text{where } p\colon u \to u' \text{ and } q\colon v' \to v \text{ are symmetries.}$$

Proof. It is enough to observe that the definition of $\underline{\mathsf{C}}$ is simply the definition of $\mathcal{Q}[N]$ enriched with the axiomatization of Sym_N^* provided by Corollary 1.3.7. ✓

The previous proposition is relevant since gives a completely axiomatic description of the structure of $\mathcal{Q}[N]$ which can be useful in many contexts. In the following, we shall time by time use as definition of $\mathcal{Q}[N]$ and Sym_N^* the one best suited for the intended application.

Next, we show that $\mathcal{Q}[\_]$ can be lifted to a functor from the category of Petri nets to an appropriate category of symmetric strict monoidal categories and symmetric strict monoidal functors. The issue is not very difficult now, since most of the work has been done in the proof of Proposition 1.3.6. We start by showing that $\mathcal{Q}[\_]$ is a *pseudo-functor* from $\underline{\mathsf{Petri}}$ to $\underline{\mathsf{SsMonCat}}$, the standard category of SSMC's, in the sense made explicit by the following proposition. Precisely, we extend $\mathcal{Q}[\_]$ to a mapping from Petri net morphisms to symmetric strict monoidal functors in such a way that *identities* are preserved *strictly*, while net morphism *composition* is preserved only up to a *monoidal natural isomorphism.* In order to do that, the key point which is still missing is to be able to embed N into $\mathcal{Q}[N]$. To this purpose, we assume for each set S a function $in_S\colon S^{\oplus} \to S^{\otimes}$ such that $\mathcal{M}(in_S(u)) = u$, i.e., a function which chooses a *"linearization"* of each $u \in S^{\oplus}$. Clearly, corresponding to different choices of the functions in_S we shall have a different—yet equivalent—extension of $\mathcal{Q}[\_]$ to a pseudo-functor. We would like to remark that this apparent arbitrariness of $\mathcal{Q}[\_]$ is not at all a concern, since the relevant fact we want to show now is that such an extension exists. Moreover, we shall see shortly that introducing the category $\underline{\mathsf{SSMC}}^{\otimes}$ one can completely dispense with the in_S. In the following, given a net N, we shall use in_N to denote in_{S_N}.

REMARK. An elegant way to express the idea of "linearization" of a multiset, would be to look for a morphism of monads $in\colon (\_)^{\oplus} \stackrel{.}{\to} (\_)^{\otimes}$. This would indeed simplify the following formal development and would actually make $\mathcal{Q}[\_]$ be a functor $\underline{\mathsf{Petri}} \to \underline{\mathsf{SsMonCat}}$. However, such a morphism does not exist. It is worth noticing that this is because it is not possibile to choose the in_S "naturally".

PROPOSITION 1.3.10 *($\mathcal{Q}[\_]\colon \underline{\mathsf{Petri}} \to \underline{\mathsf{SsMonCat}}$)*
Let $\langle f, g\rangle\colon N_0 \to N_1$ be a morphism in $\underline{\mathsf{Petri}}$. Then, there exists a symmetric strict monoidal functor $\mathcal{Q}[\langle f, g\rangle]\colon \mathcal{Q}[N_0] \to \mathcal{Q}[N_1]$ which extends $\langle f, g\rangle$. Moreover, $\mathcal{Q}[id_N] = id_{\mathcal{Q}[N]}$ and $\mathcal{Q}[\langle f_1, g_1\rangle \circ \langle f_0, g_0\rangle] \cong \mathcal{Q}[\langle f_1, g_1\rangle] \circ \mathcal{Q}[\langle f_0, g_0\rangle]$.

Proof. Let $\langle f, g\rangle\colon N_0 \to N_1$ be a morphism of Petri nets. Since g is a monoid homomorphism from the free monoid $S^{\oplus}_{N_0}$ to $S^{\oplus}_{N_1}$, it corresponds to a unique function $g \circ \eta_{S_{N_0}}$ from S_{N_0} to $S^{\oplus}_{N_1}$, where η is the unit of the "commutative monoids" monad, whence we obtain $\hat{g} = in_{N_1} \circ g \circ \eta_{S_{N_0}}\colon S_{N_0} \to S^{\otimes}_{N_1}$, i.e., a function from S_{N_0} to the set of objects of $\mathcal{Q}[N_1]$. Then, from Proposition 1.3.6, we have the symmetric strict monoidal functor $\mathsf{F}'\colon Sym_{S_{N_0}} \to \mathcal{Q}[N_1]$. Clearly, the objects component of F' is $\bar{\mu}_{S_{N_1}} \circ \hat{g}^{\otimes}$, where $\bar{\mu}$ is the multiplication of the "monoids" monad. Finally, we extend F' to a functor F from $\mathcal{Q}[N_0]$ to $\mathcal{Q}[N_1]$ by considering the symmetric strict monoidal functor which coincides with F' on Sym_{N_0} and maps $t_{u,v}\colon u \to v$ to $f(t)_{\mathsf{F}(u),\mathsf{F}(v)}\colon \mathsf{F}(u) \to \mathsf{F}(v)$. Since monoidal functors map symmetries to symmetries, and since $f(t)$ is transition of N_1, it follows immediately that F preserves axiom (Φ), i.e., that F is well defined.

Next, we have to show that the above definition makes $\mathcal{Q}[\_]$ into a pseudo-functor. First of all, observe that whatever in_N, the function $S_N \hookrightarrow S^{\oplus}_N \xrightarrow{in_N} S^{\otimes}_N$ is the inclusion of S_N in $S^{\otimes}_N$. It follows easily from the uniqueness part of the universal property stated in Proposition 1.3.6 that $\mathcal{Q}[id_N]\colon \mathcal{Q}[N] \to \mathcal{Q}[N]$ is the identity functor. Now consider $\langle f_0, g_0\rangle\colon N_0 \to N_1$ and $\langle f_1, g_1\rangle\colon N_1 \to N_2$ and, for $i = 0, 1$, let F_i be $\mathcal{Q}[\langle f_i, g_i\rangle]\colon \mathcal{Q}[N_i] \to \mathcal{Q}[N_{i+1}]$ and F be $\mathcal{Q}[\langle f_1 \circ f_0, g_1 \circ g_0\rangle]$. We have to show that $\mathsf{F} \cong \mathsf{F}_1\mathsf{F}_0$. Let $u \in S^{\otimes}_{N_0}$. By definition, we have that $\mathsf{F}(u_i) = in_{N_2} \circ g_1 \circ g_0(u_i)$ is a permutation of $\mathsf{F}_1\mathsf{F}_0(u_i) = \bar{\mu}_{S_{N_2}} \circ \hat{g}_1^{\otimes} \circ \hat{g}_0(u_i)$ and, therefore, there exists a symmetry $s_i\colon \mathsf{F}(u_i) \to \mathsf{F}_1\mathsf{F}_0(u_i)$ in $\mathcal{Q}[N_2]$. Then, we take s_u to be $s_1 \otimes \cdots \otimes s_n\colon \mathsf{F}(u) \to \mathsf{F}_1\mathsf{F}_0(u)$, where n is the lenght of the string u. We shall prove that the family of the s_u, for $u \in S^{\otimes}_{N_0}$ is a natural transformation $\mathsf{F} \stackrel{.}{\to} \mathsf{F}_1\mathsf{F}_0$. Since s is clearly a monoidal transformation and each s_u is an isomorphism, this concludes the proof.

We must show that for any $\alpha\colon u \to v$ in $\mathcal{Q}[N_0]$ we have $\mathsf{F}(\alpha); s_v = s_u; \mathsf{F}_1\mathsf{F}_0(\alpha)$. Exploiting the characterization of $\mathcal{Q}[N_0]$ given by Proposition 1.3.9, we proceed by induction on the structure of α. The key of the proof is that s is monoidal, i.e., $s_{u\otimes v} = s_u \otimes s_v$. If α is an identity, then the claim is obvious. Moreover, if α is a transition $t_{u,v}$, then we have $\mathsf{F}(\alpha) = f_1 \circ f_0(t)_{\mathsf{F}(u),\mathsf{F}(v)}$ and $\mathsf{F}_1\mathsf{F}_0(\alpha) = f_1(f_0(t))_{\mathsf{F}_1\mathsf{F}_0(u),\mathsf{F}_1\mathsf{F}_0(v)}$ and the thesis follows immediately from axiom (Φ). Let us consider now $\alpha = \gamma(u,v)$. Since F and $\mathsf{F}_1\mathsf{F}_0$ are symmetric strict monoidal functors, the equation we have to prove reduces to $\gamma(\mathsf{F}(u),\mathsf{F}(v)); s_v \otimes s_u = s_u \otimes s_v; \gamma(\mathsf{F}_1\mathsf{F}_0(u), \mathsf{F}_1\mathsf{F}_0(v))$ which certainly holds since the family $\{\gamma(u,v)\}_{u,v\in S_0^{\otimes}}$ is a natural transformation $x_1 \otimes x_2 \xrightarrow{\cdot} x_2 \otimes x_1$. If $\alpha = \alpha' \otimes \alpha''$, where $\alpha'\colon u' \to v'$ and $\alpha''\colon u'' \to v''$ then, by induction, we have $\mathsf{F}(\alpha'); s_{v'} = s_{u'}; \mathsf{F}_1\mathsf{F}_0(\alpha')$ and $\mathsf{F}(\alpha''); s_{v''} = s_{u''}; \mathsf{F}_1\mathsf{F}_0(\alpha'')$. Therefore, we deduce $\mathsf{F}(\alpha') \otimes \mathsf{F}(\alpha''); s_{v'} \otimes s_{v''} = s_{u'} \otimes s_{u''}; \mathsf{F}_1\mathsf{F}_0(\alpha') \otimes \mathsf{F}_1\mathsf{F}_0(\alpha'')$, i.e., $\mathsf{F}(\alpha); s_v = s_u; \mathsf{F}_1\mathsf{F}_0(\alpha)$. Finally, in the case $\alpha = \alpha'; \alpha''$, where $\alpha'\colon u \to v$ and $\alpha''\colon u \to w$, the induction is mantained by pasting the two commutative squares in following diagrams, which exist by the induction hypothesis

$$\begin{array}{ccc}
\mathsf{F}(u) & \xrightarrow{\;s_u\;} & \mathsf{F}_1\mathsf{F}_0(u) \\
{\scriptstyle \mathsf{F}(\alpha')}\downarrow & & \downarrow{\scriptstyle \mathsf{F}_1\mathsf{F}_0(\alpha')} \\
\mathsf{F}(v) & \xrightarrow[\;s_v\;]{} & \mathsf{F}_1\mathsf{F}_0(v) \\
{\scriptstyle \mathsf{F}(\alpha'')}\downarrow & & \downarrow{\scriptstyle \mathsf{F}_1\mathsf{F}_0(\alpha'')} \\
\mathsf{F}(w) & \xrightarrow[\;s_w\;]{} & \mathsf{F}_1\mathsf{F}_0(w)
\end{array}$$

Thus, $\mathsf{F}(\alpha); s_v = s_u; \mathsf{F}_1\mathsf{F}_0(\alpha)$, which concludes the proof. ✓

Therefore, due to technical reasons concerned with the naturality of choice of the functions *in*, $\mathcal{Q}[\_]$ fails to be a functor from $\underline{\mathsf{Petri}}$ to $\underline{\mathsf{SsMonCat}}$. It is only a *pseudo-functor*. However, it is worth remarking that this failure is *intrinsically* different from the situation of $\mathcal{P}[\_]$ and that the pseudo-functoriality of $\mathcal{Q}[\_]$ is already a *valuable* result. In fact, in the case of $\mathcal{P}[\_]$, we *cannot* lift net morphisms to functors between the categories of processes, a failure which may possibly rise doubts on the structure chosen to represent the processes of a single net, while in the case of $\mathcal{Q}[\_]$, we just cannot define arrow composition better that "up to isomorphism". This simply brings us to the conclusion that $\underline{\mathsf{SsMonCat}}$ is not the correct target category for the functorial construction we are searching. Indeed, as we shall see in the following, it is easy to identify a category $\underline{\mathsf{SSMC}}^{\otimes}$ of symmetric strict monoidal categories such that $\mathcal{Q}[\_]$ is a functor $\underline{\mathsf{Petri}} \to \underline{\mathsf{SSMC}}^{\otimes}$. Actually, this construction is already implicit in Proposition 1.3.10 and consists of taking an appropriate quotient of $\underline{\mathsf{SsMonCat}}$. Moreover, we shall provide $\mathcal{Q}[\_]$ with a "backwards" functor $\underline{\mathsf{SSMC}}^{\otimes} \to \underline{\mathsf{Petri}}$.

Definition 1.3.11 *(Symmetric Petri Categories)*
A symmetric Petri category is a symmetric strict monoidal category $\underline{\mathsf{C}}$ *whose monoid of objects is the free monoid* $S^{\otimes}$ *for some set* S.

For any pair $\underline{\mathsf{C}}$ and $\underline{\mathsf{D}}$ of symmetric Petri categories, consider the binary relation $\mathcal{R}_{\underline{\mathsf{C}},\underline{\mathsf{D}}}$ on the symmetric strict monoidal functors from $\underline{\mathsf{C}}$ to $\underline{\mathsf{D}}$ defined as $\mathsf{F}\ \mathcal{R}_{\underline{\mathsf{C}},\underline{\mathsf{D}}}\ \mathsf{G}$ if and only if there exists an *monoidal natural isomorphism* $\sigma\colon \mathsf{F} \cong \mathsf{G}$ whose components are all *symmetries*. Clearly, $\mathcal{R}_{\underline{\mathsf{C}},\underline{\mathsf{D}}}$ is an equivalence relation. Moreover, if $\mathsf{F}'\colon \underline{\mathsf{C}}' \to \underline{\mathsf{C}}$ and $\mathsf{G}'\colon \underline{\mathsf{D}} \to \underline{\mathsf{D}}'$ are symmetric strict monoidal functors, then whenever $\mathsf{F}\ \mathcal{R}_{\underline{\mathsf{C}},\underline{\mathsf{D}}}\ \mathsf{G}$ we have $\mathsf{G}'\mathsf{F}\mathsf{F}'\ \mathcal{R}_{\underline{\mathsf{C}}',\underline{\mathsf{D}}'}\ \mathsf{G}'\mathsf{G}\mathsf{F}'$. In fact, if $\sigma\colon \mathsf{F} \cong \mathsf{G}$ then $\mathsf{G}'\sigma\mathsf{F}'\colon \mathsf{F}'\mathsf{F}\mathsf{G}' \cong \mathsf{F}'\mathsf{G}\mathsf{G}'$, where $\mathsf{G}'\sigma\mathsf{F}'$ is clearly monoidal and all its components are symmetries. In other words, the family $\mathcal{R}$ is a congruence with respect to functor composition. Therefore, the following definition makes sense.

DEFINITION 1.3.12 *($\underline{\mathsf{SSMC}}^{\otimes}$)*
Let $\underline{\mathsf{SSMC}}^{\otimes}$ be the quotient of the full subcategory of $\underline{\mathsf{SsMonCat}}$ consisting of symmetric Petri categories modulo the congruence $\mathcal{R}$. We shall refer to $\underline{\mathsf{SSMC}}^{\otimes}$ as the category of the symmetric Petri catogeries.

Of course, concerning $\underline{\mathsf{SSMC}}^{\otimes}$ there is the following easy result.

PROPOSITION 1.3.13 *($\mathcal{Q}[\_]\colon \underline{\mathsf{Petri}} \to \underline{\mathsf{SSMC}}^{\otimes}$)*
$\mathcal{Q}[\_]$ extends to a functor from $\underline{\mathsf{Petri}}$ to $\underline{\mathsf{SSMC}}^{\otimes}$.

Proof. For $\langle f, g\rangle\colon N_0 \to N_1$, let $\mathcal{Q}[\langle f, g\rangle]$ be the equivalence class of the symmetric strict monoidal functor from $\mathcal{Q}[N_0]$ to $\mathcal{Q}[N_1]$ described in Proposition 1.3.10.

Then, by the cited proposition, for any PT net N, we have that $\mathcal{Q}[id_N] = [id_{\mathcal{Q}[N]}]_{\mathcal{R}}$, which is the identity of $\mathcal{Q}[N]$. Moreover, we have proved that, given $\langle f_0, g_0\rangle\colon N_0 \to N_1$ and $\langle f_1, g_1\rangle\colon N_1 \to N_2$ in $\underline{\mathsf{Petri}}$, then there exists a monoidal natural isomorphism $s\colon \mathcal{Q}[\langle f_1 \circ f_0, g_1 \circ g_0\rangle] \cong \mathcal{Q}[\langle f_1, g_1\rangle] \circ \mathcal{Q}[\langle f_0, g_0\rangle]$ whose components are symmetries. Then, $\mathcal{Q}[\langle f_1 \circ f_0, g_1 \circ g_0\rangle] = \mathcal{Q}[\langle f_1, g_1\rangle] \circ \mathcal{Q}[\langle f_0, g_0\rangle]$ in $\underline{\mathsf{SSMC}}^{\otimes}$, i.e., $\mathcal{Q}[\_]$ is a functor from $\underline{\mathsf{Petri}}$ to $\underline{\mathsf{SSMC}}^{\otimes}$. ✓

Observe that, when describing $\mathcal{Q}[\langle f, g\rangle]$ in $\underline{\mathsf{SSMC}}^{\otimes}$, there is no need to consider the family of functions in, since the extensions of $\langle f, g\rangle$ to a symmetric strict monoidal functor corresponding to different choices of in_S yield the same functor in $\underline{\mathsf{SSMC}}^{\otimes}$. Interestingly enough, the following proposition identifies a functor from $\underline{\mathsf{SSMC}}^{\otimes}$ to $\underline{\mathsf{Petri}}$. It is worth remarking that this is only a first step towards a result needed to answer to the legitimate possible criticism about $\underline{\mathsf{SSMC}}^{\otimes}$. In fact, in principle, the functoriality result for $\mathcal{Q}[\_]$ could be due to a very tight choice of the category $\underline{\mathsf{SSMC}}^{\otimes}$. We shall further discuss this issue shortly.

PROPOSITION 1.3.14 *($\mathcal{G}[\_]\colon \underline{\mathsf{SSMC}}^{\otimes} \rightharpoonup \underline{\mathsf{Petri}}$)*
Let $\underline{\mathsf{C}}$ belong to $\underline{\mathsf{SSMC}}^{\otimes}$ and let A be the set of arrows of $\underline{\mathsf{C}}$. Consider the equivalence relation $\sim$ on A defined by

$$\alpha \sim \beta \quad \textit{if and only if} \quad \exists s, s' \textit{ in } Sym_{\underline{\mathsf{C}}} \quad \begin{array}{ccc} u & \xrightarrow{\;s\;} & u' \\ {\scriptstyle\alpha}\big\downarrow & & \big\downarrow{\scriptstyle\beta} \\ v & \xrightarrow[\;s'\;]{} & v' \end{array} \quad \textit{commutes.}$$

Then, the structure $\mathcal{G}[\underline{\mathsf{C}}] = (\partial^0, \partial^1 : A/\!\sim\, \to S^{\oplus})$*, where*

- S *is the set of generators of the objects of* $\underline{\mathsf{C}}$*;*
- $\partial^0([\alpha : u \to v]_{\sim}) = \mathcal{M}(u)$*;*
- $\partial^1([\alpha : u \to v]_{\sim}) = \mathcal{M}(v)$*;*

is a Petri net. In addition, given a morphism $\mathsf{F} : \underline{\mathsf{C}} \to \underline{\mathsf{D}}$ *in* $\underline{\mathsf{SSMC}}^{\otimes}$*, consider the pair of mappings* $\langle f, g\rangle$ *of arrows, respectively objects, of* $\underline{\mathsf{C}}$ *to arrows, respectively objects, of* $\underline{\mathsf{D}}$ *such that* $f([\alpha]_{\sim}) = [\mathsf{F}(\alpha)]_{\sim}$ *and* $g(\mathcal{M}(u)) = \mathcal{M}(\mathsf{F}(u))$*. Then, defining* $\mathcal{G}[\mathsf{F}]$ *to be* $\langle f, g\rangle$ *makes* $\mathcal{G}$ *into a functor from* $\underline{\mathsf{SSMC}}^{\otimes}$ *to* $\underline{\mathsf{Petri}}$*.*

Proof. First of all, observe that $\sim$ is an equivalence relation on A. In fact, since the identities are symmetries, $\sim$ is reflexive, since the symmetries are invertible arrows, $\sim$ is symmetric, and finally since the composition of symmetries is a symmetry, $\sim$ is transitive. Therefore, $A/\!\sim$ is well defined.

Recall that, by definition, the symmetries of $\underline{\mathsf{C}}$ are generatared by tensoring and composing instances of the symmetry isomorphism γ and identities. Then, since $\gamma_{u,v} : u \otimes v \to v \otimes u$, it follows immediately that whenever $s : u \to v$ is a symmetry, then $\mathcal{M}(u) = \mathcal{M}(v)$. Then, if $(\alpha : u \to v) \sim (\beta : u' \to v')$, we have $\mathcal{M}(u) = \mathcal{M}(u')$ and $\mathcal{M}(v) = \mathcal{M}(v')$. Therefore, ∂^0 and ∂^1 are well defined functions and, thus, $\mathcal{G}[\underline{\mathsf{C}}]$ is a Petri net.

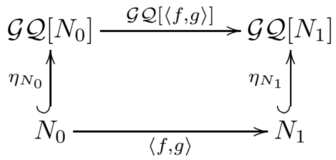
Now let $\mathsf{F} : \underline{\mathsf{C}} \to \underline{\mathsf{D}}$ be a morphism of symmetric Petri categories, let S_0 and S_1 be, respectively, the sets of generators of the objects of $\underline{\mathsf{C}}$ and $\underline{\mathsf{D}}$, and let $\langle f, g\rangle$ be the pair of mappings defined in hypothesis. Since the monoid of objects of $\underline{\mathsf{C}}$ is free, the object component of F is determined by its behaviour on the generators, i.e., for each $u \in S_0^{\otimes}$, we have $\mathsf{F}(u) = \bigotimes_i \mathsf{F}(u_i)$. Then, if $\mathcal{M}(u) = \mathcal{M}(v)$, we clearly have $\mathcal{M}(\mathsf{F}(u)) = \mathcal{M}(\mathsf{F}(v))$, which implies that $g : S_0^{\oplus} \to S_1^{\oplus}$ is well defined. In addition, $g(0) = g(\mathcal{M}(0)) = \mathcal{M}(\mathsf{F}(0)) = \mathcal{M}(0) = 0$ and $g(u \oplus v) = g(\mathcal{M}(u \otimes v)) = \mathcal{M}(\mathsf{F}(u) \otimes \mathsf{F}(v)) = \mathcal{M}(\mathsf{F}(u)) \oplus \mathcal{M}(\mathsf{F}(v)) = g(u) \oplus g(v)$, i.e., g is a monoid homomorphism. Concerning f, let A, $\sim$, A' and $\sim'$ be the set of arrows and the equivalence on them of, respectively, $\underline{\mathsf{C}}$ and $\underline{\mathsf{D}}$. Then, consider $\alpha : u \to v$ and $\beta : u' \to v'$ in $\underline{\mathsf{C}}$ and suppose that $\alpha \sim \beta$. Then, there exist symmetries s and s' such that $\alpha; s' = s; \beta$, and since symmetric strict monoidal functors preserves symmetries, we have that $\mathsf{F}(\alpha) \sim' \mathsf{F}(\beta)$, i.e., $f : A/\!\sim\, \to A'/\!\sim'$ is well defined. Then, since $\langle f, g\rangle$ clearly respects the source and target functions, we have that $\langle f, g\rangle : \mathcal{G}[\underline{\mathsf{C}}] \to \mathcal{G}[\underline{\mathsf{D}}]$ is a morphism in $\underline{\mathsf{Petri}}$.

Suppose now that $\mathsf{F}\ \mathcal{R}\ \mathsf{F}'$ and let $\langle f, g\rangle$ and $\langle f', g'\rangle$ be, respectively, $\mathcal{G}[\mathsf{F}]$ and $\mathcal{G}[\mathsf{F}']$. Since there is a monoidal transformation $\sigma : \mathsf{F} \to \mathsf{F}'$ whose components are symmetries, for all $u \in S_0^{\otimes}$, we have a symmetry $\sigma_u : \mathsf{F}(u) \to \mathsf{F}'(u)$, whence $\mathcal{M}(\mathsf{F}(u)) = \mathcal{M}(\mathsf{F}'(u))$, i.e., $g = g'$. Moreover, for all $\alpha : u \to v$, we have $\mathsf{F}(\alpha); \sigma_v = \sigma_u; \mathsf{F}'(\alpha)$, whence $\mathsf{F}(\alpha) \sim' \mathsf{F}'(\alpha)$, i.e., $f = f'$. Thus, $\mathcal{G}[\_]$ is a well defined mapping of functors in $\underline{\mathsf{SSMC}}^{\otimes}$ to arrows in $\underline{\mathsf{Petri}}$. Let us demonstrate that such a mapping is actually a functor. Clearly, $\mathcal{G}[\mathbf{1}_{\underline{\mathsf{C}}}] = id_{\mathcal{G}[\underline{\mathsf{C}}]}$. Then, consider $\mathsf{F}_0 : \underline{\mathsf{C}}_0 \to \underline{\mathsf{C}}_1$ and $\mathsf{F}_1 : \underline{\mathsf{C}}_1 \to \underline{\mathsf{C}}_2$, and suppose that the objects of $\underline{\mathsf{C}}_i$ are generated by the set S_i, for $i = 0, \ldots, 2$. Moreover, let

$\langle f_i, g_i\rangle: \mathcal{G}[\underline{\mathsf{C}}_i] \to \mathcal{G}[\underline{\mathsf{C}}_{i+1}]$ be $\mathcal{G}[\mathsf{F}_i]$ for $i = 0, 1$, and let $\langle f, g\rangle = \mathcal{G}[\mathsf{F}_1\mathsf{F}_0]$. Then, for all $u \in S_0^{\otimes}$, we have $g_1(g_0(\mathcal{M}(u))) = g_1(\mathcal{M}(\mathsf{F}_0(u)) = \mathcal{M}(\mathsf{F}_1\mathsf{F}_0(u))$, i.e., $g_1 \circ g_0 = g$. In the same way, $f_1(f_0([\alpha]_\sim)) = f_1([\mathsf{F}_0(\alpha)]_\sim) = [\mathsf{F}_1\mathsf{F}_0(\alpha)]_\sim$, i.e., $f_1 \circ f_0 = f$, which concludes the proof that $\mathcal{G}[\_]$ is a functor. ✓

As already mentioned, a legitimate question about $\underline{\mathsf{SSMC}}^{\otimes}$, and therefore about the functoriality of $\mathcal{Q}[\_]$, concerns whether the congruence $\mathcal{R}$ which defines $\underline{\mathsf{SSMC}}^{\otimes}$ induces too many isomorphisms of categories. In other words, one may wonder whether $\mathcal{Q}[\_]$ makes undesired identifications of nets. The best possible answer to such a question, of course, would be a result showing that $\mathcal{Q}[\_]$ has a *right adjoint* functor. However, it is not difficult to verify that $\mathcal{G}[\_]$ is not such a functor. Nevertheless, we think that a right adjoint to (a functor whose object component coincides with that of) $\mathcal{Q}[\_]$ may exist, perhaps modulo some *minor* modifications to $\underline{\mathsf{SSMC}}^{\otimes}$ (and therefore to the arrow component of $\mathcal{Q}[\_]$).

However, for the time being, Proposition 1.3.14 provides a basis for the following observation on our construction. First of all, notice that the relation $\sim$ will never equate arrows corresponding to different transitions, since, if $t \neq t' \in T_N$, there do not exist any symmetries such that $s; t_{u,v}; s' = t'_{u',v'}$. Therefore, the inclusion $\eta_N: N \to \mathcal{G}\mathcal{Q}[N]$ which is the identity on the places and maps a transition to its equivalence class is *injective* (more formally is *mono*). Then, by the very definition of $\mathcal{G}[\_]$, we have that for any $\langle f, g\rangle: N_0 \to N_1$ in $\underline{\mathsf{Petri}}$ the following diagram commutes.



In other words, any functor $\mathcal{G}\mathcal{Q}[\langle f, g\rangle]: \mathcal{G}\mathcal{Q}[N_0] \to \mathcal{G}\mathcal{Q}[N_1]$ in $\underline{\mathsf{SSMC}}^{\otimes}$ restricts to a mapping between N_0 and N_1 (viewed as substructures of, respectively, $\mathcal{G}\mathcal{Q}[N_0]$ and $\mathcal{G}\mathcal{Q}[N_1]$ via the inclusions η_N) whose place component is g and whose transition component sends $[t]_\sim$ to $[f(t)]_\sim$. It follows that if $\mathcal{G}\mathcal{Q}[\langle f, g\rangle]: \mathcal{G}\mathcal{Q}[N_0] \to \mathcal{G}\mathcal{Q}[N_1]$ is an isomorphism, then $\langle f, g\rangle: N_0 \to N_1$ has to be so.

1.4 Strong Concatenable Processes

In this section we introduce a slight refinement of concatenable processes and we show that they can be abstractly represented as the arrows of the category $\mathcal{Q}[N]$. In other words, we find a process-like representation for the arrows of $\mathcal{Q}[N]$. This yields a functorial construction for the category of the processes of a net N. Once again most of the work has already been done in the proof of Proposition 1.3.6 and therefore our task now easy.

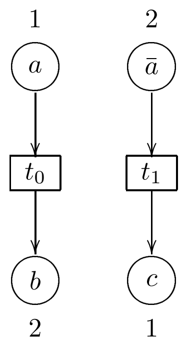


Figure 1.8: A strong concatenable process for the net of Figure 1.2

DEFINITION 1.4.1 *(Strong Concatenable Processes)*
Given a net N in Petri*, a strong concatenable process of N is a tuple (π, ℓ, L) where $\pi\colon \Theta \to N$ is a finite process of N, and $\ell\colon \min(\Theta) \to \{1, \ldots, |\min(\Theta)|\}$ and $L\colon \max(\Theta) \to \{1, \ldots, |\max(\Theta)|\}$ are isomorphisms, i.e., total orderings of, respectively, the minimal and the maximal places of Θ.*

An isomorphism of strong concatenable processes is an isomorphism of the underlying processes which, in addition, preserves the orderings ℓ and L. As usual, we shall identify isomorphic strong concatenable processes.

So, a strong concatenable process is a non-sequential process where the minimal and maximal places are linearly ordered. Graphically, we shall represent strong concatenable processes by the usual representation of non-sequential processes enriched by labelling the minimal and the maximal places with the value of, respectively, ℓ and L. An example is shown in Figure 1.8.

As for concatenable processes, it is easy to define an operation of concatenation of strong concatenable processes. We associate a source and a target in $S_N^{\otimes}$ to each strong concatenable process *SCP* by taking the string corresponding to the linear ordering of, respectively, $\min(\Theta)$ and $\max(\Theta)$. Then, the concatenation of $(\pi_0\colon \Theta_0 \to N, \ell_0, L_0)\colon u \to v$ and $(\pi_1\colon \Theta_1 \to N, \ell_1, L_1)\colon v \to w$ is the strong concatenable process $(\pi\colon \Theta \to N, \ell, L)\colon u \to w$ defined as follows (see also Figure 1.9), where, in order to simplify notations, we assume that S_{Θ_0} and S_{Θ_1} are disjoint.

- Let A be the set of pairs (x, y) such that $x \in \max(\Theta_0)$, $y \in \min(\Theta_1)$ and $\ell(y) = L(x)$. By the definitions of concatenable processes and of their sources and targets, each element of $\max(\Theta_0)$ belongs exactly to one pair of A, and of course the same happens to $\min(\Theta_1)$. Consider $S_0 = S_{\Theta_0} \setminus \max(\Theta_0)$ and $S_1 = S_{\Theta_1} \setminus \min(\Theta_1)$. Then, let $in_0\colon S_{\Theta_0} \to S_0 \cup A$ be the function which is the identity on $x \in S_0$ and maps $x \in \max(\Theta_1)$ to the corresponding pair in A.

Define $in_1: S_{\Theta_1} \to S_1 \cup A$ analogously. Then,

$$\Theta = (\partial^0, \partial^1 : T_{\Theta_0} + T_{\Theta_1} \to (S_0 \cup S_1 \cup A)^{\oplus}),$$

where

– $\partial^0 = in_0^{\oplus} \circ \partial^0_{\Theta_0} + in_1^{\oplus} \circ \partial^0_{\Theta_1}$;

– $\partial^1 = in_0^{\oplus} \circ \partial^1_{\Theta_0} + in_1^{\oplus} \circ \partial^1_{\Theta_1}$;

- Suppose $\pi_i = \langle f_i, g_i \rangle$, for $i = 0, 1$ and consider the function $g(x) = g_i(x)$ if $a \in S_i$ and $g((x, y)) = g_0(x) = g_1(y)$ otherwise. Then $\pi = \langle f_0 + f_1, g \rangle$.
- $\ell(x) = \ell_0(x)$ if $x \in \min(\Theta_0)$ and $\ell((x, y)) = \ell_0(x)$ if $(x, y) \in \min(\Theta)$.
- $L(y) = L_1(y)$ if $y \in \max(\Theta_1)$ and $L((x, y)) = L_1(y)$ if $(x, y) \in \max(\Theta)$.

PROPOSITION 1.4.2
Under the above defined operation of sequential composition, the strong concatenable processes of N form a category $\mathcal{CQ}[N]$ with identities those processes consisting only of places, which therefore are both minimal and maximal, and such that $\ell = L$.

Strong concatenable processes admit also a tensor operation $\otimes$ such that, given $SCP_0 = (\pi_0 : \Theta_0 \to, \ell_0, L_0) : u \to v$ and $SCP_1 = (\pi_1 : \Theta_1 \to N, \ell_1, L_1) : u' \to v'$, $SCP_0 \otimes SCP_1$ is the strong concatenable process $(\pi : \Theta \to N, \ell, L) : u \otimes u' \to v \otimes v'$ given below (see also Figure 1.9).

- $\Theta = (\partial^0_{\Theta_0} + \partial^0_{\Theta_1}, \partial^1_{\Theta_0} + \partial^1_{\Theta_1} : T_{\Theta_0} + T_{\Theta_1} \to (S_{\Theta_0} + S_{\Theta_1})^{\oplus})$,
- $\pi = \pi_0 + \pi_1$;
- $\ell(in_0(x)) = \ell_0(x)$ and $\ell(in_1(y)) = |\min(\Theta_0)| + \ell_1(y)$.
- $L(in_0(x)) = L_0(x)$ and $L(in_1(y)) = |\max(\Theta_1)| + L_1(y)$.

Observe that $\otimes$ is a functor $\mathcal{CQ}[N] \times \mathcal{CQ}[N] \to \mathcal{CQ}[N]$. The strong concatenable processes consisting only of places are the analogous in $\mathcal{CQ}[N]$ of the permutations of $\mathcal{Q}[N]$. In particular, for any $u, v \in S_N^{\otimes}$, the strong concatenable process which consists of places in one-to-one correspondence with the elements of the string $u \otimes v$ mapped by π to the corresponding places of N, and such that $\ell(u_i) = i$, $\ell(v_i) = |u| + i$, $L(u_i) = |v| + i$ and $L(v_i) = i$, plays in $\mathcal{CQ}[N]$ the role played in $\mathcal{Q}[N]$ by the permutation $\gamma(u, v)$ (see also Figure 1.10).

PROPOSITION 1.4.3
The lluf subcategory of $\mathcal{CQ}[N]$ consisting of the processes with only places, which we call permutations, is a symmetric strict monoidal category isomorphic to Sym_N^*.

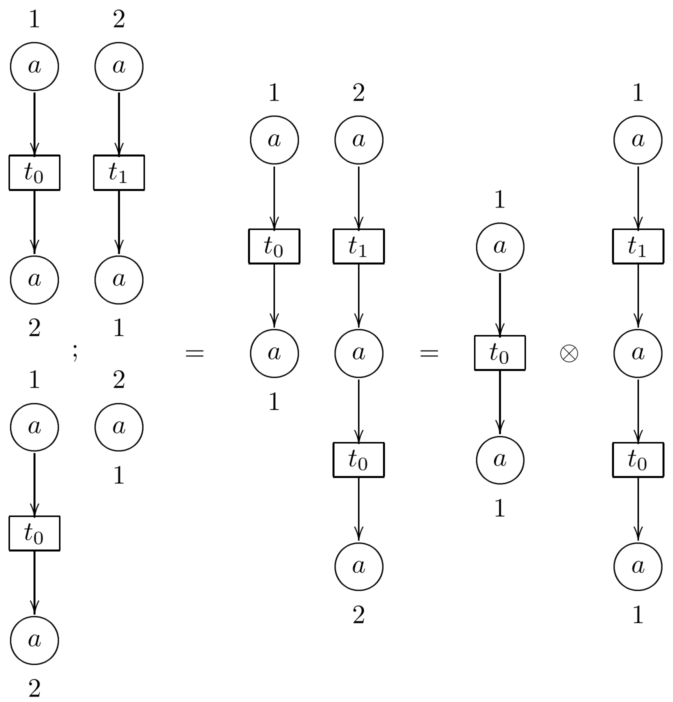


Figure 1.9: An example of the algebra of concatenable processes

Proof. Let *Sym* be the subcategory of the permutations of $\mathcal{CQ}[N]$. In order to show that it is a symmetric strict monoidal category, we need to verify that $\otimes$ restricts to a functor from $Sym \times Sym \to Sym$ which satisfies axiom (1.2)–(1.4) and to check that *Sym* satisfies axioms (1.5) and (1.6) with respect to the symmetries $\bar{\gamma}(u, v)$ defined above. Moreover, it must be verified that the collection of the $\bar{\gamma}(u, v)$, $u, v \in S_N^{\otimes}$, is a natural transformation. These tasks are really immediate and thus omitted.

Concerning the relationships of *Sym* with Sym_N^*, observe that, by Proposition 1.3.6, there exists a functor F from Sym_N^* to *Sym*, corresponding to the identity function on $S_N^{\otimes}$, which is the identity on the objects. Moreover, since for any $u, v \in S_N^{\otimes}$, the strong concatenable processes from u to v in *Sym* are clearly isomorphic to the permutations $p\colon u \to v$ in Sym_N^*, we have that F is also full and faithful. Therefore, F is an isomorphism. ✓

Finally, the transitions t of N are faithfully represented in the obvious way by processes with a unique transition which is in the post-set of any minimal place and in the pre-set of any maximal place, minimal and maximal places being in one-to-one correspondence, respectively, with $\partial_N^0(t)$ and $\partial_N^1(t)$. Thus, varying ℓ and L on

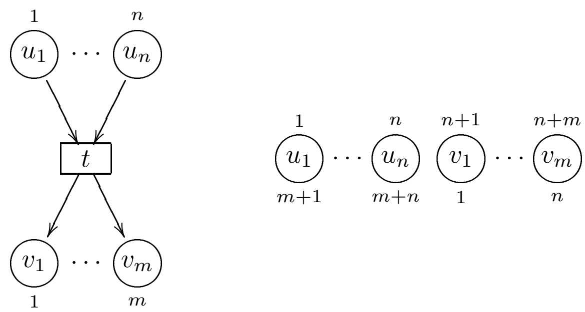


Figure 1.10: A transitions $t_{u,v}\colon u \to v$ and the symmetry $\gamma(u, v)$ in $\mathcal{CQ}[N]$

the process corresponding to a transition we obtain a representative in $\mathcal{CQ}[N]$ of each instance $t_{u,v}$ in $\mathcal{Q}[N]$ of t (see also Figure 1.10). In the following $\underline{\mathsf{C}}^n$ denotes the n-th power of $\underline{\mathsf{C}}$, i.e., the cartesian product of n copies of $\underline{\mathsf{C}}$. Moreover, for $n \geq 2$, we use $\otimes^n\colon \underline{\mathsf{C}}^n \to \underline{\mathsf{C}}$ to indicate $\otimes \circ (\mathbf{1}_{\underline{\mathsf{C}}} \times \otimes) \circ \cdots \circ (\mathbf{1}_{\underline{\mathsf{C}}^{n-2}} \times \otimes)$.

LEMMA 1.4.4
Let $\underline{\mathsf{C}}$ be a symmetric strict monoidal category. For each permutation σ of n elements, $n \geq 2$, let $\mathsf{F}_\sigma\colon \underline{\mathsf{C}}^n \to \underline{\mathsf{C}}^n$ the functor which "swaps" is arguments according to σ, i.e.,

$$\begin{array}{ccc} \underline{\mathsf{C}}^n & \xrightarrow{\mathsf{F}_\sigma} & \underline{\mathsf{C}}^n \\ (x_1, \dots, x_n) & \longmapsto & (x_{\sigma(1)}, \dots, x_{\sigma(n)}) \\ \big\downarrow{\scriptstyle (f_1,\dots,f_n)} & & \big\downarrow{\scriptstyle (f_{\sigma(1)},\dots,f_{\sigma(n)})} \\ (y_1, \dots, y_n) & \longmapsto & (y_{\sigma(1)}, \dots, y_{\sigma(n)}) \end{array}$$

Then, there exists a natural isomorphism $\gamma_\sigma\colon \otimes^n \stackrel{\bullet}{\to} \otimes^n \circ \mathsf{F}_\sigma$. We shall call γ_σ the "σ-interchange" symmetry.

Proof. From Lemma 1.2.3 we know that each permutation of n elements can be written as a composition of *transpositions*, where, for $i = 1, \dots, n-1$, the transposition τ_i is the permutation which leaves fixed all the elements but i and $i+1$, which are (of course) exchanged. This formalizes the intuitive fact that a permutation can always be achieved by performing a sequence of *"swappings"* of adjacent integers. Then, assume that σ is $\tau_{i_k} \circ \cdots \circ \tau_{i_1}$. We show the thesis by induction on k.

base case. If $k = 0$ then $\sigma = id$, and thus $\mathbf{1}_{\otimes^n}$ is the isomorphism looked for.

inductive step. Let σ' be $\tau_{i_{k-1}} \circ \cdots \circ \tau_{i_1}$. Then, by inductive hypothesis, we have a σ'-interchange symmetry $\gamma_{\sigma'}\colon \otimes^n \to \otimes^n \circ \mathsf{F}_{\sigma'}$. Now, let i_k be $\sigma'(i)$ and consider the natural isomorphism

$$\bar{\tau} = id_{x_{\sigma'(1)}} \otimes \cdots \otimes id_{x_{\sigma'(i-1)}} \otimes \gamma_{x_{\sigma'(i)}, x_{\sigma'(i+1)}} \otimes id_{x_{\sigma'(i+2)}} \otimes \cdots \otimes id_{x_{\sigma'(n)}}$$

from $\otimes^n \circ \mathsf{F}_{\sigma'}$ to $\otimes^n \circ \mathsf{F}_{\sigma' \circ \tau_{i_1}}$. Of course, since $\tau_{i_k} \circ \sigma' = \sigma$, we have that γ_σ is the (vertical) composition $\bar{\tau} \cdot \gamma_{\sigma'} \colon \otimes^n \to \otimes^n \circ \mathsf{F}_\sigma$.

Observe that, since σ admits several factorizations in terms of transpositions, in principle many different γ_σ may exist. However it is worth noticing that this is not the case. In particular, there exists a unique σ-interchange symmetry, as follows from the Kelly-MacLane coherence theorem (see [87, 62]) which, informally speaking, states that, given any pair of functors built up from identity functors and $\otimes$, there is at most one natural transformation built up from identities and components of the symmetry γ between them. Of course, the uniqueness of γ_σ can also be shown directly by exploiting the coherence axioms (1.5) and (1.6) of SSMC's and the axioms (1.14) which characterize the symmetric group $\Pi(n)$. ✓

Observe that in case $\underline{\mathsf{C}}$ in the proposition above is $\mathcal{Q}[N]$, then the σ-interchange symmetry is exactly the family $\{\sigma(u_1, \ldots, u_n)\}_{u_1,\ldots,u_n \in S_N^\otimes}$ of the interchange permutations as defined in Proposition 1.3.4. To show this, it is enough to verify that the given family of arrows is a natural transformation from $x_1 \otimes \cdots \otimes x_n$ to $x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)}$. Then, the claim follows from the uniqueness of γ_σ. The σ-interchange symmetry plays a relevant role in the proof of the following proposition.

Proposition 1.4.5
$\mathcal{CQ}[N]$ and $\mathcal{Q}[N]$ are isomorphic.

Proof. First of all observe that $\mathcal{CQ}[N]$ satisfies axiom (Φ) of Definition 1.3.8, the symmetries and the (instances of) transitions being as explained earlier. In order to prove this claim, let $T_{u,v} = (\pi_0, \ell_0, L_0)$ and $T_{u',v'} = (\pi_1, \ell_1, L_1)$ be different instances of some transition t and let $S \colon u \to u'$ and $S' \colon v \to v'$ be symmetries of $\mathcal{CQ}[N]$. Moreover, suppose that S^{-1} and S' correspond, respectively, to the permutations $p \colon u' \to u$ and $q \colon v \to v'$ in $\mathcal{Q}[N]$. Then, $S^{-1}; T_{u,v}; S$ is (isomorphic to) $(\pi_0, p \circ \ell_0, q \circ L_0)$. Consider now the function $g \colon S_{\Theta_0} \to S_{\Theta_1}$ such that $g(x) = \ell_1^{-1}(p(\ell_0(x)))$ if $x \in \min(\Theta_0)$ and $g(x) = L_1^{-1}(q(L_0(x)))$ if $x \in \max(\Theta_1)$. Clearly, by definition of Θ_0 and Θ_1, g is an isomorphism. Moreover, since for each $x \in \min(\Theta_0)$ and $y \in \max(\Theta_0)$ we have $u_{\ell_0(x)} = u'_{p(\ell_0(x))}$ and $v_{L_0(y)} = u'_{q(L_1(y))}$, it follows that $\pi_1(g(x)) = u'_{\ell_1(g(x))} = u'_{p(\ell_0(x))} = u_{\ell_0(x)} = \pi_0(x)$ and that $\pi_1(g(y)) = u'_{L_1(g(y))} = u'_{q(L_0(y))} = u_{L_0(y)} = \pi_0(y)$. Therefore, we have an isomorphism $\langle f, g^\otimes \rangle \colon \Theta_0 \to \Theta_1$, where $g^\otimes \colon S_{\Theta_0}^\otimes \to S_{\Theta_1}^\otimes$ is the free monoidal extension of g and f is the function which maps the unique transition in Θ_0 to the unique transition in Θ_1. Then, $S^{-1}; T_{u,v}; S' = T_{u',v'}$, i.e., (Φ) holds.

Thus, since by definition $\mathcal{Q}[N]$ is the free symmetric strict monoidal category built on Sym_N^* plus the additional arrows in T_N and which satisfies axiom (Φ), there is a strict monoidal functor $\mathcal{H} \colon \mathcal{Q}[N] \to \mathcal{CQ}[N]$ which is the identity on the objects and sends the generators, i.e., symmetries and transitions, to the corresponding strong concatenable processes. We want to show that $\mathcal{H}$ is an isomorphism.

<u>fullness</u>. It is completely trivial to see that any strong concatenable process SCP may be obtained as a concatenation $SCP_0; \ldots; SCP_n$ of strong concatenable processes

SCP_i of depth one. Now, each of these SCP_i may be split into the concatenation of a symmetry S_0^i, the tensor of the (processes representing the) transitions which appear in it plus some identities, say $u_i \otimes \bigotimes_j T_j^i$ and finally another symmetry S_1^i. The intuition about this factorization is as follows. We take the tensor of the transitions which appear in SCP_i in any order and multiply the result by an identity concatenable process in order to get the correct source and target. Then, in general, we need a pre-concatenation and a post-concatenation with a symmetry in order to get the right indexing of minimal and maximal places. Then, we finally have

$$SCP = S_0^0; (u_1 \otimes \bigotimes_j T_j^1); (S_1^0; S_0^1); \dots; (S_1^{n-1}; S_0^n); (u_n \otimes \bigotimes_j T_j^n); S_1^n$$

which shows that every strong concatenable process is in the image of $\mathcal{H}$.

faithfulness. The arrows of $\mathcal{Q}[N]$ are equivalence classes modulo the axioms stated in Definition 1.3.9 of terms built by applying tensor and sequentialization to the identities id_u, the symmetries $c_{u,v}$, and the transitions $t_{u,v}$. We have to show that, given two such terms α and β, whenever $\mathcal{H}(\alpha) = \mathcal{H}(\beta)$ we have $\alpha =_\mathcal{E} \beta$, where $=_\mathcal{E}$ is the equivalence induced by the axioms (1.21), (1.22) and (Φ).

First of all, observe that if $\mathcal{H}(\alpha)$ is a strong process SCP of depth n, then α can be proved equal to a term

$$\alpha' = s_0; (id_{u_1} \otimes \bigotimes_j \tau_j^1); s_1; \dots; s_{n-1}; (id_{u_n} \otimes \bigotimes_j \tau_j^n); s_n$$

where, for $1 \leq i \leq n$, $\tau_j^i = (t_j^i)_{u_j^i, v_j^i}$ and the transitions t_j^i, for $1 \leq j \leq n_i$, are exactly the transitions of SCP at depth i and where $s_0, \dots, s_n$ are symmetries. Moreover, we can assume that in the i-th tensor product $\bigotimes_j \tau_j^i$ the transitions are indexed according to a global ordering $\leq$ of T_N assumed for the purpose of this proof, i.e., $t_1^i \leq \cdots \leq t_{n_i}^i$, for $1 \leq i \leq n$. Let us prove our claim. It is easily shown by induction on the structure of terms that using axioms (1.21) α can be rewritten as $\alpha_1; \dots; \alpha_h$, where $\alpha_i = \bigotimes_k \xi_k^i$ and ξ_k^i is either a transition or a symmetry. Now, observe that by functoriality of $\otimes$, for any $\alpha' : u' \to v'$, $\alpha'' : u'' \to v''$ and $s : u \to u$, we have $\alpha' \otimes s \otimes \alpha'' = (id_{u'} \otimes s \otimes id_{u''}); (\alpha' \otimes id_u \otimes \alpha'')$, and thus, by repeated applications of (1.21), we can prove that α is equivalent to $\bar{s}_0; \bar{\alpha}_1; \bar{s}_1 \dots; \bar{s}_{h-1}; \bar{\alpha}_h$, where $\bar{s}_0, \dots, \bar{s}_{h-1}$ are symmetries and each $\bar{\alpha}_i$ is a tensor $\bigotimes_k \bar{\xi}_k^i$ of transitions and identities. The fact that the transitions at depth i can be brought to the i-th tensor product, follows intuitively from the facts that they are "disjointly enabled", i.e., concurrent to each other, and that they depend causally on some transition at depth $i-1$. In particular, the sources of the transitions of depth 1 can be target only of symmetries. Therefore, reasoning formally as above, they can be pushed up to $\bar{\alpha}_1$ exploiting axioms (1.21). Then, the same happens for the transitions of depth 2, which can be brought to $\bar{\alpha}_2$. Proceeding in this way, eventually we show that α is equivalent to the composition $\bar{\bar{s}}_0; \bar{\bar{\alpha}}_1; \bar{\bar{s}}_1 \dots; \bar{\bar{s}}_{n-1}; \bar{\bar{\alpha}}_n; \bar{\bar{s}}_n$ of the symmetries $\bar{\bar{s}}_0, \dots, \bar{\bar{s}}_n$ and the products $\bar{\bar{\alpha}}_i = \bigotimes_k \bar{\bar{\xi}}_k^i$ of transitions at depth i and identities. Finally, exploiting Lemma 1.4.4, the order of the $\bar{\bar{\xi}}_k^i$ can be permuted in the way required by $\leq$. This is achieved by pre- and post-composing each product by appropriate σ-interchange symmetries. More precisely, let σ be a permutation such that $\bigotimes_k \bar{\bar{\xi}}_{\sigma(k)}^i$ coincides with $id_{u_i} \otimes \bigotimes_j \tau_j^i$, and suppose that $\bar{\bar{\xi}}^i_k: u^i_k \to v^i_k$, for $1 \le k \le k_i$. Then, by Lemma 1.4.4, we have that

$$\sigma(u^i_1,\ldots,u^i_{k_i}); (\bigotimes_k \bar{\bar{\xi}}^i_{\sigma(k)}) = (\bigotimes_k \bar{\bar{\xi}}^i_k); \sigma(v^i_1,\ldots,v^i_{k_i}),$$

and therefore, since $\sigma(u^i_1,\ldots,u^i_{k_i})$ is an isomorphism, we have that

$$(id_{u_i} \otimes \bigotimes_j \tau^i_j) = \sigma(u^i_1,\ldots,u^i_{k_i})^{-1}; (\bigotimes_k \bar{\bar{\xi}}^i_k); \sigma(v^i_1,\ldots,v^i_{k_i}).$$

Now, applying the same argument to β, one proves that it is equivalent to a term $\beta' = p_0;\beta_0;p_1;\ldots p_{n-1};\beta_n;p_n$, where $p_0,\ldots,p_n$ are symmetries and β_i is the product of (instances of) the transitions at depth i in $\mathcal{H}(\beta)$ and of identities. Then, since $\mathcal{H}(\alpha) = \mathcal{H}(\beta)$, and since the transitions occurring in β_i are indexed in a predetermined way, we conclude that $\beta_i = (id_{u_i} \otimes \bigotimes_j \bar{\tau}^i_j)$, where $\bar{\tau}^i_j = (t^i_j)_{\bar{u}^i_j,\bar{v}^i_j}$ i.e.,

$$\begin{aligned}
\alpha' &= s_0; (id_{u_1} \otimes \textstyle\bigotimes_j (t^1_j)_{u^1_j,v^1_j}); s_1; \ldots; s_{n-1}; (id_{u_n} \otimes \textstyle\bigotimes_j (t^n_j)_{u^n_j,v^n_j}); s_n \\
\beta' &= p_0; (id_{u_1} \otimes \textstyle\bigotimes_j (t^1_j)_{\bar{u}^1_j,\bar{v}^1_j}); p_1; \ldots; p_{n-1}; (id_{u_n} \otimes \textstyle\bigotimes_j (t^n_j)_{\bar{u}^n_j,\bar{v}^n_j}); p_n
\end{aligned} \tag{1.23}$$

In other words, the only possible differences between α' and β' are the symmetries and the sources and targets of the corresponding instances of transitions. Observe now that the steps which led from α to α' and from β to β' have been performed by using the axioms which define $\mathcal{Q}[N]$ and since such axioms hold in $\mathcal{CQ}[N]$ as well and $\mathcal{H}$ preserves them, we have that $\mathcal{H}(\alpha') = \mathcal{H}(\alpha) = \mathcal{H}(\beta) = \mathcal{H}(\beta')$. Thus, we conclude the proof by showing that, if α and β are terms of the form given in (1.23) which differ only by the intermediate symmetries and if $\mathcal{H}(\alpha) = \mathcal{H}(\beta)$, then α and β are equal in $\mathcal{Q}[N]$.

We proceed by induction on n. Observe that if n is zero then there is nothing to show: since we know that $\mathcal{H}$ is an isomorphism on the symmetries, s_0 and p_0, and thus α and β, must coincide. To provide a correct basis for the induction, we need to prove the thesis also for $n = 1$.

depth 1. In this case, we have

$$\begin{aligned}
\alpha &= s_0; (id_u \otimes \textstyle\bigotimes_j (t_j)_{u_j,v_j}); s_1 \\
\beta &= p_0; (id_u \otimes \textstyle\bigotimes_j (t_j)_{\bar{u}_j,\bar{v}_j}); p_1.
\end{aligned}$$

Without loss of generality, we may assume that p_0 and p_1 are identities. In fact, we can multiply both terms by p_0^{-1} on the left and by p_1^{-1} on the right and obtain a pair of terms whose images through $\mathcal{H}$ still coincide and whose equality implies the equality in $\mathcal{Q}[N]$ of the original α and β.

Let $(\pi: \Theta \to N, \ell, L)$ and $(\bar{\pi}: \bar{\Theta} \to N, \bar{\ell}, \bar{L})$ be, respectively, the strong concatenable processes $\mathcal{H}(id_u \otimes \bigotimes_j (t_j)_{u_j,v_j})$ and $\mathcal{H}(id_u \otimes \bigotimes_j (t_j)_{\bar{u}_j,\bar{v}_j})$. Clearly, we can assume that $\mathcal{H}(s_0)$ and $\mathcal{H}(s_1)$ are respectively $(\pi_0: \Theta_0 \to N, \ell', \ell)$ and $(\pi_1: \Theta_1 \to N, L, L')$, where Θ_0 is $\min(\Theta)$, Θ_1 is $\max(\Theta)$, π_0 and π_1 are the corresponding restrictions of π, ℓ' and L' are the orderings respectively of the minimal and the maximal places of Θ.

Then, we have that $\mathcal{H}(s_0; (id_u \otimes \bigotimes_j (t_j)_{u_j,v_j}); s_1)$ is (π, ℓ', L'), and by hypothesis there is an isomorphism $\varphi: \Theta \to \bar{\Theta}$ such that $\bar{\pi} \circ \varphi = \pi$ and which respects all the orderings, i.e., $\bar{\ell}(\varphi(a)) = \ell'(a)$ and $\bar{L}(\varphi(b)) = L'(b)$, for all $a \in \Theta_0$ and $b \in \Theta_1$. Let us write $id_u \otimes \bigotimes_j (t_j)_{u_j,v_j}$ as $\bigotimes_k \xi_k$ and $id_u \otimes \bigotimes_j (t_j)_{\bar{u}_j,\bar{v}_j}$ as $\bigotimes_k \bar{\xi}_k$, where ξ_k, respectively $\bar{\xi}_k$, is either a transition $(t_j)_{u_j,v_j}$, respectively $(t_j)_{\bar{u}_j,\bar{v}_j}$, or the identity of a place in u. Clearly, φ induces a permutation, namely the permutation σ such that $\bar{\xi}_{\sigma(k)} = \varphi(\xi_k)$. In order for φ to be a morphism of nets, it must map the (places corresponding to the) pre-set, respectively post-set, of $(t_j)_{u_j,v_j}$ to (the places corresponding to the) pre-set, respectively post-set, of $(t_{\sigma(j)})_{\bar{u}_{\sigma(j)},\bar{v}_{\sigma(j)}}$. It follows that (π_1, L, L'), which is $\mathcal{H}(s_1)$, must be a symmetry obtained by post-concatenating the component at $(\bar{v}_1, \ldots, \bar{v}_{k_i})$ of the σ-interchange symmetry in $\mathcal{CQ}[N]$ to a tensor product $\bigotimes_j S^1_j$ of symmetries, one for each t occurring in α, where $S^1_j \colon v_j \to \bar{v}_j$, whose role is to reorganize the tokens in the post-sets of each transitions. Reasoning along the same lines, we can conclude that (π_0, ℓ, ℓ'), which is $\mathcal{H}(s_0)^{-1}$, must be a symmetry obtained by concatenating a tensor product $\bigotimes_j S^0_j$, where $S^0_j \colon u_j \to \bar{u}_j$ is a symmetry and the component at $(\bar{u}_1, \ldots, \bar{u}_{k_i})$ of the σ-interchange symmetry. Then, since $\mathcal{H}$ is an isomorphism between $Sym_{\mathcal{Q}[N]}$ and $Sym_{\mathcal{CQ}[N]}$, s_0 and s_1 must necessarily be, respectively, $\sigma(\bar{u}_1, \ldots, \bar{u}_{k_i})^{-1}; (id_u \otimes \bigotimes_j s^0_j)$, and $(id_u \otimes \bigotimes_j s^1_j); \sigma(\bar{v}_1, \ldots, \bar{v}_{k_i})$, where $s^0_j \colon \bar{u}_j \to u_j$ and $s^1_j \colon v_j \to \bar{v}_j$ are symmetries.

Then, by distributing the tensor of symmetries on the transitions and using axiom (Φ), we show that

$$\begin{aligned} \alpha &= \sigma(\bar{u}_1, \ldots, \bar{u}_{k_i})^{-1}; (id_u \otimes \textstyle\bigotimes_j s^0_j; (t_j)_{u_j,v_j}; s^1_j); \sigma(\bar{v}_1, \ldots, \bar{v}_{k_i}) \\ &= \sigma(\bar{u}_1, \ldots, \bar{u}_{k_i}); (id_u \otimes \textstyle\bigotimes_j (t_j)_{\bar{u}_j,\bar{v}_j}); \sigma(\bar{v}_1, \ldots, \bar{v}_{k_i}), \end{aligned}$$

which, by definition of σ-interchange symmetry, is $(id_u \otimes \bigotimes_j (t_j)_{\bar{u}_j,\bar{v}_j})$. Thus, we have $\alpha =_\mathcal{E} \beta$ as required.

Inductive step. Suppose that $n > 1$ and let $\alpha = \alpha'; \alpha''$ and $\beta = \beta'; \beta''$, where

$$\begin{aligned} \alpha' &= s_0; (id_{u_1} \otimes \textstyle\bigotimes_j \tau^1_j); s_1; \ldots; s_{n-1} \quad \text{and} \quad \alpha'' = (id_{u_n} \otimes \bigotimes_j \tau^n_j); s_n \\ \beta' &= p_0; (id_{u_1} \otimes \textstyle\bigotimes_j \bar{\tau}^1_j); p_1; \ldots; p_{n-1} \quad \text{and} \quad \beta'' = (id_{u_n} \otimes \bigotimes_j \bar{\tau}^n_j); p_n \end{aligned}$$

We show that there exists a symmetry s in $\mathcal{Q}[N]$ such that $\mathcal{H}(\alpha'; s) = \mathcal{H}(\beta')$ and $\mathcal{H}(s^{-1}; \alpha'') = \mathcal{H}(\beta'')$. Then, by the induction hypothesis, we have $(\alpha'; s) =_\mathcal{E} \beta'$ and $(s^{-1}; \alpha'') =_\mathcal{E} \beta''$. Therefore, we conclude that $(\alpha'; s; s^{-1}; \alpha'') =_\mathcal{E} (\beta'; \beta'')$, i.e., that $\alpha = \beta$ in $\mathcal{Q}[N]$.

Let $(\pi \colon \Theta \to N, \ell, L)$ be the strong concatenable process $\mathcal{H}(\alpha) = \mathcal{H}(\beta)$. Without loss of generality we may assume that the strong processes $\mathcal{H}(\alpha')$ and $\mathcal{H}(\beta')$ are, respectively, $(\pi \colon \Theta' \to N, \ell', L^{\alpha'})$ and $(\pi' \colon \Theta' \to N, \ell', L^{\beta'})$, where Θ' is the subnet of depth $n-1$ of Θ, ℓ' is the appropriate restriction of ℓ and finally $L^{\alpha'}$ and $L^{\beta'}$ are orderings of the places at depth $n-1$ of Θ. Consider the symmetry $S = (\bar{\pi} \colon \bar{\Theta} \to N, \bar{\ell}, \bar{L})$ in $\mathcal{CQ}[N]$, where

- $\bar{\Theta}$ is the process nets consisting of the maximal places of Θ';
- $\bar{\pi} \colon \bar{\Theta} \to N$ is the restriction of π to $\bar{\Theta}$;
- $\bar{\ell} = L^{\alpha'}$;
- $\bar{L} = L^{\beta'}$.

Then, by definition, we have $\mathcal{H}(\alpha'); S = \mathcal{H}(\beta')$. Let us consider now α'' and β''. Clearly, we can assume that $\mathcal{H}(\alpha'')$ and $\mathcal{H}(\beta'')$ are respectively $(\pi'' : \Theta'' \to N, \ell^{\alpha''}, L'')$ and $(\pi'' : \Theta'' \to N, \ell^{\beta''}, L'')$, where Θ'' is the process net obtained by removing from Θ the subnet Θ', L'' is the restriction of L to Θ'', and $\ell^{\alpha''}$ and $\ell^{\beta''}$ are orderings of the places at depth $n-1$ of Θ. Now, in our hypothesis, it must be $L^{\alpha'} = \ell^{\alpha''}$ and $L^{\beta'} = \ell^{\beta''}$, which shows directly that $S^{-1}; \mathcal{H}(\alpha'') = \mathcal{H}(\beta'')$. Then, $s = \mathcal{H}^{-1}(S)$ is the required symmetry of $\mathcal{Q}[N]$.

Then, since $\mathcal{H}$ is full and faithful and is an isomorphism on the objects, it is an isomorphism and the proof is concluded. ✓

1.5 Place/Transition Nets

In this section we introduce the category PTNets of *Place/Transition (PT) nets*, as appeared in [98] and the (by now) classical categories Safe of *safe nets* and Occ of *occurrence nets* [143]. These are basically categories of *marked pointed nets* as defined in Section 1.1 with a major difference: *infinite markings* in which, however, each place has a *finite* multiplicity are allowed. Moreover, we shall not allow *always-enabled* transitions, i.e., transitions with empty pre-set. Finally, a slight restriction is imposed on the morphisms in PTNets. The final subsection is devoted to the study of compositional properties in PTNets.

Remark. As remarked earlier in Section 1.1, these generalizations and restrictions are required in order to define the unfolding semantics for nets. We should like to stress that, in particular, they are needed only to enforce the existence of a left adjoint to the unfolding functor $\mathcal{U}[\_]$: PTNets $\to$ Occ, the latter not depending at all on them. We shall make use of this observation later on in Section 1.9.

The Categories PTNets, Safe and Occ

Given a set S, let $S^{\mathcal{M}}$ denote the set of *multisets* of S, i.e., the set of all functions from S to the set of natural numbers ω, and by $S^{\mathcal{M}_\infty}$ the set of *multisets* with (possibly) *infinite multiplicities*, i.e., the functions from S to $\omega_\infty = \omega \cup \{\infty\}$. We keep using the notation $[\![\mu]\!]$ also for $\mu \in S^{\mathcal{M}_\infty}$ to denote the subset of S consisting of those elements s such that $\mu(s) > 0$.

Since the set of multisets does not give rise to a monad, some care must be used in order to define nets with infinite markings. Clearly, a multiset $\mu \in S^{\mathcal{M}_\infty}$ can be represented as a formal sum $\bigoplus_{s \in S} \mu(s) \cdot s$. Given an arbitrary index set I and $\{\eta_i \in \omega_\infty \mid i \in I\}$, we define $\Sigma_{i \in I} \eta_i$ to be the usual sum in ω if only finitely many η_i are nonzero and ∞ otherwise. Then, we can give meaning to linear combinations of multisets, i.e., multisets of multisets, by defining

$$\bigoplus_{\mu \in S^{\mathcal{M}_\infty}} \eta_\mu \cdot \mu = \bigoplus_{\mu \in S^{\mathcal{M}_\infty}} \eta_\mu \cdot \left(\bigoplus_{s \in S} \mu(s) \cdot s \right) = \bigoplus_{s \in S} \Big(\sum_{\mu \in S^{\mathcal{M}_\infty}} \eta_\mu \mu(s) \Big) \cdot s.$$

A $(\_)^{\mathcal{M}_\infty}$–homomorphism from $S_0^{\mathcal{M}_\infty}$ to $S_1^{\mathcal{M}_\infty}$ is a function $g\colon S_0^{\mathcal{M}_\infty} \to S_1^{\mathcal{M}_\infty}$ such that

$$g(\mu) = \bigoplus_{s\in S_0} \mu(s)\cdot g(1\cdot s),$$

where $1\cdot s$ is the formal sum corresponding to the function which yields 1 on s and zero otherwise. Actually, it is worth noticing that $(\_)^{\mathcal{M}_\infty}$ defines a monad on <u>Set</u> which sends S to $S^{\mathcal{M}_\infty}$, whose multiplication is the operation of linear combination of multisets and whose unit maps $s \in S$ to $1\cdot s$. In these terms, $S^{\mathcal{M}_\infty}$ is a $(\_)^{\mathcal{M}_\infty}$–algebra and a $(\_)^{\mathcal{M}_\infty}$–homomorphism is a homomorphism between $(\_)^{\mathcal{M}_\infty}$-algebras.

<u>NOTATION</u>. All the notations we used in the case of finite multisets are extended to $S^{\mathcal{M}}$ and $S^{\mathcal{M}_\infty}$. In particular, as in the case of $S^\oplus$, we shall regard $S^{\mathcal{M}}$ also as a pointed set whose pointed element is the empty multiset 0. We shall often denote a multiset $\mu \in S^{\mathcal{M}_\infty}$ by $\bigoplus_{i\in I}\eta_i s_i$ where $\{s_i \mid i \in I\} = [\![\mu]\!]$ and $\eta_i = \mu(s_i)$, i.e., as a sum whose summands are all nonzero. In case of multisets in $S^{\mathcal{M}}$, instead of η_i, we will use $n_i, m_i, \ldots$, the standard variables for natural numbers. Finally, given $S' \subseteq S$, $\bigoplus S'$ stands for for $\bigoplus_{s\in S'} 1\cdot s = \bigoplus_{s\in S'} s$.

DEFINITION 1.5.1 *(PT Nets)*
A PT net is a structure $N = \big(\partial_N^0, \partial_N^1\colon (T_N, 0) \to S_N^{\mathcal{M}}, u_N\big)$ *where* S_N *is a set of places;* T_N *is a pointed set of transitions;* $\partial_N^0, \partial_N^1$ *are pointed set morphisms; and* $u_N \in S_N^{\mathcal{M}}$ *is the initial marking. Moreover, we assume the standard constraint that* $\partial_N^0(t) = 0$ *if and only if* $t = 0$.

A morphism of PT nets from N_0 *to* N_1 *consists of a pair of functions* $\langle f, g\rangle$ *such that:*

i) $f\colon T_{N_0} \to T_{N_1}$ *is a pointed set morphism;*

ii) $g\colon S_{N_0}^{\mathcal{M}_\infty} \to S_{N_1}^{\mathcal{M}_\infty}$ *is a* $(\_)^{\mathcal{M}_\infty}$*–homomorphism;*

iii) $\partial_{N_1}^0 \circ f = g \circ \partial_{N_0}^0$ *and* $\partial_{N_1}^1 \circ f = g \circ \partial_{N_0}^1$, *i.e.,* $\langle f, g\rangle$ *respects source and target;*

iv) $g(u_{N_0}) = u_{N_1}$, *i.e.,* $\langle f, g\rangle$ *respects the initial marking;*

v) $\forall b \in [\![u_{N_1}]\!]$, $\exists! a \in [\![u_{N_0}]\!]$ *such that* $b \in [\![g(a)]\!]$
$\forall t \in T_{N_0}, \forall b \in [\![\partial_{N_1}^1(f(t))]\!]$, $\exists! a \in [\![\partial_{N_0}^1(t)]\!]$ *such that* $b \in [\![g(a)]\!]$.

This, with the obvious componentwise composition of morphisms, defines the category <u>PTNets</u>.

<u>TERMINOLOGY</u>. The reader will have noticed that we have used the name "PT nets" for both the objects of <u>Petri</u> and <u>PTNets</u>. In order not to confuse the reader, in the rest of this part, unless differently specified, PT net, or simply net, will indicate an object of <u>PTNets</u>.

A PT net is thus a graph whose arcs are the transitions and whose nodes are the multisets on the set of places, i.e., *markings* of the net. Transitions are restricted to have pre- and post-sets, i.e., sources and targets, in which each place has only finitely many tokens, i.e., finite multiplicity. The same is required for the initial marking. To be consistent with the use of zero transitions as a way to treat partial mappings, they are required to have empty pre- and post-sets. Moreover, they are the only transitions which can have empty pre-sets. Observe that <u>PTNets</u> contains also the empty net, i.e., in our setting the net with empty set of places and having the unique transition 0. This is an interesting net, since it is the terminal object in the category and can be useful for defining other nets recursively. The initial object of <u>PTNets</u> is the net consisting of a unique place s, of no transitions, and whose initial marking is $1 \cdot s$.

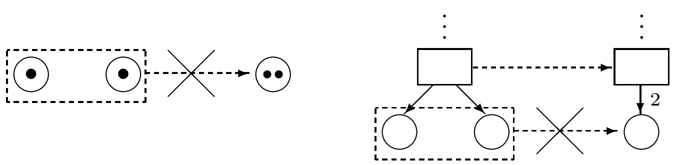
Morphisms of PT nets are graph morphisms in the precise sense of respecting source and target of transitions, i.e., they make the two rectangles obtained by choosing the upper or lower arrows in the parallel pairs of the diagram below

$$\begin{array}{ccccc}
T_{N_0} & \underset{\partial^1_{N_0}}{\overset{\partial^0_{N_0}}{\rightrightarrows}} & S^{\mathcal{M}}_{N_0} & \hookrightarrow & S^{\mathcal{M}_\infty}_{N_0} \\
\downarrow f & & & & \downarrow g \\
T_{N_1} & \underset{\partial^1_{N_1}}{\overset{\partial^0_{N_1}}{\rightrightarrows}} & S^{\mathcal{M}}_{N_1} & \hookrightarrow & S^{\mathcal{M}_\infty}_{N_1}
\end{array}$$

commute. Moreover they map initial markings to initial markings.

A $(\_)^{\mathcal{M}_\infty}$–homomorphism $g\colon S^{\mathcal{M}_\infty}_{N_0} \to S^{\mathcal{M}_\infty}_{N_1}$, which constitutes the place component of a morphism $\langle f, g\rangle$, is defined by its behaviour on S_{N_0}, the generators of $S^{\mathcal{M}_\infty}_{N_0}$. Therefore, as in the case of $(\_)^{\oplus}$ morphisms, we will often define morphisms by giving their transition components and simply a map $g\colon S_{N_0} \to S^{\mathcal{M}_\infty}_{N_1}$ for their place components, which should be considered lifted to $(\_)^{\mathcal{M}_\infty}$–homomorphisms.

The last condition in the definition means that morphisms are not allowed to map two different places in the initial marking or in the post-set of some transition to two multisets having a place in common. This is pictorially described in the figure below, where dashed arrows represent the forbidden morphisms and the initial marking is given by the number of "tokens" in the places. Recall that sources and targets are directed arcs whose weights represent multiplicities and that unitary weights are omitted.



Such a condition will play an important role while establishing the adjunction between PTNets and DecOcc. In fact, it is crucial for showing the *universality* of the *counit* of the adjunction. Moreover, it is the part of this condition concerning the places in the initial marking which makes coproducts exist in PTNets.

The set $S[N]$ of steps for a net N in PTNets is defined exactly as for the nets in Petri (see Section 1.1). Concerning step sequences, only the sequences leaving from the initial marking are considered. Thus, in this case, the set of step sequences, denoted as $\mathcal{SS}[N]$, is given by the rule:

$$\frac{u_N[\alpha_0\rangle v_0, \ldots, u_n[\alpha_n\rangle v_n \text{ in } S[N] \text{ and } u_i = v_{i-1},\ i = 1, \ldots, n}{u_N[\alpha_0\rangle[\alpha_1\rangle \cdots [\alpha_n\rangle v_n \text{ in } \mathcal{SS}[N]}.$$

The set $\mathcal{R}[N]$ of *reachable markings* of N is the set of markings which are target of some step sequence, i.e.,

$$\mathcal{R}[N] = \{v \mid \exists\, (u_N[\alpha_0\rangle \cdots [\alpha_n\rangle v) \text{ in } \mathcal{SS}[N]\}.$$

Since step sequences are of finite length, and each step consists of finitely many transitions, from the conditions on u_N, ∂_N^0 and ∂_N^1 in Definition 1.5.1, it is easy to see that $\mathcal{R}[N] \subseteq S_N^{\mathcal{M}}$. Now, we recall the definition of a well-known class of nets: *safe nets*.

Definition 1.5.2 *(Safe Nets)*
A PT net N is safe if and only if

$$\begin{aligned} \forall t \in T_N,\ \bigoplus [\![\partial_N^i(t)]\!] &= \partial_N^i(t),\ i = 0, 1, \quad \textit{and} \\ \forall v \in \mathcal{R}[N],\ \bigoplus [\![v]\!] &= v. \end{aligned}$$

This defines the category Safe *as a full subcategory of* PTNets.

Observe that $\bigoplus [\![v]\!] = v$ is a compact way of saying that each $s \in S$ has multiplicity at most one in v. Therefore this definition is exactly the classical definition of safe nets. Since in the rest of the chapter we will often state and check conditions on both ∂_N^0 and ∂_N^1, we will use ∂_N^i ranging over them.

Morphisms of safe nets have a nice characterization in terms of their behaviour on the elements of pre- and post-sets.

Proposition 1.5.3 *(Characterization of Safe Net Morphisms)*
Let N_0 and N_1 be safe nets. Then $\langle f, g\rangle: N_0 \to N_1$ is a morphism in Safe *if and only if f is a morphism of pointed sets from T_{N_0} to T_{N_1}, g is a $(\_)^{\mathcal{M}_\infty}$–homomorphism from $S_{N_0}^{\mathcal{M}_\infty}$ to $S_{N_1}^{\mathcal{M}_\infty}$ such that $\forall t \in T_{N_0}, \forall b \in [\![u_{N_0}]\!] \cup [\![\partial_{N_0}^i(t)]\!]$, $\bigoplus [\![g(b)]\!] = g(b)$, and*

i) $[\![g(u_{N_0})]\!] \subseteq [\![u_{N_1}]\!]$ *and* $\forall b_1 \in [\![u_{N_1}]\!],\ \exists! b_0 \in [\![u_{N_0}]\!]$ *such that* $b_1 \in [\![g(b_0)]\!]$*;*

ii) $\forall t \in T_{N_0},\ [\![g(\partial^i_{N_0}(t))]\!] \subseteq [\![\partial^i_{N_1}(f(t))]\!]$ *and*
$\forall b_1 \in [\![\partial^i_{N_1}(f(t))]\!],\ \exists! b_0 \in [\![\partial^i_{N_0}(t)]\!]$ *such that* $b_1 \in [\![g(b_0)]\!]$.

Proof. ($\Rightarrow$) The proof is completely trivial, but observe that there is no need to use condition *(v)* in the definition of PT net morphisms: in fact if there were b_1 in $[\![u_{N_1}]\!]$ or $[\![\partial^i_{N_1}(f(t))]\!]$ and b, b' respectively in $[\![u_{N_0}]\!]$ or $[\![\partial^i_{N_0}(t)]\!]$ such that $b_1 \in [\![g(b)]\!] \cap [\![g(b')]\!]$, by definition of morphism, it would be $g(u_{N_0})(b_1) = u_{N_1}(b_1) \geq 2$ or $g(\partial^i_{N_0}(t))(b_1) = \partial^i_{N_1}(f(t))(b_1) \geq 2$. But this is impossible, since N_1 is a safe net.

($\Leftarrow$) Conditions *(i)*, *(ii)* and *(v)* in the definition of PT net morphisms are already present. Points *(i)* and *(ii)* above imply that

$$[\![u_{N_1}]\!] = \bigcup \{ [\![g(a)]\!] \mid a \in [\![u_{N_0}]\!] \} \text{ and } [\![\partial^i_{N_1}(f(t))]\!] = \bigcup \left\{ [\![g(a)]\!] \mid a \in [\![\partial^i_{N_0}(t)]\!] \right\}.$$

Now, since $\bigoplus [\![g(a)]\!] = g(a)$ and all the $[\![g(a)]\!]$ in the unions are disjoint, we obtain $g(u_{N_0}) = u_{N_1}$ and $g(\partial^i_{N_0}(t)) = \partial^i_{N_1}(f(t))$. ✓

Corollary 1.5.4 *(Correspondence with Winskel's Safe Nets)*
Winskel's category of safe nets [143], called <u>Net</u>*, is a full subcategory of* <u>Safe</u>.

Proof. The conditions given in the above proposition are a characterization of morphisms in <u>Net</u> [143, Proposition 3.1.9], while the objects in <u>Safe</u> strictly contain the objects in <u>Net</u>. In fact, the objects of <u>Net</u> are the objects of <u>Safe</u> with sets of places, initial markings and post-sets which are non-empty, and without *isolated places*—places belonging neither to the initial marking nor to the pre- or post-set of any transition. ✓

Since in the proof of Proposition 1.5.3 the demonstration of the $\Rightarrow$ implication never used the fifth axiom of PT net morphisms, we have shown an easier characterization of morphisms between safe nets: $\langle f, g\rangle: N_0 \to N_1$ is a morphism in <u>Safe</u> between the safe nets N_0 and N_1 if and only if conditions *(i)*, *(ii)*, *(iii)* and *(iv)* of Definition 1.5.1 hold, i.e., condition *(v)* is already implied by the structure of safe nets.

Another important class of nets is that of *occurrence nets*. They are safe nets with a nice stratified structure whose minimal element constitute the initial marking. The definition is very close to that of process nets, the difference being that here we allow "forward branching", i.e., non-determinism.

Definition 1.5.5 *(Occurrence Nets)*
A occurrence net is a safe net Θ *such that*

i) $a \in [\![u_\Theta]\!]$ *if and only if* ${}^\bullet a = \varnothing$*;*

ii) *for all* $a \in S_\Theta$, $|{}^\bullet a| \leq 1$, *where* $|\_|$ *gives the cardinality of sets;*

iii) $\prec$ *is irreflexive, where* $\prec$ *is the transitive closure of the relation*

$$\prec^1 = \{(a,t) \mid a \in S_\Theta,\, t \in T_\Theta,\, t \in a^\bullet\} \cup \{(t,a) \mid a \in S_\Theta,\, t \in T_\Theta,\, t \in {}^\bullet a\};$$

moreover, for all $t \in T_\Theta$, $\{t' \in T_\Theta \mid t' \prec t\}$ *is finite;*

iv) *the binary "conflict" relation* $\#$ *on* $T_\Theta \cup S_\Theta$ *is irreflexive, where*

$$\forall t_1, t_2 \in T_\Theta,\ t_1 \#_m t_2 \Leftrightarrow [\![\partial^0_\Theta(t_1)]\!] \cap [\![\partial^0_\Theta(t_2)]\!] \neq \varnothing \text{ and } t_1 \neq t_2,$$

$$\forall x, y \in T_\Theta \cup S_\Theta,\ x \# y \Leftrightarrow \exists t_1, t_2 \in T_\Theta:\ t_1 \#_m t_2 \text{ and } t_1 \preccurlyeq x \text{ and } t_2 \preccurlyeq y,$$

where $\preccurlyeq$ *is the reflexive closure of* $\prec$.

This defines the category <u>Occ</u> *as a full subcategory of* <u>Safe</u>.

From Definition 1.5.5 and Corollary 1.5.4, it is immediate to see that Winskel's category of occurrence nets [143], here called $\underline{\mathsf{Occ}}_W$, is a full subcategory of <u>Occ</u>. However, since all the results in [143] easily extend to <u>Safe</u> and <u>Occ</u>, in the following we will ignore any difference between <u>Safe</u> and <u>Net</u> and between <u>Occ</u> and $\underline{\mathsf{Occ}}_W$.

COMPOSITION OF PT NETS

Products and coproducts are importante constructions for nets, and generally in categories of models for concurrency, due to their natural role, respectively, in the operations of parallel and non-deterministic composition [144]. In this section, we show that the category <u>PTNets</u> has both products and coproducts and, studying the relationships between the computations of the composed nets and those of the original nets, we clarify in what sense products and coproducts are related to the operations of parallel and non-deterministic composition.

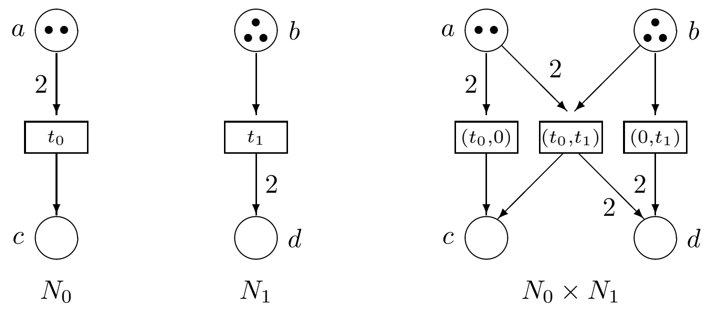
While in the categories of safe nets and of occurrence nets products and coproducts exist [144], the category of PT nets with initial markings introduced in [144] has products but does not have coproducts. In [97], it is shown that coproducts exist in the full subcategory of PT nets whose initial markings are sets rather than multisets. However, due to the additional condition *(v)* in Definition 1.5.1, we can prove the existence of coproducts of any pair of objects in <u>PTNets</u>.

Given the PT nets N_0 and N_1, we define

$$N_0 \times N_1 = \left(\partial^0_{N_0} \times \partial^0_{N_1}, \partial^1_{N_0} \times \partial^1_{N_1} \colon (T_{N_0} \times T_{N_1}, (0,0)) \to S^{\mathcal{M}}_{N_0} \times S^{\mathcal{M}}_{N_1}, (u_{N_0}, u_{N_1})\right).$$

Since $S^{\mathcal{M}}_{N_0} \times S^{\mathcal{M}}_{N_1} \cong (S_{N_0} + S_{N_1})^{\mathcal{M}}$, and $S^{\mathcal{M}_\infty}_{N_0} \times S^{\mathcal{M}_\infty}_{N_1} \cong (S_{N_0} + S_{N_1})^{\mathcal{M}_\infty} \cong S^{\mathcal{M}_\infty}_{N_0} \oplus S^{\mathcal{M}_\infty}_{N_1}$, where $+$ is the disjoint union of sets and $\oplus$ the coproduct in the category of multisets (with possibly infinite multiplicities) and $(\_)^{\mathcal{M}_\infty}$–homomorphisms, we have that $N_0 \times N_1$ is indeed a net with places $S_{N_0} + S_{N_1}$.

Example 1.5.6



The PT nets N_0, N_1, and their product $N_0 \times N_1$

Now, consider the projections $\pi_0\colon N_0 \times N_1 \to N_0$ and $\pi_1\colon N_0 \times N_1 \to N_1$ defined by

$$\pi_i\left((t_0, t_1)\right) = t_i \quad \text{and} \quad \pi_i\left((u_0, u_1)\right) = u_i.$$

It is easy to see that they are morphisms in $\underline{\mathsf{PTNets}}$.

Proposition 1.5.7 *(Product of Nets)*
$N_0 \times N_1$, with projections π_0 and π_1, is the product of N_0 and N_1 in $\underline{\mathsf{PTNets}}$.

Proof. Observe that, given any PT net N and morphisms $h_0\colon N \to N_0$ and $h_1\colon N \to N_1$, the map $\langle h_0, h_1\rangle\colon N \to N_0 \times N_1$ defined by

$$\langle h_0, h_1\rangle(t) = \big(h_0(t), h_1(t)\big) \text{ and } \langle h_0, h_1\rangle(u) = \big(h_0(u), h_1(u)\big)$$

is a PT net morphism. So, clearly, $\pi_i \circ \langle h_0, h_1\rangle = h_i$, and $\langle h_0, h_1\rangle$ is the unique morphism for which that happens. ✓

The product of the nets N_0 and N_1 is their *parallel* composition with *synchronization*, in the precise sense that each step sequence of $N_0 \times N_1$ is the parallel composition of a step sequence of N_0 and a step sequence of N_1, and viceversa. Since transitions of $N_0 \times N_1$ are of the forms $(t_0, 0)$, $(0, t_1)$ or (t_0, t_1), for $t_i \in T_{N_i}$, $i = 0, 1$, the product models both *asynchronous* and *synchronous* interactions of N_0 and N_1, where transitions of the form $(t_0, 0)$ or $(0, t_1)$ correspond to either N_1 or N_0 staying idle, while transitions of the form (t_0, t_1) correspond to steps in which both N_0 and N_1 proceed together, synchronizing to each other. This result, formally stated in the next proposition, coincides with those in [144, 97].

In the following, given a PT net morphism $h\colon N_0 \to N_1$, we will denote by $h^\oplus$ the unique $(\_)^{\mathcal{M}_\infty}$–homomorphism from $T_{N_0}^{\mathcal{M}_\infty}$ to $T_{N_1}^{\mathcal{M}_\infty}$ generated by the transition component of h. Observe that, since such a component is a function, $h^\oplus$ maps finite multisets to finite multisets. In particular, $h^{\oplus}$ can be defined on such multisets simply by:

$$h^{\oplus}(t) = h(t) \text{ and } h^{\oplus}(\alpha \oplus \beta) = h^{\oplus}(\alpha) \oplus h^{\oplus}(\beta).$$

Proposition 1.5.8 *(Product and Parallel Composition)*
The sequence $u_{N_0 \times N_1}[\alpha_0\rangle \cdots [\alpha_n\rangle v$ *belongs to* $\mathcal{SS}[N_0 \times N_1]$ *if and only if, for* $i = 0, 1$, *the sequence* $\pi_i(u_{N_0 \times N_1})[\pi_i^{\oplus}(\alpha_0)\rangle \cdots [\pi_i^{\oplus}(\alpha_n)\rangle \pi_i(v)$ *belongs to* $\mathcal{SS}[N_i]$.

Therefore, $v \in \mathcal{R}[N_0 \times N_1]$ *if and only if* $\pi_i(v) \in \mathcal{R}[N_i]$ *for* $i = 0, 1$.

Proof. It suffices to show that $u[\alpha\rangle v$ is in $\mathcal{S}[N_0 \times N_1]$ if and only if $\pi_i(u)[\pi_i^{\oplus}(\alpha)\rangle \pi_i(v)$ is in $\mathcal{S}[N_i]$, $i = 0, 1$.

Suppose $\alpha = \bigoplus_{j \in J}(t_0^j, t_1^j)$. The 'only if' implication follows directly from the fact that π_0 and π_1 are PT net morphisms. In fact, by definition of $\mathcal{S}[N]$, $u[\alpha\rangle v$ in $\mathcal{S}[N_0 \times N_1]$ if and only if $u = w \oplus \bigoplus_{j \in J} u_j$, $v = w \oplus \bigoplus_{j \in J} v_j$, and the $(t_0^j, t_1^j) \colon u_j \to v_j$ are transitions of $N_0 \times N_1$. Thus, we have that $\pi_i(t_0^j, t_1^j) \colon \pi_i(u_j) \to \pi_i(v_j)$ is a transition (possibly 0) of N_i, $i = 0, 1$, $j \in J$. Therefore, for $i = 0, 1$,

$$\Big(\pi_i(w) \oplus \bigoplus_{j \in J} \pi_i(u_j)\Big) \Big[\bigoplus_{j \in J} \pi_i(t_0^j, t_1^j)\Big\rangle \Big(\pi_i(w) \oplus \bigoplus_{j \in J} \pi_i(v_j)\Big) \text{ is in } \mathcal{S}[N_i],$$

i.e., $\pi_i(u)[\pi_i^{\oplus}(\alpha)\rangle \pi_i(v)$ belongs to $\mathcal{S}[N_i]$, $i = 0, 1$.

In order to show the 'if' implication, observe that $t_0 \colon \pi_0(u) \to \pi_0(v)$ in N_0 and $t_1 \colon \pi_1(u) \to \pi_1(v)$ in N_1 implies $(t_0, t_1) \colon u \to v$ in $N_0 \times N_1$. In fact, by definition $(t_0, t_1) \colon (\pi_0(u), \pi_1(u)) \to (\pi_0(v), \pi_1(v))$, and since $S_{N_0 \times N_1}^{\mathcal{M}_\infty}$ is the product of $S_{N_0}^{\mathcal{M}_\infty}$ and $S_{N_1}^{\mathcal{M}_\infty}$ with projections the place components of π_0 and π_1, we have $(\pi_0(u), \pi_1(u)) = u$ and $(\pi_0(v), \pi_1(v)) = v$.

Assume $\pi_i(u)[\pi_i^{\oplus}(\alpha)\rangle \pi_i(v)$ in $\mathcal{S}[N_i]$, $i = 0, 1$, and $u = (u^0, u^1), v = (v^0, v^1)$. Then, by definition of $\mathcal{S}[N_i]$, we have that $u^i = w^i \oplus \bigoplus_{j \in J} u_j^i$, $v^i = w^i \oplus \bigoplus_{j \in J} v_j^i$ and $\pi_i(t_0^j, t_1^j) \colon u_j^i \to v_j^i$ in N_i, $i = 0, 1$, $j \in J$. Now consider $u_j = (u_j^0, u_j^1)$, $v_j = (v_j^0, v_j^1)$ and $w = (w^0, w^1)$. Clearly, we have $u = w \oplus \bigoplus_{j \in J} u_j$, $v = w \oplus \bigoplus_{j \in J} v_j$ and $\pi_i(t_0^j, t_1^j) \colon \pi_i(u_j) \to \pi_i(v_j)$ in N_i. Therefore, $(t_0^j, t_1^j) \colon u_j \to v_j$ in N_i, $i = 0, 1$, $j \in J$ and we have

$$\Big(w \oplus \bigoplus_{j \in J} u_j\Big) \Big[\bigoplus_{j \in J} (t_0^j, t_1^j)\Big\rangle \Big(w \oplus \bigoplus_{j \in J} v_j\Big) \text{ in } \mathcal{S}[N_0 \times N_1],$$

i.e., $u[\alpha\rangle v$ in $\mathcal{S}[N_0 \times N_1]$. ✓

Example 1.5.9 *(Parallel Computations)*
Consider again the nets of Example 1.5.6.

The step $(2a, 3b)[(t_0, t_1)\rangle (c, 2b \oplus 2d)$ *of* $N_0 \times N_1$ *corresponds to the steps*

$$2a[t_0\rangle c \text{ of } N_0 \text{ and } 3b[t_1\rangle 2b \oplus 2d \text{ of } N_1,$$

$(2a, 3b)[(t_0, 0)\rangle [(0, t_1)\rangle (c, 2b \oplus 2d)$ *corresponds to* $2a[t_0\rangle [0\rangle c$ *and* $3b[0\rangle [t_1\rangle 2b \oplus 2d$.

We now consider coproducts in PTNets. Suppose $u_{N_0} = \bigoplus_i n_i a_i$ and $u_{N_1} = \bigoplus_j m_j b_j$. Consider $S = (S_{N_0} - [\![u_{N_0}]\!]) + (S_{N_1} - [\![u_{N_1}]\!]) + ([\![u_{N_0}]\!] \times [\![u_{N_1}]\!])$ and the $(\_)^{\mathcal{M}_\infty}$–homomorphisms $\alpha_l : ([\![u_{N_l}]\!])^{\mathcal{M}_\infty} \to ([\![u_{N_0}]\!] \times [\![u_{N_1}]\!])^{\mathcal{M}_\infty}$, $l = 0,1$, defined by

$$\begin{aligned} \alpha_0(a_i) &= \bigoplus_j \frac{\mathrm{lcm}(n_i, m_j)}{n_i}(a_i, b_j) \\ \alpha_1(b_j) &= \bigoplus_i \frac{\mathrm{lcm}(n_i, m_j)}{m_j}(a_i, b_j), \end{aligned}$$

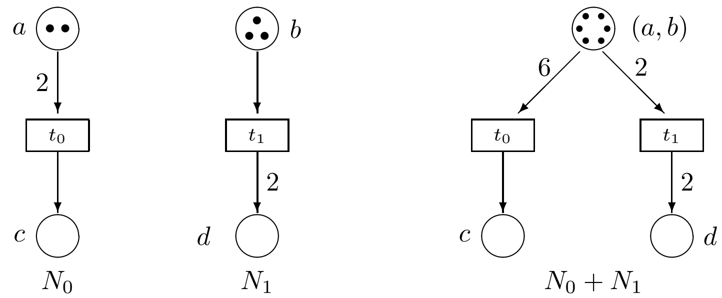
where $\mathrm{lcm}(n_i, m_j)$ is the lowest common multiple of n_i and m_j.

Let us define $\gamma_i = (\alpha_i \oplus \beta_i) : S_{N_i}^{\mathcal{M}_\infty} \to S^{\mathcal{M}_\infty}$, $i = 0,1$, where β_i is the injection of $(S_{N_i} - [\![u_{N_i}]\!])^{\mathcal{M}_\infty}$ in $(S_{N_0} - [\![u_{N_0}]\!] + S_{N_1} - [\![u_{N_1}]\!])^{\mathcal{M}_\infty}$. Observe that γ_i restricts to a pointed set morphism from $S_{N_i}^{\mathcal{M}}$ to $S^{\mathcal{M}}$. Let $\delta_{N_j}^i$ be $\gamma_j \circ \partial_{N_j}^i : (T_{N_j}, 0) \to S^{\mathcal{M}}$, for $i, j = 0,1$. Now, define

$$N_0 + N_1 = \left([\delta_{N_0}^0, \delta_{N_1}^0], [\delta_{N_0}^1, \delta_{N_1}^1] : (T, 0) \to S^{\mathcal{M}}, \gamma_0(u_{N_0}) = \gamma_1(u_{N_1})\right),$$

where $(T, 0)$ is the coproduct of pointed sets $(T_{N_0}, 0)$ and $(T_{N_1}, 0)$, i.e., the quotient of their disjoint union obtained by identifying the two pointed elements, and $[\delta_{N_0}^i, \delta_{N_1}^i]$ denotes the unique pointed set morphism induced from the coproduct $(T, 0)$ by $\delta_{N_0}^i$ and $\delta_{N_1}^i$.

Example 1.5.10



The PT nets N_0, N_1, and their coproduct $N_0 + N_1$

The injections $in_i : N_i \to N_0 + N_1$, $i = 0,1$, are defined as

$$in_i = \langle \kappa_i, \gamma_i \rangle,$$

where κ_i is the injection of $(T_{N_i}, 0)$ in $(T, 0)$. It is immediate to see that the in_i are PT net morphisms.

Given any PT net N and a pair of morphisms $h_0 \colon N_0 \to N$ and $h_1 \colon N_1 \to N$, let $[h_0, h_1] \colon N_0 + N_1 \to N$ be the map such that

$$
\begin{array}{rcll}
[h_0,h_1](t) & = & h_i(t') & \text{if } t = in_i(t') \text{ for } t' \in T_{N_i} \\
[h_0,h_1](c) & = & h_i(c') & \text{if } c = in_i(c') \text{ for } c' \in S_{N_i} - [\![u_{N_i}]\!] \\
[h_0,h_1](a_i,b_j) & = & \bigoplus \left\{ \dfrac{n_k}{\mathrm{lcm}(n_i,m_j)} c_k \;\middle|\; c_k \in [\![h_0(a_i)]\!] \cap [\![h_1(b_j)]\!] \right\} &
\end{array}
$$

where n_k is the coefficient of c_k in u_N. To simplify the notation, in the proof of the following proposition we will denote $[\![h_0(a_i)]\!] \cap [\![h_1(b_j)]\!]$ by $[\![\Im(a_i,b_j)]\!]$.
Remember that $u \in S^{\mathcal{M}}$ is a function from S to ω. Therefore, $u(a)$, for $a \in S$, is the multiplicity of a in u.

PROPOSITION 1.5.11 *(Coproducts of Nets)*
$N_0 + N_1$, with injections in_0 and in_1, is the coproduct of N_0 and N_1 in <u>PTNets</u>.

Proof. We show that for any PT net N and for any pair of morphisms $h_0 \colon N_0 \to N$, $h_1 \colon N_1 \to N$, $[h_0,h_1]$ is the unique morphism in <u>PTNets</u> such that $[h_0,h_1] \circ in_i = h_i$. First we have to show that $[h_0,h_1]$ is well-defined, i.e., that $\dfrac{n_k}{\mathrm{lcm}(n_i,m_j)}$ is actually a natural number. If $c_k \in [\![h_0(a_i)]\!]$ then $h_0(a_i) = r_k c_k \oplus u$ and so $h_0(n_i a_i) = n_i r_k c_k \oplus n_i u$. Thus, by definition of PT net morphisms, we know that $u_N(c_k) = n_i r_k$ and so it must be $n_i r_k = n_k$. In the same way, there exists q_k such that $m_j q_k = n_k$. Therefore n_k is divisible by $\mathrm{lcm}(n_i,m_j)$.

Now, observe that $[h_0,h_1] \circ in_i = h_i$. This is clear for transitions and for places in $S_{N_i} - [\![u_{N_i}]\!]$. So, consider $a_i \in [\![u_{N_0}]\!]$. We have

$$
\begin{array}{rcl}
[h_0,h_1](in_0(a_i)) & = & [h_0,h_1] \left(\bigoplus_j \dfrac{\mathrm{lcm}(n_i,m_j)}{n_i}(a_i,b_j) \right) \\
& = & \bigoplus_j \dfrac{\mathrm{lcm}(n_i,m_j)}{n_i} [h_0,h_1]((a_i,b_j)) \\
& = & \bigoplus_j \dfrac{\mathrm{lcm}(n_i,m_j)}{n_i} \bigoplus \left\{ \dfrac{n_k}{\mathrm{lcm}(n_i,m_j)} c_k \;\middle|\; c_k \in [\![\Im(a_i,b_j)]\!] \right\} \\
& = & \bigoplus_j \left\{ \frac{n_k}{n_i} c_k \;\middle|\; c_k \in [\![\Im(a_i,b_j)]\!] \right\}.
\end{array}
$$

Since for each $c_k \in [\![h_0(a_i)]\!]$ there exists a unique b_j such that $c_k \in [\![h_1(b_j)]\!]$, the last term is equal to

$$
\bigoplus \left\{ \frac{n_k}{n_i} c_k \;\middle|\; c_k \in [\![h_0(a_i)]\!] \right\} = \bigoplus \left\{ r_k c_k \mid h_0(a_i) = r_k c_k \oplus u' \right\} = h_0(a_i).
$$

The same argument goes through for $b_j \in [\![u_{N_1}]\!]$.

To prove uniqueness, suppose that there exists h such that $h \circ in_i = h_i$. Clearly, $h = [h_0,h_1]$ on the transitions and on places in $S_{N_0} - [\![u_{N_0}]\!]$ and in $S_{N_1} - [\![u_{N_1}]\!]$. Therefore, in order to show that h coincides with $[h_0,h_1]$ we need to show that it does so for $(a_i, b_j) \in [\![u_{N_0}]\!] \times [\![u_{N_1}]\!]$. Since $h(in_0(a_i)) = h_0(a_i)$ and h is a morphism, we have

$$\begin{aligned}
\bigoplus_j \frac{\mathrm{lcm}(n_i, m_j)}{n_i} h(a_i, b_j) &= \bigoplus \left\{ \tfrac{n_k}{n_i} c_k \;\middle|\; c_k \in [\![h_0(a_i)]\!] \right\} \\
&= \bigoplus_j \left\{ \tfrac{n_k}{n_i} c_k \;\middle|\; c_k \in [\![\Im(a_i, b_j)]\!] \right\} \\
&= \bigoplus_j \frac{\mathrm{lcm}(n_i, m_j)}{n_i} \\
&\qquad \cdot \left(\bigoplus \left\{ \frac{n_k}{\mathrm{lcm}(n_i, m_j)} c_k \;\middle|\; c_k \in [\![\Im(a_i, b_j)]\!] \right\} \right).
\end{aligned}$$

In the same way, we obtain that

$$\bigoplus_i \frac{\mathrm{lcm}(n_i, m_j)}{m_j} h(a_i, b_j) = \bigoplus_i \frac{\mathrm{lcm}(n_i, m_j)}{m_j} \bigoplus \left\{ \frac{n_k}{\mathrm{lcm}(n_i, m_j)} c_k \middle| c_k \in [\![\Im(a_i, b_j)]\!] \right\}.$$

Now fix i and j. Since $c_k \in [\![\Im(a_i, b_j)]\!]$ for a unique pair (a_i, b_j), the summands in the above equalities are all distinct except for $\bigoplus\{\frac{n_k}{\mathrm{lcm}(n_i, m_j)} c_k \mid c_k \in [\![\Im(a_i, b_j)]\!]\}$, which appears in both. Therefore it must be

$$h(a_i, b_j) = \bigoplus \left\{ \frac{n_k}{\mathrm{lcm}(n_i, m_j)} c_k \;\middle|\; c_k \in [\![\Im(a_i, b_j)]\!] \right\} \text{ which is } [h_0, h_1](a_i, b_j).$$

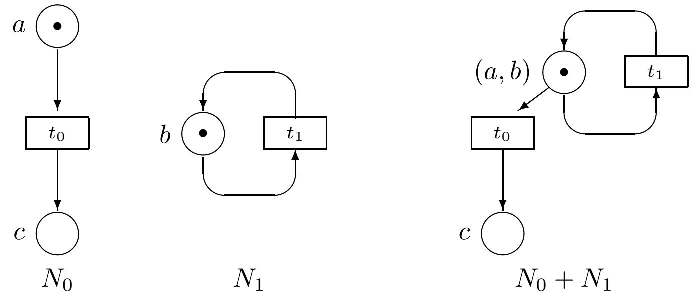
The last thing we have left to show is that $[h_0, h_1]$ is a morphism in PTNets. But now this task is trivial and is therefore omitted. ✓

The coproduct of N_0 and N_1 is their *non-deterministic* composition in the sense that the two nets are put side by side to compete for common resources (tokens).

Differently from the CCS non-deterministic operator [102], the result of such a composition cannot be thought of simply as the system which performs an initial choice between passing the control to N_0 or to N_1 and discards the net which has not been chosen. Nevertheless, we think that it gives the right notion of non-deterministic composition of PT nets. In fact, since a resource can be consumed and produced several times during a single computation, it is possible that the composed net returns several times to a state in which common resources are present and the two nets compete for them. Clearly, there is no reason why the outcome of such competitions should always favor the same net. This is illustrated by the following example.

Example 1.5.12
Consider the simple PT nets N_0 and N_1 given in the picture below together with

their coproduct $N_0 + N_1$.

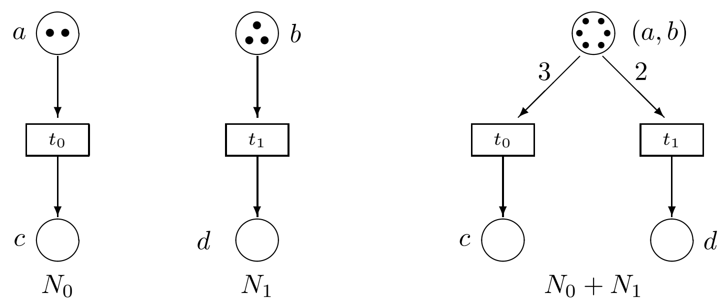


The initial marking (a, b) *of* N_0+N_1 *is a state in which a resource is non-deterministically assigned either to* t_0 *or to* t_1. *This state is reached again and again, and each time the choice is repeated.*

This kind of behaviour is characteristic of the coproduct in many categories of models which admit cyclic behaviours, like, for instance, transition systems.

In addition, since the resources are present in multiple instances (multiple tokens in a place), while the computations of N_0 and N_1 are also computations of N_0+N_1, they are not the only computations that the coproduct net can perform: the non-deterministic *interaction* between N_0 and N_1 gives rise to joint computations which are not purely *injections* of computations from one of the original nets. In other words, since providing N_0 with the resources it needs does not necessarily consume all the available instances of such resources, it is possible that N_1 can also have, at the same time, other instances of the same resources. This is shown by the following example.

EXAMPLE 1.5.13 *(Non-deterministic Computations)*



The steps $6(a, b)[2t_0\rangle 2c$ *and* $6(a, b)[3t_1\rangle 3d$ *of* $N_0 + N_1$ *correspond, respectively, to the step* $2a[2t_0\rangle 2c$ *of* N_0 *and to the step* $3b[3t_1\rangle 3d$ *of* N_1.

The step $6(a,b)[t_0 \oplus t_1\rangle c \oplus d$ is a computation of $N_0 + N_1$ which is not the image of a computation in one of the original nets.

However, as anticipated above, all the computations which either N_0 or N_1 can perform are computations which $N_0 + N_1$ can perform; viceversa, all the computations of N_0+N_1 consisting of markings and steps from N_i are actually computations of N_i. This is stated in the next proposition, whose proof simply follows from the fact that in_0 and in_1 are PT net morphisms and is, therefore, omitted.

Proposition 1.5.14 *(Coproduct and Non-deterministic Composition)*
Let i belong to $\{0,1\}$. Then, the step sequence

$$in_i(u_{N_i})[in_i^{\oplus}(\alpha_0)\rangle \cdots [in_i^{\oplus}(\alpha_n)\rangle in_i(v)$$

belongs to $\mathcal{SS}[N_0 + N_1]$ if and only if the step sequence $u_{N_i}[\alpha_0\rangle \cdots [\alpha_n\rangle v$ belongs to $\mathcal{SS}[N_i]$.

To strengthen the intuition about the coproduct construction, it is worth recalling that in the case of safe nets all the resources are present in a unique copy. This fact can be thought of as forcing a choice between the two nets in the assignment of resources. Therefore, for safe nets, the computations of $N_0 + N_1$ are alternating sequences of computations of the original nets, i.e., each step is either a step of N_0 or a step of N_1. This is stated in the next proposition, which is a rephrasing in the present context of [144, Theorem 5.11, pg. 219] and whose proof is, therefore, omitted.

Proposition 1.5.15 *(Coproduct and Safe Nets)*
Let N_0 and N_1 be safe nets.

Then $u[\alpha\rangle v$ belongs to $\mathcal{S}[N_0 + N_1]$, for some $u \in \mathcal{R}[N_0 + N_1]$, if and only if there exist i in $\{0,1\}$, u' in $\mathcal{R}[N_i]$ and $(u'[\alpha'\rangle v')$ in $\mathcal{S}[N_i]$ such that $in_i(u') = u$, $in_i(v') = v$ and $in_i^{\oplus}(\alpha') = \alpha$.

Therefore, all the step sequences of $N_0 + N_1$ are of the form

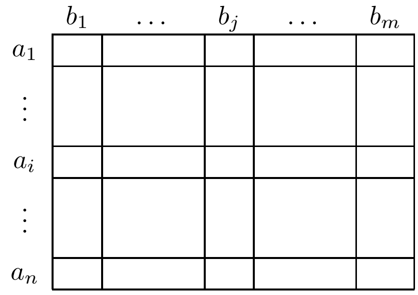
$$in_{i_0}(u_{N_{i_0}})[in_{i_0}^{\oplus}(\alpha_0)\rangle[in_{i_1}^{\oplus}(\alpha_1)\rangle \cdots [in_{i_{(k-1)}}^{\oplus}(\alpha_{(k-1)})\rangle[in_{i_k}^{\oplus}(\alpha_k)\rangle in_{i_k}(v_{i_k}),$$

where $i_1,\ldots,i_k \in \{0,1\}$, $v_{i_k} \in \mathcal{R}[N_{i_k}]$, and α_j is a step of N_{i_j}, $j = 1,\ldots,k$.

Then, $u \in \mathcal{R}[N_0 + N_1]$ if and only if $u = in_i(u_i)$ for $u_i \in \mathcal{R}[N_i]$ and $i \in \{0,1\}$.

It is interesting to observe how in this case the standard coproduct construction actually implements a sophisticated mechanism of *distributed* choice. Consider two safe nets N_0 and N_1 whose initial markings are respectively $a_1 \oplus \cdots \oplus a_n$ and

$b_1 \oplus \cdots \oplus b_m$. Then, the initial marking of $N_0 + N_1$ can be thought of as an $n \times m$ matrix whose (i, j)-th entry represents the token (a_i, b_j).



From the definition of $\partial^0_{N_0+N_1}$, it is immediate to see that if a_i is the pre-set of a transition t_0 in N_0, then the pre-set of t_0 in $N_0 + N_1$ contains $(a_i, b_1) \oplus \cdots \oplus (a_i, b_m)$, i.e., a whole row of the matrix. Now, since a transition t_0 of N_0 enabled at u_{N_0} requires at least one of the tokens in u_{N_0} in order to fire, say a_i, the firing of t_0 in $N_0 + N_1$ will result in consuming all the tokens in the i-th row of the matrix. It follows that no transition of N_1 can be enabled, since for any $j = 1, \ldots, m$, the token (a_i, b_j) is missing. Therefore, the firing of t_0 prevents any transition of N_1 from firing until the possible cyclic behaviour of $N_0 + N_1$ eventually generates again the tokens in $u_{N_0+N_1}$.

We conclude this discussion about coproducts considering the case of occurrence nets. Since cyclic behaviours are not possible in occurrence nets, the coproduct net, after having performed the first step, cannot reach anymore a state in which common resources are available. In this case, therefore, the coproduct net can be seen as the system which performs an initial choice between the original nets—by assigning to one of them the resources it needs—and forgets about the other. This is formally stated in the following proposition.

Proposition 1.5.16 *(Coproduct and Occurrence Nets)*
Let Θ_0 and Θ_1 be occurrence nets.
Then, all the step sequences of $\Theta_0 + \Theta_1$ from the initial marking are of the form

$$in_i(u_{\Theta_i})[in_i^{\oplus}(\alpha_0)\rangle \cdots [in_i^{\oplus}(\alpha_k)\rangle in_i(v_i),$$

where $i \in \{0, 1\}$, $v_i \in \mathcal{R}[\Theta_i]$, and α_j is a step of Θ_i, $j = 1, \ldots k$.

We conclude this section with some remarks about the relationships between products and coproducts in the other categories of nets introduced in Section 1.5.

It is easy to see that products and coproducts of *safe* nets viewed as objects in PTNets are again safe nets. Therefore, we have that products and coproducts exist in $\underline{\mathsf{Safe}}$ and that they are given by the same constructions just defined for PT nets. The same applies for *coproducts* in $\underline{\mathsf{Occ}}$.

However, the product of two occurrence nets in $\underline{\mathsf{PTNets}}$ is *not* necessarily an occurrence net. This can be easily seen by looking back at Example 1.5.6, which shows that condition *(ii)* in Definition 1.5.5 of occurrence nets is not preserved by the product construction. Nevertheless, products exist in $\underline{\mathsf{Occ}}$ and the result that the *product* of two occurrence nets is (isomorphic to) the *unfolding* of their product as safe nets [144] can be immediately extended to our setting: considering that the unfolding of PT nets defined here coincides on safe nets with Winskel's (see the following Theorem 1.7.8), we have that the product of two occurrence nets is (isomorphic to) the unfolding of their product in $\underline{\mathsf{PTNets}}$.

1.6 Decorated Occurrence Nets

In this section, we introduce $\underline{\mathsf{DecOcc}}$, the category of *decorated occurrence nets*, a type of occurrence nets in which places are grouped into families. They allow a convenient treatment of multiplicity issues in the unfolding of PT nets. We will use the following notational conventions:

$[n,m]$	for the segment $\{n,\dots,m\}$ of ω;
$[n]$	for $[1,n]$;
$[k]_i$	for the i-th block of length k of $\omega - \{0\}$, i.e., $[ik] - [(i-1)k]$.

Definition 1.6.1 *(Block Functions)*
We call a function $f\colon[n] \to [m]$ a block function if $n = km$ and $f([k]_i) = \{i\}$, for $i = 1,\dots,m$.

In other words, a block function from $[n] = [km]$ to $[m]$ is a function making the diagram

$$
\begin{array}{ccccccc}
 & & \multicolumn{5}{c}{\overbrace{\qquad\qquad\qquad\qquad}^{m \text{ times}}} \\
[n] & \cong & [k] & + & \cdots & + & [k] \\
{\scriptstyle f}\big\downarrow & & {\scriptstyle !}\big\downarrow & & & & \big\downarrow{\scriptstyle !} \\
[m] & \cong & [1] & + & \cdots & + & [1]
\end{array}
$$

commute, where the upper isomorphism maps the segment $[k]_i$ to the i-th copy of $[k]$, and the lower maps i to the i-th copy of $[1]$.

The place component g of a PT net morphism $\langle f,g\rangle\colon N_0 \to N_1$ can be thought of as a *multirelation* (with possibly infinite multiplicities) between S_{N_0} and S_{N_1}, namely the multirelation g such that $a\, g\, \eta b$ if and only if $g(a)(b) = \eta$. Indeed, this is a (generalization of a) widely used formalization of net morphisms due to Winskel [141, 144]. In the case of morphisms between occurrence nets, since by definition such nets have no *isolated places*—i.e., places belonging neither to the initial marking nor to any pre- or post-set—as an immediate corollary to Proposition 1.5.3, we have that g is a *relation* and that the inverse relation g^{op}, defined by $bg^{op}a$ if and only if agb, restricts to (total) functions $g^{op}_{\varnothing}: [\![u_{N_1}]\!] \to [\![u_{N_0}]\!]$ and $g^{op}_{\{t\}}: [\![\partial^1_{N_1}(f(t))]\!] \to [\![\partial^1_{N_0}(t)]\!]$ for each $t \in T_{N_0}$. We will use these functions in the next definition.

DEFINITION 1.6.2 *(Decorated Occurrence Nets)*
A decorated occurrence net is an occurrence net Θ such that:

i) S_Θ *is of the form* $\bigcup_{a \in A_\Theta} \{a\} \times [n_a]$, *where the set* $\{a\} \times [n_a]$ *is called the family of a. We will use a^F to denote the family of a regarded as a multiset;*

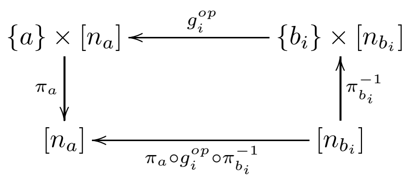
ii) $\forall a \in A_\Theta,\ \forall x, y \in \{a\} \times [n_a],\ {}^\bullet x = {}^\bullet y$.

A morphism of decorated occurrence nets $\langle f, g\rangle: \Theta_0 \to \Theta_1$ is a morphism of occurrence nets which respects families, i.e., for each $[\![a^F]\!] \subseteq S_{\Theta_0}$, given $x = {}^\bullet[\![a^F]\!]$—which is a singleton set or the empty set by ii above and the definition of occurrence nets—we have:

i) $g(a^F) = \bigoplus_{i \in I_a} b^F_i$, *for some index set* I_a*;*

ii) $\pi_a \circ g^{op}_i \circ in_{b_i}$ *is a block function, where*
π_a *is the projection of* $\{a\} \times [n_a]$ *to* $[n_a]$,
π^{-1}_a *is the inverse bijection from* $[n_a]$ *to* $\{a\} \times [n_a]$, *and*
$g^{op}_i: \{b_i\} \times [n_{b_i}] \to \{a\} \times [n_a]$ *is* g^{op}_x *restricted to* $\{b_i\} \times [n_{b_i}]$.

The composition in (ii) can be summarized by means of the diagram



This defines the category <u>DecOcc</u>.

A family is thus a collection of finitely many places with the same pre-set, and a decorated occurrence net is an occurrence net where each place belongs to exactly one family. Families, and therefore decorated occurrence nets, are capable of describing relationships between places by grouping them together. We will use families to relate places which are instances of the same place obtained in a process of unfolding. Therefore, morphisms treat families in a special way: they map families to families (condition *(i)*) and they do that in a unique pre-determined way (condition *(ii)*). This is because what we want to describe is that a^F is mapped to b^F. Hence, since the way to map a family to another family is fixed by definition, in the following we will often define morphisms by just saying what families are sent to what families.

Observe that the full subcategory of DecOcc consisting of all nets Θ such that $S_\Theta = \bigcup_{a\in A_\Theta}\{a\}\times[1]$ is (isomorphic to) Occ. Observe also that, since the initial marking consists exactly of the elements with empty pre-set and, by point *(ii)* in Definition 1.6.2, elements of a family have the same pre-set, for a decorated occurrence net u_Θ is of the form $\bigoplus_{i\in I} a_i^F$.

The following is a useful property of decorated occurrence net morphisms which directly follows from their definition.

Proposition 1.6.3 *(Decorated Occurrence Net Morphisms)*
Let Θ_0 and Θ_1 be decorated occurrence nets and $\langle f,g\rangle\colon\Theta_0\to\Theta_1$ a morphism in DecOcc. Then

$$\forall [\![b^F]\!]\subseteq[\![u_{N_1}]\!],\ \exists! [\![a^F]\!]\subseteq[\![u_{N_0}]\!] \text{ such that } [\![b^F]\!]\subseteq[\![g(a^F)]\!] \text{ and}$$
$$\forall [\![b^F]\!]\subseteq[\![\partial^1_{N_1}(f(t))]\!],\ \exists! [\![a^F]\!]\subseteq[\![\partial^1_{N_0}(t)]\!] \text{ such that } [\![b^F]\!]\subseteq[\![g(a^F)]\!].$$

We have seen that for occurrence nets and decorated occurrence nets simple concepts of causal dependence ($\prec$) and conflict ($\#$) can be defined. The orthogonal concept is that of concurrency.

Definition 1.6.4 *(Concurrent Elements)*
Given a (decorated) occurrence net Θ (which defines $\prec$, $\preccurlyeq$ and $\#$), we can define

- *For $x,y\in T_\Theta\cup S_\Theta$, x co y iff $\neg(x\prec y$ or $y\prec x$ or $x\ \#\ y)$;*
- *For $X\subseteq T_\Theta\cup S_\Theta$, $Co(X)$ iff*
  $(\forall x,y\in X,\ x \text{ co } y)$ *and* $|\{t\in T_\Theta \mid \exists x\in X,\ t\preccurlyeq x\}|\in\omega$.

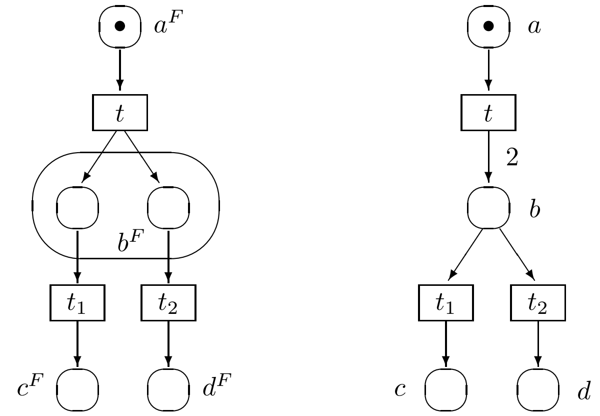
As a first step in relating the categories DecOcc and PTNets, we define a functor from decorated occurrence nets to PT nets.

Definition 1.6.5 *($(\_)^+$: from DecOcc to PTNets)*
For $\Theta=\big(\partial^0_\Theta,\partial^1_\Theta\colon(T_\Theta,0)\to(\bigcup_{a\in A_\Theta}\{a\}\times[n_a])^{\mathcal{M}},u_\Theta\big)$, let $(\_)^+$ denote the $(\_)^{\mathcal{M}_\infty}$–homomorphism from $S_\Theta^{\mathcal{M}_\infty}$ to $A_\Theta^{\mathcal{M}_\infty}$ such that $(a,j)^+=a$.
Then, we define Θ^+ to be the net $\big((\_)^+\circ\partial^0_\Theta,\ (\_)^+\circ\partial^1_\Theta\colon(T_\Theta,0)\to A_\Theta^{\mathcal{M}},(u_\Theta)^+\big)$.

Given a morphism $\langle f,g\rangle\colon\Theta_0\to\Theta_1$, let $\langle f,g\rangle^+\colon\Theta_0^+\to\Theta_1^+$ be $\langle f,(\_)^+\circ g\circ\rho\rangle$ where $\rho\colon A_{\Theta_0}^{\mathcal{M}_\infty}\to S_{\Theta_0}^{\mathcal{M}_\infty}$ is the $(\_)^{\mathcal{M}_\infty}$–homomorphism such that $\rho(a)=(a,1)$.

The following example shows the result of applying $(\_)^+$ to a decorated occurrence net. In all the pictures to follow, a family is represented by drawing its elements from left to right in accordance with its ordering, and enclosing them into an oval. Families of cardinality one are not explicitly indicated.

EXAMPLE 1.6.6



A decorated occurrence net Θ and the net Θ^+

PROPOSITION 1.6.7 *($(\_)^+$ is well-defined)*
Θ^+ is a PT net and $\langle f, g\rangle^+$ is a PT net morphism.

Proof. The first statement is completely clear. Let us show the second. Conditions *(i)* and *(ii)* are trivial while condition *(v)* derives directly from Proposition 1.6.3.
Observe now that, if $g(a,i) = \bigoplus_l(\oplus\{b_l\} \times [k_l]_i)$, by definition of g, we have $g(a,j) = \bigoplus_l(\oplus\{b_l\} \times [k_l]_j)$ and so $g(a,i)^+ = \bigoplus_l k_l b^l = g(a,j)^+$.
Therefore, $g(u)^+ = ((\_)^+ \circ g \circ \rho)(u^+)$.

(iii)
$$\begin{aligned} ((\_)^+ \circ g \circ \rho)(\partial^i_{\Theta_0^+}(t)) &= ((\_)^+ \circ g \circ \rho)(\partial^i_{\Theta_0}(t)^+) = g(\partial^i_{\Theta_0}(t))^+ \\ &= \partial^i_{\Theta_1}(f(t))^+ = \partial^i_{\Theta_1^+}(f(t)). \end{aligned}$$

(iv) $((\_)^+ \circ g \circ \rho)(u_{\Theta_0^+}) = ((\_)^+ \circ g \circ \rho)((u_{\Theta_0})^+) = g(u_{\Theta_0})^+ = (u_{\Theta_1})^+ = u_{\Theta_1^+}$. ✓

PROPOSITION 1.6.8 *($(\_)^+$: <u>DecOcc</u> $\to$ <u>PTNets</u>)*
$(\_)^+$: <u>DecOcc</u> $\to$ <u>PTNets</u> is a functor.

Proof. It is completely clear that $\langle id_{T_\Theta}, id_{S_\Theta}\rangle^+ = \langle id_{T_\Theta}, id_{A_\Theta}\rangle$. Moreover, given $\langle h, k\rangle \circ \langle f, g\rangle : \Theta_0 \to \Theta_1$, we have that for each $u \in A^{\mathcal{M}_\infty}_{\Theta_0}$

$$\begin{aligned} ((\_)^+ \circ k \circ \rho) \circ ((\_)^+ \circ g \circ \rho)(u) &= ((\_)^+ \circ k \circ \rho)\big((g \circ \rho)(u)^+\big) \\ &= k((g \circ \rho)(u))^+ = \big((\_)^+ \circ (k \circ g) \circ \rho\big)(u). \end{aligned}$$

So, $(\langle h, k\rangle \circ \langle f, g\rangle)^+ = \langle h, k\rangle^+ \circ \langle f, g\rangle^+$. ✓

Nets obtained via $(\_)^+$ from decorated occurrence nets have a structure very similar to that of occurrence nets. We will denote by $\underline{\mathsf{DecOcc}^+}$ the full subcategory of $\underline{\mathsf{PTNets}}$ consisting of (nets isomorphic to) nets of the form Θ^+.

Proposition 1.6.9 *(Structure of Decorated Occurrence Nets)*
If Θ is a decorated occurrence net, then Θ^+ is a PT net such that:

i) $a \in [\![u_{\Theta^+}]\!]$ *if and only if* ${}^\bullet a = \varnothing$;

ii) $\forall a \in S_{\Theta^+},\ |{}^\bullet a| \leq 1$;

iii) the relation $\prec$ *is irreflexive and* $\forall t \in T_{\Theta^+},\ \{t' \in T_{\Theta^+} \mid t' \prec t\}$ *is finite.*

Moreover, if Θ is (isomorphic to) an occurrence net, then Θ^+ is an occurrence net isomorphic to Θ.

Proof. Obvious from the definition of $(\_)^+$. ✓

Let $\underline{\mathcal{B}}$ range over $\underline{\mathsf{Occ}}$, $\underline{\mathsf{DecOcc}}$ and $\underline{\mathsf{DecOcc}^+}$. For any net in $\underline{\mathcal{B}}$, we can define the concept of *depth* of an element of the net, thanks to their nicely stratified structure.

Definition 1.6.10 *(Depth)*
Let Θ be a net in $\underline{\mathcal{B}}$. The depth of an element in $T_\Theta \cup S_\Theta$ is inductively defined by

- $\mathrm{depth}(x) = 0$ *if* $b \in S_\Theta$ and ${}^\bullet b = \varnothing$;
- $\mathrm{depth}(x) = \max\{\mathrm{depth}(b) \mid b \prec x\} + 1$ *if* $x \in T_\Theta$;
- $\mathrm{depth}(x) = \mathrm{depth}(t)$ *if* $x \in S_\Theta$ and ${}^\bullet x = \{t\}$.

Definition 1.6.11 *(Subnets of a Net)*
Given a net Θ in $\underline{\mathcal{B}}$ define its subnet of depth n, $\Theta^{(n)}$, as

- $T_{\Theta^{(n)}} = \{t \in T_\Theta \mid \mathrm{depth}(t) \leq n\}$;
- $S_{\Theta^{(n)}} = \{b \in S_\Theta \mid \mathrm{depth}(b) \leq n\}$;
- $\partial^0_{\Theta^{(n)}}$ and $\partial^1_{\Theta^{(n)}}$ *are the restrictions of* ∂^0_Θ and ∂^1_Θ *to* $T_{\Theta^{(n)}}$;
- $u_{\Theta^{(n)}} = u_\Theta$.

It is easy to see that $\Theta^{(n)}$ is a net in $\underline{\mathcal{B}}$.

Clearly, for each $n \leq m$ there is a morphism $in_{n,m}: \Theta^{(n)} \rightarrow \Theta^{(m)}$ whose components are both set inclusions. In the following we will call such net morphisms simply *inclusions*. Obviously, if $\langle f, g\rangle: \Theta_0 \rightarrow \Theta_1$ is an inclusion, then we have $u_{\Theta_0} = u_{\Theta_1}$ and, for each $t \in T_{\Theta_0}$, $\partial^i_{\Theta_0}(t) = \partial^i_{\Theta_1}(t)$, $i = 0, 1$.

Now, consider the category $\underline{\omega} = \{0 \rightarrow 1 \rightarrow 2 \rightarrow 3 \cdots\}$ and the class of diagrams $D: \underline{\omega} \rightarrow \underline{\mathcal{B}}$ such that $D(n \rightarrow n+1) = in_n: D(n) \rightarrow D(n+1)$ is an inclusion. For such a class we have the following results. The reader is referred to [90, III.3] (and to Appendix A.1) for the definition of the categorical concepts involved.

PROPOSITION 1.6.12 *(*Colim(D) *exists)*
For any $D \in \mathcal{D}$, the colimit of D in $\underline{\mathcal{B}}$ exists.

Proof. Consider the net $\Theta = \left(\partial^0_\Theta, \partial^1_\Theta: (T_\Theta, 0) \rightarrow S^{\mathcal{M}}_\Theta, u_\Theta\right)$ where

$$\begin{array}{rcl} T_\Theta & = & \bigcup_n T_{D(n)} \\ S_\Theta & = & \bigcup_n S_{D(n)} \\ u_\Theta & = & u_{D(0)} \\ \partial^i_\Theta(t) & = & \partial^i_{D(n)}(t) \text{ for } n \text{ such that } t \in T_{D(n)}. \end{array}$$

Clearly, Θ is well-defined, is a net, and belongs to $\underline{\mathcal{B}}$.

Now, for any n, let $\mu_n: D(n) \rightarrow \Theta$ be the obvious inclusion. By definition we have $\mu_n = \mu_{n+1} \circ in_n$. Now consider a family of morphisms $\tau_n: D(n) \rightarrow \Theta'$, $n \in \omega$, such that $\tau_n = \tau_{n+1} \circ in_n$. Define $\sigma: \Theta \rightarrow \Theta'$ by:

$$\begin{array}{rcl} \sigma(t) & = & \tau_n(t) \text{ for } n \text{ such that } t \in T_{D(n)}; \\ \sigma(a) & = & \tau_n(a) \text{ for } n \text{ such that } a \in S_{D(n)}. \end{array}$$

σ is clearly a morphism in $\underline{\mathcal{B}}$. Now consider $\sigma \circ \mu_n: D(n) \rightarrow \Theta'$. We have that, for any $t \in T_{D(n)}$, $\sigma(t) = \tau_n(t)$ and for any $a \in S_{D(n)}$, $\sigma(a) = \tau_n(a)$. Therefore, since μ_n is an inclusion, we have $\sigma \circ \mu_n = \tau_n$ for each $n \in \omega$.

Given any $\sigma': \Theta \rightarrow \Theta'$, suppose that $\sigma' \circ \mu_n = \tau_n$ for each $n \in \omega$. Since $t \in T_\Theta$ ($a \in S_\Theta$) belongs to $T_{D(n)}$ (to $S_{D(n)}$) for some n, we have that $\sigma'(t) = \tau_n(t) = \sigma(t)$ ($\sigma'(a) = \tau_n(a) = \sigma(a)$). Therefore, $\sigma' = \sigma$. ✓

PROPOSITION 1.6.13 *(Θ is the colimit of its subnets)*
Given a net Θ in $\underline{\mathcal{B}}$, let $D_\Theta: \underline{\omega} \rightarrow \underline{\mathcal{B}}$ be the functor such that $D_\Theta(n) = \Theta^{(n)}$ and $D_\Theta(n \rightarrow n+1) = in_{n,n+1}: \Theta^{(n)} \rightarrow \Theta^{(n+1)}$. Then $\Theta =$ Colim(D_Θ).

Proof. Since $D_\Theta \in \mathcal{D}$, we are in the conditions of the previous proposition. So, it is enough to observe that the colimit construction for diagrams in $\mathcal{D}$ in the proof of that proposition gives a family $\mu_n: D(n) \rightarrow \Theta$, $n \in \omega$, where $\mu_n: \Theta^{(n)} \rightarrow \Theta$ is the inclusion of $\Theta^{(n)}$ in Θ. ✓

PROPOSITION 1.6.14 *($(\_)^+$ preserves the colimit of D_Θ)*
If $\underline{\mathcal{B}} = \underline{\text{DecOcc}}$, then Colim$(D_\Theta)^+ =$ Colim(D^+_Θ).

Proof. Since the previous proposition states that each Θ is completely identified by the diagram D_Θ, it is enough to observe that D_Θ^+ is exactly D_{Θ^+}.
So, $\mathrm{Colim}(D_\Theta^+) = \mathrm{Colim}(D_{\Theta^+}) = \Theta^+ = \mathrm{Colim}(D_\Theta)^+$. ✓

1.7 PT Net Unfoldings

In this section, we define the *unfolding* of PT nets in term of decorated occurrence nets and show that it is a functor from <u>PTNets</u> to <u>DecOcc</u> which is right adjoint to $(\_)^+$.

We start by giving the object component of such a functor. To this aim, given a net N, we define a family of decorated occurrence nets, one for each $n \in \omega$, where the n-th net approximates the unfolding of N up to depth n, i.e., it reflects the behaviour of the original net up to step sequences of length at most n. Clearly, the unfolding of N will be defined to be the colimit of an appropriate $\boldsymbol{\omega}$-shaped diagram built on the approximant nets. We will use the following notation: given $s \in X_1 \times \cdots \times X_n$, we denote by $s \downarrow X_i$ the projection of s on the X_i component. Moreover, given $S = \{s_j \mid j \in J\}$, $S \downarrow X_i$ will be $\{s_j \downarrow X_i \mid j \in J\}$ and $S \overset{\oplus}{\downarrow} X_i$ will denote $\bigoplus_{j \in J}(s_j \downarrow X_i)$.

Definition 1.7.1 *(PT Nets Unfoldings: $\mathcal{U}[\_]^{(k)}$)*
Let $N = \big(\partial_N^0, \partial_N^1 \colon (T_N, 0) \to S_N^{\mathcal{M}}, u_N\big)$ *be a net in* <u>PTNets</u>.
We define the nets $\mathcal{U}[N]^{(k)} = \big(\partial_k^0, \partial_k^1 \colon (T_k, 0) \to S_k^{\mathcal{M}}, u_k\big)$, *for* $k \in \omega$, *where:*

- $S_0 = \bigcup \{\, \{(\varnothing, b)\} \times [n] \mid u_N(b) = n\}$;
- $T_0 = \{0\}$, *and the* ∂_0^i *with the obvious definitions;*
- $u_0 = \bigoplus S_0$;

and for $k > 0$,

- $T_k = T_{k-1} \cup \left\{ (B, t) \;\middle|\; B \subseteq S_{k-1},\ Co(B),\ t \in T_N,\ B \overset{\oplus}{\downarrow} S_N = \partial_N^0(t) \right\}$;
- $S_k = S_{k-1} \cup \left(\bigcup_{\substack{t_0 \in T_k,\ b \in S_N \\ \partial_N^1(t_0 \downarrow T)(b) = n}} \left\{ \left\{ \big(\{t_0\}, b\big) \right\} \times [n] \right\} \right)$;
- $\partial_k^0(B, t) = \bigoplus B$, *and* $\partial_k^1(B, t) = \bigoplus \left\{ \big((\{(B, t)\}, b), i\big) \in S_k \right\}$;
- $u_k = \bigoplus \left\{ \big((\varnothing, b), i\big) \in S_k \right\} = \bigoplus S_0 = u_0$.

Therefore, informally speaking, the net $\mathcal{U}[N]^{(0)}$ is obtained by exploding in families the initial marking of N, and $\mathcal{U}[N]^{(n+1)}$ is obtained, inductively, by generating a new transition for each possible subset of concurrent places of $\mathcal{U}[N]^{(n)}$ whose corresponding multiset of places of N constitutes the source of some transition t of N; the target of t is also exploded in families which are added to $\mathcal{U}[N]^{(n+1)}$. As a consequence, the transitions of the n-th approximant net are instances of transitions of N, in the precise sense that each of them corresponds to a unique occurrence of a transition of N in one of its step sequences of length at most n.

LEMMA 1.7.2 *($\mathcal{U}[N]^{(n)}$ is a decorated occurrence net)*
For all $n \in \omega$, $\mathcal{U}[N]^{(n)}$ is a decorated occurrence net of depth n. Moreover, for each $n \in \omega$ there is an inclusion $in_n : \mathcal{U}[N]^{(n)} \to \mathcal{U}[N]^{(n+1)}$.

Proof. That $\mathcal{U}[N]^{(n)}$ has depth n and that there exists an inclusion from $\mathcal{U}[N]^{(n)}$ to $\mathcal{U}[N]^{(n+1)}$ is obvious from the definition. We have to show that $\mathcal{U}[N]^{(n)}$ is a decorated occurrence net. For each $t \in T_n$, $\partial_n^i(t)$ is a multiset where all the elements have multiplicity one, i.e., a set. The same happens for u_n.

i) Observe that for each $((x,b),i) \in S_n$, ${}^\bullet((x,b),i) = x$ which is the empty set or a singleton. So $|{}^\bullet((x,b),i)| \leq 1$.

ii) Moreover, $((x,b),i) \in [\![u_n]\!]$ iff $x = \varnothing$ iff ${}^\bullet((x,b),i) = \varnothing$.

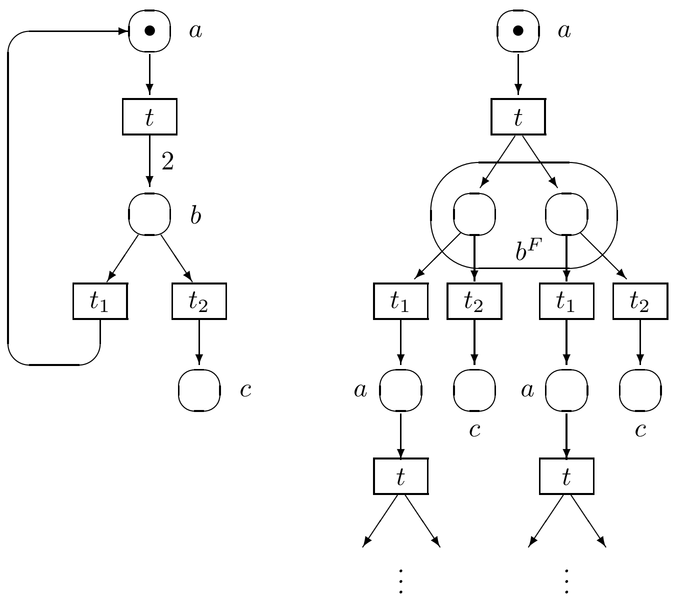
iii) By definition of $\mathcal{U}[N]^{(n)}$, whenever $x \prec^1 y \prec^1 z$ we have that $\text{depth}(z) = \text{depth}(x) + 1$. Since $x, z \in T_n$ or $x, z \in S_n$ implies that there exists at least one y such that $x \prec y \prec^1 z$ we have that $\text{depth}(x) < \text{depth}(z)$ and so $x \neq z$. Therefore, $\prec$ is irreflexive. This, together with *(i)* and *(ii)*, implies that, in any reachable marking, each place has multiplicity at most one. In fact, that being true in u_n, each place having only one pre-event and each transition occurring at most once in any computation, there is no way to generate more than one token in a place. Moreover, $\forall t \in T_n$, $\{t' \in T_n \mid t' \prec t\}$ is finite, because of the definition of Co.

iv) $\#$ is irreflexive. Recall that $x \# x$ iff $\exists t, t' \in T_n$, $t \neq t'$ and $t \#_m t'$ such that $t \preccurlyeq_n x$ and $t' \preccurlyeq_n x$. So, by point *(i)*, x cannot be a place, otherwise we would have backward branching. This means that there exist $b, b' \in [\![\partial_n^0(x)]\!]$, $b \neq b'$ such that $b \; co \; b'$, i.e., $x = (B,t)$ and $\neg Co(B)$. This is impossible.

The other conditions of decorated occurrence nets are obviously true. ✓

DEFINITION 1.7.3 *(PT Net Unfoldings: $\mathcal{U}[\_]$)*
We define $\mathcal{U}[N]$ to be the colimit of the diagram $D: \underline{\omega} \to \underline{\mathsf{DecOcc}}$ such that $D(n) = \mathcal{U}[N]^{(n)}$ and $D(n \to n+1) = in_n$. By Lemma 1.7.2 D belongs to $\mathcal{D}$ and so, by Proposition 1.6.12, the colimit exists and is a decorated occurrence net.

Example 1.7.4



A PT Net N and (part of) its unfolding $\mathcal{U}[N]$

The correspondence between elements of the unfolding and elements of the original net is formalized by the folding morphism, which will also be the counit of the adjunction.

Proposition 1.7.5 *(Folding Morphism)*
Consider the map $\epsilon_N = \langle f_\epsilon, g_\epsilon\rangle \colon \mathcal{U}[N]^+ \to N$ defined by

- $f_\epsilon(B,t) = t$ *and* $f_\epsilon(0) = 0$;
- $g_\epsilon(\bigoplus_i (x_i, y_i)) = \bigoplus_i y_i$.

Then, ϵ_N is a morphism in <u>PTNets</u>, *called the folding of $\mathcal{U}[N]$ into N.*

Proof. Recall that a transition in $\mathcal{U}[N]^+$ is of the form $(B,t)\colon (\bigoplus B)^+ \to ((t,b)^F)^+$ where $B \subseteq S_{\mathcal{U}[N]}$, $t \in T_N$, $B \overset{\oplus}{\downarrow} S_N = \partial^0_N(t)$, and $[\![(t,b)^F]\!] \overset{\oplus}{\downarrow} S_N = \partial^1_N(t)$.

So, $t\colon B \overset{\oplus}{\downarrow} S_N \to [\![(t,b)^F]\!] \overset{\oplus}{\downarrow} S_N$. Now, since $\forall B \subseteq S_{\mathcal{U}[N]}$, $g_\epsilon((\bigoplus B)^+) = B \overset{\oplus}{\downarrow} S_N$, we have that

$$g_\epsilon\left(\partial^i_{\mathcal{U}[N]^+}(B,t)\right) = \partial^i_N\left(f_\epsilon(B,t)\right).$$

Moreover, we have that $u_{\mathcal{U}[N]^+} = \bigoplus_{b \in S_N} u_N(b) \cdot (\varnothing, b)$.

So $g_\epsilon(u_{\mathcal{U}[N]^+}) = \bigoplus_{b \in S_N} u_N(b) \cdot b = u_N$. Concerning condition *(v)* in Definition 1.5.1, observe that $[\![g_\epsilon(x,a)]\!] \cap [\![g_\epsilon(y,b)]\!] \neq \varnothing$ implies $a = b$. So, if $(x,a) \neq (y,b)$ we must have $(x,a) \notin [\![u_{\mathcal{U}[N]^+}]\!]$ or $(y,b) \notin [\![u_{\mathcal{U}[N]^+}]\!]$, because either x or y must be non-empty, and ${}^\bullet(x,a) \cap {}^\bullet(y,b) = x \cap y = \varnothing$. ✓

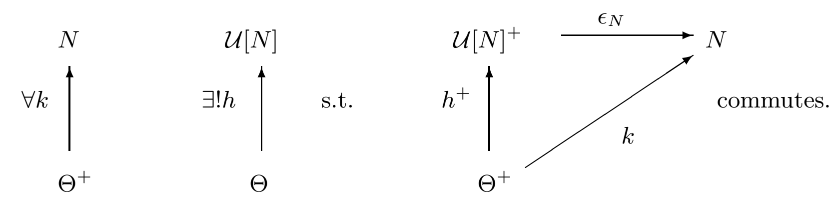
Lemma 1.7.6 *(Occurrence Net Morphisms preserve Concurrency)*
Let Θ_0 and Θ_1 be (decorated) occurrence nets and let $\langle f,g\rangle : \Theta_0 \to \Theta_1$ be a PT net morphism. Then, for each $t_0 \in T_{\Theta_0}$, $Co\left([\![\partial^0_{\Theta_0}(t_0)]\!]\right)$ and $Co\left([\![g(\partial^0_{\Theta_0}(t_0))]\!]\right)$.

Proof. Since, by definition of (decorated) occurrence nets, $\{t' \preccurlyeq t\}$ is finite, we have $\neg Co([\![\partial^0_{\Theta_0}(t_0)]\!])$ iff $\exists b, b' \in [\![\partial^0_{\Theta_0}(t_0)]\!]$ such that $b \;\#\; b'$. This would mean that $\exists t, t' \in T_{\Theta_0}$, $t \neq t'$ and $t \;\#_m\; t'$ such that $t \preccurlyeq b$ and $t' \preccurlyeq b'$. Thus, since $t \preccurlyeq t_0$ and $t' \preccurlyeq t_0$, we would have $t_0 \;\#\; t_0$ which is impossible since Θ_0 is a (decorated) occurrence net. Furthermore, $g(\partial^0_{\Theta_0}(t_0)) = \partial^0_{\Theta_1}(f(t_0))$, which is the pre-set of a transition of a (decorated) occurrence net and so, by the first part of this proposition, $Co([\![g(\partial^0_{\Theta_0}(t_0))]\!])$. ✓

Finally, we are ready to prove that $\mathcal{U}[\_]$ is right adjoint to $(\_)^+$.

Theorem 1.7.7 *($(\_)^+ \dashv \mathcal{U}[\_]$)*
The pair $\langle (\_)^+, \mathcal{U}[\_]\rangle : \underline{\mathsf{DecOcc}} \rightharpoonup \underline{\mathsf{PTNets}}$ constitutes an adjunction.

Proof. Let N be a PT Net and $\mathcal{U}[N]$ its unfolding. By [90, Theorem 2, pg. 81], it is enough to show that the folding $\epsilon_N : \mathcal{U}[N]^+ \to N$ is universal from $(\_)^+$ to N, i.e., for any decorated occurrence net Θ and any morphism $k : \Theta^+ \to N$ in $\underline{\mathsf{PTNets}}$, there exists a unique $h : \Theta \to \mathcal{U}[N]$ in $\underline{\mathsf{DecOcc}}$ such that $k = \epsilon_N \circ h^+$.



Consider the diagram in $\underline{\mathsf{DecOcc}}$ given by $D_\Theta(n) = \Theta^{(n)}$, the subnet of Θ of depth n and $D_\Theta(n \to n+1) = in_n : \Theta^{(n)} \to \Theta^{(n+1)}$. We define a sequence of morphisms of nets $h_n : \Theta^{(n)} \to \mathcal{U}[N]$, such that for each n, $h_n = h_{n+1} \circ in_n$.

Since $\Theta = \mathrm{Colim}(D_\Theta)$, there is a unique $h : \Theta \to \mathcal{U}[N]$ such that $h \circ \mu_n = h_n$ for each n. At the same time, we show that

$$\forall n \in \omega,\; k \circ \mu_n^+ = \epsilon_N \circ h_n^+ \tag{1.24}$$

and that the h_n are the unique sequence of morphisms $h_n : \Theta^{(n)} \to \mathcal{U}[N]$ such that equation (1.24) holds. Now, by functoriality of $(\_)^+$, we have that

$$\forall n \in \omega,\; k \circ \mu_n^+ = \epsilon_N \circ h^+ \circ \mu_n^+.$$

Therefore, since by Proposition 1.6.14 $(\_)^+ \circ D_\Theta = D_{\Theta^+}$, and, by Proposition 1.6.13, $\Theta^+ = \mathrm{Colim}(D^+_\Theta) = \mathrm{Colim}((\_)^+ \circ D_\Theta)$, by the universal property of the colimit we must have $k = \epsilon_N \circ h^+$.

To show the uniqueness of h, let h' be such that $k = \epsilon_N \circ h'^+$. Then we have $k \circ \mu^+_n = \epsilon_N \circ h'^+ \circ \mu^+_n$. But h_n is the unique morphism for which this happens. Therefore, for each n, $h_n = h' \circ \mu_n$ and so, by the universal property of the colimit, $h = h'$.

Let us now define h_n and therefore $h\colon \Theta \to \mathcal{U}[N]$, and show that the h_n, $n \in \omega$ are the unique sequence of morphisms such that (1.24) holds.

depth 0. Suppose $u_{\Theta^+} = \bigoplus_i n_i a_i$. So $u_\Theta = \bigoplus_i (\oplus\{a_i\} \times [n_i])$. Suppose $k(a_j) = \bigoplus_l m^j_l b^j_l$. By definition of k, since k does not merge different places in the initial marking and $k(u_{\Theta^+}) = u_N$, we have $u_N = v \oplus \bigoplus_l n_j m^j_l b^j_l$, with $b^j_l \notin [\![v]\!]$. Thus, in $\mathcal{U}[N]$ we have the places $\bigcup_l \{(\varnothing, b^j_l)\} \times [n_j m^j_l]$. So, we define

$$h_0(a_j, i) = \bigoplus_l \Big(\oplus\{(\varnothing, b^j_l)\} \times [m^j_l]_i \Big).$$

We have
$$\begin{aligned} h^+_0(a_j) &= (h_0(a_j,i))^+ = \textstyle\bigoplus_j m^j_l (\varnothing, b^j_l) \text{ and} \\ \epsilon_N \circ h^+_0(a_j) &= \textstyle\bigoplus_l m^j_l b^j_l = k(a_j) = k \circ \mu^+_0(a_j). \end{aligned}$$

Observe that h_0 so defined, lifting its place component to a $(\_)^{\mathcal{M}_\infty}$–homomorphism, is a morphism $\Theta^{(0)} \to \mathcal{U}[N]$ and that it is completely determined by k and the conditions of decorated occurrence net morphisms.

depth n+1. Let us suppose that we have defined $h_n\colon \Theta^{(n)} \to \mathcal{U}[N]$ and that it is a morphism. Suppose that for each $m \le n$, h_m is the unique morphism such that $\epsilon_N \circ h^+_m = k \circ \mu_m$. Let h_{n+1} be h_n on the elements of depth less or equal to n. Now, we define h_{n+1} on the elements of depth $n+1$.

Let $t_1 \in T_\Theta$ such that $\mathrm{depth}(t_1) = n+1$ and $k(t_1) = t$.

Since $[\![\partial^0_\Theta(t_1)]\!]$ is a set of elements of depth less or equal to n, $h_n(\partial^0_\Theta(t_1))$ is defined. Since h_n is a morphism, by Lemma 1.7.6, we have $Co([\![h_n(\partial^0_\Theta(t_1))]\!])$.

Moreover, since $\epsilon_N \circ h^+_n = k \circ \mu^+_n$,

$$\begin{aligned} \partial^0_N(t) = k(\partial^0_\Theta(t)) &= \epsilon_N \circ (h_n)^+ \Big(\partial^0_{\Theta^+}(t_1)\Big) = \epsilon_N \circ (h_n)^+ \Big((\partial^0_\Theta(t_1))^+\Big) \\ &= \epsilon_N \circ h_n(\partial^0_\Theta(t_1))^+ = [\![h_n(\partial^0_\Theta(t_1))]\!] \overset{\oplus}{\downarrow} S_N \end{aligned}$$

Therefore $t_0 = \Big([\![h_n(\partial^0_\Theta(t_1))]\!], t\Big) = \Big([\![h_{n+1}(\partial^0_\Theta(t_1))]\!], t\Big) \in T_{\mathcal{U}[N]}$.

Now, since h_{n+1} has to make the diagram commute, $h_{n+1}(t_1)$ must be of the form (B, t) and, since it has to be a morphism, it must be $\partial^0_{\mathcal{U}[N]}((B,t)) = \bigoplus B = h_{n+1}(\partial^0_\Theta(t_1))$. Therefore $h_{n+1}(t_1) = t_0$. Observe that there is only one choice for $h_{n+1}(t_1)$, given k and h_n by inductive hypothesis.

Obviously, $\epsilon_N \circ h^+_{n+1}(t_1) = t = k(t_1) = k \circ \mu^+_{n+1}(t_1)$.

Now, let $\partial^1_{\Theta^+}(t_1) = \bigoplus_i n_i a_i$. So $\partial^1_\Theta(t_1) = \bigoplus_i (\oplus\{a_i\} \times [n_i])$ in Θ. Suppose $k(a_j) = \bigoplus_l m^j_l b^j_l$. By definition of k, since it does not merge different places in the post-set of a transition and $k(\partial^1_{\Theta^+}(t_1)) = \partial^1_N(k(t_1))$, we have $\partial^1_N(k(t_1)) = v \oplus \bigoplus_l n_j m^j_l b^j_l$, with $b^j_l \notin [\![v]\!]$. Thus in $\mathcal{U}[N]$ we have the places $\bigcup_l \{(\{t_0\}, b^j_l)\} \times [n_j m^j_l]$. We define

$$h_{n+1}(a_j, i) = \bigoplus_l \Big(\oplus \Big\{ (\{t_0\}, b^j_l) \Big\} \times [m^j_l]_i \Big).$$

So
$$\begin{aligned} h^+_{n+1}(a_j) &= (h_{n+1}(a_j, i))^+ = \textstyle\bigoplus_l m^j_l(\{t_0\}, b^j_l) \text{ and} \\ \epsilon_N \circ h^+_{n+1}(a_j) &= \textstyle\bigoplus m^j_l b^j_l = k(a_j) = k \circ \mu^+_{n+1}(a_j). \end{aligned}$$

Observe that $h_{n+1}(a_j, i)$ is completely determined by k and by the conditions of decorated occurrence net morphisms.

Now we have to show that h_{n+1} is a morphism $\Theta^{(n+1)} \to \mathcal{U}[N]$. But this task is really trivial because, by its own construction, h_{n+1} preserves source, target and initial marking and respects families. ✓

THEOREM 1.7.8 *(Correspondence with Winskel's Safe Net Unfoldings [143])*
Let N be a safe net. Then $\mathcal{U}[N]$ is (isomorphic to) an occurrence net and therefore, by Proposition 1.6.9, $\mathcal{U}[N]^+ \cong \mathcal{U}[N]$. Moreover, $\mathcal{U}[N]$ is (isomorphic to) Winskel's unfolding of N. Finally, whenever N is (isomorphic to) an occurrence net, the unit of the adjunction $(\_)^+ \dashv \mathcal{U}[\_]$, $\eta_N : N \to \mathcal{U}[N^+] \cong \mathcal{U}[N]$, is an isomorphism.

Therefore, the adjunction $\langle (\_)^+, \mathcal{U}[\_] \rangle : \underline{\mathsf{DecOcc}} \rightharpoonup \underline{\mathsf{PTNets}}$ restricts to Winskel's coreflection $\langle (\_)^+_{\underline{\mathsf{Occ}}}, \mathcal{U}[\_]_{\underline{\mathsf{Safe}}} \rangle : \underline{\mathsf{Occ}} \rightharpoonup \underline{\mathsf{Safe}}$.

Proof. Concerning the claimed correspondence, it is enough to observe that, when N is safe, our definition of $\mathcal{U}[N]$ is such that $(b, 1)$ is a place in $\mathcal{U}[N]$ if and only if b is a condition in Winskel's unfolding. So $\mathcal{U}[N]^+$ and ϵ_N are exactly Winskel's unfolding and folding morphism for N. The other statements are evident. ✓

COMPOSITION OF DECORATED OCCURRENCE NETS

In this subsection, we give the definitions of products and coproducts in $\underline{\mathsf{DecOcc}}$. Proofs are mostly omitted because they are similar to those given for the corresponding constructions in $\underline{\mathsf{PTNets}}$. The characterizations of the behaviours of composed nets in terms of those of the original nets are not discussed here, since they obviously coincide with the correspondent characterizations given in Section 1.5 (Proposition 1.5.8 and Proposition 1.5.16), for the case of occurrence nets.

Given decorated occurrence nets Θ_0 and Θ_1, we can consider them inside $\underline{\mathsf{PTNets}}$, form their product $\Theta_0 \times \Theta_1$, and then form the decorated occurrence net $\mathcal{U}[\Theta_0 \times \Theta_1]$. Recall that, although $\Theta_0 \times \Theta_1$ is not necessarily an occurrence net, it is of course a safe net. So, by definition of $\mathcal{U}[\_]$ and by Theorem 1.7.8, places in $\mathcal{U}[\Theta_0 \times \Theta_1]$ have the form $\Big((x, (a, i)), 1\Big)$, where $(a, i) \in S_{\Theta_0} + S_{\Theta_1}$ and x is of the form $x = \Big\{(B, (t_0, t_1))\Big\}$, for $(t_0, t_1) \in T_{\Theta_0} \times T_{\Theta_1}$ and B a subset of places of $\mathcal{U}[\Theta_0 \times \Theta_1]$. Therefore, we can recover the structure of the families originally present in Θ_0 and Θ_1 by replacing each place $\Big((x,(a,i)),1\Big)$ by $\Big((x,a),i\Big)$.

A little care must be taken in doing such a replacement, since (names of) places also appear as part of transitions in the sets denoted by B. We define the replacement $[\_]^R$ inductively on the depth on elements in $\mathcal{U}[\Theta_0 \times \Theta_1]$ as follows:

$$\begin{aligned}\Big((\varnothing,(a,i)),1\Big)^R &= \Big((\varnothing,a),i\Big);\\ \Big(B,(t_0,t_1)\Big)^R &= \Big(B^R,(t_0,t_1)\Big);\\ \Big((x,(a,i)),1\Big)^R &= \Big((x^R,a),i\Big);\end{aligned}$$

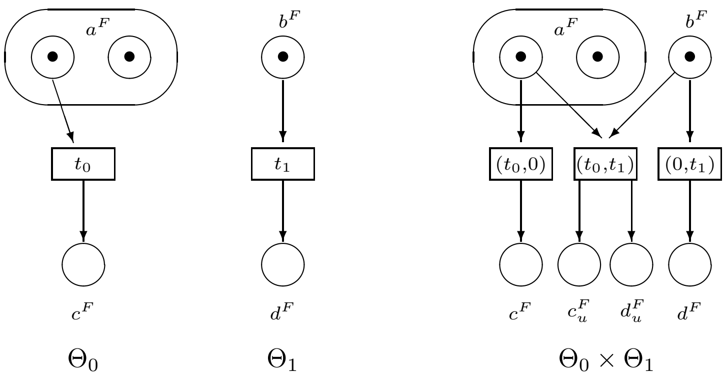
where X^R for a set X is the set $\{x^R \mid x \in X\}$. Let $\mathcal{U}[\Theta_0 \times \Theta_1]^R$ be the decorated occurrence net obtained from $\mathcal{U}[\Theta_0 \times \Theta_1]$ by applying the above described replacement to places and transitions.

Formally, if $\mathcal{U}[\Theta_0 \times \Theta_1] = \big(\partial^0, \partial^1 \colon (T,0) \to S^{\mathcal{M}}, u\big)$, then we have

$$\mathcal{U}[\Theta_0 \times \Theta_1]^R = \Big((\partial^0)^R, (\partial^1)^R \colon (T,0)^R \to (S^R)^{\mathcal{M}}, \bigoplus([\![u]\!]^R)\Big),$$

where $(\partial^i)^R\Big(B^R,(t_0,t_1)\Big) = \bigoplus\Big([\![\partial^i(B,(t_0,t_1))]\!]^R\Big)$.

Example 1.7.9



Θ_0 and Θ_1 and their product $\Theta_0 \times \Theta_1$

Consider the maps $\pi_i^R \colon \mathcal{U}[\Theta_0 \times \Theta_1]^R \to \Theta_i$ given by

$$\pi_i^R\Big(B,(t_0,t_1)\Big) = t_i \quad \text{and} \quad \pi_i^R\Big((x,a),j\Big) = \pi_i(a,j),$$

where π_0 and π_1 are the projections of the product $\Theta_0 \times \Theta_1$. Obviously, π_0^R and π_1^R are morphisms in $\underline{\mathsf{DecOcc}}$.

Now, consider any decorated occurrence net Θ, and $h_0 \colon \Theta \to \Theta_0$, $h_1 \colon \Theta \to \Theta_1$, morphisms in $\underline{\mathsf{DecOcc}}$. Define, inductively on the depth of the elements of Θ, the map $\langle h_0, h_1 \rangle \colon \Theta \to \mathcal{U}[\Theta_0 \times \Theta_1]^R$ so as to have:

$$\begin{aligned}
\langle h_0, h_1 \rangle(c) &= \bigoplus \{\varnothing\} \times [\![h_0(c) \oplus h_1(c)]\!] \text{ if } \mathrm{depth}(c) = 0 \\
\langle h_0, h_1 \rangle(t) &= \Big([\![\langle h_0, h_1 \rangle(\partial^0_\Theta(t))]\!], \big(h_0(t), h_1(t)\big) \Big) \\
\langle h_0, h_1 \rangle(c) &= \bigoplus \big\{ \langle h_0, h_1 \rangle(\bigoplus {}^\bullet c) \big\} \times [\![h_0(c) \oplus h_1(c)]\!]
\end{aligned}$$

It can be shown that $\langle h_0, h_1 \rangle$ is a morphism in $\underline{\mathsf{DecOcc}}$. Moreover, it is easy to show, by induction on the depth, that $\pi_i^R \circ \langle h_0, h_1 \rangle = h_i$ and that $\langle h_0, h_1 \rangle$ is the unique such decorated occurrence net morphism. So we have that:

Proposition 1.7.10 *(Product of Decorated Occurrence Nets)*
$\mathcal{U}[\Theta_0 \times \Theta_1]^R$, with projections π_0^R and π_1^R, is the product of Θ_0 and Θ_1 in $\underline{\mathsf{DecOcc}}$.

To define the coproduct of Θ_0 and Θ_1 in $\underline{\mathsf{DecOcc}}$, suppose $u_{\Theta_0} = \bigoplus_i a_i^F$ and $u_{\Theta_1} = \bigoplus_j b_j^F$, where $|[\![a_i^F]\!]| = n_i$ and $|[\![b_j^F]\!]| = m_j$.

Let $S = \Big(S_{\Theta_0} - [\![u_{\Theta_0}]\!]\Big) + \Big(S_{\Theta_1} - [\![u_{\Theta_1}]\!]\Big) + \Big(\bigcup_{ij} \{(a_i, b_j)\} \times [\mathrm{lcm}(n_i, m_j)]\Big)$ and consider the $(\_)^{\mathcal{M}_\infty}$–homomorphisms $\alpha_l \colon ([\![u_{\Theta_l}]\!])^{\mathcal{M}_\infty} \to \Big(\bigcup_{ij} \{(a_i, b_j)\} \times [\mathrm{lcm}(n_i, m_j)]\Big)^{\mathcal{M}_\infty}$, $l = 0, 1$, defined by:

$$\begin{aligned}
\alpha_0(a_i, k) &= \bigoplus_j \{(a_i, b_j)\} \times \Big[\frac{\mathrm{lcm}(n_i, m_j)}{n_i}\Big]_k \\
\alpha_1(b_j, k) &= \bigoplus_i \{(a_i, b_j)\} \times \Big[\frac{\mathrm{lcm}(n_i, m_j)}{m_j}\Big]_k .
\end{aligned}$$

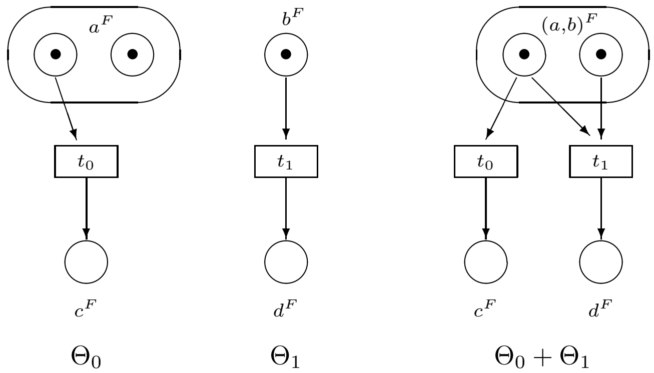
Consider $\gamma_i = (\alpha_i \oplus \beta_i) \colon S_{\Theta_i}^{\mathcal{M}_\infty} \to S^{\mathcal{M}_\infty}$, where β_i are the injections already introduced for coproducts in $\underline{\mathsf{PTNets}}$, and $\delta^i_{\Theta_j} = \gamma_j \circ \partial^i_{\Theta_j}$. Now, we define

$$\Theta_0 + \Theta_1 = \big([\delta^0_{\Theta_0}, \delta^0_{\Theta_1}], [\delta^1_{\Theta_0}, \delta^1_{\Theta_1}] \colon (T, 0) \to S^{\mathcal{M}}, \gamma_0(u_{\Theta_0}) = \gamma_1(u_{\Theta_1}) \big) ,$$

where $(T, 0)$ is the coproduct of pointed sets $(T_{\Theta_0}, 0)$ and $(T_{\Theta_1}, 0)$, with the obvious injections $in_i \colon \Theta_i \to \Theta_0 + \Theta_1$ given by κ_i and γ_i as in the case of $\underline{\mathsf{PTNets}}$.

The following is a simple example of coproduct in $\underline{\mathsf{DecOcc}}$.

EXAMPLE 1.7.11



Θ_0 and Θ_1 and their coproduct $\Theta_0 + \Theta_1$

PROPOSITION 1.7.12 *(Coproducts of Decorated Occurrence Nets)*
$\Theta_0 + \Theta_1$, with injections in_0 and in_1, is the coproduct of Θ_0 and Θ_1 in DecOcc.

Proof. Given any decorated occurrence net Θ, and morphisms $h_0 \colon \Theta_0 \to \Theta$, $h_1 \colon \Theta_1 \to \Theta$, let $[h_0, h_1] \colon \Theta_0 + \Theta_1 \to \Theta$ be the map defined by:

$$\begin{aligned} [h_0,h_1](t) &= h_i(t') && \text{if } t = in_i(t') \text{ for } t' \in T_{\Theta_i} \\ [h_0,h_1](c) &= h_i(c') && \text{if } c = in_i(c') \text{ for } c' \in S_{\Theta_i} - [\![u_{\Theta_i}]\!] \\ [h_0,h_1]((a_i,b_j),k) &= \bigoplus \Big\{ \{c\} \times \Big[\frac{n_c}{\mathrm{lcm}(n_i,m_j)} \Big]_k \;\Big|\; (c,l) \in [\![h_0(a_i^F)]\!] \cap [\![h_1(b_j^F)]\!] \Big\} \end{aligned}$$

where n_c is the coefficient of c in u_Θ. Following a scheme similar to that used in the case of PTNets, it is easy to show that $[h_0, h_1]$ is a morphism in DecOcc and that it is the unique such that $[h_0, h_1] \circ in_i = h_i$. ✓

1.8 PT Nets, Event Structures and Domains

In this section, we show an adjunction between occurrence nets and decorated occurrence nets. Composing this adjunction with that given in Section 1.7, we obtain an adjunction between Occ and PTNets. Moreover, exploiting Winskel's coreflections in [143], we obtain adjunctions between PES and PTNets and between Dom and PTNets, as explained in the Introduction. For the sake of completeness, at the end of this section, we give, in an appendix-like style, the basic guidelines of the coreflection of Occ in PES and of the equivalence of PES and Dom.

We first define a functor from decorated occurrence nets to occurrence nets. It is simply the *forgetful* functor which forgets about the structure of families.

Definition 1.8.1 *($\mathcal{F}[\_]$: from $\underline{\mathrm{DecOcc}}$ to $\underline{\mathrm{Occ}}$)*
Given a decorated occurrence net Θ define $\mathcal{F}[\Theta]$ to be the occurrence net Θ. Furthermore, given $\langle f, g\rangle\colon \Theta_0 \to \Theta_1$, define $\mathcal{F}[\langle f, g\rangle]$ to be $\langle f, g\rangle$.

In order to define a left adjoint for $\mathcal{F}[\_]$, we need to identify, for any occurrence net Θ, a decorated occurrence net $\mathcal{D}[\Theta]$ which is, informally speaking, a "saturated" version of Θ, in the sense that it can match in a *unique* way the structure of any decorated occurrence net whose transitions are "similar" to those of Θ. More precisely, the existence of an adjunction requires $\mathcal{D}[\Theta]$ to be such that, for any occurrence net Θ',

$$\underline{\mathrm{Occ}}[\Theta, \mathcal{F}[\Theta']] \cong \underline{\mathrm{DecOcc}}[\mathcal{D}[\Theta], \Theta']$$

i.e., the set of morphisms from Θ to $\mathcal{F}[\Theta']$ in $\underline{\mathrm{Occ}}$ and the set of morphisms from $\mathcal{D}[\Theta]$ to Θ' in $\underline{\mathrm{DecOcc}}$ are isomorphic. It follows from this condition that each transition of $\mathcal{D}[\Theta]$ must have enough families in its post-set to "cover" those in the post-set of any transition of Θ' to which it could be mapped by an occurrence net morphism and, at the same time, it must not have too many of them so that such a covering is realized by a unique decorated occurrence net morphism from $\mathcal{D}[\Theta]$ to Θ'.

Because of the uniqueness requirement, saturating occurrence nets is a delicate matter: we need to identify a suitable set of families which can "represent" *uniquely* all the possible others. To this aim are devoted the following definition and lemma, where the relation $\mapsto$ is introduced to capture the behaviour of decorated occurrence net morphisms on families—which will be represented as strings on appropriate alphabets—and *prime strings* are meant to represent—in a sense that will be clear later—exactly the families which we must add to Θ in order to saturate it.

In the following, given a string s on an alphabet Σ, as usual we denote the i-th element of s by s_i and its length by $|s|$. Moreover, σ^n, for $\sigma \in \Sigma$ and $n \in \omega$, will denote the string consisting of the symbol σ repeated n times.

Definition 1.8.2 *(Prime Strings)*
Let Σ be an alphabet, i.e., a set of symbols. Define the binary relation $\mapsto$ on Σ^+, the language of non-empty strings on Σ, by

$$\sigma_1^{n_1} \cdots \sigma_k^{n_k} \mapsto \sigma_1^{m_1} \cdots \sigma_k^{m_k} \Leftrightarrow \sigma_i \neq \sigma_{i+1} \text{ and } \exists q \in \omega \text{ s.t. } qn_i = m_i,\ i = 1, \ldots, k.$$

Define the language of prime strings on Σ to be

$$\Sigma^P = \Sigma^+ - \{\sigma_1^{n_1}\sigma_2^{n_2} \cdots \sigma_k^{n_k} \mid \sigma_i \in \Sigma,\ \sigma_i \neq \sigma_{i+1},\ \gcd(n_1, \ldots, n_k) > 1\},$$

where gcd *is the greatest common divisor.*

Lemma 1.8.3 *(Prime Strings are primes)*
Given $s' \in \Sigma^+$ there exists a unique $s \in \Sigma^P$ such that $s \mapsto s'$.

Proof. Let $s' = \sigma_1^{m_1} \cdots \sigma_k^{m_k}$, where $\sigma_i \neq \sigma_{i+1}$. Consider $h = \gcd(m_1, \ldots, m_k)$. Since h is the unique integer such that m_i is divisible by h for $1 \leq i \leq k$ and $\gcd(\frac{m_1}{h}, \ldots, \frac{m_k}{h}) = 1$, and since h always exists (possibly $h = 1$) we have that $s = \sigma_1^{m_1/h} \cdots \sigma_k^{m_k/h}$ is the unique prime string such that $s \mapsto s'$. ✓

We start relating strings and nets by looking at sets of places as alphabets and at families as strings on such alphabets. Given a (decorated) occurrence net Θ and a transition $t \in T_\Theta$, we denote by $\Sigma_{\{t\}}$ the alphabet $[\![\partial^1_\Theta(t)]\!]$. By analogy, since the places in the initial marking are in the post-set of no transition, $\Sigma_\varnothing$ will consist of the places $[\![u_\Theta]\!]$; following the analogy, in the rest of the section u_Θ will also be denoted by $\partial^1_\Theta(\varnothing)$.

Since a family b^F of a decorated occurrence net Θ is nothing but an ordered subset of the initial marking or of the post-set of a transition, it corresponds naturally to a string in Σ_x^+ where $x = {}^\bullet[\![b^F]\!]$, namely, the string of length $|[\![b^F]\!]|$ whose i-th element is (b, i). We will write $\hat{b}^F$ to indicate such a string.

Now, we can define the saturated net corresponding to an occurrence net Θ. It is the net $\mathcal{D}[\Theta]$ whose transitions are the transitions of Θ, and whose families in the post-set of a transition t are the prime strings on the alphabet defined by t in Θ. It is immediate to see that this construction is well-defined, i.e., that $\mathcal{D}[\Theta]$ is a decorated occurrence net.

Definition 1.8.4 *($\mathcal{D}[\_]$: from $\underline{\mathsf{Occ}}$ to $\underline{\mathsf{DecOcc}}$)*
Let Θ be a net in $\underline{\mathsf{Occ}}$. We define the decorated occurrence net

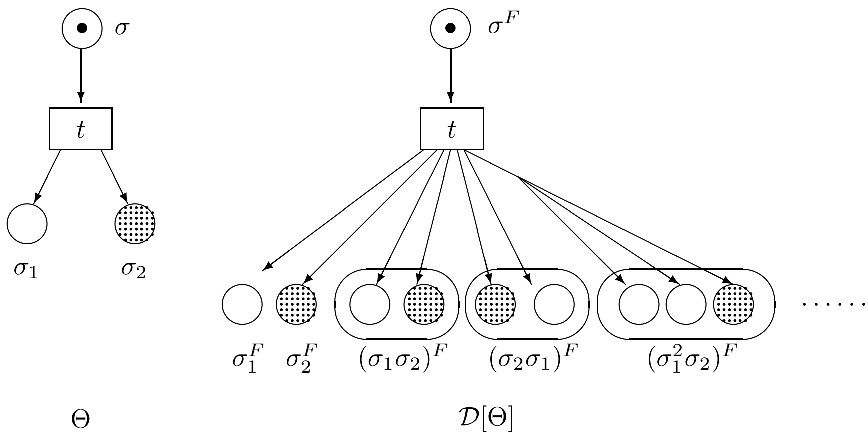
$$\mathcal{D}[\Theta] = \left(\partial^0_{\mathcal{D}[\Theta]}, \partial^1_{\mathcal{D}[\Theta]} \colon (T_\Theta, 0) \to S^{\mathcal{M}}_{\mathcal{D}[\Theta]}, u_{\mathcal{D}[\Theta]}\right),$$

where

- $S_{\mathcal{D}[\Theta]} = \bigcup \Big\{ \{s\} \times \Big[|\, s \,| \Big] \;\Big|\; s \in \Sigma_x^P \text{ and } (x = \{t\} \subseteq T_\Theta \text{ or } x = \varnothing) \Big\}$;
- $\partial^0_{\mathcal{D}[\Theta]}(t) = \bigoplus \Big\{ (s, i) \in S_{\mathcal{D}[\Theta]} \mid s_i \in [\![\partial^0_\Theta(t)]\!] \Big\}$;
- $\partial^1_{\mathcal{D}[\Theta]}(t) = \bigoplus \Big\{ (s, i) \in S_{\mathcal{D}[\Theta]} \mid s_i \in [\![\partial^1_\Theta(t)]\!] \Big\} = \bigoplus \Big\{ s^F \mid s \in \Sigma^P_{\{t\}} \Big\}$;
- $u_{\mathcal{D}[\Theta]} = \bigoplus \Big\{ s^F \mid s \in \Sigma^P_\varnothing \Big\}$.

The following example shows the decorated occurrence net corresponding to a very simple occurrence net Θ. The place σ_2 is filled to the purpose of making the difference between the family $(\sigma_1\sigma_2)^F$ and the family $(\sigma_2\sigma_1)^F$ of $\mathcal{D}[\Theta]$ graphically suggestive. Of course, only part of $\mathcal{D}[\Theta]$ is shown.

EXAMPLE 1.8.5



An occurrence net Θ and (part of) the decorated occurrence net $\mathcal{D}[\Theta]$

We now select a candidate for the unit of the adjunction.

PROPOSITION 1.8.6 *(Unit Morphism)*
Given an occurrence net Θ consider the map $\eta_\Theta: \Theta \to \mathcal{FD}[\Theta]$ defined by:

$$\begin{array}{rcl}\eta_\Theta(t) & = & t; \\ \eta_\Theta(a) & = & \bigoplus\{(s,i) \in S_{\mathcal{D}[\Theta]} \mid s_i = a\}.\end{array}$$

Then η_Θ is a morphism in <u>Occ</u>.

Proof. The only non-trivial case is that of condition *(iii)* in the definition of morphisms:

$$\begin{array}{rcl}\eta_\Theta(\partial^i_\Theta(t)) & = & \bigoplus \eta_\Theta\Big(\{a \mid a \in [\![\partial^i_\Theta(t)]\!]\}\Big) \\ & = & \bigoplus\Big\{(s,i) \mid s_i = a \text{ and } a \in [\![\partial^i_\Theta(t)]\!]\Big\} \quad = \quad \partial^i_{\mathcal{FD}[\Theta]}(t).\end{array}$$

✓

In order to illustrate the above definition, consider again the net Θ of Example 1.8.5. For such a net we have that

$$\eta_\Theta(\sigma_1) = (\sigma_1, 1) \oplus (\sigma_1\sigma_2, 1) \oplus (\sigma_2\sigma_1, 2) \oplus (\sigma_1^2\sigma_2, 1) \oplus (\sigma_1^2\sigma_2, 2) \oplus \cdots;$$

$$\eta_\Theta(\sigma_2) = (\sigma_2, 1) \oplus (\sigma_1\sigma_2, 2) \oplus (\sigma_2\sigma_1, 1) \oplus (\sigma_1^2\sigma_2, 2) \oplus \cdots .$$

Before showing that η_Θ is universal, we need to develop further the relation between nets and strings. Since a morphism maps post-sets to post-sets, it naturally induces a (contravariant) mapping between the languages associated to transitions related by the morphism. To simplify the exposition, in the rest of this section, for k a morphism of nets, $k(\{t\})$ and $k(\varnothing)$, denote, respectively, $\{k(t)\}$ and $\varnothing$; moreover, $\partial^1_\Theta(\{t\})$ denotes $\partial^1_\Theta(t)$.

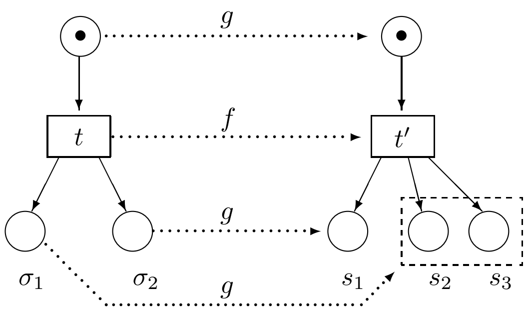
Definition 1.8.7 *($\mathcal{S}^x_k$: from $\Sigma^+_{k(x)}$ to Σ^+_x)*
Let Θ_0 and Θ_1 be (decorated) occurrence nets, let $k = \langle f, g\rangle\colon \Theta_0 \to \Theta_1$ be a morphism and let $x = \{t\} \subseteq T_{\Theta_0}$ or $x = \varnothing$ and y be such that $f(x) = y$. Then k induces a unique semigroup homomorphism $\mathcal{S}^x_k$ from Σ^+_y to Σ^+_x defined on the generators $b \in [\![\partial^1_{\Theta_1}(y)]\!]$ by

$$\mathcal{S}^x_k(b) = a \in [\![\partial^1_{\Theta_0}(x)]\!] \quad \textit{such that} \quad g(a) = b.$$

From the properties of safe net morphisms in Proposition 1.5.3, it is easy to see that $\mathcal{S}^x_k$ is well-defined, i.e., there exists one and only one $a \in [\![\partial^1_{\Theta_0}(x)]\!]$ such that $g(a) = b$.

To clarify the relation between $\mapsto$ and decorated occurrence net morphisms, observe that, in the condition of the previous definition, if Θ is a decorated occurrence net and k is a decorated occurrence net morphism, then $\hat{a}^F \mapsto \mathcal{S}^x_k(\hat{b}^F)$ if and only if $k(a^F) = b^F$.

Example 1.8.8
Consider the following figure, where the morphism $\langle f, g\rangle$ is such that $g(\sigma_1) = s_2 \oplus s_3$ and $g(\sigma_2) = s_1$.

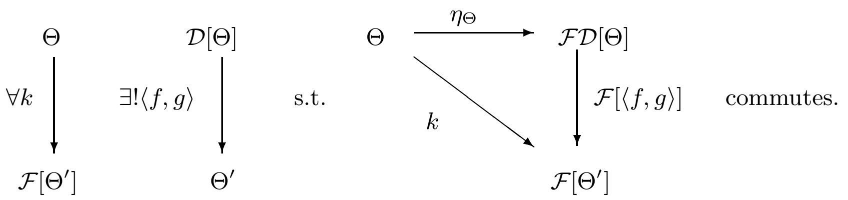


Then, for instance, we have that $\mathcal{S}^{\{t\}}_{\langle f,g\rangle}(s_1 s_2 s_3 s_2 s_1) = \sigma_2 \sigma_1^3 \sigma_2$.

Finally, we show that $\mathcal{D}[\_]$ extends to a functor which is left adjoint to $\mathcal{F}[\_]$.

Theorem 1.8.9 *($\mathcal{D} \dashv \mathcal{F}$)*
The pair $\langle \mathcal{D}, \mathcal{F}\rangle : \underline{\mathsf{Occ}} \rightharpoonup \underline{\mathsf{DecOcc}}$ constitutes an adjunction.

Proof. Let Θ be an occurrence net. By [90, Theorem 2, pg. 81] it is enough to show that the morphism $\eta_\Theta\colon \Theta \to \mathcal{FD}[\Theta]$ is universal from Θ to $\mathcal{F}$, i.e., for any decorated occurrence net Θ' and any $k\colon \Theta \to \mathcal{F}[\Theta']$ in $\underline{\mathsf{Occ}}$, there exists a unique $\langle f, g\rangle\colon \mathcal{D}[\Theta] \to \Theta'$ in $\underline{\mathsf{DecOcc}}$ such that $k = \mathcal{F}[\langle f, g\rangle] \circ \eta_\Theta$.



Given Θ' and k, we define $\langle f, g\rangle\colon \mathcal{D}[\Theta] \to \Theta'$ as follows:

$$f(t) = k(t)$$

$$[\![b^F]\!] \subseteq [\![g(s^F)]\!] \Leftrightarrow s \mapsto \mathcal{S}^x_k(\hat{b}^F), \text{ where } x = {}^\bullet[\![s^F]\!] \text{ and } k(x) = {}^\bullet[\![b^F]\!]$$

First note that $\langle f, g\rangle$ is well-defined: if $s = \sigma_1^{n_1} \cdots \sigma_r^{n_r} \mapsto \mathcal{S}^x_k(\hat{b}^F)$ then there is one and only one way to have $[\![b^F]\!] \subseteq [\![g(s^F)]\!]$, namely

$$g(s, i) = \bigoplus \{b\} \times [q]_i,$$

where q is the unique integer such that $\sigma_1^{qn_1} \cdots \sigma_r^{qn_r} = \mathcal{S}^x_k(\hat{b}^F)$.

Let $x = \{t_0\}$ or $x = \varnothing$. Observe that $\forall a \in [\![\partial^1_{\Theta'}(x)]\!]$

$$\forall (b, j) \in [\![k(a)]\!]\ \exists!(s, i) \text{ such that } (s, i) \in [\![\partial^1_{\mathcal{D}[\Theta]}(x)]\!] \text{ and } (b, j) \in [\![g(s, i)]\!]. \qquad (1.25)$$

Moreover, (s, i) is the unique place in $\mathcal{D}[\Theta]$ such that $s_i = a$ and $(b, j) \in [\![g(s, i)]\!]$.

In fact, given $x = {}^\bullet a$, by Lemma 1.8.3, there exists a unique $s \in \Sigma^P_x$ such that $s \mapsto \mathcal{S}^x_k(\hat{b}^F)$. If $(b, j) \in [\![k(a)]\!]$ then, since k is a morphism, $k(x) = {}^\bullet[\![b^F]\!]$ and so there exists a unique s^F in $\partial^1_{\mathcal{D}[\Theta]}(x)$ such that $[\![b^F]\!] \subseteq [\![g(s^F)]\!]$, i.e., $\exists!(s, i) \in [\![\partial^1_{\mathcal{D}[\Theta]}(x)]\!]$ such that $(b, j) \in [\![g(s, i)]\!]$.

Obviously $s_i = a$, by definition of $\mathcal{S}^x_k$ and $\mapsto$. Moreover if there were another such (s', j), then $s' \in \Sigma^P_x$ since a belongs only to Σ_x. So by the previous lemma $s' = s$ and, since g respects families, $j = i$.

Now, if $(b, j) \in [\![g(s, i)]\!]$ then $s \mapsto \mathcal{S}^x_k(\hat{b}^F)$ and therefore, by definition of $\mapsto$, we have $\mathcal{S}^x_k(\hat{b}^F)_{(i-1)q+1}, \ldots, \mathcal{S}^x_k(\hat{b}^F)_{iq} = s_i$. Thus, by definition of $\mathcal{S}^x_k$, $\{b\} \times [q]_i \subseteq [\![k(s_i)]\!]$. So we have $\bigcup\{[\![g(s, i)]\!] \mid s_i = a\} = [\![k(a)]\!]$. Obviously, all the $[\![g(s, i)]\!]$ are disjoint and $\bigoplus [\![g(s, i)]\!] = g(s, i)$, since the families are disjoint. Therefore,

$$\bigoplus \{g(s, i) \mid s_i = a\} = k(a).$$

It is now easy to see that the diagram commutes. For transitions this is clear. Concerning places, we have:

$$\mathcal{F}[\langle f, g\rangle] \circ \eta_\Theta(a) = \bigoplus g\Big(\{s_i \mid s_i = a\}\Big) = k(a).$$

Now, consider any morphism $h\colon \mathcal{D}[\Theta] \to N$ which makes the diagram commute. Because of the definition of η_Θ on the transitions, h must be of the form $\langle f, g' \rangle$. We have to show that, necessarily $g = g'$.

Let $[\![b^F]\!] \subseteq [\![g'(s^F)]\!]$. So, the family of b must be $\{b\} \times [qn]$ for some q, where $n = |s|$. Since $\langle f, g' \rangle$ is a morphism, given $x = {}^\bullet[\![s^F]\!]$ and $y = {}^\bullet[\![b^F]\!]$, it must be $f(x) = y$. Since the diagram must commute and s^F is the unique family in $\partial^1_{\mathcal{D}[\Theta]}(x)$ whose image contains b^F, it must be $\{b\} \times [q]_i \subseteq [\![k(s_i)]\!]$ for $i = 1, \dots, n$. Therefore, we have $\mathcal{S}^x_k(\hat{b}^F) = s_1^q \cdots s_n^q$ and so $s \mapsto \mathcal{S}^x_k(\hat{b}^F)$, which means, by definition of g, that $[\![b^F]\!] \subseteq [\![g(s^F)]\!]$. Hence, we have $[\![g'(s^F)]\!] \subseteq [\![g(s^F)]\!]$.

On the other hand, suppose $[\![b^F]\!] \subseteq [\![g(s^F)]\!]$. Then $s \mapsto \mathcal{S}^x_k(\hat{b}^F)$, for some x. Necessarily, it must exist s'^F with $s' \in \Sigma^P_x$ such that $[\![b^F]\!] \subseteq [\![g'(s'^F)]\!]$. Then, by Lemma 1.8.3, such an s' cannot be anything but s. Therefore $[\![g(s^F)]\!] \subseteq [\![g'(s^F)]\!]$ and as done before $g'(s^F) = g(s^F)$. Therefore, we conclude that $g' = g$ and $h = \langle f, g \rangle$.

Let us now show that $\langle f, g \rangle$ is a morphism. It is enough to verify conditions *(i)* and *(ii)* in Proposition 1.5.3.

Let $x = \{t_0\}$ or $x = \varnothing$ and $f(x) = y$. If $[\![b^F]\!] \subseteq [\![g(\partial^1_{\mathcal{D}[\Theta]}(x))]\!]$, then by definition of g, we have $[\![b^F]\!] \subseteq [\![\partial^1_{\Theta'}(y)]\!]$. So $[\![g(\partial^1_{\mathcal{D}[\Theta]}(x))]\!] \subseteq [\![\partial^1_{\Theta'}(y)]\!]$. Observe that this, together with property (1.25), proves the required conditions both on the initial marking and on $\partial^1_{\mathcal{D}[\Theta]}$. We still have to check that $\langle f, g \rangle$ respects sources.

Suppose $f(t_0) = t_1$. Let $(s, i) \in [\![\partial^0_{\mathcal{D}[\Theta]}(t_0)]\!]$ and $(b, j) \in [\![g(s, i)]\!]$. Then $s_i \in [\![\partial^0_\Theta(t_0)]\!]$ and since $\bigoplus\{g(s, i) \mid s_i = a\} = k(a)$, we have $(b, j) \in [\![k(s_i)]\!] \subseteq [\![\partial^0_{\Theta'}(t_1)]\!]$, since k is a morphism and $k(t_0) = t_1$. So $[\![g(\partial^0_{\mathcal{D}[\Theta]}(t_0))]\!] \subseteq [\![\partial^0_{\Theta'}(f(t_0))]\!]$.

Now, if $(b, j) \in [\![\partial^0_{\Theta'}(t_1)]\!]$ there exists a unique $a \in [\![\partial^0_\Theta(t_0)]\!]$ such that $(b, j) \in [\![k(a)]\!]$. Therefore, there exists a unique (s, i) such that $(b, j) \in [\![g(s, i)]\!]$ and $s_i = a$. Thus, $(s, i) \in [\![\partial^0_{\mathcal{D}[\Theta]}(t_0)]\!]$. Now, if $(s', j) \in [\![\partial^0_{\mathcal{D}[\Theta]}(t_0)]\!]$ is such that $(b, j) \in [\![g(s', j)]\!]$ it must be $s'_j = a$, otherwise a would not be the unique element in $[\![\partial^0_\Theta(t_0)]\!]$ whose image contains (b, j). Therefore, $(s', j) = (s, i)$. ✓

The next corollary summarizes the results we have by means of the adjunction $\langle \mathcal{D}[\_], \mathcal{F}[\_] \rangle : \underline{\mathsf{Occ}} \rightharpoonup \underline{\mathsf{DecOcc}}$ and by means of Winskel's coreflections $\langle \mathcal{N}[\_], \mathcal{E}[\_] \rangle : \underline{\mathsf{PES}} \rightharpoonup \underline{\mathsf{Occ}}$ and $\langle \mathcal{P}r[\_], \mathcal{L}[\_] \rangle : \underline{\mathsf{Dom}} \rightharpoonup \underline{\mathsf{PES}}$.

COROLLARY 1.8.10 *(Extensions of Winskel's coreflections [143])*
The following are adjunctions whose right adjoints relate PT nets to, respectively, occurrence nets, prime event structures and prime algebraic domains.

- $\langle (\_)^+ \mathcal{D}[\_], \mathcal{FU}[\_] \rangle : \underline{\mathsf{Occ}} \rightharpoonup \underline{\mathsf{PTNets}}$;
- $\langle (\_)^+ \mathcal{DN}[\_], \mathcal{EFU}[\_] \rangle : \underline{\mathsf{PES}} \rightharpoonup \underline{\mathsf{PTNets}}$;
- $\langle (\_)^+ \mathcal{DNP}r[\_], \mathcal{LEFU}[\_] \rangle : \underline{\mathsf{Dom}} \rightharpoonup \underline{\mathsf{PTNets}}$.

In addition, $\mathcal{FU}[\_]_{\underline{\mathsf{Safe}}} = \mathcal{U}_w[\_]$ *and, therefore,* $\mathcal{EFU}[\_]_{\underline{\mathsf{Safe}}} = \mathcal{EU}_w[\_]$ *and* $\mathcal{LEFU}[\_]_{\underline{\mathsf{Safe}}} = \mathcal{LEU}_w[\_]$*, i.e., the semantics given to safe nets by the chain of adjunctions presented in this work coincides with the semantics given by Winskel's chain of coreflections.* ✓

Occurrence Nets, Event Structures and Domains

This subsection contains a brief summary of Winskel's work on the coreflection of $\underline{\mathsf{PES}}$ in $\underline{\mathsf{Occ}}$ and on the equivalence of the category of prime event structures (with binary conflict) and the category of finitary, (coherent), prime algebraic domains [143] (see also [20, 148]).

Prime event structures [143] are the simplest event based model of computation. They consist of a set of events, intended as indivisible *quanta* of computation, which are related to each other by two binary relation: *causality*, modelled by a partial order relation $\leq$, and conflict, modelled by an irreflexive, symmetric and hereditary relation $\#$.

Definition 1.8.11 *(Prime Event Structures and* $\underline{\mathsf{PES}}$*)*
A prime event structure is a structure $ES = (E, \#, \leq)$ *consisting of a set of events* E *partially ordered by* $\leq$*, and a symmetric, irreflexive relation* $\# \subseteq E \times E$*, the conflict relation, such that*

$$\{e' \in E \mid e' \leq e\} \text{ is finite for each } e \in E$$

$$e \# e' \leq e'' \text{ implies } e \# e'' \text{ for each } e, e', e'' \in E.$$

For an event $e \in E$*, define* $\lfloor e \rfloor = \{e' \in E \mid e' \leq e\}$.

A prime event structure morphism $\theta\colon (E_0, \leq_0, \#_0) \to (E_1, \leq_1, \#_1)$ *is a partial function* $\theta\colon E_0 \rightharpoonup E_1$ *such that*

$$\theta(e) \text{ is defined implies } \lfloor \theta(e) \rfloor \subseteq \theta(\lfloor e \rfloor)$$

$$(\theta(e) \#_1 \theta(e') \text{ or } \theta(e) = \theta(e')) \text{ implies } (e \#_0 e' \text{ or } e = e')$$

This defines the category $\underline{\mathsf{PES}}$ *of prime event structures.*

The computational intuition behind event structures is really simple: an event e can occur when all its *causes*, i.e. $\lfloor e \rfloor$, have occurred and no event which it is in conflict with has already occurred. This is formalized by the following notion of *configuration*, which gives the computations of event structures.

Definition 1.8.12 *(Configurations)*
Given a prime event structure $(E, \#, \leq)$*, define its configurations to be those subsets*

$x \subseteq E$ which are

$$\textit{Conflict Free: } \forall e_1, e_2 \in x,\ \mathit{not}(e_1 \# e_2)$$
$$\textit{Left Closed: } \forall e \in x\ \forall e' \le e,\ e' \in x$$

Let $\mathcal{L}(ES)$ denote the set of configurations of the prime event structure E.

We first recall the relationships between PES and Occ.

Definition 1.8.13
Let Θ be an occurrence net. Then, $\mathcal{E}[\Theta]$ is the event structure $(T_\Theta, \preccurlyeq, \#)$, where $\preccurlyeq$ and $\#$ are the restriction to the set of transitions of Θ of, respectively, the flow ordering and the conflict relation implicitly defined by Θ.

Of course, for $\langle f, g\rangle : N_0 \to N_1$, we define $\mathcal{E}[\langle f, g\rangle] = f : \mathcal{E}[N_0] \to \mathcal{E}[N_1]$, which clearly gives a functor $\mathcal{E}$: Occ $\to$ PES.

Consider now the event structure $ES = (E, \le, \#)$. As a notation, for a subset A in E, we write $\#A$ to mean that for all $a, a' \in A$ if $a \neq a'$ then $a \# a'$. Similarly, $e < A$ means that $e < e'$ for all $e' \in A$. Then, define

$$\mathcal{N}[ES] = (\partial^0, \partial^1 : E \to (M \cup B)^\oplus, \bigoplus M),$$

where

- $M = \{(\varnothing, A) \mid A \subseteq E \text{ and } \#A\}$;
- $B = \{(e, A) \mid e \in E,\ \#A \text{ and } e < A\}$;
- $\partial^0(e) = \bigoplus\{(c, A) \mid (c, A) \in B \cup M \text{ and } e \in A\}$;
- $\partial^1(e) = \bigoplus\{(e, A) \mid (e, A) \in B\}$.

Then, we have the following.

Proposition 1.8.14
$\mathcal{N}[ES]$ is an occurrence net such that $\mathcal{E}\mathcal{N}[ES] = ES$. Moreover, the identity function $ES \to \mathcal{E}\mathcal{N}[ES] = ES$ is universal from ES to $\mathcal{E}[\_]$. Therefore, $\mathcal{N}[\_]$ extends to a functor left adjoint to $\mathcal{E}[\_]$.

Finitary prime algebraic domains or dI-domains—introduced by G. Berry while studying sequentiality of functions [8]—are particular Scott's domains which are distributive and in which each finite element is preceded only by a finite number of elements of the domain.

DEFINITION 1.8.15 *(Finitary (Coherent) Prime Algebraic Domains)*
Let $(D, \sqsubseteq)$ be a partial order. Recall that a set $X \subseteq D$ is directed *if all the pairs $x, y \in X$ have an upper bound in X, is* compatible *if there exists $d \in D$ such that $x \sqsubseteq d$ for all $x \in X$ and is* pairwise compatible *if $\{x, y\}$ is compatible for all $x, y \in X$. We say that D is a (coherent) domain if it is* pairwise complete*, i.e., if for all pairwise compatible $X \subseteq D$ there exists the least upper bound of X, in symbols $\bigsqcup X$, in D.*

A complete prime *of D is an element $p \in D$ such that, for any compatible $X \subseteq D$, if $p \sqsubseteq \bigsqcup X$, then there exists $x \in X$ such that $p \sqsubseteq x$. We say that a domain D is* prime algebraic *if for all $d \in D$ we have $d = \bigsqcup\{p \sqsubseteq d \mid p \text{ is a complete prime}\}$.*

Moreover, an element $e \in D$ is finite *if for any directed $S \subseteq D$, if $e \sqsubseteq \bigsqcup S$, then there exists $s \in S$ such that $e \sqsubseteq s$. We say that D is* finitary *if for all finite elements $e \in D$, $|\{d \sqsubseteq e \mid d \in D\}| \in \omega$.*

In the following we shall refer to pairwise complete, prime algebraic and finitary partial orders simply as *domains*. In order to simplify the definition of domain morphisms, the relation of immediate precedence $\prec$ is needed.

DEFINITION 1.8.16 *(Immediate Precedence)*
Given a domain D, let $\prec$ be the binary relation on D defined by

$$d \preccurlyeq d' \quad \text{if} \quad d \sqsubseteq d' \ \wedge \ (\forall z \ d \sqsubseteq z \sqsubseteq d' \Rightarrow d = z \ \vee \ d' = z);$$
$$d \prec d' \quad \text{if} \quad d \preccurlyeq d' \ \wedge \ d \neq d'.$$

DEFINITION 1.8.17 *(Domain Morphisms and $\underline{\mathsf{Dom}}$)*
Let D_0 and D_1 be domains. A domain morphism *$f \colon D_0 \to D_1$ is a function which is*

$$\textit{Additive:} \quad \forall X \subseteq D_0, \ X \text{ pairwise compatible}, \ f(\bigsqcup X) = \bigsqcup f(X);$$
$$\textit{Stable:} \quad \forall X \subseteq D_0, \ X \neq \varnothing, \ f(\sqcap X) = \sqcap f(X);$$
$$\preccurlyeq\textit{-preserving:} \quad \forall x, y \in D_0, \ x \prec y \quad \Rightarrow \quad f(x) \preccurlyeq f(y).$$

The category $\underline{\mathsf{Dom}}$ is the category whose objects are domains and whose arrows are domain morphisms.

The categories $\underline{\mathsf{PES}}$ and $\underline{\mathsf{Dom}}$ are related by an *adjoint equivalence*, i.e., they are equivalent categories. Now, let us recall the functors which constitute such an equivalence.

PROPOSITION 1.8.18 *($\mathcal{L}$ from $\underline{\mathsf{PES}}$ to $\underline{\mathsf{Dom}}$)*
The mapping $\mathcal{L}[\_]$ which maps a prime event structure ES to $(\mathcal{L}(ES), \subseteq)$, its set

of configurations ordered by inclusion, and which maps a prime event structure morphism $\theta\colon ES_0 \rightharpoonup ES_1$ *to the function* $\mathcal{L}[\theta]\colon (\mathcal{L}(ES_0), \subseteq) \to (\mathcal{L}(ES_1), \subseteq)$*, defined by*

$$\mathcal{L}(\theta)(x) = \theta(x),$$

is a functor from <u>PES</u> *to* <u>Dom</u>*.*

Moreover, the complete primes of $\mathcal{L}[ES]$ *are the elements* $\lfloor e \rfloor$ *for* $e \in E$*.*

The definition on the objects of the "quasi-inverse" functor from <u>Dom</u> to <u>PES</u> is as easy as that of $\mathcal{L}$.

Proposition 1.8.19 *($\mathcal{P}r$: from* <u>Dom</u> *to* <u>PES</u>*, part I)*
Given a domain D*, let* P_D *denote its set of complete primes. Then* $\mathcal{P}r(D)$ *is the prime event structure* $(P_D, \#, \leq)$*, where*

$$p \leq p' \quad \text{if} \quad p \sqsubseteq p' \qquad \text{and} \qquad p \# p' \quad \text{if} \quad \text{not } \exists\, p \sqcup p'$$

is a prime event structure.

However, as far a morphisms is concerned, the behaviour of $\mathcal{P}r$ is a bit more complex. In order to be able to define it, we are required to look more closely to domain morphisms.

Definition 1.8.20 *(Prime Intervals)*
A prime interval of a domain D *is a pair* $[d, d']$ *such that* $d \prec d'$*. Define*

$$[c, c'] \leq [d, d'] \quad \text{if} \quad (c = c' \sqcap d) \text{ and } (c' \sqcup d = d'),$$

and let $\sim$ *be the equivalence relation obtained as the transitive and symmetric closure of (the preorder)* $\leq$*.*

Then, the following result can be shown.

Proposition 1.8.21 *($\sim$-classes are complete primes and viceversa)*
Given a domain D*, the map*

$$[d, d']_\sim \mapsto p,$$

where $\{p\} = \phi(d') \setminus \phi(d)$*, is an isomorphism of the* $\sim$*–classes of prime intervals of* D *and the complete primes* P_D *of* D*, whose inverse is the function*

$$p \mapsto \Big[\bigsqcup\{c \sqsubseteq p \ \wedge\ c \neq p\}, p\Big]_\sim.$$

LEMMA 1.8.22
Let $f: D_0 \to D_1$ be a morphism in $\underline{\mathsf{Dom}}$. Then

$$[c, c'] \sim [d, d'] \quad \text{and} \quad f(c) \prec f(c') \quad \Rightarrow \quad f(d) \prec f(d') \text{ and } [f(c), f(c')] \sim [f(d), f(d')]$$

It is now easy to define $\mathcal{P}r$ on domain morphisms.

PROPOSITION 1.8.23 *($\mathcal{P}r$: from $\underline{\mathsf{Dom}}$ to $\underline{\mathsf{PES}}$, part II)*
For $f: D_0 \to D_1$ in $\underline{\mathsf{Dom}}$ define $\mathcal{P}r(f): \mathcal{P}r(D_0) \to \mathcal{P}r(D_1)$ by

$$\mathcal{P}r(f)(p) = p' \quad \text{if} \quad p \mapsto [d, d']_\sim \text{ and } f(d) \prec f(d') \text{ and } [f(d), f(d')]_\sim \mapsto p'.$$

The lemma above guarantees that this definition is well given. Moreover, it makes $\mathcal{P}r: \underline{\mathsf{Dom}} \to \underline{\mathsf{PES}}$ into a functor.

The functors $\mathcal{L}$ and $\mathcal{P}r$ are adjoints via the unit and counit of such adjunction and established by the following.

THEOREM 1.8.24 *(ψ, θ, ψ, η are isomorphisms)*
Given a domain D, the function $\phi: D \to \mathcal{L}\mathcal{P}r(D)$, defined by

$$\phi(d) = \{p \sqsubseteq d \mid p \in P_D\},$$

is a domain morphism which is an isomorphism with inverse $\theta: \mathcal{L}\mathcal{P}r(D) \to D$, given by $\theta(x) = \bigsqcup x$. Given a prime event structure E the map $\psi: \mathcal{P}r\mathcal{L}(E) \to E$, defined by

$$\psi(\lfloor e \rfloor) = e,$$

is a morphism of prime event structures which is an isomorphism whose inverse $\eta: E \to \mathcal{P}r\mathcal{L}(E)$ is given by $\eta(e) = \lfloor e \rfloor$.

THEOREM 1.8.25 *(ψ, θ, ψ, η are natural)*
ϕ is a natural isomorphism $\mathbf{1} \cong \mathcal{L}\mathcal{P}r$ with inverse θ.
ψ is a natural isomorphism $\mathcal{P}r\mathcal{L} \cong \mathbf{1}$ with inverse η.

Therefore, we can conclude that

$$\mathcal{L}: \underline{\mathsf{PES}} \to \underline{\mathsf{Dom}} \quad \text{and} \quad \mathcal{P}r: \underline{\mathsf{Dom}} \to \underline{\mathsf{PES}}$$

are equivalences of categories and, equivalently, that

$$\langle \mathcal{P}r, \mathcal{L}, \phi, \psi \rangle: \underline{\mathsf{Dom}} \rightharpoonup \underline{\mathsf{PES}} \text{ and } \langle \mathcal{L}, \mathcal{P}r, \eta, \theta \rangle: \underline{\mathsf{PES}} \rightharpoonup \underline{\mathsf{Dom}}$$

are adjoint equivalences.

1.9 Process versus Unfolding Semantics of Nets

The semantics obtained via the unfolding yields an explanation of the behaviour of nets in terms of event structures, that is, in terms of domains. Domains can be unambiguously thought as partial orderings of computations, where a computation is represented by a configuration, which, in our context, is a "downward" closed, conflict free set of occurrences of transitions. On the other hand, processes are by definition left closed and conflict free (multi)sets of transition. Moreover, the processes from a given initial marking are naturally organized in a preorder-like fashion via a comma category construction. The question which therefore arises spontaneously concerns the relationships between these two notions, and this is the question addressed in this section.

It is worth noticing that in the case of safe nets the question is readily answered exploiting Winskel's coreflection $\langle \hookrightarrow, \mathcal{U}[\_] \rangle : \underline{\mathsf{Occ}} \rightharpoonup \underline{\mathsf{Safe}}$. In fact, by definition an adjunction $\langle \mathsf{F}, \mathsf{G} \rangle : \underline{\mathsf{C}} \rightharpoonup \underline{\mathsf{D}}$ determines an isomorphism between arrows of the kind $\mathsf{F}(c) \to d$ in $\underline{\mathsf{D}}$ and the arrows of the kind $c \to \mathsf{G}(d)$ in $\underline{\mathsf{C}}$. Then, in the case of safe nets, we have a one-to-one correspondence

$$\pi : \Theta \to N \quad \Longleftrightarrow \quad \pi' : \Theta \to \mathcal{U}[N]$$

for each safe net N and each occurrence net Θ. Therefore, since such correspondence is easily seen to map processes to processes, in this special case, the correspondence between process and unfolding semantics of N is very tidy: they are the same notion in the precise sense that there is an isomorphism between the processes of N and the processes of $\mathcal{U}[N]$, i.e., the deterministic finite subnets of the unfolding of N, i.e., the finite configurations of $\mathcal{EU}[N]$.

In our context, however, we have that the unfolding of N is strictly more concrete than the processes of N. For example, consider the simple net N and its unfolding $\mathcal{FU}[N]$ shown in Figure 1.11. Clearly, there is a unique process of N in which a single instance of t has occurred. Nevertheless, there are two deterministic subnets of $\mathcal{FU}[N]$ which correspond to such process, namely those obtained by choosing respectively the left and the right instance of t. It is worth noticing that such subnets are isomorphic and that this is not a fortunate case, since it is easy to show that two finite deterministic subnets of $\mathcal{FU}[N]$ correspond to the same process of N if and only if they are isomorphic via an isomorphism which sends instances of an element of N to instances of the same element. More interestingly, the results in this paper will prove that this is the exact relationship between the two semantics of N: the unfolding contains several copies of the same process which, as illustrated in the preceding Sections 1.7 and 1.8 , are needed to provide a fully *causal* explanation of the behaviour of N, i.e., to obtain an occurrence net whose transitions represent exactly the instances of the transitions of N in all the possible causal contexts and which can therefore account for concurrent multiple instances

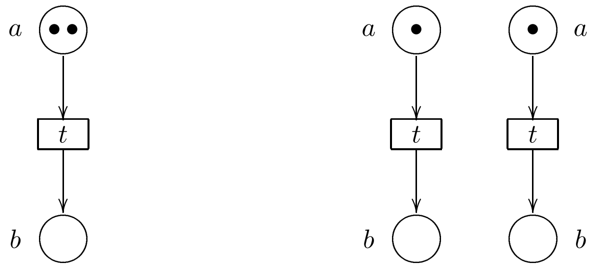


Figure 1.11: A net N and its unfolding $\mathcal{FU}[N]$

of the same element of N, that is for *autoconcurrency*. More precisely, we shall see that the finite deterministic subnets of the unfolding of N can be characterized by appropriately *decorating* the processes of N, which shows directly that the difference between process and unfolding semantics of N is due only to the replication of data needed in the latter.

In the following, we shall also see that the difference between the concatenable processes of N and the concatenable processes of $\mathcal{FU}[N]$ arises from a single axiom, namely the part $t; s = t$ of axiom (Ψ) of Definition 1.1.16. Then, it is easy to observe that the evident modification of $\mathcal{P}[\_]$ identifies a *symmetric strict monoidal category* which gives a full account of the unfolding semantics. In other words, we introduce a new notion of process $\mathcal{DP}[\_]$, in the style of [16], which gives a *process-oriented* account of the unfolding construction, in the precise sense that, as already stated in the introduction, for each PT net N and for each initial marking u we have

$$\begin{array}{ccccc} N & \longmapsto & \mathcal{DP}[N] & & \\ \big\downarrow & & & \searrow & \\ & & & & \mathcal{L}_F\mathcal{EFU}[(N,u)] \cong \langle u{\downarrow}\mathcal{DP}[N]\rangle \\ & & & \nearrow & \\ (N,u) & \longmapsto & \mathcal{EFU}[(N,u)] & & \end{array} \qquad (1.26)$$

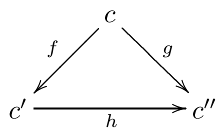
It is also worthwhile to notice that the part $t; s = t$ of axiom (Ψ) captures the essential difference between occurrence nets and decorated occurrence nets. Therefore, decorated (*deterministic*) occurrence nets—we shall call them *decorated process* nets—which at first seem to be just a convenient technical solution to establish the adjunction from PT nets to occurrence nets, provide *both* the process and the algebraic counterpart of the unfolding semantics. This also indicates that decorated process nets and their algebraic formalization $\mathcal{DP}[\_]$ are structures of interest on their own, being the minimal refinement of Goltz-Reisig processes which guarantees the identity of all tokens. In fact, in order to achieve this it is necessary to disambiguate both the tokens in the same place of the initial marking and the tokens which are multiple instances of the same place, and, therefore, to introduce the notion of *families*.

NOTATION. Observe that the categories $\underline{\mathsf{Petri}}_0$ and $\underline{\mathsf{PTNets}}$ are incomparable, since $\underline{\mathsf{Petri}}_0$ has less objects but more morphisms. However, as we already noticed in the remark in page 65 at the beginning of Section 1.5, dropping the restriction of morphisms given by axiom *(v)* in Definition 1.5.1 has no consequences on the definition of $\mathcal{U}[\_]$ and, therefore, on the definition of $\mathcal{FU}[\_]$. The same of course applies to the existence of the pointed element. Thus, in the following we shall consider $\mathcal{U}[\_]$ on $\underline{\mathsf{MPetri}}^*$, the full subcategory of $\underline{\mathsf{MPetri}}$ consisting of the marked nets which belong also to $\underline{\mathsf{PTNets}}$, i.e., those nets N such that $\partial^0_N(t) \neq 0$ for all t.

Moreover, to simplify notation, we shall use a single letter to denote net morphisms. The context will nearly always say to which component of the morphism we are referring to. However, when confusion is likely to arise, given $f\colon N_0 \to N_1$, we shall write use f_t and f_p to indicate, respectively, the transition and the place components of f.

To the sake of readability, we recall the following simple notion from category theory, explained in further details in Appendix A.1.

DEFINITION 1.9.1 *(Comma Categories)*
Let $\underline{\mathsf{C}}$ be a category and c an object of $\underline{\mathsf{C}}$. Then, the comma category $\langle c{\downarrow}\underline{\mathsf{C}}\rangle$, also called the category of elements under c, is the category whose objects are the arrows $f\colon c \to c'$ of $\underline{\mathsf{C}}$ and whose arrows $h\colon (f\colon c \to c') \to (g\colon c \to c'')$ are commutative diagrams



Identities and arrows composition are inherited in the obvious way from $\underline{\mathsf{C}}$.

We start by observing that finite configurations of $\mathcal{EFU}[N]$ coincide with finite deterministic subnets of $\mathcal{FU}[N]$. We extract from the theory in Section 1.5 the following easy useful properties of morphisms between occurrence nets.

LEMMA 1.9.2
Let $f\colon \Theta_0 \to \Theta_1$ be a morphism of marked PT nets between the occurrence nets Θ_0 and Θ_1 which maps places to places. Then, for all $x \in T_{\Theta_0} \cup S_{\Theta_0}$ we have $\mathrm{depth}(f(x)) = \mathrm{depth}(x)$.

Proof. By induction on the depth of x. Since marked PT net morphisms map initial markings to initial markings, the thesis holds in the base case, i.e., if $\mathrm{depth}(x) = 0$.

<u>inductive step</u>. Let n be the depth of x and suppose that x is a transition. Then, by definition of depth, we have that $\mathrm{depth}(y) \leq n-1$ for all $y \in {}^\bullet x$ and that there exists

$z \in {}^\bullet x$ such that $\mathrm{depth}(z) = n-1$. Then, since $f({}^\bullet x) = {}^\bullet f(x)$, the thesis follows immediately by induction. If instead x is a place we have that $\mathrm{depth}(t) = n$, where t is the unique element in ${}^\bullet x$. Then, as we just proved, $\mathrm{depth}(f(t)) = n$ and since $f(x) \in f(t)^\bullet$ the proof is concluded. ✓

LEMMA 1.9.3
Let $f\colon \Theta_0 \to \Theta_1$ be a morphism of marked PT nets between the occurrence nets Θ_0 and Θ_1 which maps places to places. Consider $x \in T_{\Theta_0} \cup S_{\Theta_0}$ and suppose that $y \preccurlyeq f(x)$ for some $y \in T_{\Theta_1} \cup S_{\Theta_1}$. Then, there exists $\bar{y} \preccurlyeq x$ such that $f(\bar{y}) = y$.

Proof. In order to show the thesis, it is enough to consider the following two cases.

i) Suppose that $a \in t^\bullet$ and $f(\bar{a}) = a$. Since a does not belong to the initial marking of Θ_1, then $\bar{a}$ cannot belong to the initial marking of Θ_0. Therefore, there exists a unique $\bar{t} \in {}^\bullet\bar{a}$ and, necessarily, $f(\bar{t}) = t$.

ii) Suppose that $a \in {}^\bullet t$ and that $f(\bar{t}) = t$. Then, since $f({}^\bullet\bar{t}) = {}^\bullet t$ and since f maps places to places, it must exists $\bar{a} \in S_{\Theta_0}$ such that $f(\bar{a}) = a$. ✓

LEMMA 1.9.4
Let $f\colon \Theta_0 \to \Theta_1$ be a morphism of marked PT nets between the occurrence nets Θ_0 and Θ_1 which maps places to places, and consider elements x and y in $T_{\Theta_0} \cup S_{\Theta_0}$. Then, if $f(x) = f(y)$ or $f(x) \# f(y)$, we have $x = y$ or $x \# y$.

Proof. We proceed by induction on the least of the depths of x and y.

base case. If $\mathrm{depth}(x) = \mathrm{depth}(y) = 0$, then $f(x) = f(y)$. In fact, in this case x and y belong to the initial marking of Θ_0 and thus, by definition of marked morphism, $f(x)$ and $f(y)$ are in the initial marking of Θ_1. It follows that they cannot be in conflict, since $\# \cap [\![u_\Theta]\!] \times [\![u_\Theta]\!] = \varnothing$. Now, if $x \neq y$, we have $f(u_{\Theta_0}) = f(x \oplus y \oplus u) = f(x) \oplus f(y) \oplus f(u) = 2f(x) \oplus f(u)$. But this is impossible, since $f(u_{\Theta_0}) = u_{\Theta_1}$ and each token in u_{Θ_1} has multiplicity one.

inductive step. Let $n \geq 1$ be the least of the depths of x and y. Without loss of generality, assume $\mathrm{depth}(x) = n$. First suppose that $f(x) = f(y)$. Then, there exist $z \in {}^\bullet x$ and $z' \in {}^\bullet y$ such that $f(z) = f(z')$. Then, if x is a transition, $\mathrm{depth}(z) < n$ and therefore, by induction, $f(z) = f(z')$ or $f(z) \# f(z')$, whence it follows that $f(x) = f(y)$ or $f(x) \# f(y)$. If instead x is a place, then z is a transition at depth n and the induction is maintained exploiting the proof given above for such a case.

Suppose instead that $f(x) \# f(y)$. By definition, this means that there exist t_0 and t_1 in T_{Θ_1} such that $t_0 \#_m t_1$, $t_0 \preccurlyeq f(x)$ and $t_1 \preccurlyeq f(y)$. Then, by Lemma 1.9.3, there exist $\bar{t}_0 \preccurlyeq x$ and $\bar{t}_1 \preccurlyeq y$ in T_{Θ_0} such that $f(\bar{t}_0) = t_0$ and $f(\bar{t}_1) = t_1$. This concludes the proof since, by induction, we have $\bar{t}_0 \# \bar{t}_1$ which implies $x \# y$. ✓

Observe that the restriction to morphisms which map places to places in the previous lemmas is not necessary to show that morphisms of occurrence nets preserve the depth of elements and reflect $\preccurlyeq$-chains and the conflict relation. However, the formulations above suffices for application in the following propositions.

PROPOSITION 1.9.5
Let N be a marked net in $\underline{\mathsf{MPetri}}^*$. There is an isomorphism between the set of finite configurations of $\mathcal{EFU}[N]$ and the set of (marked) processes π of $\mathcal{FU}[N]$.

Proof. Let ϕ be the function which maps a process $\pi\colon \Theta \to \mathcal{FU}[N]$ to the set of transitions $\pi(T_\Theta)$. Recall that π is a marked net morphism between occurrence nets which maps places to places. Then, by Lemma 1.9.4, we have that π maps concurrent transitions to concurrent transitions. Since Θ is a process net, and thus deterministic, $\pi(T_\Theta)$ is conflict free. Consider now $t \in \pi(T_\Theta)$ and let $t' \in T_{\mathcal{FU}[N]}$ be such that $t' \preccurlyeq t$. Then, by Lemma 1.9.3, there exists $x \in T_\Theta$ such that $\pi(x) = t'$, i.e., $\pi(T_\Theta)$ is downwards closed and, thus, a finite configuration of $\mathcal{EFU}[N]$.

On the contrary, let X be a finite configuration of $\mathcal{EFU}[N]$. By depth of an element x of X we mean the length of the shortest chain in X whose maximal element is x; the depth of X is the greatest of the depths of its elements. We show by induction on the depth of X that there exists a unique (up to isomorphism) process $\pi\colon \Theta \to \mathcal{FU}[N]$ such that $\pi(T_\Theta) = X$.

<u>base case</u>. If $X = \varnothing$, let Θ be the subnet of depth zero of $\mathcal{FU}[N]$, i.e., the net consisting of the minimal places of $\mathcal{FU}[N]$, and let π be the inclusion $\mathcal{FU}[N]^{(0)} \hookrightarrow \mathcal{FU}[N]$. Clearly, π is the unique (marked) process of $\mathcal{FU}[N]$ such that $\phi(\pi) = \varnothing$.

<u>inductive step</u>. Suppose that the depth of X is $n+1$. Let Z be the set of elements of X at depth $n+1$. Since the elements of Z are necessarily maximal in X, the set $Y = X \setminus Z$ is a configuration of $\mathcal{EFU}[N]$. Moreover, the depth of Z is n. Then, by induction, there exists a unique $\pi\colon \Theta \to \mathcal{FU}[N]$ such that $\pi(T_\Theta) = Y$. Let $t \in Z$ and consider $a \in \partial^0_{\mathcal{FU}[N]}(t)$. We show that there exists a unique place $x_a \in S_\Theta$, which in addition is maximal, such that $\pi(x_a) = a$. The following two cases are possible.

i) ${}^\bullet a = \varnothing$. Then, a belongs to the initial marking of $\mathcal{FU}[N]$ and thus, by definition of marked net morphism, there exists a unique $x_a \in u_\Theta$ such that $\pi(x_a) = a$. Moreover, since by Lemma 1.9.2 π preserves the depth of elements, there is no other $x \in S_\Theta$ such that $\pi(x) = a$.

ii) ${}^\bullet a = \{t'\}$. Then, $t' \prec t$ and thus, since X is downwards closed, there exists $x \in T_\Theta$ such that $\pi(x) = t'$. It follows that we can find a unique $x_a \in x^\bullet$ such that $\pi(x_a) = a$. Now, since by Lemma 1.9.4 π maps concurrent transitions to concurrent transitions, x is the unique transition of Θ mapped to t'. Therefore, x_a is the unique place of Θ mapped to a.

Observe that x_a must be maximal in Θ. In fact, is there were $x \in x_a^\bullet$, there would be $\pi(x) \in X$ with $\pi(x) \# t$, which is impossible since X is a configuration.

Now, it is easy to see that π can be extended to a process π' such that $\phi(\pi') = X$ in essentially a unique way. To this purpose, consider the net Θ' obtained by adding to Θ, for each $t \in Z$, a new transition x_t and a new place $\bar{a}$ for each $a \in \partial^1_{\mathcal{FU}[N]}(t)$ with

$$\partial^0_{\Theta'}(x_t) = \bigoplus\{x_a \mid a \in \partial^0_{\mathcal{FU}[N]}(t)\} \quad \text{and} \quad \partial^1_{\Theta'}(x_t) = \bigoplus\{\bar{a} \mid a \in \partial^1_{\mathcal{FU}[N]}(t)\}.$$

Since $\mathcal{FU}[N]$ is an occurrence net, we have that $\partial^1_{\mathcal{FU}[N]}(t_0) \cap \partial^1_{\mathcal{FU}[N]}(t_1) = \varnothing$, for $t_0 \neq t_1$ in Z, and therefore, by definition, Θ' is an occurrence net. Moreover, since Z

is a set of concurrent transitions, we also have $\partial^0_{\mathcal{FU}[N]}(t_0) \cap \partial^0_{\mathcal{FU}[N]}(t_1) = \varnothing$. Then, considering also that each x_a is maximal in Θ, we conclude that Θ' is deterministic. Therefore, π' defined as

$$\pi'(x) = \begin{cases} \pi(x) & \text{if } x \in T_\Theta \cup S_\Theta \\ t & \text{if } x = x_t \text{ for } t \in Z \\ a & \text{if } x = \bar{a} \text{ for } a \in \partial^1_{\mathcal{FU}[N]}(t) \text{ and } t \in Z \end{cases}$$

is a process of $\mathcal{FU}[N]$ such that $\phi(\pi') = \pi'(T_{\Theta'}) = X$. Observe that, given the uniqueness of x_a, the only possible variation in the construction of π' is in the choice of "names" for the transitions and the places added to Θ. Then, since π is by inductive hypothesis the unique process such that $\pi(T_\Theta) = Y$, we conclude that π' is (up to isomorphism) the unique process such that $\pi'(T_{\Theta'}) = X$.

Therefore, ϕ is an isomorphism. ✓

Our next task is to characterize the processes of $\mathcal{FU}[N]$ in terms of processes of N. We shall do it by means of the following notion of decorated process.

DEFINITION 1.9.6 *(Decorated Processes)*
Let $N \in \underline{\mathsf{MPetri}}^*$. A decorated process of N is a triple $DP = (\pi, \ell, \tau)$ where

- *$\pi: \Theta \to N$ is a finite (marked) process of N;*
- *ℓ is a π-indexed ordering of $\min(\Theta)$;*
- *τ is a family $\{\tau(t)\}$ indexed by the non maximal transitions t of Θ, where each $\tau(t)$ is a π-indexed ordering of the post-set of t in Θ.*

The decorated processes $(\pi: \Theta \to N, \ell, \tau)$ and $(\pi': \Theta' \to N, \ell', \tau')$ are isomorphic, and then identified, if their underlying processes are isomorphic via an isomorphism φ which respects all the orderings, i.e., $\ell'_{\pi'(\varphi(a))}(\varphi(a)) = \ell_{\pi(a)}(a)$ for all $a \in \min(\Theta)$, and $\tau'(\varphi(t))_{\pi'(\varphi(a))}(\varphi(a)) = \tau(t)_{\pi(a)}(a)$ for all non maximal $t \in T_\Theta$ and for all $a \in t^\bullet$.

We say that $(\pi: \Theta \to N, \ell, \tau) \le (\pi': \Theta' \to N, \ell', \tau')$ if there exists $\varphi: \Theta \to \Theta'$ which preserves all the orderings and such that $\pi = \pi' \circ \varphi$. Clearly, the set of decorated processes of N is preordered by $\le$. We shall write $DP[N]$ to indicate such preordering.

PROPOSITION 1.9.7
$DP[N]$ is a partial order.

Proof. Consider $DP = (\pi: \Theta \to N, \ell, \tau)$ and $DP' = (\pi': \Theta' \to N, \ell', \tau')$, and suppose that $DP \le DP'$ and $DP' \le DP$. Then, by definition, there exist $\varphi: \Theta \to \Theta'$ and $\varphi': \Theta' \to \Theta$ which respect all the orderings and such that $\pi = \pi' \circ \varphi$ and $\pi' = \pi \circ \varphi'$.

Since we identify isomorphic decorated processes, to conclude the proof it is enough to show that φ is an isomorphism. Observe however that, since π and π' map places to places and since $\pi = \pi' \circ \varphi$, it follows that φ has to map places to places. The same of course holds for φ'. Then, we show the thesis by showing the following more general fact: whenever the process nets Θ and Θ' are linked by marked PT net morphisms $\varphi\colon \Theta \to \Theta'$ and $\varphi'\colon \Theta' \to \Theta$ which map places to places, then φ (φ') is an isomorphism. Observe that, because of the aforesaid property of its place component, in order to show that φ (φ') is an isomorphism it is enough to show that it is injective and surjective on both places and transitions.

injectivity. Since Θ is deterministic, it follows immediately in virtue of Lemma 1.9.2 and Lemma 1.9.4 that φ is injective. Of course, for the same reason, also φ' is injective.

surjectivity. By Lemma 1.9.2, we know that, for each $n \geq 1$ ($n \geq 0$), φ and φ' restrict to functions between the sets of transitions (places) at depth n of Θ and Θ'. Moreover, by definition of process nets, we have that such sets are finite. Then, the surjectivity of φ follows immediately from the injectivity of φ and φ' and from the following general fact: if $f\colon A \to B$ is an injective function between the *finite* sets A and B, and if there is injective function $g\colon B \to A$, then f (g) is surjective. We prove this claim by induction on the cardinality of A.

base case. If $A = \varnothing$, then necessarily $B = \varnothing$ and the thesis holds.

inductive step. Let $|A| = n+1$ and consider $x \in B$. We have to show that there exists $y \in A$ such that $f(x) = y$. Consider $y = g(x)$ in A and $x' = f(y)$ in B. If $x = x'$ we are done. Otherwise, since f and g are injective, they restrict, respectively, to injective functions $f\colon A \setminus \{y\} \to B \setminus \{x'\}$ and $g\colon B \setminus \{x'\} \to A \setminus \{y\}$. Then, by induction hypothesis, there exists $y' \in A \setminus \{y\}$, i.e., $y' \in A$, such that $f(y') = x$. ✓

Consider now the marked net morphism $\epsilon_N\colon \mathcal{FU}[N] \to N$ given by

$$((x,a),i) \mapsto a \quad \text{and} \quad (B,t) \mapsto t.$$

By construction of $\mathcal{FU}[N]$, this is clearly a morphism in MPetri* (though not in PTNets). Given its similarity with the folding morphism of $\mathcal{U}[N]$ into N defined in Proposition 1.7.5, i.e., the counit of the adjunction DecOcc ⇀ PTNets, we use the same notation for both the morphisms. Moreover, we shall refer to $\epsilon_N\colon \mathcal{FU}[N] \to N$ as the *folding* of $\mathcal{FU}[N]$ into N. However, the reader is warned *not* to confuse the two notions. The folding ϵ_N provides an obvious way to map a process $\pi\colon \Theta \to \mathcal{FU}[N]$ to a process of N, namely $\epsilon_N \circ \pi\colon \Theta \to N$. Moreover, we also have the following natural way of finding ℓ and τ which decorate this process and make it be a decorated process $P(\pi) = (\epsilon \circ \pi, \ell, \tau)$ of N.

- Let b be in $\min(\Theta)$ and suppose that $\pi(b) = ((\varnothing, a), i)$. Then, taking $\ell_a(b) = i$ clearly gives a $\epsilon\pi$-indexed ordering of $\min(\Theta)$.
- Let t be a transition of Θ, and consider $b \in t^\bullet$. Since π is a process morphism, its image through π must be a place in the post-set of $\pi(t)$, i.e., a component

of some family in $\pi(t)^\bullet$, say $\pi(b) = ((\pi(t), a), j)$. Then, taking $\tau(t)_a(b) = j$ clearly gives a $\epsilon\pi$-indexed ordering of $t^\bullet$.

In the opposite direction, we define a mapping F as follows. Let (π, ℓ, τ) be a decorated process of N with $\pi\colon \Theta \to N$. Then, $F(\pi, \ell, \tau)$ is $f\colon \Theta \to \mathcal{FU}[N]$ defined inductively as follows.

depth 0. For $b \in \min(\Theta)$, consider $f_p(b) = ((\varnothing, a), i)$ with $a = \pi(b)$ and $i = \ell_a(b)$, while, of course, for $t \in T_\Theta$, $f_t(t)$ is $([\![f_p({}^\bullet t)]\!], \pi(t))$.

depth n+1. Let t be an element of depth $n+1$ in T_Θ. Again, $f_t(t)$ is $([\![f_p({}^\bullet t)]\!], \pi(t))$. Consider now $b \in t^\bullet$. If t is not maximal, then $f_p(b) = ((\{f_t(t)\}, \pi(b)), i)$ for $i = \tau(t)_a(b)$. If instead t is maximal, take $g(b)$ to be any place of $\mathcal{FU}[N]$ (corresponding in $\mathcal{U}[N]$ to an element) in the family $(\pi(t), \pi(b))^F$ in such a way that the restriction of f_p to $\pi^{-1}(b) \cap t^\bullet$ is an isomorphism. It follows immediately from the definition of $\mathcal{U}[N]$ and from the fact that, by definition of process, π maps places to places, that this is always possible. Moreover, observe that all the different possible choices for $f_p(b)$ give rise to *isomorphic* decorated processes. For this, it is enough to consider the isomorphism of process nets $\varphi\colon \Theta \to \Theta$ which permutes appropriately the places in $\pi^{-1}(b) \cap t^\bullet$ and is the identity elsewhere.

Informally, the behaviour of P and F may be explained by saying that P and F just move the information about families, respectively, in ℓ and τ from π and back in π from ℓ and τ. Of course, we have that $FP(\pi) = \pi$ and it shows clearly in the construction of $F(\pi, \ell, \tau)$ that $PF(\pi, \ell, \tau)$ is (up to isomorphism) again (π, ℓ, τ). Therefore, we have shown the announced correspondence.

PROPOSITION 1.9.8
The set of decorated processes of N is isomorphic to the set of (marked) processes of $\mathcal{FU}[N]$ via the maps F and P given above.

The correspondence above can be easily lifted to the partial orders of decorated processes and finite configurations of $\mathcal{EFU}[N]$.

PROPOSITION 1.9.9
For any $N \in$ Petri*, $DP[N]$ is isomorphic to $\mathcal{L}_F\mathcal{EFU}[N]$.

Proof. We only need to show that, given the decorated processes $DP = (\pi\colon \Theta \to N, \ell, \tau)$ and $DP' = (\pi'\colon \Theta' \to N, \ell', \tau')$, we have $DP \leq DP'$ if and only if $\phi F(DP) \subseteq \phi F(DP')$, where ϕF gives the configuration corresponding to a marked decorated process as described by Proposition 1.9.5 and Proposition 1.9.8.

If $DP \leq DP'$, then there exists $\varphi\colon \Theta \to \Theta'$ which preserves the labellings and such that $\pi = \pi' \circ \varphi$. It follows immediately that φ is a morphism between the process nets underlying $F(DP)$ and $F(DP')$, and therefore $\phi F(DP) \subseteq \phi F(DP')$. The other implication comes along the same lines: if $\phi F(DP) \subseteq \phi F(DP')$, then there is a morphism φ from the process net underlying $F(DP)$, i.e., Θ, to the process net underlying $F(DP')$, i.e., Θ', such that $F(DP) = F(DP') \circ \varphi$. Clearly, φ is the marked net morphisms which maps the element x of Θ to the unique element of Θ' in $\mathsf{F}(DP')^{-1}(F(DP)(x))$. Then, φ is a morphism from Θ to Θ' which preserves the labellings ℓ and τ and such that $\pi = \pi' \circ \varphi$. Therefore, φ shows that $DP \leq DP'$. ✓

Exploiting further the idea of decorated processes, the same conceptual step which led from non-sequential processes to concatenable processes suggests the following definition.

Definition 1.9.10 *(Decorated Concatenable Processes)*
A decorated concatenable process of the (unmarked) net N in <u>Petri</u>*, is a quadruple (π, ℓ, τ, L) where (π, ℓ, L) is a concatenable process of N and τ is a family $\{\tau(t)\}$ indexed by all transitions t of Θ, where each $\tau(t)$ is a π-indexed ordering of the post-set of t in Θ.*

An isomorphism of decorated concatenable processes is an isomorphism of the underlying concatenable processes which, in addition, preserves all the orderings given by τ, i.e., $\tau'(\varphi(t))_{\pi'(\varphi(a))}(\varphi(a)) = \tau(t)_{\pi(a)}(a)$ for all $t \in T_\Theta$ and $a \in t^\bullet$.

So, a decorated concatenable process is a concatenable process where the post-sets of all transitions are π-indexed ordered. Observe that in the definition above (π, ℓ, τ) is different from a decorated process in two respects: first of all, π is *unmarked* and secondly, and more importantly, there is a component of τ also for the maximal transitions. It follows that a place in $\max(\Theta)$ has in every case a double ordering: derived from L and ℓ if it is also minimal, and derived from L and τ otherwise.

Since decorated concatenable processes are concatenable processes, they can be given a source and a target, namely those of the underlying concatenable process. Moreover, the concatenation of concatenable processes can be lifted to an operation on decorated concatenable processes. The concatenation of $(\pi_0, \ell_0, \tau_0, L_0): u \to v$ and $(\pi_1, \ell_1, \tau_1, L_1): v \to w$ is the decorated concatenable process $(\pi, \ell, \tau, L): u \to w$ defined as follows (see also Figure 1.12, where $\tau(t)$ is depicted by decorating the arcs outgoing from t). In order to simplify notation, we assume that the process nets corresponding to π_0 and π_1, say Θ_0 and Θ_1, are disjoint.

- Let A be the set of pairs (y, x) such that $x \in \max(\Theta_0)$, $y \in \min(\Theta_1)$, $\pi_0(x) = \pi_1(y)$ and $(\ell_1)_{\pi_1(y)}(y) = (L_0)_{\pi_0(x)}(x)$. By the definitions of decorated concatenable processes and of their sources and targets, A determines an isomorphism $A: \min(\Theta_1) \to \max(\Theta_0)$. Consider $S_1 = S_{\Theta_1} \setminus \min(\Theta_1)$, and let $in: S_{\Theta_1} \to S_{\Theta_0} \cup S_1$ be the function which is the identity on S_1 and maps $y \in \min(\Theta_1)$ to $A(y)$. Then,

$$\Theta = (\partial^0_\Theta, \partial^1_\Theta: T_{\Theta_0} \cup T_{\Theta_1} \to (S_{\Theta_0} \cup S_1)^\oplus),$$

where

– $\partial^0_\Theta(t) = \partial^0_{\Theta_0}(t)$ if $t \in T_{\Theta_0}$ and $\partial^0_\Theta(t) = in^\oplus(\partial^0_{\Theta_1}(t))$ if $t \in T_{\Theta_1}$;
– $\partial^1_\Theta(t) = \partial^1_{\Theta_i}(t)$ if $t \in T_{\Theta_i}$.

Then, $\pi: \Theta \to N$ coincides with π_0 on $S_{\Theta_0} \cup T_{\Theta_0}$ and with π_1 on $S_1 \cup T_{\Theta_1}$.

- $\ell = \ell_0$.
- $\tau(t) = \tau_i(t)$ if $t \in T_{\Theta_i}$.
- $L_a(y) = (L_1)_a(y)$ if $y \in S_1$, $L_a(x) = (L_1)_a(A^{-1}(x))$ if $x \in \max(\Theta_0)$.

Therefore, we can consider the category $\mathcal{DCP}[N]$ whose objects are the finite multisets on S_N and whose arrows are the decorated concatenable processes.

Proposition 1.9.11
Under the above defined operation of sequential composition, $\mathcal{DCP}[N]$ is a category with identities those decorated concatenable processes consisting only of places, which therefore are both minimal and maximal, and such that $\ell = L$.

Decorated concatenable processes admit also a tensor operation $\otimes$ such that, given $DCP_0 = (\pi_0, \ell_0, \tau_0, L_0): u \to v$ and $DCP_1 = (\pi_1, \ell_1, \tau_1, L_1): u' \to v'$, its tensor $DCP_0 \otimes DCP_1$ is the decorated concatenable process $(\pi, \ell, \tau, L): u \oplus u' \to v \oplus v'$ given below (see also Figure 1.12), where again we suppose that Θ_0 and Θ_1, the underlying process nets, are disjoint.

- $\Theta = (\partial^0_\Theta, \partial^1_\Theta: T_{\Theta_0} \cup T_{\Theta_1} \to (S_{\Theta_0} \cup S_{\Theta_1})^\oplus)$, where
  – $\partial^0_\Theta(t) = \partial^0_{\Theta_i}(t)$ if $t \in T_{\Theta_i}$;
  – $\partial^1_\Theta(t) = \partial^1_{\Theta_i}(t)$ if $t \in T_{\Theta_i}$.

  Then, $\pi: \Theta \to N$ is obviously given by $\pi(x) = \pi_i(x)$ for $x \in T_{\Theta_i} \cup S_{\Theta_i}$.
- $\ell_a(x) = (\ell_0)_a(x)$ if $x \in S_{\Theta_0}$, and $\ell_a(x) = |\pi_0^{-1}(a) \cap \min(\Theta_0)| + (\ell_1)_a(x)$ otherwise.
- $\tau(t) = \tau_i(t)$ if $t \in T_{\Theta_i}$.
- $L_a(x) = (L_0)_a(x)$ if $x \in S_{\Theta_0}$, and $L_a(x) = |\pi_1^{-1}(a) \cap \max(\Theta_1)| + (L_1)_a(x)$ otherwise.

It is easy to see that $\otimes$ is a functor from $\mathcal{DCP}[N] \times \mathcal{DCP}[N] \to \mathcal{DCP}[N]$. Moreover, as in the case of concatenable and strong concatenable processes, we have that the decorated concatenable processes consisting only of places play the role of the symmetries of monoidal categories. In particular, for any $u = n_1a_1 \oplus \cdots \oplus n_ka_k$ and $v = m_1b_1 \oplus \cdots \oplus m_hb_h$, the concatenable process having as many places as elements in the multiset $u \oplus v$ mapped by π to the corresponding places of N and

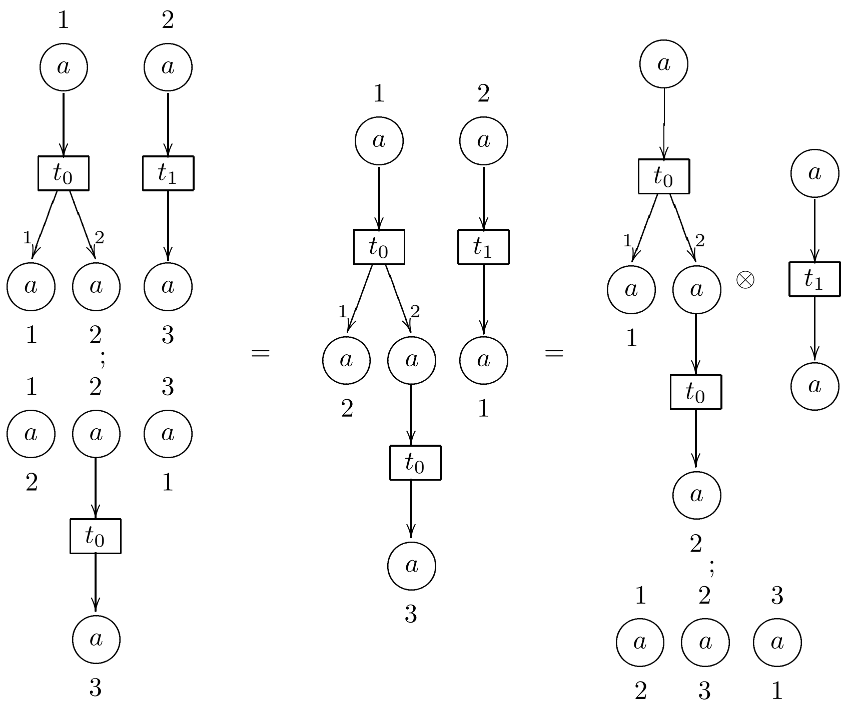


Figure 1.12: An example of the algebra of decorated concatenable processes

such that $L_{a_i}(x) = v(a_i) + \ell_{a_i}(x)$ and $\ell_{b_i}(x) = L_{b_i}(x) - u(b_i)$ (see also Figure 1.13) is the symmetry coherence isomorphism $\gamma_{u,v}$ with respect to which $\mathcal{DCP}[N]$ is a *symmetric monoidal category*, i.e., equations (1.2)–(1.6) hold in $\mathcal{DCP}[N]$ for the given family of $\gamma_{u,v}$ which, moreover, is a natural transformation. Therefore, we have the following.

Proposition 1.9.12
$\mathcal{DCP}[N]$ is a symmetric strict monoidal category with the symmetry isomorphism $\{\gamma_{u,v}\}_{u,v \in S_N^{\oplus}}$ given above.

Observe that, since the decorated concatenable processes consisting only of places are just concatenable processes, in fact the subcategory $Sym_{\mathcal{DCP}[N]}$ of symmetries of $\mathcal{DCP}[N]$ coincides with the corresponding one of $\mathcal{CP}[N]$. Such observation will be useful later on. Observe also that the transitions t of N are represented by decorated concatenable processes with a unique transition and two layers of places: the minimal, in one-to-one correspondence with $\partial^0_N(t)$, and the maximal, in one-to-one correspondence with $\partial^1_N(t)$ (see also Figure 1.13). The decoration, of course, consists in taking $\tau(t) = L$.

Recalling that the concatenable processes of N correspond to the arrows of $\mathcal{P}[N]$, and observing that the π-indexed orderings of the post-sets of the transitions of decorated concatenable processes is manifestly linked to the $t; s = t$ part of axioms (Ψ) in Definition 1.1.16, we are led to the following definition of the symmetric monoidal category $\mathcal{DP}[N]$ which captures the *algebraic essence* of decorated (concatenable) processes, and thus of the unfolding construction, simply by dropping that axiom in the definition of $\mathcal{P}[N]$.

DEFINITION 1.9.13 *(The category $\mathcal{DP}[N]$)*
Let N be a PT net in <u>Petri</u>. *Then $\mathcal{DP}[N]$ is the monoidal quotient of the free symmetric strict monoidal category on N modulo the axioms*

$$\begin{array}{rcll} \gamma_{a,b} & = & id_{a\oplus b} & \text{if } a,b \in S_N \text{ and } a \neq b \\ s;t & = & t & \text{if } t \in T_N \text{ and } s \text{ is a symmetry.} \end{array} \tag{1.27}$$

Explicitly, the category $\mathcal{DP}[N]$ is the category whose objects are the elements of $S_N^\oplus$ and whose arrows are generated by the inference rules

$$\frac{u \in S_N^\oplus}{id_u\colon u \to u \text{ in } \mathcal{DP}[N]} \qquad \frac{u, v \text{ in } S_N^\oplus}{c_{u,v}\colon u \oplus v \to u \oplus v \text{ in } \mathcal{DP}[N]} \qquad \frac{t\colon u \to v \text{ in } T_N}{t\colon u \to v \text{ in } \mathcal{DP}[N]}$$

$$\frac{\alpha\colon u \to v \text{ and } \beta\colon u' \to v' \text{ in } \mathcal{DP}[N]}{\alpha \otimes \beta\colon u \oplus u' \to v \oplus v' \text{ in } \mathcal{DP}[N]} \qquad \frac{\alpha\colon u \to v \text{ and } \beta\colon v \to w \text{ in } \mathcal{DP}[N]}{\alpha;\beta\colon u \to w \text{ in } \mathcal{DP}[N]}$$

modulo the axioms expressing that $\mathcal{DP}[N]$ is a strict monoidal category, namely,

$$\begin{array}{rcl} \alpha; id_v = \alpha = id_u; \alpha & \text{and} & (\alpha;\beta);\delta = \alpha;(\beta;\delta), \\ (\alpha\otimes\beta)\otimes\delta = \alpha\otimes(\beta\otimes\delta) & \text{and} & id_0 \otimes \alpha = \alpha = \alpha \otimes id_0, \\ id_u \otimes id_v = id_{u\oplus v} & \text{and} & (\alpha\otimes\alpha');(\beta\otimes\beta') = (\alpha;\beta)\otimes(\alpha';\beta'), \end{array} \tag{1.28}$$

the latter whenever the righthand term is defined, the following axioms expressing that $\mathcal{DP}[N]$ is symmetric with symmetry isomorphism c

$$\begin{array}{rcll} c_{u,v\oplus w} & = & (c_{u,v} \otimes id_w); (id_v \otimes c_{u,w}), & \\ c_{u,u'}; (\beta \otimes \alpha) & = & (\alpha \otimes \beta); c_{v,v'} & \text{for } \alpha\colon u \to v,\ \beta\colon u' \to v', \\ c_{u,v}; c_{v,u} & = & id_{u\oplus v}, & \end{array} \tag{1.29}$$

and the following axioms corresponding to axioms (1.27)

$$\begin{array}{rcll} c_{a,b} & = & id_{a\oplus b} & \text{if } a,b \in S_N \text{ and } a \neq b \\ (id_u \otimes c_{a,a} \otimes id_v); t & = & t & \text{if } t \in T. \end{array} \tag{1.30}$$

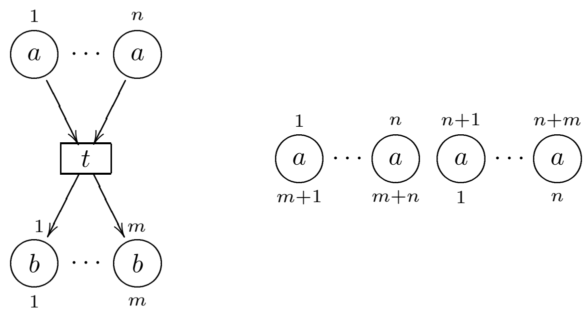


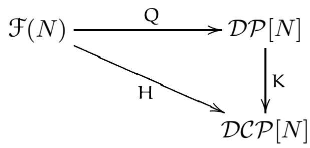
Figure 1.13: A transition $t\colon n \cdot a \rightarrow m \cdot b$ and the symmetry $\gamma_{n\cdot a, m\cdot a}$ in $\mathcal{DCP}[N]$

It is worthwhile to remark again that in the definition above axioms (1.28) and (1.29) define $\mathcal{F}(N)$, the free symmetric strict monoidal category on N. Moreover, as we already know from Section 1.2, exploiting the coherence axiom, i.e., the first of (1.29), a symmetry in $\mathcal{F}(N)$ can always be written as a composition of symmetries of the kind $(id_u \otimes c_{a,b} \otimes id_v)$ for $a, b \in S_N$. Then, since we have $c_{a,b} = id_{a\oplus b}$ if $a \neq b$, the second of (1.27) takes the particular form stated in (1.30).

The following expected result is now a simple revision of Proposition 1.1.19 and Proposition 1.4.5. Actually, the scheme of the demonstration is the one used in the proof of Proposition 1.4.5, which in turn matches that of the proof of Proposition 1.1.19 in [16, 17]. The differences reside of course in the use of the axioms and, in particular, in the proof of the case depth 1 of the induction, where the differences between the categories $\mathcal{P}[N]$, $\mathcal{Q}[N]$ and $\mathcal{DP}[N]$ play an evident role.

Proposition 1.9.14
$\mathcal{DCP}[N]$ and $\mathcal{DP}[N]$ are isomorphic.

Proof. Let $\mathcal{F}(N)$ be the free symmetric strict monoidal category on N (see the remark following Definition 1.9.13). Corresponding to the inclusion morphism $N \hookrightarrow \mathcal{DCP}[N]$, i.e., to the PT net morphism whose place component is the identity and whose transition component sends $t \in T_N$ to the corresponding decorated concatenable process (see Figure 1.13), there is a symmetric strict monoidal functor $\mathsf{H}\colon \mathcal{F}(N) \rightarrow \mathcal{DCP}[N]$. Observe that $\mathcal{DCP}[N]$ satisfies axioms (1.27), the symmetries and the transitions being as explained before (see Figure 1.13). In fact, if $a, b \in S_N$ and $a \neq b$, by definition of f-indexed ordering, we have $\gamma_{a,b} = id_{a\oplus b}$. Moreover, for S a symmetry and T a transition in $\mathcal{DCP}[N]$, it follows easily from the definition of $\_;\_$ that $S;T$ is (isomorphic to) T. Therefore, we have $\mathsf{H}(c_{a,b}) = \gamma_{a,b} = id_{a\oplus b}$ when $a \neq b \in S_N$, and since any symmetric monoidal functor sends symmetries to symmetries, we have $\mathsf{H}(s;t) = \mathsf{H}(s);\mathsf{H}(t) = S;\mathsf{H}(t) = \mathsf{H}(t)$, i.e., taking (1.27) as our set $\mathcal{E}$ of equations, H satisfies condition in *(ii)* of Proposition 1.2.2. Therefore, denoting by Q the quotient functor from $\mathcal{F}(N)$ to $\mathcal{DP}[N]$ induced by equations (1.27), by Proposition 1.2.2, there exists a (unique) symmetric strict monoidal functor $\mathsf{K}\colon \mathcal{DP}[N] \rightarrow \mathcal{DCP}[N]$ such that the diagram below commutes.



In the following we shall prove that K is an isomorphism. Observe that, by definition, for any $u \in S_N^{\oplus}$, we have $\mathsf{K}(u) = \mathsf{K}(\mathsf{Q}(u)) = \mathsf{H}(u) = u$, i.e., K is the identity on the objects. Moreover, we can easily conclude that it is an isomorphism on the symmetries. In fact, as already remarked, the decorated concatenable process of depth zero, i.e., the symmetries of $\mathcal{DCP}[N]$, are exactly the concatenable processes of depth zero, i.e., the symmetries of $\mathcal{CP}[N]$. Therefore, we have $Sym_{\mathcal{DCP}[N]} = Sym_{\mathcal{CP}[N]}$. Now observe that, by Proposition 1.2.5, we know that $\mathcal{P}[N]$ is the monoidal quotient of $\mathcal{DP}[N]$ modulo the axiom $t; s = t$. Then, since composing and tensoring terms containing transitions will never yield a symmetry, this axiom does not induce any equality on the symmetries. Therefore, we have that $Sym_{\mathcal{P}[N]} = Sym_{\mathcal{DP}[N]}$. Moreover, Proposition 1.1.19 shows that $Sym_{\mathcal{P}[N]}$ and $Sym_{\mathcal{CP}[N]}$ are isomorphic via a functor whose object component is the identity (see also Section 1.2 and [17, 121]). Now observe that, once the object component is fixed, there can be at most one symmetric strict monoidal functor F between two categories of symmetries. In fact, on the one hand we have that, by definition, the symmetries of a symmetric strict monoidal category are generated by the identities and the components of the isomorphism γ, while on the other hand, it must necessarily be $\mathsf{F}(id_u) = id_{\mathsf{F}(u)}$ and $\mathsf{F}(\gamma_{u,v}) = \gamma_{\mathsf{F}(u),\mathsf{F}(v)}$ (see axioms (1.7)–(1.9)). Then, since K is a symmetric strict monoidal functor whose object component is the identity, its restriction to $Sym_{\mathcal{DP}[N]}$ is an isomorphism $Sym_{\mathcal{DP}[N]} = Sym_{\mathcal{P}[N]} \cong Sym_{\mathcal{CP}[N]} = Sym_{\mathcal{DCP}[N]}$.

We proceed now to show that K is full and faithful.

<u>fullness</u>. It is completely obvious that any decorated concatenable process DCP may be obtained as a concatenation $DCP_0; \ldots; DCP_n$ of decorated concatenable processes DCP_i of depth one. Now, each of these DCP_i may be split into the concatenation of a symmetry S_0^i, the tensor of the (processes representing the) transitions which appear in DCP_i plus some identities, say $id_{u_i} \otimes \bigotimes_j \mathsf{K}(t_j^i)$, and finally another symmetry S_1^i. The intuition about this factorization is as follows. We take the tensor of the transitions which appear in DCP_i in any order and multiply the result by an identity concatenable process in order to get the correct source and target. Then, in general, we need a pre- and a post-concatenation with a symmetry in order to get the right indexing of minimal and maximal places and of the post-sets of each $\mathsf{K}(t_j^i)$. Thus, we finally have

$$DCP = S_0^0; (id_{u_1} \otimes \textstyle\bigotimes_j \mathsf{K}(t_j^1)); (S_1^0; S_0^1); \ldots; (S_1^{n-1}; S_0^n); (id_{u_n} \otimes \textstyle\bigotimes_j \mathsf{K}(t_j^n)); S_1^n$$

which shows that every decorated concatenable process is in the image of K.

<u>faithfulness</u>. The arrows of $\mathcal{DP}[N]$ are equivalence classes modulo the axioms stated in Definition 1.9.13 of terms built by applying tensor and sequentialization to the identities id_u, the symmetries $c_{u,v}$, and the transitions t. We have to show that, given

two such terms α and β, whenever $\mathsf{K}(\alpha) = \mathsf{K}(\beta)$ we have $\alpha =_{\mathcal{E}} \beta$, where $=_{\mathcal{E}}$ is the equivalence induced by (1.28), (1.29) and (1.30).

First of all, observe that if $\mathsf{K}(\alpha)$ is a decorated process DCP of depth n, then α can be proved equal to a term

$$\alpha' = s_0; (id_{u_1} \otimes \bigotimes_j t_j^1); s_1; \dots; s_{n-1}; (id_{u_n} \otimes \bigotimes_j t_j^n); s_n$$

where, for $1 \leq i \leq n$, the transitions t_j^i, for $1 \leq j \leq n_i$, are exactly the transitions of DCP at depth i and where $s_0, \dots, s_n$ are symmetries. Moreover, we can assume that in the i-th tensor product $\bigotimes_j t_j^i$ the transitions are indexed according to a global ordering $\leq$ of T_N assumed for the purpose of this proof, i.e., $t_1^i \leq \dots \leq t_{n_i}^i$, for $1 \leq i \leq n$. Let us prove our claim. It is easily shown by induction on the structure of terms that using axioms (1.28) α can be rewritten as $\alpha_1; \dots; \alpha_h$, where $\alpha_i = \bigotimes_k \xi_k^i$ and ξ_k^i is either a transition or a symmetry. Now, observe that by functoriality of $\otimes$, for any $\alpha': u' \to v'$, $\alpha'': u'' \to v''$ and $s: u \to u$, we have $\alpha' \otimes s \otimes \alpha'' = (id_{u'} \otimes s \otimes id_{u''}); (\alpha' \otimes id_u \otimes \alpha'')$, and thus, by repeated applications of (1.28), we can prove that α is equivalent to $\bar{s}_0; \bar{\alpha}_1; \bar{s}_1 \dots; \bar{s}_{h-1}; \bar{\alpha}_h$, where $\bar{s}_0, \dots, \bar{s}_{h-1}$ are symmetries and each $\bar{\alpha}_i$ is a tensor $\bigotimes_k \bar{\xi}_k^i$ of transitions and identities. The fact that the transitions at depth i can be brought to the i-th tensor product, follows intuitively from the facts that they are "disjointly enabled", i.e., concurrent to each other, and that they depend causally on some transition at depth $i-1$. In particular, the sources of the transitions of depth 1 can be target only of symmetries. Therefore, reasoning formally as above, they can be pushed up to $\bar{\alpha}_1$ exploiting axioms (1.28). Then, the same happens for the transitions of depth 2, which can be brought to $\bar{\alpha}_2$. Proceeding in this way, eventually we show that α is equivalent to the composition $\bar{\bar{s}}_0; \bar{\bar{\alpha}}_1; \bar{\bar{s}}_1 \dots; \bar{\bar{s}}_{n-1}; \bar{\bar{\alpha}}_n; \bar{\bar{s}}_n$ of the symmetries $\bar{\bar{s}}_0, \dots, \bar{\bar{s}}_n$ and the products $\bar{\bar{\alpha}}_i = \bigotimes_k \bar{\bar{\xi}}_k^i$ of transitions at depth i and identities. Finally, exploiting Lemma 1.4.4, the order of the $\bar{\bar{\xi}}_k^i$ can be permuted in the way required by $\leq$. This is achieved by pre- and post-composing each product by appropriate σ-interchange symmetries. More precisely, let σ be a permutation such that $\bigotimes_k \bar{\bar{\xi}}_{\sigma(k)}^i$ coincides with $id_{u_i} \otimes \bigotimes_j t_j^i$, suppose that $\bar{\bar{\xi}}_k^i: u_k^i \to v_k^i$, for $1 \leq k \leq k_i$ and let γ_σ be the σ-interchange symmetry guaranteed by Lemma 1.4.4 in $\mathcal{DP}[N]$. Then, since γ_σ is a natural transformation, we have that

$$\gamma_{\sigma\, u_1^i, \dots, u_{k_i}^i}; (\bigotimes_k \bar{\bar{\xi}}_{\sigma(k)}^i) = (\bigotimes_k \bar{\bar{\xi}}_k^i); \gamma_{\sigma\, v_1^i, \dots, v_{k_i}^i},$$

and then, since γ_σ is an isomorphism, we have that

$$(id_{u_i} \otimes \bigotimes_j t_j^i) = \gamma_{\sigma\, u_1^i, \dots, u_{k_i}^i}^{-1}; (\bigotimes_k \bar{\bar{\xi}}_k^i); \gamma_{\sigma\, v_1^i, \dots, v_{k_i}^i}.$$

Now, applying the same argument to β, one proves that it is equivalent to a term $\beta' = p_0; \beta_0; p_1; \dots p_{n-1}; \beta_n; p_n$, where $p_0, \dots, p_n$ are symmetries and β_i is the product of the transitions at depth i in $\mathsf{K}(\beta)$ and of identities. Then, since $\mathsf{K}(\alpha) = \mathsf{K}(\beta)$, and since the transitions occurring in β_i are indexed in a predetermined way, we conclude that $\beta_i = (id_{u_i} \otimes \bigotimes_j t^i_j)$, i.e.,

$$\begin{aligned}\alpha' &= s_0; (id_{u_1} \otimes \textstyle\bigotimes_j t^1_j); s_1; \ldots; s_{n-1}; (id_{u_n} \otimes \bigotimes_j t^n_j); s_n \\ \beta' &= p_0; (id_{u_1} \otimes \textstyle\bigotimes_j t^1_j); p_1; \ldots; p_{n-1}; (id_{u_n} \otimes \bigotimes_j t^n_j); p_n. \end{aligned} \tag{1.31}$$

In other words, the only possible differences between α' and β' are the symmetries. Observe now that the steps which led from α to α' and from β to β' have been performed by using the axioms which define $\mathcal{DP}[N]$ and since such axioms hold in $\mathcal{DCP}[N]$ as well and K preserves them, we have that $\mathsf{K}(\alpha') = \mathsf{K}(\alpha) = \mathsf{K}(\beta) = \mathsf{K}(\beta')$. Thus, we conclude the proof by showing that, if α and β are terms of the form given in (1.31) which differ only by the intermediate symmetries and if $\mathsf{K}(\alpha) = \mathsf{K}(\beta)$, then α and β are equal in $\mathcal{DP}[N]$.

We proceed by induction on n. Observe that if n is zero then there is nothing to show: since we know that K is an isomorphism on the symmetries, s_0 and p_0, and thus α and β, must coincide. To provide a correct basis for the induction, we need to prove the thesis also for $n = 1$.

depth 1. In this case, we have

$$\begin{aligned}\alpha &= s_0; (id_u \otimes \textstyle\bigotimes_j t_j); s_1 \\ \beta &= p_0; (id_u \otimes \textstyle\bigotimes_j t_j); p_1.\end{aligned}$$

Without loss of generality, we may assume that p_0 and p_1 are identities. In fact, we can multiply both terms by p_0^{-1} on the left and by p_1^{-1} on the right and obtain a pair of terms whose images through K still coincide and whose equality implies the equality in $\mathcal{DP}[N]$ of the original α and β.

Let $(\pi\colon \Theta \to N, \ell, \tau, L)$ be the decorated concatenable process $\mathsf{K}(id_u \otimes \bigotimes_j t_j)$. Of course, we can assume that $\mathsf{K}(s_0)$ and $\mathsf{K}(s_1)$ are respectively $(\pi_0\colon \Theta_0 \to N, \ell', \varnothing, \ell)$ and $(\pi_1\colon \Theta_1 \to N, L, \varnothing, L')$, where Θ_0 is $\min(\Theta)$, Θ_1 is $\max(\Theta)$, π_0 and π_1 are the corresponding restrictions of π, and ℓ' and L' are π-indexed orderings respectively of the minimal and the maximal places of Θ.

Then, we have that $\mathsf{K}(s_0; (id_u \otimes \bigotimes_j t_j); s_1)$ is $(\pi\colon \Theta \to N, \ell', \tau, L')$, and by hypothesis there is an isomorphism $\varphi\colon \Theta \to \Theta$ such that $\pi \circ \varphi = \pi$ and which respects all the orderings, i.e., $\ell'_{\pi(\varphi(a))}(\varphi(a)) = \ell_{\pi(a)}(a)$ and $L'_{\pi(\varphi(b))}(\varphi(b)) = L_{\pi(b)}(b)$, for all $a \in \Theta_0$ and $b \in \Theta_1$, and $\tau(\varphi(t))_{\pi\varphi(a)}(\varphi(a)) = \tau(t)_{\pi(a)}(a)$ for all $t \in \Theta$ and $a \in t^\bullet$. Let us write $id_u \otimes \bigotimes_j t_j$ as $\bigotimes_k \xi_k$, where ξ_k is either a transition t_j or the identity of a place in u. Moreover, let $\xi_k\colon u_k \to v_k$, for $1 \le k \le k_i$. Clearly, φ induces a permutation of the symbols ξ_k, namely the permutation σ such that $\xi_{\sigma(k)} = \varphi(\xi_k)$. Then, in order to be a morphism of nets, φ must map the (places corresponding to the) pre-set, respectively post-set, of t_j to (the places corresponding to the) pre-set, respectively post-set, of $t_{\sigma(j)}$. Observe now that this identifies φ uniquely on the maximal places of Θ, which implies that $\mathsf{K}(s_1)$ is completely determined. In fact, if a maximal place x is also minimal, then the corresponding ξ_k is the identity id_{u_k} and thus x must be mapped to the object for which $\xi_{\sigma(k)}$ is the identity. If instead x is in the post-set of t_j then x must be mapped to the post-set of $t_{\sigma(j)}$ in the unique way compatible with the family of π-indexed orderings τ. In other words, $\mathsf{K}(s_1)$ is the component at

$(v_1, \ldots, v_{k_i})$ of the σ-interchange symmetry. Then, since K is an isomorphism between $Sym_{\mathcal{DP}[N]}$ and $Sym_{\mathcal{DCP}[N]}$, s_1 must necessarily be the corresponding component of the σ-interchange symmetry in $\mathcal{DP}[N]$.

Concerning $\mathsf{K}(s_0)$, we cannot be so precise. However, since we know that the pre-sets of transitions are mapped by φ according to σ, reasoning as above we can conclude that $(\pi_0, \ell, \varnothing, \ell')$, which is $\mathsf{K}(s_0)^{-1}$, must be a symmetry obtained by post-concatenating the component at $(u_1, \ldots, u_{k_i})$ of the σ-interchange symmetry to some product $\bigotimes_j S_j$ of symmetries, one for each t occurring in α, whose role is to reorganize the tokens in the pre-sets of each transitions. It follows that s_0 is $\gamma^{-1}_{\sigma\, u_1,\ldots,u_{k_i}}; (id_u \otimes \bigotimes_j s_j)$, where s_j is a symmetry on the source of t_j.

Then, by distributing the tensor of symmetries on the transitions and using the second of (1.30), we show that $\alpha = \gamma^{-1}_{\sigma\, u_1,\ldots,u_{k_i}}; (id_u \otimes \bigotimes_j t_j); \gamma_{\sigma\, v_1,\ldots,v_{k_i}}$, which, by definition of σ-interchange symmetry, is $(id_u \otimes \bigotimes_j t_j)$. Thus, we have $\alpha =_{\mathcal{E}} \beta$ as required.

Inductive step. Suppose that $n > 1$ and let $\alpha = \alpha'; \alpha''$ and $\beta = \beta'; \beta''$, where

$$\begin{array}{rclcrcl} \alpha' & = & s_0; (id_{u_1} \otimes \bigotimes_j t^1_j); s_1; \ldots; s_{n-1} & \text{and} & \alpha'' & = & (id_{u_n} \otimes \bigotimes_j t^n_j); s_n \\ \beta' & = & p_0; (id_{u_1} \otimes \bigotimes_j t^1_j); p_1; \ldots; p_{n-1} & \text{and} & \beta'' & = & (id_{u_n} \otimes \bigotimes_j t^n_j); p_n \end{array}$$

We show that there exists a symmetry s in $\mathcal{DP}[N]$ such that $\mathsf{K}(\alpha'; s) = \mathsf{K}(\beta')$ and $\mathsf{K}(s^{-1}; \alpha'') = \mathsf{K}(\beta'')$. Then, by the induction hypothesis, we have $(\alpha'; s) =_{\mathcal{E}} \beta'$ and $(s^{-1}; \alpha'') =_{\mathcal{E}} \beta''$. Therefore, we conclude that $(\alpha'; s; s^{-1}; \alpha'') =_{\mathcal{E}} (\beta'; \beta'')$, i.e., that $\alpha = \beta$ in $\mathcal{DP}[N]$.

Let $(\pi\colon \Theta \to N, \ell, \tau, L)$ be the decorated concatenable process $\mathsf{K}(\alpha) = \mathsf{K}(\beta)$. Without loss of generality we may assume that the decorated occurrence nets $\mathsf{K}(\alpha')$ and $\mathsf{K}(\beta')$ are, respectively, $(\pi'\colon \Theta' \to N, \ell', \tau', L^{\alpha'})$ and $(\pi'\colon \Theta' \to N, \ell', \tau', L^{\beta'})$, where Θ' is the subnet of depth $n-1$ of Θ, ℓ' and τ' are the appropriate restrictions of ℓ and τ and finally $L^{\alpha'}$ and $L^{\beta'}$ are π-indexed orderings of the places at depth $n-1$ of Θ. Consider the symmetry $S = (\bar{\pi}, \bar{\ell}, \varnothing, \bar{L})$ in $\mathcal{DCP}[N]$, where

- $\bar{\Theta}$ is the process nets consisting of the maximal places of Θ';
- $\bar{\pi}\colon \bar{\Theta} \to N$ is the restriction of π to $\bar{\Theta}$;
- $\bar{\ell} = L^{\alpha'}$;
- $\bar{L} = L^{\beta'}$.

Then, by definition, we have $\mathsf{K}(\alpha'); S = \mathsf{K}(\beta')$. Let us consider now α'' and β''. Clearly, we can assume that $\mathsf{K}(\alpha'')$ and $\mathsf{K}(\beta'')$ are $(\pi''\colon \Theta'' \to N, \ell^{\alpha''}, \tau'', L'')$ and $(\pi''\colon \Theta'' \to N, \ell^{\beta''}, \tau'', L'')$, where Θ'' is the process net obtained by removing from Θ the subnet Θ', τ'' and L'' are respectively the restrictions of τ and L to Θ'', and $\ell^{\alpha''}$ and $\ell^{\beta''}$ are π-indexed orderings of the places at depth $n-1$ of Θ. Now, in our hypothesis, it must be $L^{\alpha'} = \ell^{\alpha''}$ and $L^{\beta'} = \ell^{\beta''}$, which shows directly that $S^{-1}; \mathsf{K}(\alpha'') = \mathsf{K}(\beta'')$. Then, $s = \mathsf{K}^{-1}(S)$ is the required symmetry of $\mathcal{DP}[N]$.

Then, since K is full and faithful and is an isomorphism on the objects, it is an isomorphism and the proof is concluded. ✓

We conclude the section by getting back to diagram (1.26). We first make the following observation on the structure of the comma category $\langle u_N \downarrow \mathcal{DP}[N]\rangle$.

PROPOSITION 1.9.15
The category $\langle u_N \downarrow \mathcal{DP}[N]\rangle$ is a preorder.

Proof. We have to show that in $\langle u_N \downarrow \mathcal{DP}[N]\rangle$ there is at most one arrow between any pair of objects $\alpha: u_N \to v$ and $\alpha': u_N \to w$. Exploiting the characterization of arrows of $\mathcal{DP}[N]$ in terms of decorated concatenable processes established by Proposition 1.9.14, the thesis can be reformulated as follows: for each pair of concatenable decorated processes $DCP_0: u_N \to v$ and $DCP_1: u_N \to w$ there exists at most one decorated concatenable process $DCP: v \to w$ such that $DCP_0; DCP = DCP_1$.

In order to show the claim, suppose that there exist DCP and DCP' from v to w such that $DCP_0; DCP = DCP_1 = DCP_0; DCP'$. Let $\bar{\pi}: \bar{\Theta} \to N$ and $\bar{\pi}': \bar{\Theta}' \to N$ be the (plain) processes underlying, respectively, $DCP_0; DCP$ and $DCP_1; DCP'$. Without loss of generality, we can assume that $\bar{\Theta}$, respectively $\bar{\Theta}'$, is formed by joining Θ_0, the process net underlying DCP_0, with Θ, the process net underlying DCP, respectively Θ', the process net underlying DCP'. Then, since $DCP_0; DCP = DCP_0; DCP'$, there exists an isomorphism $\varphi: \bar{\Theta} \to \bar{\Theta}'$ which respects all the orderings and such that $\bar{\pi} = \bar{\pi}' \circ \varphi$. Since, we can assume that φ restricts to the identity of Θ_0 (as a subnet of $\bar{\Theta}$ and $\bar{\Theta}'$), it follows that it restricts to an isomorphism $\varphi': \Theta \to \Theta'$ which shows $DCP = DCP'$. ✓

The next proposition essentially shows that, for any $(N, u_N) \in \underline{\mathsf{MPetri}}^*$, the canonical partial order associated to $\langle u_N \downarrow \mathcal{DP}[N]\rangle$ is $DP[(N, u_N)]$.

PROPOSITION 1.9.16
For any marked PT net (N, u_N) in $\underline{\mathsf{MPetri}}^*$,

$$\langle u_N \downarrow \mathcal{DP}[N]\rangle \cong DP[(N, u_N)] \cong \mathcal{L}_F \mathcal{EFU}[(N, u_N)].$$

Proof. Consider the mapping from the objects of $\langle u_N \downarrow \mathcal{DP}[N]\rangle$ to $DP[(N, u_N)]$ given by $(\pi, \ell, \tau, L) \mapsto (\pi, \ell, \bar{\tau})$, where $\bar{\tau}$ is the restriction of τ to the non maximal transitions of process net underlying π. Now, observe that there is a morphism from $DCP = (\pi: \Theta \to N, \ell, \tau, L)$ to $DCP' = (\pi': \Theta' \to N, \ell', \tau', L')$ in $\langle u_N \downarrow \mathcal{DP}[N]\rangle$ if and only if there exists a decorated concatenable process DCP'' such that $DCP; DCP'' = DCP'$ if and only if there exists $\varphi: \Theta \to \Theta'$ such that $\pi = \pi' \circ \varphi$ and which preserves all orderings, i.e., if and only if $(\pi, \ell, \bar{\tau}) \leq (\pi', \ell', \bar{\tau}')$ in $DP[(N, u_N)]$. Thus, since from Proposition 1.9.15 we know that $\langle u_N \downarrow \mathcal{DP}[N]\rangle$ is a preorder, the mapping above is clearly a full and faithful functor. Moreover, since such a mapping is surjective on the objects, it is an equivalence of categories.

Observe that the second equivalence is actually an isomorphism, as shown by Proposition 1.9.9. ✓

It is worth reminding that diagram (1.26) commutes only at the object level, since, concerning functoriality, $\mathcal{DP}[\_]$ has the same problems as $\mathcal{P}[\_]$. It is an open question whether exploiting idea similar to those used in Sections 1.3 and 1.4 yields a functorial diagram.

Conclusions and Further Work

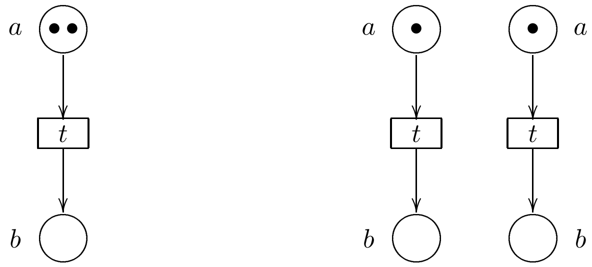
In this chapter we have presented the basic elements of net theory using the algebraic approach of [97, 16].

Moreover, we have shown that strong concatenable processes are the least extension of concatenable processes which yields functoriality, i.e., the least extension of Goltz-Reisig processes yield an operation of concatenation and admit a functorial treatment. Then, in order to identify an axiomatization of the category of (the categories of) net behaviours, we have introduced the category $\underline{\mathsf{SSMC}}^{\otimes}$, an appropriate quotient of the category of the SSMC's whose objects form free (non-commutative) monoids, together with a forgetful functor $\mathcal{G}[\_]\colon \underline{\mathsf{SSMC}}^{\otimes} \to \underline{\mathsf{Petri}}$ that, to a certain extent, shows the adequacy of our proposal. An interesting closing result for the theory of strong concatenable and their algebraic/categorical/functorial counterpart $\mathcal{Q}[\_]$ would be to prove the existence of a right adjoint to $\mathcal{Q}[\_]$.

We also introduced an extension to the case of PT nets of Winskel's semantics for safe nets [143]. This extended semantics is given by a chain of adjunctions between the categories of finitary prime algebraic domains, of prime event structures and of occurrence nets. These results have been achieved by identifying a suitable adjunction between the category of PT nets and the category of occurrence nets, and by exploiting the existing adjunctions between occurrence nets, prime event structures and prime algebraic domains.

Finally, we have proced how such unfolding semantics given can be reconciled with a process-oriented semantics and with the algebraic paradigm. The key of this formal achievements is the notion of decorated occurrence nets. Although $\underline{\mathsf{DecOcc}}$ arose from the need of factorizing the involved adjunction from $\underline{\mathsf{PTNets}}$ to $\underline{\mathsf{Occ}}$, and, thus, decorated occurrence nets might at first seem to be just a convenient technical solution, we have shown that there in fact are some insights on the semantics of nets given by the present unfolding construction and the associated notion of decorated occurrence nets. In fact, decorated deterministic occurrence nets, suitably axiomatized as arrows of the symmetric monoidal category $\mathcal{DP}[N]$ provide both the process-oriented and the algebraic counterpart of the unfolding semantics. Moreover, they can be characterized as the minimal refinement of Goltz-Reisig processes which guarantees the identity of all tokens, i.e., as the minimal refinement of occurrence nets which guarantees the existence of an unfolding for PT nets.

We conclude this chapter discussing an observation about functors $\mathcal{U}[\_]$ and $\mathcal{FU}[\_]$ introduced in the previous sections, namely the observation that both places and transitions of the unfolded nets are implicitly considered *labelled* by elements of the original net. Of course, this understanding is not completely implicit, since the role of the folding morphism $\mathcal{FU}[N] \to N$ is to provide such labelling. However, the "labelling" is *certainly* forgotten when we consider morphisms between unfolded nets. As an example, consider the unfolding Θ of the net consisting of the unique transition $t\colon a \to b$ with initial marking $2a$, as illustrated by the picture below.



As we have already noticed, due to the individualities of the transitions underlying the label t, there are two different endomorphisms on Θ which map t to t, and this fact has *dramatic* consequences on the unfolding construction, since it prevents $\mathcal{FU}[\_]$ to be right adjoint to the inclusion functor (and to be right adjoint at all, without the restriction we imposed on the morphisms of <u>PTNets</u>), i.e., to form a coreflection. However, the point, in our opinion, is that the individualities of those transitions should not matter at all! After all, they are just place holders put there to mean "t has occurred".

It appears evident therefore that an interesting framework for studying the features of the unfolding may be found, for instance by looking at categories of labelled occurrence nets in which the underlying events are given weaker individualities. It is worthwhile to observe that, although categories of labelled objects have often been considered in literature [145, 146], to the best of our knowledge, no such category has been provided with such mechanisms. In particular, we claim that considering the labelling also at the level of morphisms, i.e., forcing some identification of morphisms, will probaly make of $\mathcal{FU}[\_]$ a coreflection.

Chapter 2

Models for Concurrency

ABSTRACT. Models for concurrency can be classified with respect to three relevant parameters: behaviour/system, interleaving/noninterleaving, linear/branching time. When modelling a process, a choice concerning such parameters corresponds to choosing the level of abstraction of the resulting semantics. The classifications are formalized through the medium of category theory.

To Thales ... the primary question is not
"What do we know", but
"How do we know it".
Aristotle

Our three dimensional space
is the only true reality we know.
Maurits Cornelis Escher

The tools we use have
a profound (and devious!)
influence on our thinking abits,
and, therefore, on our thinking ability.
Edsges W. Dijkstra

This chapter is based on joint work with Mogens Nielsen and Glynn Winskel [124, 125, 126].

Introduction

Since its beginning, many efforts in the development of the *theory of concurrency* have been devoted to the study of suitable models for concurrent and distributed processes, and to the formal understanding of their semantics.

As a result, in addition to standard models like languages, automata and transition systems [60, 113], models like *Petri nets* [109], *process algebras* [102, 47], *Hoare traces* [48], *Mazurkiewicz traces* [94], *synchronization trees* [142] and *event structures* [106, 143] have been introduced.

The idea common to the models above is that they are based on atomic units of change—be they called transitions, actions, events or symbols from an alphabet—which are *indivisible* and constitute the steps out of which computations are built.

The difference between the models may be expressed in terms of the parameters according to which models are often classified. For instance, a distinction made explicitly in the theory of Petri nets, but sensible in a wider context, is that between so-called *"system"* models allowing an explicit representation of the (possibly repeating) states in a system, and *"behaviour"* models abstracting away from such information, which focus instead on the behaviour in terms of patterns of occurrences of actions over time. Prime examples of the first type are transition systems and Petri nets, and of the second type, trees, event structures and traces. Thus, we can distinguish among models according to whether they are *system* models or *behaviour* models, in this sense; whether they can faithfully take into account the difference between *concurrency* and *nondeterminism*; and, finally, whether they can represent the *branching structure* of processes, i.e., the points in which choices are taken, or not. Therefore, relevant parameters when looking at models for concurrency are

Behaviour or System model;
Interleaving or Noninterleaving model;
Linear or Branching Time model.

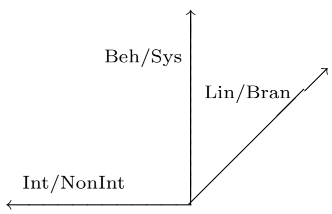
These parameters correspond to choices of the *level of abstraction* at which we examine processes and which are not necessarily fixed for a process once and for all. It is the actual application one has in mind for the formal semantics which *time by time* guides the choice of the abstraction level. It can therefore be of value to be able to move back and forth between the representation of a process in one model and its representation in another, if possible in a way which respects its structure. In other words, it is relevant to study translations between models, and particularly with respect to the three parameters above.

This chapter presents a classification of models for concurrency based on the three parameters, which represent a further step towards the identification of systematic connections between transition based models. In particular, we study a

Beh./Int./Lin.	Hoare languages	<u>HL</u>
Beh./Int./Bran.	synchronization trees	<u>ST</u>
Beh./Nonint./Lin.	deterministic labelled event structures	<u>dLES</u>
Beh./Nonint./Bran.	labelled event structures	<u>LES</u>
Sys./Int./Lin.	deterministic transition systems	<u>dTS</u>
Sys./Int./Bran.	transition systems	<u>TS</u>
Sys./Nonint./Lin.	deterministic transition systems with independence	<u>dTSI</u>
Sys./Nonint./Bran.	transition systems with independence	<u>TSI</u>

Table 2.1: The models

representative for each of the eight classes of models obtained by varying the parameters *behaviour/system*, *interleaving/noninterleaving* and *linear/branching* in all the possible ways. Intuitively, the situation can be graphically represented, as in the picture below, by a three-dimensional frame of reference whose coordinate axes represent the three parameters.

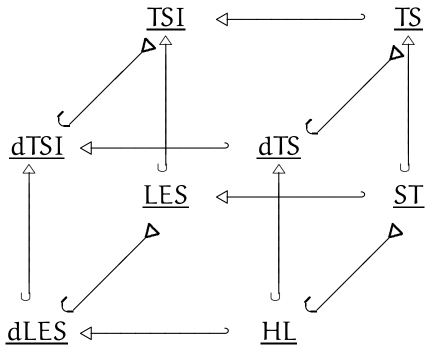


Our choices of models are summarized in Table 2.1. It is worth noticing that, with the exception of the new model of *transition systems with independence*, each model is well-known.

The formal relationships between models are studied in a *categorical* setting, using the standard categorical tool of *adjunctions*. The "translations" between models we shall consider are *coreflections* or *reflections*. These are particular kinds of adjunctions between two categories which imply that one category is *embedded*, fully and faithfully, in another.

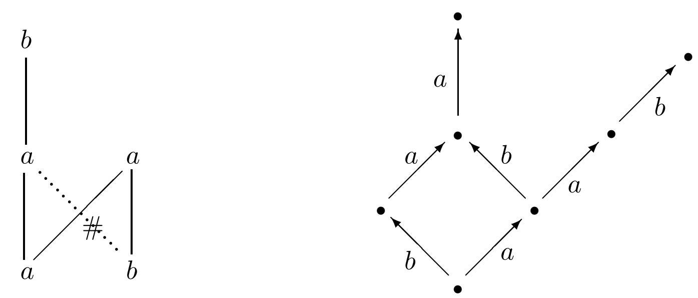
Here we draw on the experience in recasting models for concurrency as categories, detailed, e.g., in [146]. Briefly the idea is that each model (transition systems are one such model) will be equipped with a notion of morphism, making it into a category in which the operations of process calculi are universal constructions. The morphisms will preserve behaviour, at the same time respecting a choice of granularity of the atomic changes in the description of processes—the morphisms are forms of *simulations*. One role of the morphisms is to relate the behaviour of a construction on processes to that of its components. The reflections and coreflections provide a way to express that one model is embedded in (is more abstract than) another, even when the two models are expressed in very different mathematical terms. One adjoint will say how to embed the more abstract model in the other, the other will abstract away from some aspect of the representation. The preservation properties of adjoints can be used to show how a semantics in one model *translates* to a semantics in another.

The picture below, in which arrows represent coreflections and the "backward" arrows reflections (see Appendix A.1), shows the "cube" of relationships (Theorem 2.7.24).



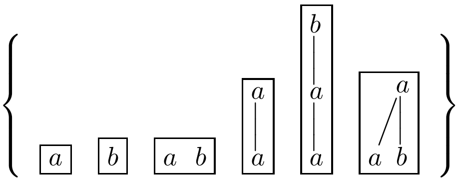
It is worthwhile to remember that we call an adjunction a (generalized) *reflection* of $\underline{A}$ in $\underline{B}$, or $\underline{B}$ is said *reflective* in $\underline{A}$, if the components of the counit are isomorphisms. Dually, it is a (generalized) *coreflection* of $\underline{B}$ in $\underline{A}$, or $\underline{A}$ is *coreflective* in $\underline{B}$, if the components of the unit are isomorphisms.

Generally speaking, the model chosen to represent a class is a canonical and universally accepted representative of that class. However, for the class of behavioural, linear-time, noninterleaving models there does not, at present, seem to be an obvious choice of a corresponding canonical model. The choice of deterministic labelled event structures is based, by analogy, on the observation that Hoare trace languages may be viewed as deterministic synchronization trees, and that labelled event structures are a canonical generalization of synchronization trees within noninterleaving models. The following picture is an example of such an event structure, together with its domain of configurations



Although not canonical, such a choice is certainly fair and, more important, it is *not* at all compelled. In order to show this, and for the sake of completeness, in Section 2.4, we investigate the relationship between this model and two of the most-studied, noninterleaving generalizations of Hoare languages in the literature: the pomsets of Pratt [114], and the traces of Mazurkiewicz [94].

Pomsets, an acronym for *p*artial *o*rdered *m*ulti*sets*, are labelled partial ordered sets. A noninterleaving representation of a system can be readily obtained by means of pomsets simply by considering the (multiset of) labels occurring in the run ordered by the *causal dependency* relation inherited from the events. The system itself is then represented by a set of pomsets. For instance, the labelled event structure given in the example discussed above can be represented by the following set of pomsets.



A simple but conceptually relevant observation about pomsets is that strings can be thought of as a particular kind of pomsets, namely those pomsets which are *finite* and *linearly* ordered. In other words, a pomset $\boxed{a_1 < a_2 < \cdots < a_n}$ represents the string $a_1 a_2 \cdots a_n$. On the other side of such correspondence, we can think of (finite) pomsets as a generalization of the notion of word (string) obtained by relaxing the constraint which imposes that the symbols in a word be linearly ordered. This is why in the literature pomsets have also appeared under the name *partial words* [38]. The analogy between pomsets and strings can be pursued to the point of defining languages of partial words, called *partial languages*, as prefix-closed—for a suitable extension of this concept to pomsets—sets of pomsets on a given alphabet of labels.

Since our purpose is to study linear-time models, which are deterministic, we shall consider only pomsets without *autoconcurrency*, i.e., pomsets such that all the elements carrying the same label are linearly ordered. Following [131], we shall refer to this kind of pomsets as *semiwords* and to the corresponding languages as *semilanguages*. We shall identify a category $\underline{\mathsf{dSL}}$ of deterministic semilanguages equivalent to the category of deterministic labelled event structures. Although pomsets have been studied extensively (see e.g. [114, 33, 38]), there are few previous results about formal relationships of pomsets with other models for concurrency.

Mazurkiewicz trace languages [94] are defined on an alphabet L together with a symmetric irreflexive binary relation I on L, called the *independence relation*. The relation I induces an equivalence on the strings of L^* which is generated by the simple rule

$$\alpha ab\beta \simeq \alpha ba\beta \quad \text{if} \quad a\ I\ b,$$

where $\alpha, \beta \in L^*$ and $a, b \in L$. A trace language is simply a subset M of L^* which is prefix-closed and $\simeq$-closed, i.e., $\alpha \in M$ and $\alpha \simeq \beta$ implies $\beta \in M$. It represents a system by representing all its possible behaviours as the sequences of (occurrences of) events it can perform. Since the independence relation can be taken to indicate the events which are *concurrent* to each other, the relation $\simeq$ does nothing but relate runs of the systems which differ only in the order in which independent events occur.

However, Mazurkiewicz trace languages are too abstract to describe faithfully labelled event structures. Consider for instance the labelled event structure shown earlier. Clearly, any trace language with alphabet $\{a, b\}$ able to describe such a labelled event structure must be such that $ab \simeq ba$. However, it cannot be such that $aba \simeq aab$. Thus, we are forced to move from the well-known model of trace languages. We shall introduce here a new notion of *generalized Mazurkiewicz trace language*, in which the independence relation is *context-dependent*. For instance, the event structure shown in the above picture will be represented by a trace language in which a is independent from b *at* ϵ, i.e., after the empty string, in symbols $a\ I_\epsilon\ b$, but a is *not* independent from b *at* a, i.e., after the string a has appeared, in symbols $a\ \not{I}_a\ b$. In particular, we shall present a category $\underline{\mathsf{GTL}}$ of generalized trace languages which is equivalent to the category $\underline{\mathsf{dLES}}$ of deterministic labelled event structures. We remark that a similar idea of generalizing Mazurkiewicz trace languages has been considered also in [50].

Summing up, Section 2.4 presents the chain of equivalences

$$\underline{\mathsf{dSL}} \overset{\cdot}{\cong} \underline{\mathsf{dLES}} \overset{\cdot}{\cong} \underline{\mathsf{GTL}}$$

which, besides identifying models which can replace $\underline{\mathsf{dLES}}$ in our classification, also introduce interesting *deterministic behavioural models* for concurrency and formalizes their mutual relationships.

Some of the results presented here will appear also in [124, 125] and [146]. In particular, we omit all the proofs concerning Section 2.4. They will appear in [125].

2.1 Preliminaries

In this section, we study the interleaving models. We start by briefly recalling some well-known relationships between languages, trees and transition systems [146], and then, we study how they relate to deterministic transition systems.

Definition 2.1.1 *(Labelled Transition Systems)*
A labelled transition system is a structure $T = (S, s^I, L, Tran)$ where S is a set of states; $s^I \in S$ is the initial state, L is a set of labels, and $Tran \subseteq S \times L \times S$ is the transition relation.

The fact that $(s, a, s') \in \mathit{Tran}_T$—also denoted by $s \xrightarrow{a} s'$, when no ambiguity is possible—indicates that the system can evolve from state s to state s' performing an action a. The structure of transition systems immediately suggests the right notion of morphism: initial states must be mapped to initial states, and for every action the first system can perform in a given state, it must be possible for the second system to perform the corresponding action—if any—from the corresponding state. This guarantees that morphisms are *simulations*.

Definition 2.1.2 *(Labelled Transition System Morphisms)*
Given the labelled transition systems T_0 and T_1, a morphism $h: T \to T'$ is a pair (σ, λ), where $\sigma: S_{T_0} \to S_{T_1}$ is a function and $\lambda: L_{T_0} \rightharpoonup L_{T_1}$ a partial function, such that[1]

i) $\sigma(s^I_{T_0}) = s^I_{T_1}$;

ii) $(s, a, s') \in \mathit{Tran}_{T_0}$ *implies* $\begin{array}{ll} \big(\sigma(s), \lambda(a), \sigma(s')\big) \in \mathit{Tran}_{T_1} & \text{if } \lambda{\downarrow}a; \\ \sigma(s) = \sigma(s') & \text{otherwise.} \end{array}$

It is immediate to see that labelled transition systems and labelled transition system morphisms, when the obvious componentwise composition of morphisms is considered, give a category, which will be referred to as $\underline{\mathsf{TS}}$.

Since we shall deal often with partial maps, we assume the standard convention that whenever a statement involves values yielded by partial functions, we implicitily assume that they are defined.

A particularly interesting class of transition systems is that of synchronization trees, i.e., the tree-shaped transition systems.

Definition 2.1.3 *(Synchronization Trees)*
A synchronization tree is an acyclic, reachable transition system S such that

$$(s', a, s),\ (s'', b, s) \in \mathit{Tran}_S \qquad \text{implies} \qquad s' = s'' \quad \text{and} \quad a = b$$

We shall write $\underline{\mathsf{ST}}$ to denote the full subcategory of $\underline{\mathsf{TS}}$ consisting of synchronization trees.

In a synchronization tree the information about the internal structure of systems is lost, and only the information about their behaviour is mantained. In other words, it is not anymore possible to discriminate between a system which reachs again and again the same state, and a system which passes through a sequence of states, as far as they are able to perform the same action. However, observe that the nondeterminism present in a state can still be expressed in full generality.

[1]We use $f{\downarrow}x$ to mean that a partial function f is defined on argument x.

In this sense, synchronization trees are branching time and interleaving models of behaviour.

A natural way of studying the behaviour of a system consists of considering its computations as a synchronization tree, or, in other words, of *"unfolding"* the transition system by decorating each state with the history of the computation which reached it.

DEFINITION 2.1.4 *(Unfoldings of Transition Systems)*
Let T be a transition system. A path π of T is ϵ, the empty path, or a sequence $t_1 \cdots t_n$, $n \geq 1$, where

i) $t_i \in \mathit{Tran}_T$, $i = 1, \ldots, n$;

ii) $t_1 = (s^I_T, a_1, s_1)$ and $t_i = (s_{i-1}, a_i, s_i)$, $i = 2, \ldots, n$.

We shall write $\mathit{Path}(T)$ to indicate the set of paths of T and π_s to denote a generic path leading to state s.

Define $ts.st(T)$ to be the synchronization tree $(\mathit{Path}(T), \epsilon, L_T, \mathit{Tran})$, where

$$\Big((t_1 \cdots t_n), a, (t_1 \cdots t_n t_{n+1})\Big) \in \mathit{Tran}$$
$$\Leftrightarrow \quad t_n = (s_{n-1}, a_n, s_n) \quad \text{and} \quad t_{n+1} = (s_n, a, s_{n+1})$$

This procedure amounts to abstracting away from the internal structure of a transition system and looking at its behaviour. It is very interesting to notice that this simple construction is functorial and, moreover, that if forms the right adjoint to the inclusion functor of <u>ST</u> in <u>TS</u>. In other words, the category of synchronization trees is coreflective in the category of transition systems. The *counit* of such adjunction is the morphism $(\phi, id_{L_T})\colon ts.st(T) \to T$, where $\phi\colon \mathit{Path}(T) \to S_T$ is given by $\phi(\epsilon) = s^I_T$, and $\phi\Big((t_1 \cdots t_n)\Big) = s$ if $t_n = (s', a, s)$.

While looking at the behaviour of a system, a further step of abstraction can be achieved forgetting also the branching structure of a tree. This leads to another well-know model of behaviour: *Hoare languages*.

DEFINITION 2.1.5 *(Hoare Languages)*
A Hoare language is a pair (H, L), where $\varnothing \neq H \subseteq L^*$, and $sa \in H \Rightarrow s \in H$.

A partial function $\lambda\colon L_0 \rightharpoonup L_1$ is a morphism of Hoare languages from (H_0, L_0) to (H_1, L_1) if for each $s \in H_0$ it is $\hat{\lambda}(s) \in H_1$, where $\hat{\lambda}\colon L_0^* \to L_1^*$ is defined by

$$\hat{\lambda}(\epsilon) = \epsilon \qquad \text{and} \qquad \hat{\lambda}(sa) = \begin{cases} \hat{\lambda}(s)\lambda(a) & \text{if } \lambda{\downarrow}a; \\ \hat{\lambda}(s) & \text{otherwise.} \end{cases}$$

These data give the category <u>HL</u> of Hoare languages.

Observe that any language (H, L) can be seen as a synchronization tree just by considering the strings of the language as states, the empty string being the initial state, and defining a transition relation where $s \xrightarrow{a} s'$ if and only if $sa = s'$. Let $hl.st\Big((H, L)\Big)$ denote such a synchronization tree.

On the contrary, given a synchronization tree S, it is immediate to see that the strings of labels on the paths of S form a Hoare language. More formally, for any transition system T and any path $\pi = (s_T^I, a_1, s_1) \cdots (s_{n-1}, a_n, s_n)$ in $Path(T)$, define $Act(\pi)$ to be the string $a_1 \cdots a_n \in L_T^*$. Moreover, let $Act(T)$ denote the set of strings

$$\Big\{ Act(\pi) \;\Big|\; \pi \in Path(T) \Big\}.$$

Then, the language associated to S is $st.hl(S) = Act(S)$, and simply by defining $st.hl\Big((\sigma, \lambda)\Big) = \lambda$, we obtain a functor $st.hl\colon \underline{\mathsf{ST}} \to \underline{\mathsf{HL}}$. Again, this constitutes the left adjoint to $hl.st\colon \underline{\mathsf{HL}} \to \underline{\mathsf{ST}}$ and given above. The situation is illustrated below, where $\hookrightarrow\!\!\rhd$ represents a coreflection and $\hookleftarrow\!\!\lhd$ a reflection.

Theorem 2.1.6

$$\underline{\mathsf{HL}} \hookleftarrow\!\!\!-\!\!\!-\!\!\!-\!\!\lhd \underline{\mathsf{ST}} \hookrightarrow\!\!\!-\!\!\!-\!\!\!-\!\!\rhd \underline{\mathsf{TS}}$$

The existence of a (co)reflection from category $\underline{\mathsf{A}}$ to $\underline{\mathsf{B}}$ tells us that there is a full subcategory of $\underline{\mathsf{B}}$ which is *equivalent* to $\underline{\mathsf{A}}$ (in the formal sense of equivalences of categories). Therefore, once we have established a (co)reflection, it is sometime interesting to indentify such subcategories. In the case of $\underline{\mathsf{HL}}$ and $\underline{\mathsf{ST}}$ such a question is answered below.

Proposition 2.1.7 *(Languages are deterministic Trees)*
The full subcategory of $\underline{\mathsf{ST}}$ consisting of those synchronization trees which are deterministic, say $\underline{\mathsf{dST}}$, is equivalent to the category of Hoare languages.

2.2 Deterministic Transition Systems

Speaking informally behaviour/system and linear/branching are independent parameters, and we expect to be able to forget the branching structure of a transition system without necessarily losing all the internal structure of the system. This leads us to identify a class of models able to represent the internal structure of processes without keeping track of their branching, i.e., the points at which the choices are actually taken. A suitable model is given by *deterministic transition systems.*

Definition 2.2.1 *(Deterministic Transition Systems)*
A transition system T is deterministic if

$$(s, a, s'),\ (s, a, s'') \in Tran_T \quad \textit{implies} \quad s' = s''.$$

Let $\underline{\mathsf{dTS}}$ *be the full subcategory of* $\underline{\mathsf{TS}}$ *consisting of those transition systems which are deterministic.*

Consider the binary relation $\simeq$ on the state of a transition system T defined as the least equivalence which is *forward closed*, i.e.,

$$s \simeq s' \text{ and } (s,a,u),\ (s',a,u') \in \mathit{Tran}_T \quad \Rightarrow \quad u \simeq u';$$

and define $\mathit{ts.dts}(T) = \Big(S/\!\simeq, [s^I_T]_\simeq, L_T, \mathit{Tran}_\simeq\Big)$, where $S/\!\simeq$ are the equivalence classes of $\simeq$ and

$$\Big([s]_\simeq, a, [s']_\simeq\Big) \in \mathit{Tran}_\simeq \quad \Leftrightarrow \quad \exists (\bar{s}, a, \bar{s}') \in \mathit{Tran}_T \quad \text{with } \bar{s} \simeq s \text{ and } \bar{s}' \simeq s'.$$

It is easy to see that $\mathit{ts.dts}(TS)$ is a deterministic transition system. Actually, this construction defines a functor which is left adjoint to the inclusion $\underline{\mathsf{dTS}} \hookrightarrow \underline{\mathsf{TS}}$. In the following we briefly sketch the proof of this fact. Since confusion is never possible, we shall not use different notations for different $\simeq$'s.

Given a transition system morphism $(\sigma,\lambda)\colon T_0 \to T_1$, define $\mathit{ts.dts}\Big((\sigma,\lambda)\Big)$ to be $(\bar{\sigma},\lambda)$, where $\bar{\sigma}\colon S_{T_0}/\!\simeq\ \to S_{T_1}/\!\simeq$ is such that

$$\bar{\sigma}\Big([s]_\simeq\Big) = [\sigma(s)]_\simeq.$$

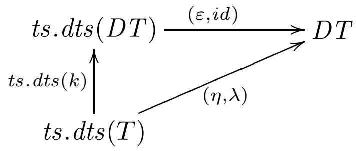
Proposition 2.2.2 *($\mathit{ts.dts}\colon \underline{\mathsf{TS}} \to \underline{\mathsf{dTS}}$ is a functor)*
The pair $(\bar{\sigma},\lambda)\colon \mathit{ts.dts}(T_0) \to \mathit{ts.dts}(T_1)$ *is a transition system morphism.*

Proof. First, we show that $\bar{\sigma}$ is well-defined.
Suppose $(s,a,s'),\ (s,a,s'') \in \mathit{Tran}_{T_0}$. Now, if $\lambda\!\uparrow\! a$, then $\sigma(s') = \sigma(s) = \sigma(s'')$. Otherwise, $\Big(\sigma(s),\lambda(a),\sigma(s')\Big)$, $\Big(\sigma(s),\lambda(a),\sigma(s'')\Big) \in \mathit{Tran}_{T_1}$. Therefore, in both cases, $\sigma(s') \simeq \sigma(s'')$. Now, since $(s,a,s') \in \mathit{Tran}_{T_0}$ implies $\Big(\sigma(s),\lambda(a),\sigma(s')\Big) \in \mathit{Tran}_{T_1}$ or $\sigma(s) = \sigma(s')$, it easily follows that $\sigma(\simeq) \subseteq\ \simeq$. It is now easy to show that $(\bar{\sigma},\lambda)$ is a morphism. ✓

It follows easily from the previous proposition that $\mathit{ts.dts}$ is a functor.

Clearly, for a deterministic transition system, say DT, since there are no pairs of transitions such that $(s,a,s'),\ (s,a,s'') \in \mathit{Tran}_{DT}$, we have that $\simeq$ is the identity. Thus, we can choose a candidate for the counit by considering, for any deterministic transition system DT, the morphism $(\varepsilon, id)\colon \mathit{ts.dts}(DT) \to DT$, where $\varepsilon([s]_\simeq) = s$. Let us show it enjoys the couniversal property.

Proposition 2.2.3 *((ε, id): $ts.dts(DT) \to DT$ is couniversal)*
For any deterministic transition system DT, any transition system T and any morphism (η, λ): $ts.dts(T) \to DT$, there exists a unique morphism k in TS *such that* $(\varepsilon, id) \circ ts.dts(k) = (\eta, \lambda)$.



Proof. The morphism k must be of the form (σ, λ), for some σ. We choose σ such that $\sigma(s) = \eta([s]_\simeq)$. With such a definition, it is immediate that k is a transition system morphism. Moreover, the diagram commutes: $(\varepsilon, id) \circ ts.dts\Big((\sigma, \lambda)\Big) = (\varepsilon \circ \bar{\sigma}, \lambda)$, and $\varepsilon\Big(\bar{\sigma}([s]_\simeq)\Big) = \varepsilon([\sigma(s)]_\simeq) = \sigma(s) = \eta([s]_\simeq)$. To show uniqueness of k, suppose that there is k' which makes the diagram commute. Necessarily, k' must be of the kind (σ', λ). Now, since $\sigma'([s]_\simeq) = [\sigma'(s)]_\simeq$, in order for the diagram to commute, it must be $\sigma'(s) = \eta([s]_\simeq)$. Therefore, $\sigma' = \sigma$ and $k' = k$. ✓

Theorem 2.2.4 *($ts.dts \dashv \hookrightarrow$)*
The functor $ts.dts$ is left adjoint to the inclusion functor dTS $\hookrightarrow$ TS*. Therefore, the adjunction is a reflection.*

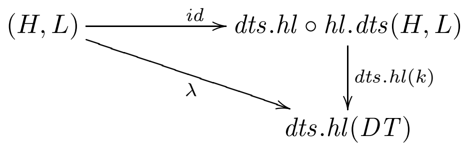
Proof. By standard results of Category Theory (see [90, chap. IV, pg. 81]). ✓

Next, we present a universal construction from Hoare languages to deterministic transition system. In particular, we show a coreflection HL $\hookrightarrow$ dTS. Let (H, L) be a language. Define $hl.dts(H, L) = (H, \epsilon, L, Tran)$, where $(s, a, sa) \in Tran$ for any $sa \in H$, which is trivially a deterministic transition system.

On the contrary, given a deterministic transition system DT, define the language $dts.hl(DT) = (Act(DT), L_{DT})$. Concerning morphisms, it is immediate to realize that if (σ, λ): $DT_0 \to DT_1$ is a transition system morphism, then the function λ: $Act(DT_0) \to Act(DT_1)$ is a morphism of Hoare languages. Therefore, defining $dts.hl\Big((\sigma, \lambda)\Big) = \lambda$, we have a functor from dTS to HL.

Now, consider the language $dts.hl \circ hl.dts(H, L)$. It contains a string $a_1 \cdots a_n$ if and only if the sequence $(\epsilon, a_1, a_1)(a_1, a_2, a_1a_2) \cdots (a_1 \cdots a_{n-1}, a_n, a_1 \cdots a_n)$ is in $Path\Big(hl.dts(T)\Big)$ if and only if $a_1 \cdots a_n$ is in H. It follows immediately that id: $(H, L) \to dts.hl \circ hl.dts(H, L)$ is a morphism of languages. We will show that id is actually the unit of the coreflection.

PROPOSITION 2.2.5 *($id\colon (H,L) \to dts.hl \circ hl.dts(H,L)$ is universal)*
For any Hoare language (H,L), any deterministic transition system DT and any morphism $\lambda\colon (H,L) \to dts.hl(DT)$, there exists a unique morphism k in <u>dTS</u> *such that $dts.hl(k) = \lambda$.*



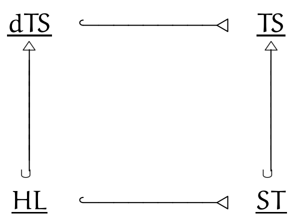
Proof. Observe that since DT is deterministic, given a string $s \in Act(DT)$, there is exactly one state in S_{DT} reachable from s^I_{DT} with a path labelled by s. We shall use $state(s)$ to denote such a state. Then, define $k = (\sigma, \lambda)\colon hl.dts(H,L) \to DT$, where $\sigma(s) = state(\hat{\lambda}(s))$. Since DT is deterministic and $\hat{\lambda}(s)$ is in $Act(DT)$, (σ, λ) is well-defined and the rest of the proof follows easily. ✓

THEOREM 2.2.6 *($hl.dts \dashv dts.hl$)*
The map $hl.dts$ extends to a functor from <u>HL</u> *to* <u>dTS</u> *which is left adjoint to $dts.hl$. Since the unit of the adjunction is an isomorphism, the adjunction is a coreflection.*

Observe that the construction of the deterministic transition system associated to a language coincides exactly with the construction of the corresponding synchronization tree. However, due to the different objects in the categories, the type of universality of the construction changes. In other words, the same construction shows that <u>HL</u> is *reflective* in <u>ST</u>—a full subcategory of <u>TS</u>—and *coreflective* in <u>dTS</u>—another full subcategory of <u>TS</u>.

Thus, we enriched the diagram at the end of the previous section and we have a square.

THEOREM 2.2.7 *(The Interleaving Surface)*



2.3 Noninterleaving vs. Interleaving Models

Event structures [106, 143] abstract away from the cyclic structure of the process and consider only events (strictly speaking event *occurrences*), assumed to be the *atomic* *computational steps*, and the cause/effect relationships between them. Thus, we can classify event structures as behavioural, branching and noninterleaving models. Here, we are interested in labelled event structures.

DEFINITION 2.3.1 *(Labelled Event Structures)*
A labelled event structure *is a structure* $ES = (E, \#, \leq, \ell, L)$ *consisting of a set of* events E *partially ordered by* $\leq$*; a symmetric, irreflexive relation* $\# \subseteq E \times E$*, the* conflict *relation, such that*

$$\{\, e' \in E \mid e' \leq e \,\} \text{ is finite for each } e \in E,$$

$$e \# e' \leq e'' \text{ implies } e \# e'' \text{ for each } e, e', e'' \in E;$$

a set of labels L *and a* labelling function $\ell: E \to L$*. For an event* $e \in E$*, define* $\lfloor e \rfloor = \{\, e' \in E \mid e' \leq e \,\}$*. Moreover, we write* $\mathrel{W}$ *for* $\# \cup \{\, (e, e) \mid e \in E_{ES} \}$*. These data define a relation of* concurrency *on events:* $co = E_{ES}^2 \setminus (\leq \cup \leq^{-1} \cup \#)$*.*

A labelled event structure morphism *from* ES_0 *to* ES_1 *is a pair of partial functions* (η, λ)*, where* $\eta: E_{ES_0} \rightharpoonup E_{ES_1}$ *and* $\lambda: L_{ES_0} \rightharpoonup L_{ES_1}$ *are such that*

i) $\lfloor \eta(e) \rfloor \subseteq \eta(\lfloor e \rfloor)$,

ii) $\eta(e) \mathrel{W} \eta(e')$ *implies* $e \mathrel{W} e'$,

iii) $\lambda \circ \ell_{ES_0} = \ell_{ES_1} \circ \eta$*, i.e., the following diagram commutes.*

$$\begin{array}{ccc} E_{ES_0} & \xrightarrow{\ \ell_{ES_0}\ } & L_{ES_0} \\ {\scriptstyle \eta}\downarrow & & \downarrow{\scriptstyle \lambda} \\ E_{ES_1} & \xrightarrow[\ \ell_{ES_1}\ ]{} & L_{ES_1} \end{array}$$

This defines the category <u>LES</u> *of labelled event structures.*

The computational intuition behind event structures is simple: an event e can occur when all its *causes*, i.e., $\lfloor e \rfloor \setminus \{e\}$, have occurred and no event which it is in conflict with has already occurred. This is formalized by the following notion of *configuration.*

DEFINITION 2.3.2 *(Configurations)*
Given a labelled event structure ES*, define the* configurations *of* ES *to be those subsets* $c \subseteq E_{ES}$ *which are*

$$\textit{Conflict Free:} \quad \forall e_1, e_2 \in c, \ \text{not } e_1 \# e_2$$

$$\textit{Left Closed:} \quad \forall e \in c\ \forall e' \leq e, \ e' \in c$$

Let $\mathcal{L}(ES)$ denote the set of configurations of ES.
We say that e is enabled at a configuration c, in symbols $c \vdash e$,

$$\textit{(i)} \quad e \notin c; \quad \textit{(ii)} \quad \lfloor e \rfloor \setminus \{e\} \subseteq c; \quad \textit{(iii)} \quad e' \in E_{ES} \text{ and } e' \# e \text{ implies } e' \notin c.$$

The occurrence of e at c transforms c in the configuration $c' = c \cup \{e\}$.

Given a finite subset c of E_{ES}, we say that a total ordering of the elements of c, say $\{e_1 < e_2 < \cdots < e_n\}$, is a *securing* for c if and only if $\{e_1, \ldots, e_{i-1}\} \vdash e_i$, for $i = 1, \ldots, n$. Clearly, c is a finite configuration if and only if there exists a securing for it. We shall write a securing for c as a string $e_1 e_2 \cdots e_n$, where $c = \{e_1, e_2, \ldots, e_n\}$ and $e_i \neq e_j$ for $i \neq j$, and, by abuse of notation, we shall consider such strings also configurations. Let $Sec(ES)$ denote the set of the securings of ES.

Definition 2.3.3 *(Deterministic Event Structures)*
A labelled event structure ES is deterministic if and only if for any $c \in \mathcal{L}(ES)$, and for any pair of events $e, e' \in E_{ES}$, whenever $c \vdash e$, $c \vdash e'$ and $\ell(e) = \ell(e')$, then $e = e'$.
This defines the category $\underline{\mathsf{dLES}}$ as a full subcategory of $\underline{\mathsf{LES}}$.

In [140], it is shown that synchronization trees and labelled event structures are related by a a coreflection from $\underline{\mathsf{ST}}$ to $\underline{\mathsf{LES}}$. As will be clear later, this gives us a way to see synchronization trees as an interleaving version of labelled event structures or, viceversa, to consider labelled event structures as a generalization of synchronization trees to the noninterleaving case. In the following subsection, we give a brief account of this coreflection.

Synchronization Trees and Labelled Event Structures

Given a tree S, define $st.les(S) = (Tran_S, \leq, \#, \ell, L_S)$, where

- $\leq$ is the least partial order on $Tran_S$ such that $(s, a, s') \leq (s', b, s'')$;
- $\#$ is the least hereditary, symmetric, irreflexive relation on $Tran_S$ such that $(s, a, s') \# (s, b, s'')$;
- $\ell\Big((s, a, s')\Big) = a$.

Clearly, $st.les(S)$ is a labelled event structure. Now, by defining $st.les\Big((\sigma, \lambda)\Big)$ to be (η_σ, λ), where

$$\eta_\sigma\Big((s, a, s')\Big) = \begin{cases} \Big(\sigma(s), \lambda(a), \sigma(s')\Big) & \text{if } \lambda{\downarrow}a \\ \uparrow & \text{otherwise,} \end{cases}$$

it is not difficult to see that $st.les$ is a functor from $\underline{\mathsf{ST}}$ to $\underline{\mathsf{LES}}$.

On the contrary, for a labelled event structure ES, define $les.st(ES)$ to be the structure $\Big(Sec(ES), \epsilon, L_{ES}, Tran\Big)$, where $(s, a, se) \in Tran$ if and only if we have s, $se \in Sec(ES)$ and $\ell_{ES}(e) = a$. Since a transition (s, a, s') implies that $|s| < |s'|$ (s is a string strictly shorter than s'), the transition system we obtain is certainly acyclic. Moreover, by definition of securing, it is reachable. Finally, if (s, a, se), $(s', a, s'e') \in Tran$ and $se = s'e'$, then obviously $s = s'$ and $e = e'$. Therefore, $les.st(ES)$ is a synchronization tree.

Concerning morphisms, for $(\eta, \lambda)\colon ES_0 \to ES_1$, define $les.st\Big((\eta, \lambda)\Big)$ to be $(\hat{\eta}, \lambda)$. This makes $les.st$ be a functor from $\underline{\mathsf{LES}}$ to $\underline{\mathsf{ST}}$.

Consider now $les.st \circ st.les(S)$. Observe that there is a transition

$$\Big((s_S^I, a_1, s_1) \cdots (s_{n-1}, a_n, s_n), a, (s_S^I, a_1, s_1) \cdots (s_{n-1}, a_n, s_n)(s_n, a, s)\Big)$$

in $Tran_{les.st \circ st.les(S)}$ if and only if $(s_S^I, a_1, s_1) \cdots (s_{n-1}, a_n, s_n)(s_n, a, s)$ is a path in S. From this fact, and since S and $les.st \circ st.les(S)$ are trees, it follows easily that there is an isomorphism between the states of S and the states of $les.st \circ st.les(S)$, and that such an isomorphism is indeed a morphism of synchronization trees.

THEOREM 2.3.4 *($st.les \dashv les.st$)*
For any synchronization tree S, the map $(\eta, id)\colon S \to les.st \circ st.les(S)$, where $\eta(s_S^I) = \epsilon$ and $\eta(s) = (s_S^I, a_1, s_1) \cdots (s_n, a, s)$, the unique path leading to s in S, is a synchronization tree isomorphism.

Moreover, $\langle st.les, les.st\rangle\colon \underline{\mathsf{ST}} \rightharpoonup \underline{\mathsf{LES}}$ is an adjuction whose unit is given by the family of isomorphisms (η, id). Thus, we have a coreflection of $\underline{\mathsf{ST}}$ into $\underline{\mathsf{LES}}$.

Consider now a synchronization tree S in $\underline{\mathsf{dST}}$, i.e., a deterministic tree. From the definition of $st.les$, it follows easily that $st.les(S)$ is a deterministic event structure; on the other hand, $les.st(ES)$ is a deterministic tree when ES is deterministic. Thus, by general reason, the coreflection $\underline{\mathsf{ST}} \hookrightarrow \underline{\mathsf{LES}}$ restricts to a coreflection $\underline{\mathsf{dST}} \hookrightarrow \underline{\mathsf{dLES}}$, whence we have the following corollary.

THEOREM 2.3.5 *($\underline{\mathsf{HL}} \hookrightarrow \underline{\mathsf{dLES}}$)*
The category $\underline{\mathsf{HL}}$ of Hoare languages is coreflective in the category $\underline{\mathsf{dLES}}$ of deterministic labelled event structures.

Proof. It is enough to recall that $\underline{\mathsf{dST}}$ and $\underline{\mathsf{HL}}$ are *equivalent*. Then, the result follows by general reasons. ✓

To conclude this subsection, we make precise our claim of labelled event structures being a generalization of synchronization trees to the noninterleaving case. Once the counits of the above coreflections have been calculated, it is not difficult to prove the following results.

Corollary 2.3.6 *(L. Event Structures = S. Trees + Concurrency)*
The full subcategory of LES *consisting of the labelled event structures ES such that* $co_{ES} = \varnothing$ *is equivalent to* ST.

The full subcategory of dLES *consisting of the deterministic labelled event structures ES such that* $co_{ES} = \varnothing$ *is equivalent to* HL.

Transition Systems with Independence

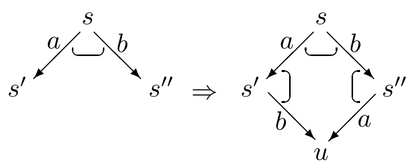
Now, on the system level we look for a way of equipping transition systems with a notion of "concurrency" or "independence", in the same way as LES may be seen as adding "concurrency" to ST. Moreover, such enriched transition systems should also represent the "system model" version of event structures. Several such models have appeared in the literature [129, 2, 132]. Here we choose a variation of these, the *transition systems with independence*.

Transition systems with independence are transition systems with an independence relation actually carried by transitions. The novelty resides in the fact that the notion of *event* becomes now a derived notion. However, four axioms are imposed in order to guarantee the consistency of this with the intuitive meaning of event.

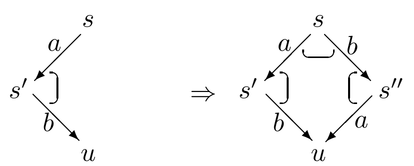
Definition 2.3.7 *(Transition Systems with Independence)*
A transition system with independence is a structure $(S, s^I, L, Tran, I)$*, where the quadruple* $(S, s^I, L, Tran)$ *is a transition system and* $I \subseteq Tran^2$ *is an irreflexive, symmetric relation, such that*

i) $(s, a, s') \sim (s, a, s'') \quad \Rightarrow \quad s' = s''$;

ii) $(s, a, s')\ I\ (s, b, s'') \quad \Rightarrow \quad \exists u.\ (s, a, s')\ I\ (s', b, u)$ and $(s, b, s'')\ I\ (s'', a, u)$;

i.e.,


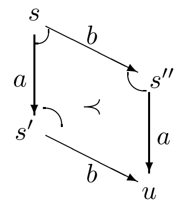
iii) $(s, a, s')\ I\ (s', b, u) \quad \Rightarrow \quad \exists s''.\ (s, a, s')\ I\ (s, b, s'')$ and $(s, b, s'')\ I\ (s'', a, u)$;

i.e.,


iv) $(s, a, s') \sim (s'', a, u) \; I \; (w, b, w') \quad \Rightarrow \quad (s, a, s') \; I \; (w, b, w')$;

where $\sim$ *is least equivalence on transitions including the relation* $\prec$ *defined by*

$$(s, a, s') \prec (s'', a, u) \quad \Leftrightarrow \quad \begin{array}{l} (s, a, s') \; I \; (s, b, s'') \text{ and} \\ (s, a, s') \; I \; (s', b, u) \text{ and} \\ (s, b, s'') \; I \; (s'', a, u). \end{array}$$



Morphisms of transition systems with independence are morphisms of the underlying transition systems which preserve independence, i.e., such that

$$(s, a, s') \; I \; (\bar{s}, b, \bar{s}') \text{ and } \lambda{\downarrow}a, \; \lambda{\downarrow}b \quad \Rightarrow \quad \Big(\sigma(s), \lambda(a), \sigma(s')\Big) \; I \; \Big(\sigma(\bar{s}), \lambda(b), \sigma(\bar{s}')\Big).$$

These data define the category <u>TSI</u> *of transition systems with independence. Moreover, let* <u>dTSI</u> *denote the full subcategory of* <u>TSI</u> *consisting of transition systems with independence whose underlying transition system is deterministic.*

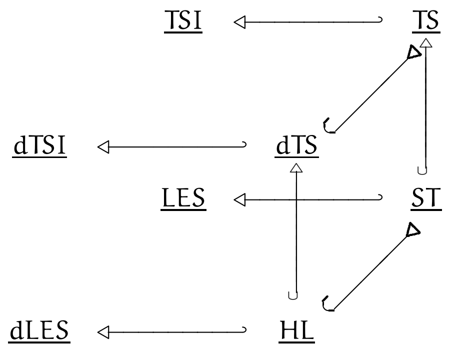
Thus, transition systems with independence are precisely standard transition systems but with an additional relation expressing when one transition is independent of another. The relation $\sim$, defined as the reflexive, symmetric and transitive closure of a relation $\prec$ which simply identifies local "diamonds" of concurrency, expresses when two transitions represent occurrences of the same event. Thus, the equivalence classes $[(s, a, s')]_\sim$ of transitions (s, a, s') are the *events* of the transition system with independence. In order to shorten notations, we shall indicate that transitions (s, a, s'), (s, b, s''), (s', b, u) and (s'', a, u) form a diamond by writing $\mathit{Diam}\Big((s, a, s'), (s, b, s''), (s', b, u), (s'', a, u)\Big)$.

Concerning the axioms, property *(i)* states that the occurrence of an event at a state yields a unique state; property *(iv)* asserts that the independence relation respects events. Finally, conditions *(ii)* and *(iii)* describe intuitive properties of independence: two independent events which can occur at the same state, can do it in any order without affecting the reached state.

Transition systems with independence admit <u>TS</u> as a coreflective subcategory. In this case, the adjunction is easy. The left adjoint associates to any transition system T the transition system with independence whose underlying transition system is T itself and whose independence relation is empty. The right adjoint simply forgets about the independence, mapping any transition system with independence to its underlying transition system. From the definition of morphisms of transition systems with independence, it follows easily that these mappings extend to functors which form a coreflection $\underline{\mathsf{TS}} \hookrightarrow \underline{\mathsf{TSI}}$. Moreover, such a coreflection trivially restricts to a coreflection $\underline{\mathsf{dTS}} \hookrightarrow \underline{\mathsf{dTSI}}$.

So, we are led to the following diagram.

Theorem 2.3.8 *(Moving along the "interleaving/noninterleaving" axis)*



2.4 Behavioural, Linear Time, Noninterleaving Models

A *labelled partial order* on L is a triple $(E, \leq, \ell)$, where E is a set, $\leq \subseteq E^2$ a partial order relation; and $\ell\colon E \to L$ is a *labelling* function. We say that a labelled partial order $(E, \leq, \ell)$ is *finite* if E is so.

Definition 2.4.1 *(Partial Words)*
A partial word on L is an isomorphism class of finite labelled partial orders, an isomorphism of labelled partial orders being an isomorphism of the underlying partial orders which, in addition, preserves the labelling. Given a finite labelled partial order p we shall denote with $[\![p]\!]$ the partial word which contains p. We shall also say that p represents the partial word $[\![p]\!]$.

A semiword is a partial word which does not exhibit autoconcurrency, i.e., such that all its subsets consisting of elements carrying the same label are linearly ordered. This is a strong simplification. Indeed, given a labelled partial order p representing a semiword on L and any label $a \in L$, such hypothesis allows us to talk *unequivocally* of the first element labelled a, of the second element labelled a, …, the n-th element labelled a. In other words, we can represent p unequivocally as a (strict) partial order whose elements are pairs in $L \times \omega$, (a, i) representing the i-th element carrying label a. Thus, we are led to the following definition, where for n a natural number, $[n]$ denote the initial segment of length n of $\omega \setminus \{0\}$, i.e., $[n] = \{1, \ldots, n\}$.

DEFINITION 2.4.2 *(Semiwords)*
A (canonical representative of a) semiword on an alphabet L is a pair $x = (A_x, <_x)$ where

- $A_x = \bigcup_{a \in L} \Big(\{a\} \times [n_a^x] \Big)$, *for some $n_a^x \in \omega$, and A_x is finite;*
- $<_x$ *is a transitive, irreflexive, binary relation on A_x such that*

$$(a, i) <_x (a, j) \quad \text{if and only if} \quad i < j,$$

where $<$ is the usual (strict) ordering on natural numbers.

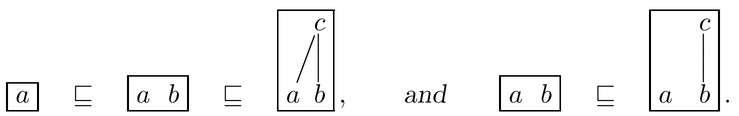
The semiword represented by x is $\Big[\!\Big[(A_x, \leq, \ell) \Big]\!\Big]$, where $(a, i) \leq (b, j)$ if and only if $(a, i) <_x (b, j)$ or $(a, i) = (b, j)$, and $\ell\Big((a, i)\Big) = a$. However, exploiting in full the existence of such an easy representation, from now on, we shall make no distinction between x and the semiword which it represents. In particular, as already stressed in Definition 2.4.2, with abuse of language, we shall refer to x as a semiword. The set of semiwords on L will be indicated by $SW(L)$. The usual set of words (strings) on L is (isomorphic to) the subset of $SW(L)$ consisting of semiwords with *total* ordering.

A standard ordering used on words is the prefix order $\sqsubseteq$, which relates α and β if and only if α is an initial segment of β. Such idea is easily extended to semiwords in order to define a prefix order $\sqsubseteq \subseteq SW(L) \times SW(L)$. Consider x and y in $SW(L)$. Following the intuition, for x to be a prefix of y, it is necessary that the elements of A_x are contained also in A_y with the same ordering. Moreover, since new elements can be added in A_y only "on the top" of A_x, no element in $A_y \setminus A_x$ may be less than an element of A_x. This is formalized by saying

$$x \sqsubseteq y \qquad \text{if and only if} \qquad A_x \subseteq A_y \;\; \text{and} \;\; <_x = <_y \cap A_x^2$$
$$\text{and} \;\; <_y \cap \big((A_y \setminus A_x) \times A_x\big) = \varnothing.$$

It is quickly realized that $\sqsubseteq$ is a partial order on $SW(L)$ and that it coincides with the usual prefix ordering on words.

EXAMPLE 2.4.3 *(Prefix Ordering)*
As a few examples of the prefix ordering of semiwords, it is



However, it is neither the case that

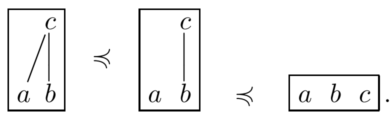
We shall use *Pref*(x) to denote the set $\{y \in SW(L) \mid y \sqsubset x\}$ of *proper prefixes* of x. The set of maximal elements in x will be denoted by *Max*(x) Semiwords with a greatest element play a key role in our development. For reasons that will be clear later, we shall refer to them as to *events*.

Another important ordering is usually defined on semiwords: the *"smoother than"* order, which takes into account that a semiword can be extended just by relaxing its ordering. More precisely, x is smoother than y, in symbols $x \preccurlyeq y$, if x imposes more order contraints on the elements of y. Formally,

$$x \preccurlyeq y \qquad \text{if and only if} \qquad A_x = A_y \quad \text{and} \quad <_x \supseteq <_y .$$

It is easy to see that $\preccurlyeq\ \subseteq SW(L) \times SW(L)$ is a partial order. We shall use *Smooth*(x) to denote the set of *smoothings* of x, i.e., $\{y \in SW(L) \mid y \preccurlyeq x\}$.

Example 2.4.4 *(Smoother than Ordering)*
The following few easy situations exemplify the smoother than ordering of semiwords.



On the other hand, neither

Semilanguages and Event Structures

Semilanguages are a straightforward generalization of Hoare languages to prefix-closed subsets of $SW(L)$.

Definition 2.4.5 *(SemiLanguages)*
A semilanguage is a pair (SW, L)*, where* L *is an alphabet and* SW *is a set of semiwords on* L *which is*

Prefix closed: $y \in SW$ *and* $x \sqsubseteq y$ *implies* $x \in SW$;
Coherent: $Pref(x) \subseteq SW$ *and* $|Max(x)| > 2$ *implies* $x \in SW$.

Semilanguage (SW, L) *is deterministic if*

$$x, y \in SW \text{ and } Smooth(x) \cap Smooth(y) \neq \varnothing \quad \text{implies} \quad x = y.$$

In order to fully understand this definition, we need to appeal to the intended meaning of semilanguages. A semiword in a semilanguage describes a (partial) run of a system in terms of the observable properties (labels) of the events which have occurred, together with the causal relationships which rule their interactions. Thus, the *prefix closedness* clause captures exactly the intuitive fact that any *initial segment* of a (partial) computation is itself a (partial) computation of the system.

In this view, the *coherence* axiom can be interpreted as follows. Suppose that there is a semiword x whose proper prefixes are in the language, i.e., they are runs of the system, and suppose that $|Max(x)| > 2$. This means that, given any pair of maximal elements in x, there is a computation of the system in which the corresponding events have both occurred. Then, in this case, the coherence axiom asks for x to be a possible computation of the system, as well. In other words, we can look at coherence as to the axiom which forces a set of events to be conflict free if it is *pairwise* conflict free, as in [106] for *prime event structures* and in [94] for *proper trace languages*.

To conclude our discussion about Definition 2.4.5, let us analyze the notion of *determinism*. Remembering our interpretation of semiwords as runs of a system, it is easy to understand how the existence of distinct x and y such that $Smooth(x) \cap Smooth(y) \neq \varnothing$ would imply nondeterminism. In fact, if there were two different runs with a common linearization, then there would be two different computations exhibiting the same observable behaviour, i.e., in other words, two *non equivalent* sequences of events with the same strings of labels.

Also the notion of morphism of semilanguages can be derived smoothly as an extension of the existing one for Hoare languages.

Any $\lambda\colon L_0 \rightharpoonup L_1$ determines a partial function $\hat{\lambda}\colon SW(L_0) \rightharpoonup SW(L_1)$ which maps x to its *relabelling* through λ, if this represents a semiword, and is undefined otherwise. Consider now semilanguages (SW_0, L_0) and (SW_1, L_1), and suppose for $x \in SW_0$ that $\hat{\lambda}$ is defined on x. Although one could be tempted to ask that $\hat{\lambda}(x)$ be a semiword in SW_1, this would be by far too strong a requirement. In fact, since in $\hat{\lambda}(x)$ the order $<_x$ is strictly preserved, morphisms would always strictly preserve causal dependency, and this would be out of tune with the existing notion of morphism for event structures, in which sequential tasks can be simulated by *"more concurrent"* ones. Fortunately enough, we have an easy way to ask for the existence of a more concurrent version of $\hat{\lambda}(x)$ in SW_1. It consists of asking that $\hat{\lambda}(x)$ be a smoothing of some semiword in SW_1.

Definition 2.4.6 *(Semilanguage Morphisms)*
Given the semilanguages (SW_0, L_0) and (SW_1, L_1), a partial function $\lambda\colon L_0 \rightharpoonup L_1$ is a morphism $\lambda\colon (SW_0, L_0) \to (SW_1, L_1)$ if

$$\forall x \in SW_0 \qquad \hat{\lambda}{\downarrow}x \quad \text{and} \quad \hat{\lambda}(x) \in \mathit{Smooth}(SW_1).$$

It is worth observing that, if (SW_1, L_1) is *deterministic*, there can be *at most one* semiword in SW_1, say x_λ, such that $\hat{\lambda}(x) \in \mathit{Smooth}(x_\lambda)$. In this case, we can think of $\lambda\colon (SW_0, L_0) \to (SW_1, L_1)$ as mapping x to x_λ.

Example 2.4.7
Given $L_0 = \{a, b\}$ and $L_1 = \{c, d\}$, consider the deterministic semilanguages below.

$$SW_0 = \left\{ \varnothing \quad \boxed{a} \quad \boxed{b} \quad \boxed{\begin{array}{c} a \\ | \\ b \end{array}} \right\} \qquad SW_1 = \left\{ \varnothing \quad \boxed{c} \quad \boxed{d} \quad \boxed{c \;\; d} \right\}.$$

Then, the function λ which maps a to c and b to d is a morphism from (SW_0, L_0) to (SW_1, L_1). For instance,

$$\hat{\lambda}\left(\boxed{\begin{array}{c} a \\ | \\ b \end{array}} \right) = \boxed{\begin{array}{c} c \\ | \\ d \end{array}} \preccurlyeq \boxed{c \;\; d}.$$

Observe that the function $\lambda'\colon L_0 \to L_1$ which sends both a and b to c is not a morphism since $\hat{\lambda}$ applied to $\boxed{b < a}$ gives $\boxed{c < c}$ which is not the smoothing of any semiword in SW_1, while $\lambda''\colon L_1 \to L_0$ which sends both c and d to a is not a morphism from (SW_1, L_1) to (SW_0, L_0) since $\hat{\lambda}$ is undefined on $\boxed{c \;\; d}$.

It can be shown that semilanguages and their morphisms, with composition that of partial functions, form a category whose full subcategory consisting of deterministic semilanguages will be denoted by $\underline{\mathsf{dSL}}$. In the following, we shall define *translation* functors between $\underline{\mathsf{dLES}}$ and $\underline{\mathsf{dSL}}$.

Given a deterministic semilanguage (SW, L) define $\mathit{dsl.dles}\big((SW, L)\big)$ to be the structure $(E, \leq, \#, \ell, L)$, where

- $E = \Big\{ e \;\Big|\; e \in SW,\ e$ is an *event*, i.e., e has a greatest element$\Big\}$;

- $\leq \, = \, \sqsubseteq \cap E^2$;
- $\# = \Big\{ (e, e') \in E^2 \;\Big|\; e \text{ and } e' \text{ are incompatible wrt } \sqsubseteq \Big\}$;
- $\ell(e)$ is the label of the greatest element of e.

Theorem 2.4.8
$dsl.dles\Big((SW, L)\Big)$ is a deterministic labelled event structure.

Consider now a deterministic labelled event structure $DES = (E, \leq, \#, \ell, L)$. Define $dles.dsl(DES)$ to be the structure (SW, L), where

$$SW = \Big\{ \Big[\!\Big[(c, \leq \cap\, c^2, \ell_{|c})\Big]\!\Big] \;\Big|\; c \text{ is a } \textit{finite} \text{ configuration of } DES \Big\}.$$

Theorem 2.4.9
$dles.dsl(DES)$ is a deterministic semilanguage.

It can be shown that *dsl.dles* and *dles.dsl* extend to functors which when composed with each other yield functors naturally isomorphic to identity functors. In other words, they form an *adjoint equivalence* [90, chap. III, pg. 91], i.e., an adjunction which is both a *reflection* and a *coreflection*. It is worthwhile noticing that this implies that the mappings *dsl.dles* and *dles.dsl* constitute a bijection between deterministic semilanguages and isomorphism classes of deterministic labelled event structures—isomorphism being identity up to the names of events.

Theorem 2.4.10
The categories $\underline{\mathsf{dSL}}$ and $\underline{\mathsf{dLES}}$ are equivalent.

In fact, dropping the axiom of coherence in Definition 2.4.5 we get semilanguages equivalent to labelled *stable event structures* [143].

Trace Languages and Event Structures

Generalized trace languages extend trace languages by considering an independence relation which may vary while the computation is progressing. Of course, we need a few axioms to guarantee the consistency of such an extension.

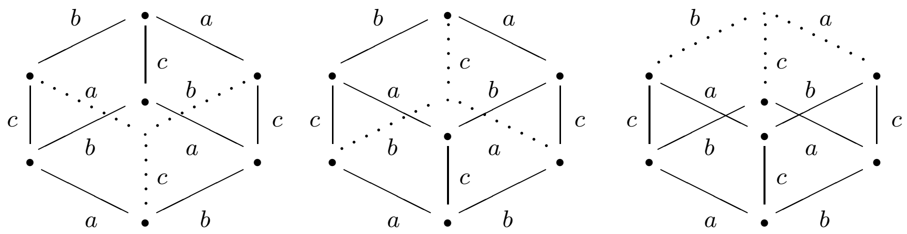
Definition 2.4.11 *(Generalized Trace Languages)*
A generalized trace language is a triple (M, I, L), where L is an alphabet, $M \subseteq L^*$

is a prefix-closed and $\simeq$-closed set of strings, $I: M \to 2^{L\times L}$ is a function which associates to each $s \in M$ a symmetric and irreflexive relation $I_s \subseteq L \times L$, such that

I is consistent: $s \simeq s'$ *implies* $I_s = I_{s'}$;
M is I-closed: $a\ I_s\ b$ *implies* $sab \in M$;
I is coherent: (i) $a\ I_s\ b$ *and* $a\ I_{sb}\ c$ *and* $c\ I_{sa}\ b$ *implies* $a\ I_s\ c$;
(ii) $a\ I_s\ c$ *and* $c\ I_s\ b$ *implies* ($a\ I_s\ b$ *if and only if* $a\ I_{sc}\ b$);

where $\simeq$ is the least equivalence relation on L^* such that $sabu \simeq sbau$ if $a\ I_s\ b$.

As in the case of trace languages, we have an equivalence relation $\simeq$ which equates those strings representing the same computation. Thus, I must be consistent in the sense that it must associate the same independence relation to $\simeq$-equivalent strings. In order to understand the last two axioms, the following picture shows in terms of computations ordered by prefix the situations which those axioms forbid. There, the dots represent computations, the labelled edges represent the prefix ordering, and the dotted lines represent the computations forced in M by the axioms.



(A) (B) (C)

It is easy to see that axiom (i) rules out the situation described by just the solid lines in (A)—impossible for stable event structures, while axiom (ii) eliminates cases (B)—which is beyond the descriptive power of *general event structures* [143] and (C)—impossible for event structures with *binary* conflict. They narrow down to those orderings of computations arising from prime event structures. It is worthwhile to observe that axiom (B) corresponds in our setting to what is called *"cube axiom"* in the setting of concurrent transition systems [132].

Definition 2.4.12 *(Generalized Trace Language Morphisms)*
Given the generalized trace languages (M, I, L) and (M', I', L'), a partial function $\lambda: L \rightharpoonup L'$ is a morphism $\lambda: (M, I, L) \to (M', I', L')$ if

λ preserves words: $s \in M$ *implies* $\lambda^*(s) \in M'$;
λ respects independence: $a\ I_s\ b$ *and* $\lambda\downarrow a,\ \lambda\downarrow b$ *implies* $\lambda(a)\ I'_{\lambda^*(s)}\ \lambda(b)$;

where λ^* *is defined by* $\lambda^*(\epsilon) = \epsilon$ *and* $\lambda^*(sa) = \begin{cases} \lambda^*(s)\lambda(a) & \text{if } \lambda{\downarrow}a \\ \lambda^*(s) & \text{otherwise.} \end{cases}$

Generalized trace languages and their morphisms, under the usual composition of partial functions, form the category $\underline{\mathsf{GTL}}$.

A derived notion of event in generalized trace languages can be captured by the relation $\sim$ defined as the least equivalence such that

$$a\ I_s\ b \quad \text{implies} \quad sa \sim sba \qquad \text{and} \qquad s \simeq s' \quad \text{implies} \quad sa \sim s'a.$$

The events occurring in $s \in M$, denoted by $Ev(s)$, are the $\sim$-classes a representative of which occurs as a non empty prefix of s, i.e.,

$$\Big\{ [u]_\sim \ \Big|\ u \text{ is a non empty prefix of } s \Big\}.$$

It can be shown that $s \simeq s'$ if and only if $Ev(s) = Ev(s')$. Extending the notation, we shall write $Ev(M)$ to denote the events of (M, I, L), i.e., the $\sim$-equivalence classes of non empty strings in M.

Now, given a generalized trace language (M, I, L) define $gtl.dles\Big((M, I, L)\Big)$ to be the structure $(Ev(M), \leq, \#, \ell, L)$, where

- $[s]_\sim \leq [s']_\sim$ if and only if $\forall u \in M,\ [s']_\sim \in Ev(u)$ implies $[s]_\sim \in Ev(u)$;
- $[s]_\sim \# [s']_\sim$ if and only if $\forall u \in M,\ [s]_\sim \in Ev(u)$ implies $[s']_\sim \notin Ev(u)$;
- $\ell\Big([s]_\sim\Big) = a$ if and only if $s = s'a$.

Theorem 2.4.13
$gtl.dles\Big((M, I, L)\Big)$ is a deterministic labelled event structure.

On the other hand, in order to define a generalized trace language from a deterministic labelled event structure $DES = (E, \leq, \#, \ell, L)$, consider

$$M = \Big\{ \ell^*(e_1 \cdots e_n) \ \Big|\ \{e_1, \ldots, e_n\} \subseteq E \quad \text{and} \quad \{e_1, \ldots, e_{i-1}\} \vdash e_i,\ i = 1, \ldots, n \Big\}.$$

Since DES is deterministic, any $s \in M$ identifies unequivocally a string of events $Sec(s) = e_1 \cdots e_n \in E^*$ such that $\{e_1, \ldots, e_{i-1}\} \vdash e_i,\ i = 1, \ldots, n$, and such that $\ell^*(e_1 \cdots e_n) = s$. Now, for any $s \in M$, take

$$I_s = \Big\{ (a, b) \ \Big|\ sab \in M,\ Sec(sab) = xe_0e_1 \text{ and } e_0\ co\ e_1 \Big\}.$$

Then, define (M, I, L) to be $dles.gtl(DES)$.

Theorem 2.4.14
$dles.gtl(DES)$ is a generalized trace language.

As in the case treated in the previous section, *dles.gtl* and *gtl.dles* extend to functors between GTL and dLES which form an adjoint equivalence. Such an equivalence restricts to an isomorphism of generalized trace languages and isomorphism classes of deterministic labelled event structures.

Theorem 2.4.15
Categories GTL and dLES are equivalent.

The result extends to labelled *stable event structures* by dropping the *'only if'* implication in part *(ii)* of the coherence axiom of Definition 2.4.11. Of course, it follows from Theorem 2.4.10 and Theorem 2.4.15 that dSL and GTL are equivalent. In [125], we also define direct translations between such categories.

2.5 Transition Systems with Independence and Labelled Event Structures

In this section, we show that transition systems with independence are an extension of labelled event structures to a system model, by showing that there exists a coreflection from LES to TSI. To simplify our task, we split such a coreflection in two parts. First, we define the *unfolding* of transition systems with independence. To this aim, we introduce the category oTSI of occurrence transition systems with independence. Later, we shall show that labelled event structures are coreflective in oTSI, thus obtaining

$$\underline{\mathsf{LES}} \hookrightarrow \underline{\mathsf{oTSI}} \hookrightarrow \underline{\mathsf{TSI}}.$$

Definition 2.5.1 *(Occurrence Transition Systems with Independence)*
An occurrence transition system with independence is a transition system with independence $OTI = (S, s^I, L, Tran, I)$ which is reachable, acyclic and such that

$$\begin{aligned}(s', a, u) \neq (s'', b, u) \in Tran \quad & \text{implies} \\ & \exists s.\ (s, b, s')\ I\ (s, a, s'') \text{ and } (s, b, s')\ I\ (s', a, u) \\ & \qquad \text{and } (s, a, s'')\ I\ (s'', b, u),\end{aligned}$$

or, in other words, (s', a, u) and (s'', b, u) form the bottom of a concurrency diamond $Diam\Big((s, a, s''), (s, b, s')(s'', b, u), (s', a, u)\Big)$.
Let oTSI denote the full subcategory of TSI whose objects are occurrence transition systems with independence.

Given a transition system with independence TI, define $\simeq \subseteq Path(TI)^2$ to be the least equivalence relation such that

$$\pi_s(s,a,s')(s',b,u)\pi_v \simeq \pi_s(s,b,s'')(s'',a,u)\pi_v \quad \text{if} \quad Diam\Big((s,a,s'),(s,b,s'')(s',b,u),(s'',a,u)\Big).$$

LEMMA 2.5.2
Given an occurrence transition system with independence OTI, let u be a state and π_u, π'_u paths leading to it. Then $\pi_u \simeq \pi'_u$.

Proof. By induction on the minimum length among those of π_u and π'_u.

If $|\pi_u| = |\pi'_u| = 0$, then $\pi_u = \epsilon = \pi'_u$.

Suppose that $\pi_u = \pi_{s'}(s',a,u)$, $\pi'_u = \pi_{s''}(s'',b,u)$ and suppose that $|\pi_{s'}| \leq |\pi_{s''}|$. Then, necessarily, it must be

$$Diam\Big((s,a,s''),(s,b,s'),(s',a,u),(s'',b,u)\Big),$$

for some $s \in S_{OTI}$.

Since OTI is reachable, there exists a path $\pi_0 = \pi_s(s,b,s')$. Since the length of $\pi_{s'}$ is $n-1$, we have that $\min\{|\pi_0|,|\pi_{s'}|\} \leq n-1$. So, we can apply the induction hypothesis and conclude that $\pi_{s'} \simeq \pi_0$. From the definition of $\simeq$, it follows that π_0 has length $n-1$. Thus, $\pi_1 = \pi_s(s,a,s'')$ has length $n-1$ and, by induction, $\pi_1 \simeq \pi_{s''}$. So, $\pi_u = \pi_{s'}(s',a,u) \simeq \pi_s(s,b,s')(s',a,u) \simeq \pi_s(s,a,s'')(s'',b,u) \simeq \pi_{s''}(s'',b,u) = \pi'_u$. ✓

Such a lemma has the following corollary.

COROLLARY 2.5.3
All the paths leading to the same state of an occurrence transition system with independence have the same length.

It is now easy to prove the following basic properties.

COROLLARY 2.5.4
Occurrence transition systems with independence do not contain infinite sequences of transitions ending in a state.

Proof. Suppose that OTI admits an infinite chain of the kind $s_0 \xleftarrow{a_1} s_1 \xleftarrow{a_2} s_2 \xleftarrow{a_3} \cdots$. Since OTI is reachable, there exists a path π_{s_0} in $Path(OTI)$. Let us suppose that $|\pi_{s_0}| = n$. Then consider the first $n+1$ elements of the chain $s_0 \xleftarrow{a_1} s_1 \cdots \xleftarrow{a_n} s_n$. Then, there exists a path π_{s_n} which when composed with $s_n \xrightarrow{a_n} \cdots \xrightarrow{a_0} s_0$ gives a path π'_{s_0} whose length is greater than n. This contradicts Lemma 2.5.2. ✓

The same technique used in the last proof shows the following corollary.

Corollary 2.5.5
Any pair of sequences of transitions leading from state s to state s' of OTI have the same length.

Proof. If there where $s \xrightarrow{a_1} \cdots \xrightarrow{a_n} s'$ and $s \xrightarrow{b_1} \cdots \xrightarrow{b_m} s'$ with $n \neq m$, since s is reachable, we would have two paths leading to s' of different length. ✓

Corollary 2.5.6
Any pair of sequences leading from state $\bar{s}$ to state $\bar{s}'$ of OTI contain the same number of representatives from any $\sim$-equivalence class.

Proof. First suppose that $\bar{s}$ is the initial state s^I_{OTI}. Then the sequences are two paths leading to the same state and therefore, by Lemma 2.5.2, they are $\simeq$-equivalent. In the case $\pi_s(s,a,s')(s',b,u)\pi_{\bar{s}'} \simeq \pi_s(s,b,s'')(s'',a,u)\pi_{\bar{s}'}$, the result is immediate, since $(s,a,s') \sim (s'',a,u)$ and $(s,b,s'') \sim (s',b,u)$. In the general case, the result follows by applying transitively the previous argument.

Now, consider two sequences from a generic $\bar{s}$ to $\bar{s}'$, say $\sigma_{\bar{s}\to\bar{s}'}$ and $\sigma'_{\bar{s}\to\bar{s}'}$. If there were a $\sim$-class whose elements occur a different number of times in $\sigma_{\bar{s}\to\bar{s}'}$ and $\sigma'_{\bar{s}\to\bar{s}'}$, then the same would happen for the paths $\pi_s\sigma_{\bar{s}\to\bar{s}'}$ and $\pi_s\sigma'_{\bar{s}\to\bar{s}'}$, and that would contradict what we have just shown in the first part of this proof. ✓

Corollary 2.5.7
If (s,a,s') and (s,b,s') are transitions of OTI, then $a = b$.

Proof. Since $\pi_s(s,a,s') \simeq \pi_s(s,b,s')$, it must be $(s,a,s') \sim (s,b,s')$ and so $a = b$. ✓

Some other interesting results which do not depend directly from Lemma 2.5.2 follow.

Lemma 2.5.8
For any $(s',a,u) \neq (s'',a,u) \in Tran_{OTI}$, it is $(s',a,u) \not\sim (s'',a,u)$.

Proof. Suppose that $(s',a,u) \sim (s'',a,u)$. Since $s' \neq s''$, by definition of occurrence transition system with independence, there exist a state s and transitions (s,a,s') and (s,a,s'') which form a diamond. Then, $(s,a,s') \sim (s'',a,u) \sim (s',a,u) \sim (s,a,s'')$, and therefore, by axiom *(i)* in Definition 2.3.7, $s' = s''$. Absurd. ✓

Corollary 2.5.9
Given $(s',a,u) \neq (s'',b,u) \in Tran_{OTI}$, there exist unique s, (s,b,s') and (s,a,s'') such that $Diam\Big((s,b,s'),(s,a,s''),(s',a,u),(s'',b,u)\Big)$.

Proof. Suppose that $Diam\Big((\bar{s},b,s'),(\bar{s},a,s''),(s',a,u),(s'',b,u)\Big)$ for $s \neq \bar{s}$. Then we have $(s,a,s'') \sim (s',a,u) \sim (\bar{s},a,s'')$, contradicting the previous lemma. Absurd. ✓

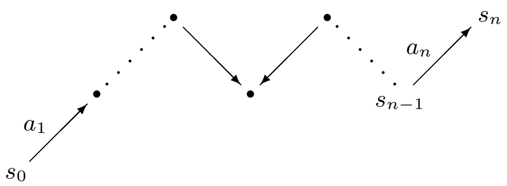
Summing up, these results tell us that occurrence transition systems with independence are particularly well structured. In particular, they imply that in an occurrence transition system with independence each diamond of concurrency is *not* degenerate, i.e., it consists of four *distinct* states.

The next step is to show that in a path of an occurrence transition system with independence at most one representative of a $\sim$-class may appear. Given a path π and an equivalence class $[(s,a,s')]_\sim$, let $\mathcal{N}\Big(\pi,[(s,a,s')]_\sim\Big)$ be the number of representatives of $[(s,a,s')]_\sim$ occurring in π. Of course, we know from Corollary 2.5.6 that such a number depends on π only by means of the state it reaches. Therefore, we shall write simply $\mathcal{N}\Big(x,[(s,a,s')]_\sim\Big)$, for $x \in S_{OTI}$. Moreover, let $s \overset{a}{\longleftrightarrow} s'$ stand for $s \overset{a}{\longrightarrow} s'$ or $s \overset{a}{\longleftarrow} s'$. Then we have the following result.

Lemma 2.5.10
Consider a sequence of states $\sigma = s_0 \overset{a_1}{\longleftrightarrow} s_1 \overset{a_2}{\longleftrightarrow} s_2 \cdots \overset{a_n}{\longleftrightarrow} s_n$. Then

$$\begin{aligned}\mathcal{N}\Big(s_n,[(s,a,s')]_\sim\Big) = \;& \mathcal{N}\Big(s_0,[(s,a,s')]_\sim\Big)\\ &+\Big|\Big\{(s_i,a_{i+1},s_{i+1}) \mid (s_i,a_{i+1},s_{i+1}) \sim (s,a,s')\Big\}\Big|\\ &-\Big|\Big\{(s_{i+1},a_{i+1},s_i) \mid (s_{i+1},a_{i+1},s_i) \sim (s,a,s')\Big\}\Big|.\end{aligned}$$

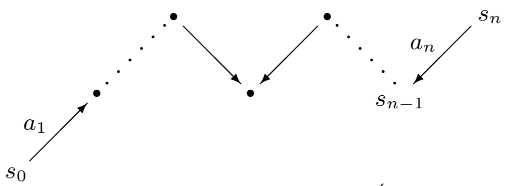
Proof. By induction on n, the length of σ. For $n = 0$, σ is empty and the thesis is trivially true. Suppose then that the thesis holds for sequences of length $n-1$. There are two cases: $s_{n-1} \overset{a_n}{\longrightarrow} s_n$ or $s_n \overset{a_n}{\longrightarrow} s_{n-1}$.



(case $s_{n-1} \overset{a_n}{\longrightarrow} s_n$.)
If $(s_{n-1},a_n,s_n) \not\sim (s,a,s')$ then $\mathcal{N}\Big(s_n,[(s,a,s')]_\sim\Big) = \mathcal{N}\Big(s_{n-1},[(s,a,s')]_\sim\Big)$, and since nothing is added to or subtracted from the right hand term, the equality holds. If otherwise $(s_{n-1},a_n,s_n) \sim (s,a,s')$, then

$$\mathcal{N}\Big(s_n,[(s,a,s')]_\sim\Big) = \mathcal{N}\Big(s_{n-1},[(s,a,s')]_\sim\Big) + 1,$$

and the equality stays since 1 is added also to the righthand term. So, the induction hypothesis is mantained.

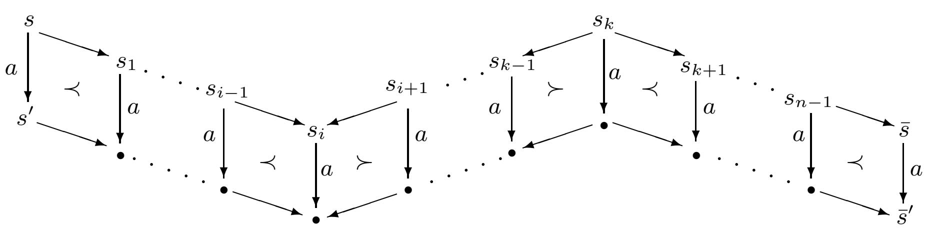


(case $s_n \xrightarrow{a_n} s_{n-1}$.)
Again, if $(s_{n-1}, a_n, s_n) \not\sim (s, a, s')$ the terms on both the sides of the equation are unchanged considering the n-th transition, and the result holds by induction. Otherwise if $(s_{n-1}, a_n, s_n) \sim (s, a, s')$, then $\mathcal{N}\Big(s_n, [(s,a,s')]_\sim\Big) = \mathcal{N}\Big(s_{n-1}, [(s,a,s')]_\sim\Big) - 1$. This time 1 is subtracted from the right hand term, and therefore the induction hypothesis is mantained. ✓

Then, we have the following important corollary.

Corollary 2.5.11
Given a path $\pi \in Path(OTI)$, at most one representative of any $\sim$-equivalence class can occur in π.

Proof. Suppose that $(s, a, s') \sim (\bar{s}, a, \bar{s}')$ occur both in π. By definition of $\sim$, it must exist a sequence $\sigma = \Big(s = s_0 \xleftrightarrow{a_0} \cdots \xleftrightarrow{a_n} s_n = \bar{s}\Big)$, as shown it the following figure.



Without loss of generality, we can assume $\pi = \pi'(s, a, s')\sigma'(\bar{s}, a, \bar{s}')\sigma''$, i.e., that (s, a, s') occurs before than $(\bar{s}, a, \bar{s}')$. Now, since (s, a, s') appears in π after state s, we have

$$\mathcal{N}\Big(s, [(s,a,s')]_\sim\Big) < \mathcal{N}\Big(\bar{s}, [(s,a,s')]_\sim\Big).$$

By the previous lemma, we have that in σ at least a representative of $[(s,a,s')]_\sim$ must occur *"positively"*, say $(s_k, a_{k+1}, s_{k+1}) \sim (s, a, s')$. Therefore, we have a diamond

$$Diam\Big((s_k, a_{k+1}, s_{k+1}), (s_k, a, \bar{s}_k), (s_{k+1}, a, \bar{s}_{k+1}), (\bar{s}_k, a_{k+1}, \bar{s}_{k+1})\Big)$$

where, for the properties already shown, it must be $s_k \neq \bar{s}_k$. This is absurd, because $(s_k, a_{k+1}, s_{k+1}) \sim (s_k, a, \bar{s}_k)$ breaks axiom *(i)* of transition systems with independence. ✓

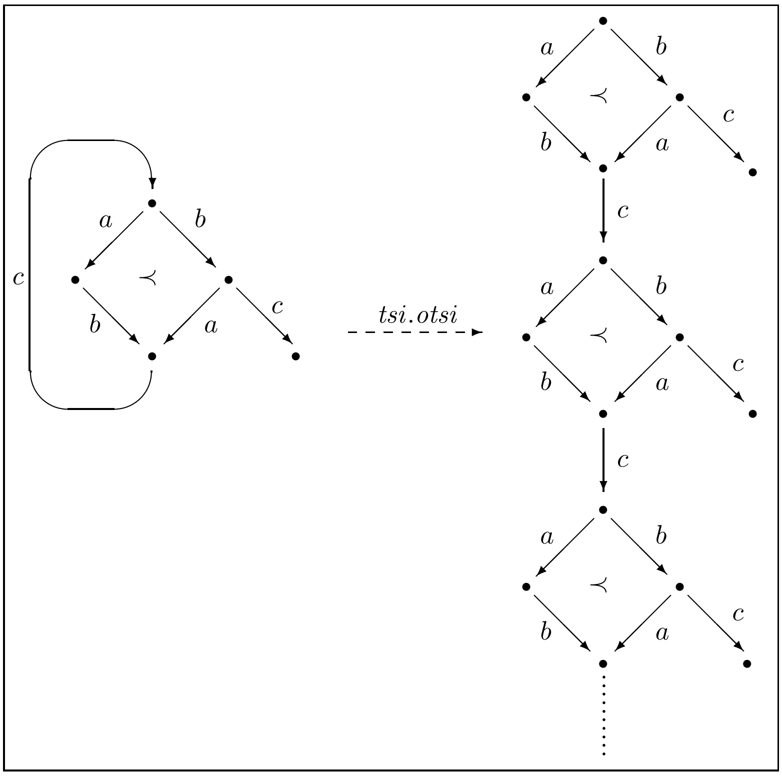


Figure 2.1: A transition system with independence TI and $tsi.otsi(TI)$.

UNFOLDING TRANSITION SYSTEMS WITH INDEPENDENCE

Given a transition system with independence $TI = (S, s^I, L, Tran, I)$, we define $tsi.otsi(TI) = \Big(\Pi_{\simeq}, [\epsilon]_{\simeq}, L, Tran_{\simeq}, I_{\simeq}\Big)$, where

- $\Pi_{\simeq}$ is the quotient of $Path(TI)$ modulo $\simeq$;
- $\Big([\pi]_{\simeq}, a, [\pi']_{\simeq}\Big) \in Tran_{\simeq} \quad \Leftrightarrow \quad \exists (s, a, s') \in Tran$ such that $\pi' \simeq \pi(s, a, s')$;
- $\Big([\pi]_{\simeq}, a, [\pi']_{\simeq}\Big)\ I_{\simeq}\ \Big([\bar{\pi}]_{\simeq}, b, [\bar{\pi}']_{\simeq}\Big) \quad \Leftrightarrow$
  $\exists (s, a, s'), (\bar{s}, b, \bar{s}') \in Tran$ such that
  $(s, a, s')\ I\ (\bar{s}, b, \bar{s}')$, $\pi' \simeq \pi(s, a, s')$, and $\bar{\pi}' \simeq \bar{\pi}(\bar{s}, b, \bar{s}')$.

Proposition 2.5.12
The transition system $tsi.otsi(TI)$ is an occurrence transition system with independence.

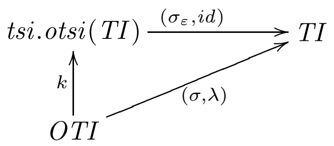
Proof. We show only the condition in Definition 2.5.1 of occurrence transition systems with independence. Suppose that $\Big([\pi']_\simeq, b, [\pi]_\simeq\Big) \neq \Big([\pi'']_\simeq, a, [\pi]_\simeq\Big)$. Then, we have $\pi \simeq \pi'(s', b, u) \simeq \pi''(s'', a, u)$ with $\pi' \neq \pi''$. By definition of $\simeq$, it must exist $\bar{\pi}$ such that $\pi'(s', b, u) \simeq \bar{\pi}(s, a, s')(s', b, u)$ and $\pi''(s'', a, u) \simeq \bar{\pi}(s, b, s'')(s'', a, u)$. Moreover, it must be $\bar{\pi}(s, a, s') \simeq \pi'$ and $\bar{\pi}(s, b, s'') \simeq \pi''$. Therefore, $\Big([\bar{\pi}]_\simeq, a, [\bar{\pi}(s, a, s')]_\simeq\Big)$ and $\Big([\bar{\pi}]_\simeq, b, [\bar{\pi}(s, b, s'')]_\simeq\Big)$ close the diamond. ✓

Figure 2.1 shows a simple example of unfolding of a transition system with independence. Next, we want to show that $tsi.otsi$ extends to a functor for <u>TSI</u> to <u>oTSI</u> which is right adjoint to the inclusion functor <u>oTSI</u> $\hookrightarrow$ <u>TSI</u>. As a candidate for the counit of such an adjunction, consider the mapping $(\sigma_\varepsilon, id)\colon tsi.otsi(TI) \to TI$ where

$$\sigma_\varepsilon(\epsilon) = s^I_{TI} \qquad \text{and} \qquad \sigma_\varepsilon\Big([\pi_s]_\simeq\Big) = s.$$

By Lemma 2.5.2, we know that σ_ε is well-defined. Then, it is not difficult to see that (σ_ε, id) is a morphism of transition systems with independence.

Proposition 2.5.13 *($(\sigma_\varepsilon, id)\colon tsi.otsi(TI) \to TI$ is couniversal)*
For any occurrence transition system with independence OTI, transition system with independence TI and morphism $(\sigma, \lambda)\colon OTI \to TI$, there exists a unique $k\colon OTI \to tsi.otsi(TI)$ in <u>oTSI</u> such that $(\sigma_\varepsilon, id) \circ k = (\sigma, \lambda)$.



Proof. Clearly, in order for the diagram to commute, k must be of the form $(\bar{\sigma}, \lambda)$. Consider the map $\bar{\sigma}(s) = [\sigma_\lambda(\pi_s)]_\simeq$, where $\sigma_\lambda\colon Path(OTI) \to Path(TI)$ is given by

$$\sigma_\lambda(\epsilon) = \epsilon; \qquad \sigma_\lambda\Big(\pi_s(s, a, s')\Big) = \begin{cases} \sigma_\lambda(\pi_s)\Big(\sigma(s), \lambda(a), \sigma(s')\Big) & \text{if } \lambda{\downarrow}a \\ \sigma_\lambda(\pi_s) & \text{otherwise.} \end{cases}$$

This definition is well-given: fixed s, let π_s and $\pi_{s'}$ be two paths leading to s. Then, since OTI is an occurrence transition system with independence, it is $\pi_s \simeq \pi_{s'}$, and since (σ, λ) is a morphism, it is $\sigma_\lambda(\pi_s) \simeq \sigma_\lambda(\pi_{s'})$. In order to show this last statement, it is enough to prove that

$$\begin{aligned}\pi_s(s,a,s')(s',b,u)\pi_v \simeq \pi_s(s,b,s'')(s'',a,u)\pi_v\\ \Rightarrow\quad \sigma_\lambda(\pi_s)\sigma_\lambda\Big((s,a,s')(s',b,u)\Big)\sigma_\lambda(\pi_v)\\ \simeq \sigma_\lambda(\pi_s)\sigma_\lambda\Big((s,b,s'')(s'',a,u)\Big)\sigma_\lambda(\pi_v).\end{aligned}$$

There are four cases:

i) $\lambda\uparrow a$, $\lambda\uparrow b$: then $\sigma_\lambda\Big((s,a,s')(s',b,u)\Big) = \epsilon = \sigma_\lambda\Big((s,b,s'')(s'',a,u)\Big)$ and the thesis follows easily.

ii) $\lambda\downarrow a$, $\lambda\uparrow b$: then

$$\begin{aligned}\sigma_\lambda\Big((s,a,s')(s',b,u)\Big) &= \Big(\sigma(s),\lambda(a),\sigma(s')\Big)\\ &= \Big(\sigma(s''),\lambda(a),\sigma(u)\Big) = \sigma_\lambda\Big((s,b,s'')(s'',a,u)\Big)\end{aligned}$$

and again the thesis follows.

iii) $\lambda\uparrow a$, $\lambda\downarrow b$: as for the previous point.

iv) $\lambda\downarrow a$, $\lambda\downarrow b$: then the thesis follows directly from the definition of morphism, since it is $Diam\Big((s,a,s')(s,b,s'')(s',b,u)(s'',a,u)\Big)$ and in this case diamonds are preserved.

Let us show that $(\bar\sigma,\lambda)$ is indeed a morphism of occurrence transition systems with independence.

i) $\bar\sigma(s^I_{OTI}) = [\epsilon]_\simeq$.

ii) Consider $(s,a,s') \in Tran_{OTI}$, and suppose $\lambda\downarrow a$. Since OTI is reachable, we have $\pi_s(s,a,s') \in Path(OTI)$, and $\sigma_\lambda(\pi_s)\Big(\sigma(s),\lambda(a),\sigma(s')\Big)$ in $Path(TI)$. Thus, $\Big([\sigma_\lambda(\pi_s)]_\simeq,\lambda(a),[\sigma_\lambda(\pi_s(s,a,s'))]_\simeq\Big) \in Tran_\simeq$, i.e., $\Big(\bar\sigma(s),\lambda(a),\bar\sigma(s')\Big)\in Tran_\simeq$.

iii) If $(s,a,s')\ I_{OTI}\ (\bar s,b,\bar s')$, $\Big(\sigma(s),\lambda(a),\sigma(s')\Big)\ I_{TI}\ \Big(\sigma(\bar s),\lambda(b),\sigma(\bar s')\Big)$, and reasoning as before, we get $\Big(\bar\sigma(s),\lambda(a),\bar\sigma(s')\Big)\ I_\simeq\ \Big(\bar\sigma(\bar s),\lambda(b),\bar\sigma(\bar s')\Big)$.

In order to show that the diagram commutes, it is enough to observe that each s is mapped to a $\simeq$-class of paths leading to $\sigma(s)$. Therefore, $\sigma_\varepsilon\circ\bar\sigma(s) = \sigma(s)$. The uniquess of $(\bar\sigma,\lambda)$ is easily obtained following the same argument. In fact, the behaviour of $\bar\sigma$ is compelled on any s: s^I_{OTI} must be mapped to $[\epsilon]_\simeq$, while a generic s must mapped to a $\simeq$-equivalence class of paths leading to $\sigma(s)$. But we know that there is a unique such class. ✓

COROLLARY 2.5.14

The construction tsi.otsi extends to a functor from <u>TSI</u> *to* <u>oTSI</u> *which is right adjoint to the inclusion* <u>oTSI</u> $\hookrightarrow$ <u>TSI</u>.

It will be useful later to notice that this coreflection cuts down to a coreflections doTSI $\hookrightarrow$ dTSI, where doTSI is the full subcategory of oTSI consisting of deterministic transition systems. In order to achieve this result, it is clearly enough to show that $tsi.otsi$ maps objects from dTSI to doTSI.

Proposition 2.5.15
If TI is deterministic, then tsi.otsi(TI) is deterministic.

Proof. Suppose that $\Big([\pi]_\simeq, a, [\pi']_\simeq\Big)$ and $\Big([\pi]_\simeq, a, [\pi'']_\simeq\Big)$ are in $Tran_\simeq$. Then, it must be $\pi' \simeq \pi_s(s,a,s')$ and $\pi'' \simeq \pi_s(s,a,s'')$, for $(s,a,s'),\ (s,a,s'') \in Tran$. Then we have $s' = s''$ and so $\pi' \simeq \pi''$. ✓

Occurrence TSI's and Labelled Event Structures

In this subsection we complete the construction of the coreflections LES $\hookrightarrow$ TSI and dLES $\hookrightarrow$ dTSI by showing the existence of coreflections LES $\hookrightarrow$ oTSI and dLES $\hookrightarrow$ doTSI.

Consider a labelled event structure $ES = (E, \leq, \#, \ell, L)$. Define $les.otsi(ES)$ to be the transition system with independence of the *finite configurations* of ES, i.e.,

$$les.otsi(ES) = \Big(\mathcal{L}_F(ES), \varnothing, L, Tran, I\Big),$$

where

- $\mathcal{L}_F(ES)$ is the set of finite configuration of ES;
- $(c,a,c') \in Tran$ if and only if $c = c' \setminus \{e\}$ and $\ell(e) = a$;
- $(c,a,c')\ I\ (\bar{c},b,\bar{c}')$ if and only if $(c' \setminus c)\ co\ (\bar{c}' \setminus \bar{c})$.

By definition, $les.otsi(ES)$ is clearly an acyclic, reachable transition system. Moreover, $I \subseteq Tran^2$ is symmetric and irreflexive, since co is such. In order to show that it is an occurrence transition system with independence, it is important the following characterization of the relation $\sim$.

Lemma 2.5.16
Given (c,a,c') and $(\bar{c},a,\bar{c}') \in Tran$, we have $(c,a,c') \sim (\bar{c},a,\bar{c}') \in Tran$ if and only if $(c' \setminus c) = (\bar{c}' \setminus \bar{c})$.

Proof. $(\Rightarrow)$. It is enough to show that

$$Diam\Big((c,a,c'), (c,b,\bar{c}), (c',b,\bar{c}'), (\bar{c},a,\bar{c}')\Big) \quad \text{implies} \quad (c' \setminus c) = (\bar{c}' \setminus \bar{c}).$$

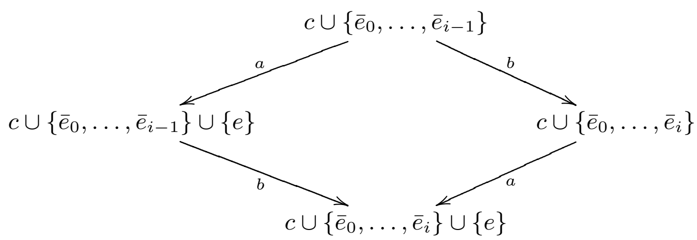
Since $(c, a, c')\ I\ (c, b, \bar{c})$, we have $(c' \setminus c)\ co\ (\bar{c} \setminus c)$. Let the event in $\bar{c}' \setminus c'$ be e'' and the one in $\bar{c}' \setminus \bar{c}$ be e'''. We have $c \cup \{e\} \cup \{e''\} = \bar{c}' = c \cup \{e'''\} \cup \{e'\}$. Thus, it must be

$$(e = e''' \text{ and } e'' = e') \text{ or } (e = e' \text{ and } e''' = e'').$$

Now, since $e\ co\ e'$, it cannot be $e = e'$ and we must discard the second hypothesis. Therefore, $e = e'''$, i.e., $(c' \setminus c) = (\bar{c}' \setminus \bar{c})$ (and necessarily $(\bar{c} \setminus c) = (\bar{c}' \setminus c')$).

($\Leftarrow$). First suppose $c \subseteq \bar{c}$. Since then event e in $(c' \setminus c) = (\bar{c}' \setminus \bar{c})$ is enabled both in c and $\bar{c}$, it means that for any $\bar{e} \in (\bar{c} \setminus c)$ we have $\bar{e}\ co\ e$. Moreover, we can order the events in $\bar{c} \setminus c$ in a chain $\bar{e}_0 \cdots \bar{e}_n$ in a such a way that $c \cup \{\bar{e}_0, \ldots, \bar{e}_{i-1}\} \vdash \bar{e}_i$, for $i = 0, \ldots, n$. To this aim, it is enough to choose at each step i one of the maximal events in $(\bar{c} \setminus c) \setminus \{\bar{e}_0, \ldots, \bar{e}_{i-1}\}$ with respect to the $\leq_{ES}$ order.

Now, since $\bar{e}_i\ co\ e$, for each $i = 0, \ldots, n$ there exists a diamond



Then, for $i = 0, \ldots, n$ we have

$$\Big(c \cup \{\bar{e}_0, \ldots, \bar{e}_{i-1}\}, a, c \cup \{\bar{e}_0, \ldots, \bar{e}_{i-1}\} \cup \{e\}\Big) \prec \Big(c \cup \{\bar{e}_0, \ldots, \bar{e}_i\}, a, c \cup \{\bar{e}_0, \ldots, \bar{e}_i\} \cup \{e\}\Big),$$

i.e., $(c, a, c') \sim (\bar{c}, a, \bar{c}')$. To complete the proof, consider $\bar{c} \cap c$. Necessarily, it enables e. So, we have that $\Big((\bar{c} \cap c), a, (\bar{c} \cap c) \cup \{e\}\Big) \in \mathit{Tran}$. Since $(\bar{c} \cap c) \subseteq \bar{c}$ and $(\bar{c} \cap c) \subseteq c$, we have $(c, a, c') \sim \Big((\bar{c} \cap c), a, (\bar{c} \cap c) \cup \{e\}\Big) \sim (\bar{c}, a, \bar{c}')$. ✓

It is now easy to show the following proposition.

Proposition 2.5.17
The transition system $les.otsi(ES)$ is an occurrence transition system with independence.

Proof. We verify just the property of occurrence transition systems with independence. Suppose that $(c', b, c) \neq (c'', a, c) \in \mathit{Tran}$. Then, we have $c = c' \cup \{e'\} = c'' \cup \{e''\}$. Since $c' \neq c''$, it must be $e' \neq e''$. Moreover, it is $e' \not\#\ e''$, since both events appear in c. It cannot be $e' < e''$ nor $e'' < e'$, because otherwise either c' or c'' would not be a configuration. So, it is $e'\ co\ e''$. It follows that $\bar{c} = c' \setminus \{e'\} = c'' \setminus \{e''\}$ is a configuration such that $\mathit{Diam}\Big((\bar{c}, a, c'), (\bar{c}, b, c''), (c', b, c), (c'', a, c)\Big)$. ✓

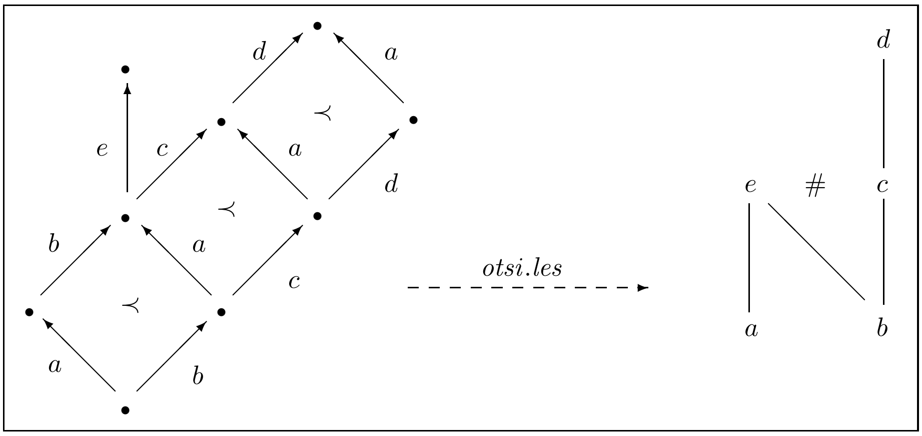


Figure 2.2: An occurrence transition system *OTI* and *otsi.les*(*OTI*).

Let us define the opposite transformation from oTSI to LES. For an occurrence transition system with independence $OTI = (S, s^I, L, \mathit{Tran}, I)$, define *otsi.les*(*OTI*) to be the structure $(\mathit{Tran}_\sim, \leq, \#, \ell, L)$, where

- $\mathit{Tran}_\sim$ is the set of the $\sim$-equivalence classes of *Tran*;
- $[(s, a, s')]_\sim < [(\bar{s}, b, \bar{s}')]_\sim$ if and only if

$$\forall \pi(\underline{\bar{s}}, b, \underline{\bar{s}}') \in \mathit{Path}(OTI) \text{ with } (\underline{\bar{s}}, b, \underline{\bar{s}}') \sim (\bar{s}, b, \bar{s}'),$$
$$\exists (\underline{s}, a, \underline{s}') \sim (s, a, s') \text{ such that } (\underline{s}, a, \underline{s}') \in \pi,$$

  and $\leq$ is the reflexive closure of $<$;
- $[(s, a, s')]_\sim \# [(\bar{s}, b, \bar{s}')]_\sim$ if and only if

$$\forall \pi \in \mathit{Path}(OTI),$$
$$\forall (\underline{\bar{s}}, b, \underline{\bar{s}}') \sim (\bar{s}, b, \bar{s}') \text{ and } \forall (\underline{s}, a, \underline{s}') \sim (s, a, s')$$
$$(\underline{s}, a, \underline{s}') \in \pi \Rightarrow (\underline{\bar{s}}, a, \underline{\bar{s}}') \notin \pi;$$

- $\ell\Big([(s, a, s')]_\sim\Big) = a;$

and we write $(s, a, s') \in \pi$ to mean that (s, a, s') occurs in the path π. Of course, *otsi.les*(*OTI*) is a labelled event structure. Figure 2.2 shows an example of the labelled event structure associated to an occurrence transition system with independence.

Next, we need to extend *otsi.les* to a functor. Given $(\sigma, \lambda)\colon OTI_0 \to OTI_1$, define $otsi.les\Big((\sigma,\lambda)\Big) = (\eta_\sigma, \lambda)$, where

$$\eta_\sigma\Big([(s,a,s')]_\sim\Big) = \begin{cases} \Big[\Big(\sigma(s), \lambda(a), \sigma(s')\Big)\Big]_\sim & \text{if } \lambda\downarrow a \\ \uparrow & \text{otherwise.} \end{cases}$$

In the proof of Proposition 2.5.13, it has been shown implicitly that $(s,a,s') \prec (\bar{s}, a, \bar{s}')$ and $\lambda\downarrow a$ implies $\Big(\sigma(s), \lambda(a), \sigma(s')\Big) \sim \Big(\sigma(\bar{s}), \lambda(a), \sigma(\bar{s}')\Big)$. Then η_σ is well-given.

PROPOSITION 2.5.18

Given a transition system with independence morphism $(\sigma, \lambda)\colon OTI_0 \to OTI_1$, $otsi.les\Big((\sigma,\lambda)\Big)\colon otsi.les(OTI_0) \to otsi.les(OTI_1)$ *is a labelled event structure morphism.*

Proof. We show the properties of labelled event structure morphisms.

i) $\lfloor \eta_\sigma(e) \rfloor \subseteq \eta_\sigma(\lfloor e \rfloor)$.
Consider $e = \Big[\Big(\sigma(s), \lambda(a), \sigma(s')\Big)\Big]_\sim$ in $otsi.les(OTI_1)$ and $[(\bar{s}, b, \bar{s}')]_\sim \le e$. Since OTI_1 is reachable there is a path $\pi_{\sigma(s)}\Big(\sigma(s), \lambda(a), \sigma(s')\Big)$, where $\pi_{\sigma(s)}$ necessarily contains $(\underline{\bar{s}}, b, \underline{\bar{s}}') \sim (\bar{s}, b, \bar{s}')$, and there exists a path $\pi_s(s,a,s') \in Path(OTI_0)$ whose image through (σ, λ) is $\pi_{\sigma(s)}\Big(\sigma(s), \lambda(a), \sigma(s')\Big)$. Then, we must have a transition $(x,c,y) \in \pi_s$ such that $\Big(\sigma(x), \lambda(c), \sigma(y)\Big) = \Big(\underline{\bar{s}}, b, \underline{\bar{s}}'\Big)$. Clearly, we have that $\eta_\sigma\Big([(x,c,y)]_\sim\Big) = [(\bar{s}, b, \bar{s}')]_\sim$. Thus, we need to show that

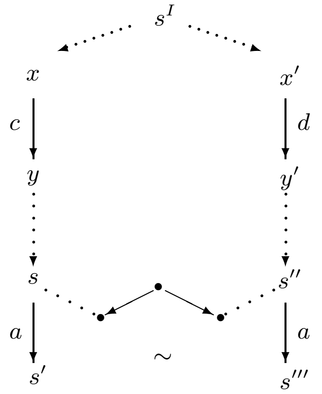
$$[(x,c,y)]_\sim < [(s,a,s')]_\sim.$$

Consider $(s'', a, s''') \sim (s, a, s')$. Since $\lambda\downarrow a$, it is $\Big(\sigma(s''), \lambda(a), \sigma(s''')\Big) \in Tran_{OTI_1}$, and since (σ, λ) is a morphism, $\Big(\sigma(s''), \lambda(a), \sigma(s''')\Big) \sim \Big(\sigma(s), \lambda(a), \sigma(s')\Big)$. It follows that for each path $\pi_{\sigma(s'')}\Big(\sigma(s''), \lambda(a), \sigma(s''')\Big)$ there exists $(\underline{\bar{s}}'', b, \underline{\bar{s}}''') \sim (\bar{s}, b, \bar{s}')$ occurring in $\pi_{\sigma(s'')}$. Therefore, there exists a path $\pi_{s''}(s'', a, s''')$ such that there is $(x', d, y') \in \pi_{s''}$ and $\Big(\sigma(x'), \lambda(d), \sigma(y')\Big) = (\underline{\bar{s}}'', b, \underline{\bar{s}}''') \sim (\bar{s}, b, \bar{s}')$. Now, since by the properties of occurrence transition systems with independence in any other path $\pi_{s'''}$ it must exist a transition $\sim$-equivalent to (x', d, y'), it is enough to show that

$$(x', d, y') \sim (x, c, y).$$

First observe that, since $\Big(\sigma(x'), \lambda(d), \sigma(y')\Big) \sim \Big(\sigma(x), \lambda(c), \sigma(y)\Big)$, no more than one element of $[(x', d, y')]_\sim \cup [(x, c, y)]_\sim$ can appear on the same path. In fact, since such a path would be mapped to a path of OTI_1, we would have a path of OTI_1 with more than one occurrence of elements from $\Big[\Big(\sigma(x), \lambda(c), \sigma(y)\Big)\Big]_\sim$.

Now suppose $(x', d, y') \not\sim (x, c, y)$. Then we are in the situation illustrated by the figure. Necessarily, it must exist



$$(\bar{x}, c, \bar{y}) \sim (x, c, y)$$

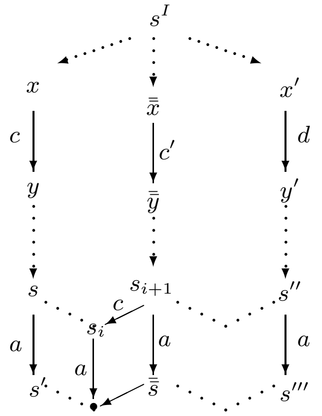
which occurs *"backward"* in the sequence

$$s \leftrightarrow s_1 \leftrightarrow \cdots s_n \leftrightarrow s''.$$

This is because the path from $s^I_{OTI_0}$ to s'' cannot contain any representative of $[(x, c, y)]_\sim$.

So suppose that $s_{i+1} = \bar{x} \xrightarrow{c} \bar{y} = s_i$.

Now consider a path $\pi_{s_{i+1}}$, and consider $\pi_{s_{i+1}}(s_{i+1}, a, \bar{\bar{s}})$, where $(s_{i+1}, a, \bar{\bar{s}}) \sim (s, a, s')$. The situation is illustrated by the figure on the side. Since $\pi_{s_{i+1}}(s_{i+1}, a, \bar{\bar{s}})$ is a path whose image ends with a element of the class $\Big[\Big(\sigma(s), \lambda(a), \sigma(s')\Big)\Big]_\sim$, namely, $\Big(\sigma(s_{i+1}), \lambda(a), \sigma(\bar{\bar{s}})\Big)$, it follows that $\pi_{s_{i+1}}$ contains a transition $\bar{\bar{x}} \xrightarrow{c'} \bar{\bar{y}}$ such that



$$\Big(\sigma(\bar{\bar{x}}), \lambda(c'), \sigma(\bar{\bar{y}})\Big) = (\underline{\bar{\bar{s}}}, b, \underline{\bar{\bar{s}}}') \sim (\bar{s}, b, \bar{s}').$$

Now consider the path

$$\pi_{s_{i+1}}(s_{i+1}, c, s_i) = \pi_{s_{i+1}}(\bar{x}, c, \bar{y}).$$

Clearly, its image through (σ, λ) contains $\Big(\sigma(\bar{\bar{x}}), \lambda(c'), \sigma(\bar{\bar{y}})\Big) = (\underline{\bar{\bar{s}}}, b, \underline{\bar{\bar{s}}}') \sim (\bar{s}, b, \bar{s}')$ and, in addition, also $\Big(\sigma(\bar{x}), \lambda(c), \sigma(\bar{y})\Big) \sim \Big(\sigma(x), \lambda(c), \sigma(y)\Big) \sim (\underline{\bar{s}}, b, \underline{\bar{s}}') \sim (\bar{s}, b, \bar{s}')$, where $(\underline{\bar{\bar{s}}}, b, \underline{\bar{\bar{s}}}') \neq (\underline{\bar{s}}, b, \underline{\bar{s}}')$. This is absurd, because no such path can exist in OTI_1.

It follows that $(x, c, y) \sim (x', d, y')$.

ii) $\eta_\sigma(e) \mathbin{\text{W}} \eta_\sigma(e') \Rightarrow e \mathbin{\text{W}} e'$.

Observe that if $\eta_\sigma(e) = \eta_\sigma(e')$ or $\eta_\sigma(e) \# \eta_\sigma(e')$, then no more than one element from $e = [(s, a, s')]_\sim \cup [(\bar{s}, b, \bar{s}')]_\sim = e'$ may occur in the same path. This is because, in such a case, there would be a path in OTI_1 in which more than one representative of the same class or two representatives of conflicting classes would appear in the same path. From such considerations, it follows that it can be neither $e < e'$ nor $e' < e$ nor $e \; co \; e'$. The only possible cases are, therefore, $e = e'$ or $e \# e'$.

iii) $\lambda(\ell_{OTI_0}(e)) = \ell_{OTI_1}(\eta_\sigma(e))$. Immediate. ✓

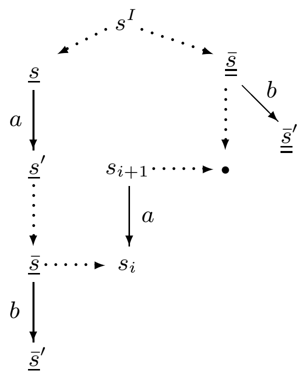
It is very easy now to show the following.

COROLLARY 2.5.19
The map otsi.les is a functor from <u>oTSI</u> *to* <u>LES</u>.

Before showing that *otsi.les* and *les.otsi* form a coreflection, we take the opportunity to show some general results about transition systems with independence, which are more general than strictly needed. However, they will be useful later on.

LEMMA 2.5.20
Whenever $[(s, a, s')]_\sim \; co \; [(\bar{s}, b, \bar{s}')]_\sim$, *then* $(s, a, s') \; I \; (\bar{s}, b, \bar{s}')$.

Proof. By hypothesis $[(s, a, s')]_\sim \not\# [(\bar{s}, b, \bar{s}')]_\sim$ and $[(s, a, s')]_\sim \not\gtrless [(\bar{s}, b, \bar{s}')]_\sim$. From the first hypothesis, it must exist a path which includes representatives of both classes, say $\pi_{\underline{s}}(\underline{s}, a, \underline{s}')\pi_{\underline{\bar{s}}}(\underline{\bar{s}}, b, \underline{\bar{s}}')$. Then, from the second condition, it must exist a path which contains a representative of $[(\bar{s}, b, \bar{s}')]_\sim$ but no representative of $[(s, a, s')]_\sim$, say $\pi_{\underline{\underline{\bar{s}}}}(\underline{\underline{\bar{s}}}, b, \underline{\underline{\bar{s}}}')$.



Now, since no representative of $[(s, a, s')]_\sim$ is in $\pi_{\underline{\underline{\bar{s}}}}$, by Lemma 2.5.10, there is a sequence

$$\underline{\bar{s}} \leftrightarrow s_1 \leftrightarrow \cdots \leftrightarrow s_n \leftrightarrow \underline{\underline{\bar{s}}}$$

such that there exists $(s_{i+1}, a, s_i) \sim (s, a, s')$, as illustrated in the figure. So,

$$(s, a, s') \sim (s_{i+1}, a, s_i) \; I \; (\underline{\underline{\bar{s}}}, b, \underline{\underline{\bar{s}}}') \sim (\bar{s}, b, \bar{s}'),$$

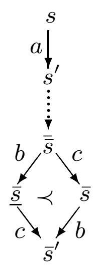
which implies, by the property *(iv)* of transition systems with independence in Definition 2.3.7, $(s, a, s') \; I \; (\bar{s}, b, \bar{s}')$. ✓

LEMMA 2.5.21
Suppose that there is a path $\pi_s(s, a, s')\pi_{\bar{s}}(\bar{s}, b, \bar{s}') \in Path(OTI)$ *and that, for each* $(x, a, y) \in \pi_{\bar{s}}$ *we have* $[(x, a, y)]_\sim \; co \; [(\bar{s}, b, \bar{s}')]_\sim$. *Then there exists a transition* $(s', b, s'') \in Tran_{OTI}$ *such that* $(s', b, s'') \sim (\bar{s}, b, \bar{s}')$.

Proof. By induction on the length of $\pi_{\bar{s}}$. If such a length is zero, then there is nothing to show. Otherwise, we have $\pi_s(s, a, s')\pi_{\bar{\bar{s}}}(\bar{\bar{s}}, c, \bar{s})(\bar{s}, b, \bar{s}')$, where

$$[(\bar{\bar{s}}, c, \bar{s})]_\sim \; co \; [(\bar{s}, b, \bar{s}')]_\sim.$$

So, by the previous lemma, we have $(\bar{\bar{s}}, c, \bar{s})\ I\ (\bar{s}, b, \bar{s}')$, that, by the general properties of transition systems with independence, must be part of a diamond of concurrency, as shown in the figure. Therefore, there exists $(\bar{\bar{s}}, b, \underline{s}) \sim (\bar{s}, b, \bar{s}')$ and thus, we have a path $\pi_s(s, a, s')\pi_{\bar{\bar{s}}}(\bar{\bar{s}}, b, \underline{s})$, where $\pi_{\bar{\bar{s}}}$ is strictly shorter than $\pi_{\bar{s}}$. Then, by inductive hypothesis, there exists a transition (s', b, s'') such that $(s', b, s'') \sim (\bar{\bar{s}}, b, \underline{s}) \sim (\bar{s}, b, \bar{s}')$, which is the thesis.



✓

LEMMA 2.5.22
Consider a path $\pi_s \in Path(OTI)$ and a class $[t]_\sim$ such that

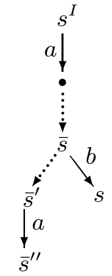
i) for each t' in π_s, it is $[t']_\sim \not\# [t]_\sim$;

ii) for each t' in π_s, we have $[t']_\sim \neq [t]_\sim$.

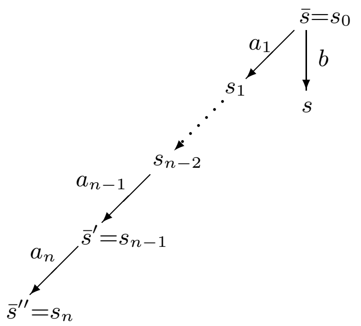
Then, there exists $\pi_s\pi_{s'}(s', a, s'') \in Path(OTI)$ with $(s', a, s'') \sim t$.

Proof. By induction on the depth of s, i.e., the length of π_s.

If $\pi_s = \epsilon$, the thesis is trivial, since OTI is reachable. Then, suppose we have $\pi_s = \pi_{\bar{s}}(\bar{s}, b, s)$. By induction hypothesis, there exists a path $\pi_{\bar{s}}\pi_{\bar{s}'}(\bar{s}', a, \bar{s}'')$, with $(\bar{s}', a, \bar{s}'') \sim t$. From the previous lemma, we can assume that $\pi_{\bar{s}'}$ does not contain any transition whose class is concurrent with $[t]_\sim$. In fact, such transitions can be pushed after the representative of $[t]_\sim$. It follows that $\pi_{\bar{s}'}$ contains only elements t' such that $[t']_\sim \leq [t]_\sim$.

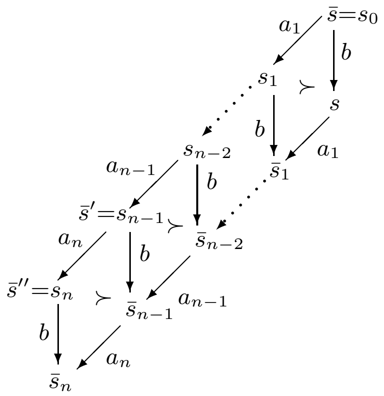


Now, if the first transition of $\pi_{\bar{s}'}$ is $(\bar{s}, b, s)$, we are done. Otherwise, we have the situation shown in the picture on the side, i.e., a chain



$$s_0 \xrightarrow{a_1} s_1 \xrightarrow{a_2} \cdots \xrightarrow{a_{n-1}} s_{n-1} \xrightarrow{a_n} s_n,$$

where $s_0 = \bar{s}$, $s_{n-1} = \bar{s}'$, $s_n = \bar{s}''$, $a_n = a$ and $s = s_i$, for $i = 1, \ldots, n$.
Since $[(s_{i-1}, a_i, s_i)]_\sim \leq [t]_\sim$ for $i = 1, \ldots, n$, and $[(\bar{s}, b, s)]_\sim \neq [t]_\sim$ and $[(\bar{s}, b, s)]_\sim \not\# [t]_\sim$, we have that $[(\bar{s}, b, s)]_\sim \not\# [(s_{i-1}, a_i, s_i)]_\sim$ for $i = 1, \ldots, n$.



In other words, we have $(\bar{s}, b, s)\ I\ (s_{i-1}, a_i, s_i)$, for $i = 1, \ldots, n$. It follows that we can complete the picture as shown in the picture and construct a sequence of diamonds of concurrency. So, we have a path

$$\pi_s(s, a_1, \bar{s}_1) \cdots (\bar{s}_{n-1}, a_n, \bar{s}_n),$$

where $(\bar{s}_{n-1}, a_n, \bar{s}_n) \sim (\bar{s}', a, \bar{s}'') \sim t$, i.e., a path $\pi_s\pi_{s'}(s', a, s'')$ as required. ✓

Lemma 2.5.23
Consider a path $\pi_s \in Path(OTI)$ and a class $[t]_\sim$ such that

i) for each t' in π_s, it is $[t']_\sim \not\#\ [t]_\sim$ and $[t']_\sim \neq [t]_\sim$,

ii) for each $[t']_\sim < [t]_\sim$, there exists a representative of $[t']_\sim$ in π_s.

Then, there exists $(s, a, s') \in Tran_{OTI}$ with $(s, a, s') \sim t$.

Proof. By the previous lemma, we find $\pi_s\pi_{\bar{s}'}(\bar{s}', a, \bar{s}'')$ with $(\bar{s}', a, \bar{s}'') \sim t$. Now, consider an element $t' \in \pi_{\bar{s}'}$. We have $[t']_\sim \not\leq [t]_\sim$, because otherwise another representative of $[t']_\sim$ would be in π_s and, by Corollary 2.5.6, this is impossible. Moreover, $[t]_\sim \not\leq [t']_\sim$, because in the path $\pi_s\pi_{\bar{s}'}(\bar{s}', a, \bar{s}'')$ transition t' occurs before than $(\bar{s}', a, \bar{s}'')$; and it is $[t']_\sim \not\#\ [t]_\sim$ because in $\pi_s\pi_{\bar{s}'}(\bar{s}', a, \bar{s}'')$ both t' and $(\bar{s}', a, \bar{s}'')$ occur. It follows that $[t']_\sim\ co\ [t]_\sim$.

Therefore, by applying Lemma 2.5.21, we find $(s, a, s') \sim (\bar{s}', a, \bar{s}'') \sim t$. ✓

Next, we show that there is a one-to-one correspondence between the states of OTI and the finite configurations of $otsi.les(OTI)$, or, in other words, states of $les.otsi \circ otsi.les(OTI)$.

Consider the map $\mathcal{C}\colon S_{OTI} \to \mathcal{L}_F\Big(otsi.les(OTI)\Big)$ given by the correspondence $s \mapsto \Big\{[t]_\sim\ \Big|\ t \in \pi_s,\ \pi_s \in Path(OTI)\Big\}$. We already know that any path leading to s contains the same equivalence classes, thus $\mathcal{C}$ is well-defined.

Lemma 2.5.24
For $s \in S_{OTI}$, the set $\mathcal{C}(s)$ is a finite configuration of $otsi.les(OTI)$.

Proof. $\mathcal{C}(s)$ is clearly finite. Moreover, it is conflict free, since all its elements have a representative belonging to the same class. Finally, if $[t']_\sim \leq [t]_\sim$ for some $[t]_\sim$ in $\mathcal{C}(s)$, there exists $\bar{t} \sim t$ in π_s and, thus, we find $\bar{t}' \sim t'$ in π_s. Thus, $[t']_\sim \in \mathcal{C}(s)$. ✓

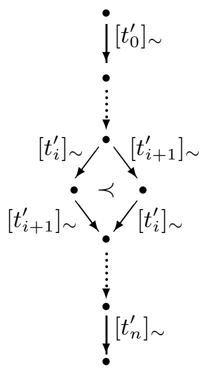
Let c be a finite configuration of $otsi.les(OTI)$ and let $\varsigma = [t_0]_\sim[t_1]_\sim \cdots [t_n]_\sim$ be a *securing* for c. Then, there is a *unique* path $\pi_\varsigma = (s_0, a_1, s_1) \cdots (s_{n-1}, a_n, s_n)$ such that $s^I_{OTI} = s_0$, $s_n = s$ and $[(s_{i-1}, a_i, s_i)]_\sim = [t_i]_\sim$, for $i = 1, \ldots, n$. The existence of π_ς is a consequence of the previous Lemma 2.5.23. It can be obtained as follows.

- (s_0, a_1, s_1) is the unique element in $[t_0]_\sim$ whose source state is s^I_{OTI}. It exists, by Lemma 2.5.23, since $\lfloor [t_0]_\sim \rfloor = \varnothing$, and it is unique because of property *(iv)* of Definition 2.3.7 of transition systems with independence.
- Inductively, (s_{i-1}, a_i, s_i) is the unique element in $[t_i]_\sim$ whose source state is s_{i-1}. Again, it exists because $(s_0, a_1, s_1) \cdots (s_{i-2}, a_n, s_{i-1})$ and $[t_i]_\sim$ satisfy the conditions of Lemma 2.5.23 and it is unique by definition of transition systems with independence.

It is important to observe that, although the actual path π_ς strictly depends on ς, the final state reached does not.

Lemma 2.5.25
Let c be a finite configuration of $otsi.les(OTI)$ and let $\varsigma = [t_0]_\sim \cdots [t_n]_\sim$ and $\varsigma' = [t'_0]_\sim \cdots [t'_n]_\sim$ be two securings for c. Then the paths π_ς and $\pi_{\varsigma'}$ obtained as illustrated above reach the same state.

Proof. It is enough to show that $\pi_\varsigma \simeq \pi_{\varsigma'}$. To this aim, we work by induction on the minimal number n of *"swappings"* of adjacent elements in ς' needed to transform it in ς. Observe that such a number exists and is finite since ς and ς' are securing of the same configuration, and, as such, they are just different permutations of the same elements.



If $n = 0$, then $\pi_\varsigma = \pi_{\varsigma'}$, since the sequences are uniquely determined by the securing. Suppose now that we need $n + 1$ swappings and that we proved the thesis for the case of n swappings. Suppose that we must swap $[t'_i]_\sim$ and $[t'_{i+1}]_\sim$. So we get $\varsigma'' = [t'_0]_\sim \cdots [t'_{i-1}]_\sim[t'_{i+1}]_\sim[t'_i]_\sim[t'_{i+2}]_\sim \cdots [t'_n]_\sim$. Now, observe that $[t'_{i+1}]_\sim$ must occur in ς before than $[t'_i]_\sim$, otherwise we would have a shorter sequence of swappings to transform ς' in ς just by avoiding the swapping of $[t'_i]_\sim$ and $[t'_{i+1}]_\sim$. It follows that $[t'_i]_\sim \not< [t'_{i+1}]_\sim$, i.e., ς'' is a securing of c. Moreover, $[t'_i]_\sim$ *co* $[t'_{i+1}]_\sim$. Therefore, we have $\pi_{\varsigma''} \simeq \pi_{\varsigma'}$. Now, ς'' can be transformed in ς with n swappings, and therefore, by induction hypothesis, $\pi_{\varsigma''} \simeq \pi_\varsigma$. So, we conclude $\pi_\varsigma \simeq \pi_{\varsigma'}$. ✓

Therefore, we can define a map $\mathcal{S}: \mathcal{L}_F(otsi.les(OTI)) \to S_{OTI}$ by saying that $c \mapsto s$, where s is the state reached by a path π_ς for a securing ς of c. Now, we can see that $\mathcal{C}$ is a set isomorphism with inverse $\mathcal{S}$.

Lemma 2.5.26
$\mathcal{S} = \mathcal{C}^{-1}$.

Proof. Consider $\mathcal{C}(s) = \{[t]_\sim \mid t \in \pi_s\}$ and consider the sequence $\varsigma = [t_0]_\sim \cdots [t_n]_\sim$ such that $\pi_s = t_0 \cdots t_n$. This is clearly a securing of $\mathcal{C}(s)$, whose associated path π_ς is π_s itself. This is because of the uniqueness of π_ς as shown earlier. So, we have $\mathcal{S}(\mathcal{C}(s)) = s$.

On the other hand, suppose $\mathcal{S}(c) = s$. Among the paths leading to s, consider π_ς, where $\varsigma = [t_0]_\sim \cdots [t_n]_\sim$ is any securing of c. Then, we may use π_ς to calculate $\mathcal{C}(\mathcal{S}(c)) = \{[t]_\sim \mid t \in \pi_\varsigma\} = \{[t_i]_\sim \mid i = 0, \ldots, n\} = c$. ✓

It is worthwhile to observe that $\mathcal{C}$ and $\mathcal{S}$ give rise to morphisms of transition systems which are each other's inverse. First observe that $\mathcal{S}(\varnothing) = s^I_{OTI}$, since the unique path associated with the unique securing of the empty configuration, is the empty path. Moreover, $\mathcal{C}(s^I_{OTI}) = \varnothing$, since the unique path leading to s^I_{OTI} in OTI is the empty path. Moreover, we have the following easy lemma.

Lemma 2.5.27
Let OTI be a transition system with independence. Then

i) If (s, a, s') is a transition of OTI, then $\Big(\mathcal{C}(s), a, \mathcal{C}(s')\Big)$ is a transition of $les.otsi \circ otsi.les(OTI)$.

ii) If (c, a, c') is a transition of $les.otsi \circ otsi.les(OTI)$, $\Big(\mathcal{S}(c), a, \mathcal{S}(c')\Big)$ is a transition of OTI.

This means that $(\mathcal{C}, id)$ from OTI to $les.otsi \circ otsi.les(OTI)$ and $(\mathcal{S}, id)$ from $les.otsi \circ otsi.les(OTI)$ to OTI are morphisms of transition systems. Moreover, $(\mathcal{S}, id) = (\mathcal{C}, id)^{-1}$. Recall that $(c, a, c')\ I\ (\bar{c}, b, \bar{c}')$ implies, by definition of *les.otsi*, that $(c' \setminus c) = [t]_\sim\ co\ [\bar{t}]_\sim = (\bar{c}' \setminus \bar{c})$. From the previous Lemma 2.5.27 we therefore have that

$$[t]_\sim = \Big[\Big(\mathcal{S}(c), a, \mathcal{S}(c')\Big)\Big]_\sim\ co\ \Big[\Big(\mathcal{S}(\bar{c}), b, \mathcal{S}(\bar{c}')\Big)\Big]_\sim = [\bar{t}]_\sim$$

and then, from Lemma 2.5.20, $\Big(\mathcal{S}(c), a, \mathcal{S}(c')\Big)\ I\ \Big(\mathcal{S}(\bar{c}), b, \mathcal{S}(\bar{c}')\Big)$. In other words, we have proved the following.

Proposition 2.5.28
$(\mathcal{S}, id)$ is a transition system with independence morphism.

However, $(\mathcal{C}, id)$ is not a morphism in <u>TSI</u>. It follows that $(\mathcal{S}, id)$, in general, is not an isomorphism of transition systems with independence. Consider now the property:

$$t\ I\ t' \quad \Rightarrow \quad \exists s.\ (s, a, s') \sim t \quad \text{and} \quad (s, b, s'') \sim t'. \tag{E}$$

Proposition 2.5.29
OTI enjoys property (E) *if and only if* $(\mathcal{C}, id)$ *is a morphism of transition systems with independence.*

Proof. ($\Rightarrow$). It is enough to show that $(\mathcal{C}, id)$ preserves independence. Suppose $(s, a, s')\ I\ (\bar{s}, b, \bar{s}')$. By condition (E), there exists

$$(s, a, s') \sim (\underline{s}, a, \underline{s}')\ I\ (\underline{s}, b, \underline{s}'') \sim (\bar{s}, b, \bar{s}'),$$

and therefore, we have $Diam\Big((\underline{s}, a, \underline{s}'), (\underline{s}, b, \underline{s}''), (\underline{s}', b, u), (\underline{s}'', a, u)\Big)$. So, we have that $[(s, a, s')]_\sim\ co\ [(\bar{s}, b, \bar{s}')]_\sim$. From Lemma 2.5.27, we have $\mathcal{C}(s') = \mathcal{C}(s) \cup \Big\{[(s, a, s')]_\sim\Big\}$ and $\mathcal{C}(\bar{s}') = \mathcal{C}(\bar{s}) \cup \Big\{[(\bar{s}, b, \bar{s}')]_\sim\Big\}$. Therefore, $\Big(\mathcal{C}(s), a, \mathcal{C}(s')\Big)\ I\ \Big(\mathcal{C}(\bar{s}), b, \mathcal{C}(\bar{s}')\Big)$.

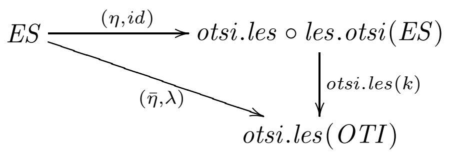
($\Leftarrow$). Suppose that $(\mathcal{C}, id)$ preserves independence. Then $(s, a, s')\ I\ (\bar{s}, b, \bar{s}')$ implies $\Big(\mathcal{C}(s), a, \mathcal{C}(s')\Big)\ I\ \Big(\mathcal{C}(\bar{s}), b, \mathcal{C}(\bar{s}')\Big)$, that is $[(s, a, s')]_\sim\ co\ [(\bar{s}, b, \bar{s}')]_\sim$. Then, by repeated applications of Lemma 2.5.23, we can find a path $\pi_{\underline{s}}(\underline{s}, a, \underline{s}')(\underline{s}', b, u)$ such that $(s, a, s,') \sim (\underline{s}, a, \underline{s}')\ I\ (\underline{s}', b, u) \sim (\bar{s}, b, \bar{s},')$. Then, by property *(iii)* of transition system with independence, there exists $\underline{s}''$ and $(\underline{s}, b, \underline{s}'') \sim (\underline{s}', b, u) \sim (\bar{s}, b, \bar{s}')$, i.e., *OTI* enjoys property (E). ✓

The next step is to define, for each labelled event structure *ES* a morphism $(\eta, id)\colon ES \to otsi.les \circ les.otsi(ES)$ as a candidate for the unit of the adjunction. Let us consider η such that

$$\eta(e) = \Big[\Big(c, a, c \cup \{e\}\Big)\Big]_\sim.$$

We have already shown in Lemma 2.5.16 that $(c, a, c') \sim (\bar{c}, a, \bar{c}')$ if and only if $(c' \setminus c) = (\bar{c}' \setminus \bar{c})$. It follows immediately that η is well-defined and is *injective*. Moreover, since any transition of $les.otsi(ES)$, say (c, a, c'), is associated with an event of *ES*, namely, $c' \setminus c$, we have that η is also *surjective*. In fact, it can be shown that (η, id) is an isomorphism of labelled event structures whose inverse is $(\bar{\eta}, id)$, where $\bar{\eta}\colon [(c, a, c')]_\sim \mapsto (c' \setminus c)$.

Proposition 2.5.30 *($(\eta, id)\colon ES \to otsi.les \circ les.otsi(ES)$ is universal)*
For any labelled event structure ES, any occurrence transition system with independence OTI and any morphism $(\bar{\eta}, \lambda)\colon ES \to otsi.les(OTI)$, there exists a unique k in <u>oTSI</u> *such that $otsi.les(k) \circ (\eta, id) = (\bar{\eta}, \lambda)$.*



Proof. Let us define $k: les.otsi(ES) \to OTI$. Clearly, in order to make the diagram commute, k must be of the form (σ, λ), for some σ. Let us consider $\sigma: c \mapsto \mathcal{S}(\bar{\eta}(c))$, i.e.,

$$(\sigma, \lambda) = (\mathcal{S}, id) \circ (\bar{\eta}, \lambda): les.otsi(ES) \to les.otsi(otsi.les(OTI)) \to OTI.$$

Then, we have immediately that σ is well-defined and that (σ, λ) is a transition system with independence morphism.

Now, we show that the diagram commutes. We verify that $\eta_\sigma \circ \eta = \eta_{\mathcal{S}} \circ \eta_{\bar{\eta}} \circ \eta = \bar{\eta}$. Consider $e \in E_{ES}$ and let a be $\ell(e)$. If $\lambda\uparrow a$, then $\bar{\eta}\uparrow a$ and $\eta_{\bar{\eta}}\uparrow a$ and, therefore, both sides of the above equality are undefined. Suppose otherwise that $\lambda\downarrow a$. We have

$$\begin{aligned} e \overset{\eta}{\mapsto} \Big[(c, a, c \cup \{e\})\Big]_\sim \ &\overset{\eta_{\bar{\eta}}}{\mapsto} \ \Big[\Big(\bar{\eta}(c), \lambda(a), \bar{\eta}(c) \cup \{\bar{\eta}(e)\}\Big)\Big]_\sim \\ &\overset{\eta_{\mathcal{S}}}{\mapsto} \ \Big[\Big(\mathcal{S}(\bar{\eta}(c)), \lambda(a), \mathcal{S}(\bar{\eta}(c) \cup \{\bar{\eta}(e)\})\Big)\Big]_\sim \\ &= \ \Big[\Big(\sigma(c), \lambda(a), \sigma(c \cup \{e\})\Big)\Big]_\sim. \end{aligned}$$

Observe that $\Big(\bar{\eta}(c), \lambda(a), \bar{\eta}(c) \cup \{\bar{\eta}(e)\}\Big)$ belongs to $les.otsi \circ otsi.les(OTI)$ and is associated with the event $\bar{\eta}(e)$ of $otsi.les(OTI)$. Then, from Lemma 2.5.27, we have that $\Big[\Big(\mathcal{S}(\bar{\eta}(c)), \lambda(a), \mathcal{S}(\bar{\eta}(c) \cup \{\bar{\eta}(e)\})\Big)\Big]_\sim = \bar{\eta}(e)$.

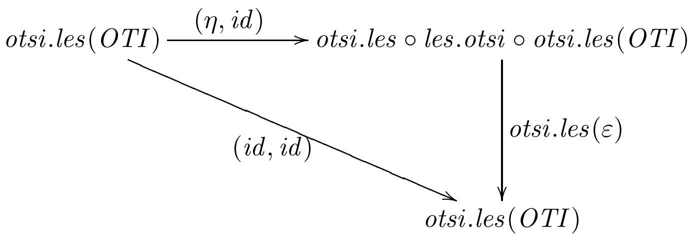
The last step to prove the universality of (η, id) is to show that k is the unique transition system with independence morphism from $les.otsi(ES)$ to OTI which makes the diagram commute. Let us suppose that there is k' which does so. It must necessarily be $k' = (\sigma', \lambda)$. Observe from the first part of the proof that in order for the diagram to commute, it must be $\eta_{\sigma'}\Big(\Big[(c, a, c \cup \{e\})\Big]_\sim\Big) = \Big[\Big(\sigma'(c), \lambda(a), \sigma'(c \cup \{e\})\Big)\Big]_\sim = \bar{\eta}(e) = \Big[\Big(\sigma(c), \lambda(a), \sigma(c \cup \{e\})\Big)\Big]_\sim$, for any e such that $\lambda\downarrow\ell(e)$. Now, it is easy to show by induction on the cardinality of c, that $\sigma' = \sigma$. ✓

Therefore, we have the following corollary.

COROLLARY 2.5.31 *($les.otsi \dashv otsi.les$)*
The map $les.otsi$ extends to a functor from <u>LES</u> *to* <u>oTSI</u> *which is left adjoint to $otsi.les$. Since the unit of the adjunction is an isomorphism, the adjunction is a coreflection.*

Next, we show that $(\mathcal{S}, id)$ is the counit of this coreflection. Actually, the task is fairly easy now: by general results in Category Theory [90, chap. IV, pg. 81], the counit of an adjunction can be determined through the unit as the unique morphism

ε: $otsi.les \circ les.otsi(OTI) \to OTI$ which makes the following diagram commute.



However, in the proof of Proposition 2.5.30, we have identified a general way to find ϵ. From it we obtain $\epsilon = (\mathcal{S}, id) \circ (id, id)$, which is $(\mathcal{S}, id)$.

The results we have shown earlier about $(\mathcal{S}, id)$ make it easy to identify the full subcategory of $\underline{\mathsf{oTSI}}$ and, therefore, of $\underline{\mathsf{TSI}}$ which is *equivalent* to $\underline{\mathsf{LES}}$, i.e., the category of those transition systems with independence which are (representations of) labelled event structures. Such a result gives yet another characterization of (the finite elements of) *coherent, finitary, prime algebraic domains*. Moreover, this axiomatization is given only in terms of conditions on the structure of transition systems.

By general results in Category Theory [90, chap. IV, pg. 91], an equivalence of categories is an adjunction whose unit and counit are both isomorphisms, i.e., which is both a reflection and a coreflection. Then, Proposition 2.5.29 gives us a candidate for the category of occurrence transition system with independence equivalent to $\underline{\mathsf{LES}}$: we consider $\underline{\mathsf{oTSI}_\mathsf{E}}$, the full subcategory of $\underline{\mathsf{oTSI}}$ consisting of those occurrence transition systems with independence satisfying condition (E). To obtain the result, it is enough to verify that $les.otsi\colon \underline{\mathsf{LES}} \to \underline{\mathsf{oTSI}}$ actually lands in $\underline{\mathsf{oTSI}_\mathsf{E}}$. In fact, this guarantees that the adjunction $\langle les.otsi, otsi.les\rangle\colon \underline{\mathsf{LES}} \rightharpoonup \underline{\mathsf{oTSI}}$ restricts to an adjunction $\underline{\mathsf{LES}} \rightharpoonup \underline{\mathsf{oTSI}_\mathsf{E}}$ whose unit and counit are again, respectively, (η, id) and $(\mathcal{S}, id)$, which are isomorphisms. It follows then, that $\underline{\mathsf{oTSI}_\mathsf{E}} \cong \underline{\mathsf{LES}}$.

Proposition 2.5.32
The occurrence transition system with independence $les.otsi(ES)$ satisfies condition (E).

Proof. Suppose $(c, a, c')\ I\ (\bar{c}, b, \bar{c}')$ and let $(c' \setminus c) = \{e\}$ and $(\bar{c}' \setminus \bar{c}) = \{\bar{e}\}$. Then, we must necessarily have $e\ co\ \bar{e}$. It follows that $\underline{c} = (\lfloor e \rfloor \setminus \{e\}) \cup (\lfloor \bar{e} \rfloor \setminus \{\bar{e}\})$ is a finite configuration of ES which enables both e and $\bar{e}$. Then, in $les.otsi(ES)$ we have $(c, a, c') \sim (\underline{c}, a, \underline{c} \cup \{e\})\ I\ (\underline{c}, b, \underline{c} \cup \{\bar{e}\}) \sim (\bar{c}, b, \bar{c}')$. ✓

Thus we have the following.

Corollary 2.5.33
The categories $\underline{\mathsf{LES}}$ and $\underline{\mathsf{oTSI}_\mathsf{E}}$ are equivalent.

We can interpret such a result as a demonstration of the claim that transition systems with independence are a generalization of labelled event structures to a model system. However, there is point a which is worth raising. The fact that just unfolding transition systems to their occurrence version does not suffice to get a category equivalent to $\underline{\mathsf{LES}}$, shows that the *independence* relation on transitions is *not* exactly a *concurrency* relation. As an intuitive explanation of this phenomenon, it is very easy to think of a transition system with independence in which independent transitions never occur in the same path, i.e., intuitively, they are in conflict. In the light of such observation, condition (E) can be seen exactly as the condition which guarantees that independence *is* concurrency. It is then that the simple unfolding of transition systems with independence yields the category $\underline{\mathsf{oTSI}_\mathsf{E}}$ equivalent to $\underline{\mathsf{LES}}$.

Next, we briefly see that the coreflection $\underline{\mathsf{LES}} \hookrightarrow \underline{\mathsf{oTSI}}$ cuts down to a coreflection $\underline{\mathsf{dLES}} \hookrightarrow \underline{\mathsf{doTSI}}$, which composes with the coreflection given earlier in this section to give a coreflection $\underline{\mathsf{dLES}} \hookrightarrow \underline{\mathsf{dTSI}}$. As a consequence, we have that $\underline{\mathsf{dLES}} \cong \underline{\mathsf{doTSI}_\mathsf{E}}$. These results are shown by the following proposition.

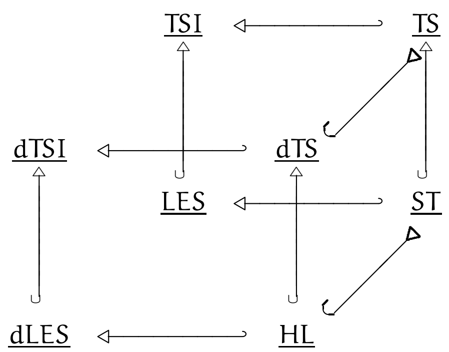
Proposition 2.5.34
If ES is deterministic, then les.otsi(ES) is deterministic. If OTI is deterministic, then otsi.les(OTI) is deterministic.

Proof. If $(c, a, c \cup \{e\})$ and $(c, b, c \cup \{\bar{e}\})$ are transitions of $les.otsi(ES)$, then $c \vdash e$ and $c \vdash \bar{e}$ and then $a \neq b$.

If $c \vdash [(s, a, s')]_\sim$ and $c \vdash [(\bar{s}, b, \bar{s}')]_\sim$, we can clearly assume that c is finite. Then, $\Big(c, a, c \cup \{[(s, a, s')]_\sim\}\Big)$, $\Big(c, b, c \cup \{[(\bar{s}, b, \bar{s}')]_\sim\}\Big)$ are in $les.otsi \circ otsi.les(OTI)$ and, therefore, $\Big(\mathcal{S}(c), a, \mathcal{S}(c \cup \{[(s, a, s')]_\sim\})\Big)$, $\Big(\mathcal{S}(c), b, \mathcal{S}(c \cup \{[(\bar{s}, b, \bar{s}')]_\sim\})\Big)$ are in OTI. Then $a \neq b$. ✓

These results are summarized in the following theorem.

Theorem 2.5.35 *(Moving along the "behaviour/system" axis)*

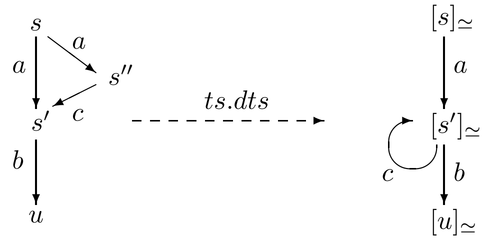


2.6 Deterministic Transition Systems with Independence

Now, we consider the relationship between dTSI and TSI, looking for a generalization of the reflection dTS $\hookrightarrow\!\!\lhd$ TS. Of course, the question to be answered is whether a left adjoint for the inclusion functor dTSI $\hookrightarrow$ TSI exists or not. This is actually a complicated issue and for a long while we could not see the answer. However, we can now answer it positively!

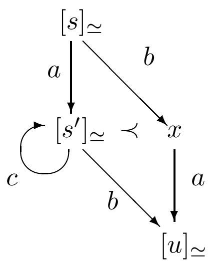
Thinking about the issue, at a first sight, one could be tempted to refine the construction given in Section 2.2 case of transition systems by defining a suitable independence relation on the deterministic transition system obtained in that way. However, this simple minded approach would not work, since, in general, no independence relation yields a transition system with independence. Let us see what happens with the following example.

Example 2.6.1
Consider the transition system T in the following figure together with its deterministic version $ts.dts(T)$.



Now, suppose that T carries the structure of a transition system with independence, i.e., an independence relation I on its transitions, and suppose further that $(s, a, s'')\ I\ (s', b, u)$. Observe now that, in order to establish the reflection at the level of transition systems with independence, since the unit would be a morphism from the original transition system to the deterministic one, the independence relation must be preserved by the construction of the deterministic transition system.

Thus, whatever the independence relation on the deterministic transition system is, it must be $([s]_\simeq, a, [s']_\simeq)\ I\ ([s']_\simeq, b, [u]_\simeq)$. Then, the transition system we obtain cannot be a transition system with independence, since axiom (iii) of Definition 2.3.7 of transition systems with independence fails. However, in the rest of this section, we will show that it is always possible to "complete" the deterministic transition system obtained by ts.dts in order to make it a transition system with independence. Moreover, such a completion will be "universal", so that it will give the reflection we are seeking. In the case of the transition system above, the *resulting transition system is shown below.*



Observe that it may also not be possible to define I to be irreflexive. Of course, this happens when there is a diamond of concurrency whose transitions carry the same label. It is easy to understand that, in this case, the only way to cope with those transitions is by eliminating them. In other words, autoconcurrency, i.e., concurrency between events carrying the same label, add a further level of difficulty to the problem.

Therefore, our task is to complete a deterministic transition system DT whose transitions carry a relation I, paying particular attention to autoconcurrency, to construct the "minimal" deterministic transition system with independence which contains DT and I, i.e., make DT into a deterministic transition system with independence by adding as few new states and transitions as possible.

DEFINITION 2.6.2 *(Pre-Transition Systems with Independence)*
A pre-transition system with independence is a transition system together with a binary and symmetric relation I on its transitions.

A morphism of pre-transition systems with independence is a transition system morphism which, in addition, preserve the relation I.

Let $\underline{\mathsf{pTSI}}$ denote the category of pre-transition systems with independence.

Given sets S and L, consider triples of the kind $(X, \equiv, I)$, where $X \subseteq S \cdot L^* = \{s\alpha \mid s \in S \text{ and } \alpha \in L^*\}$, and $\equiv$ and I are binary relations on X. On such triples, the following closure properties can be considered.

(Cl1) $x \equiv z$ and $za \in X$ implies $xa \in X$ and $xa \equiv za$;

(Cl2) $x \equiv z$ and $za\ I\ yc$ implies $xa\ I\ yc$;

(Cl3) $xab \equiv xba$ and $xa\ I\ xb$ or $xa\ I\ xab$ implies
$$xa\ I\ yc \Leftrightarrow xba\ I\ yc.$$

We say that $(X, \equiv, I)$ is *suitable* if $\equiv$ is an equivalence relation, I is a symmetric relation and it enjoys properties *(Cl1)*, *(Cl2)* and *(Cl3)*. Suitable triples are meant to represent deterministic (pre) transition systems with independence, the elements in X representing both states and transitions. Namely, xa represents the state reached from (the state corresponding to) x with an a-labelled transition, and that transition itself. Thus, equivalence $\equiv$ relate paths which lead to the same state and relation I expresses independence of transitions. With this understanding, *(Cl1)* means that from any state there is at most one a-transition, while *(Cl2)* says that I acts on transitions rather than on their representation. Finally, *(Cl3)*—the analogous of axiom *(iv)* of transition systems with independence—tells that transitions on the opposite edges of a diamond behave the same with respect to I.

For $x \in S \cdot L^*$ and $a \in L$, let $x\lceil a$ denote the pruning of x with respect to a. Formally,

$$s\lceil a = s \qquad \text{and} \qquad (xb)\lceil a = \begin{cases} x\lceil a & \text{if } a = b \\ (x\lceil a)b & \text{otherwise} \end{cases}$$

Of course, $(x\lceil a)\lceil b = (x\lceil b)\lceil a$ and thus it is possibile to use unambiguously $x\lceil A$ for $A \subseteq L$. Given $X \subseteq S \cdot L^*$, we use $X\lceil A$ to denote the set $\{x\lceil A \mid x \in X\}$ and for a binary relation R on X $R\lceil A$ will be $\{(x\lceil A, y\lceil A) \mid (x,y) \in R\}$.

For a transition system with independence $TI = (S, s^I, L, \mathit{Tran}, I)$, we define the sequence a triples $(S_i, \equiv_i, I_i)$, for $i \in \omega$, inductively as follows. For $i = 0$, $(S_0, \equiv_0, I_0)$ is the least (with respect to componentwise set inclusion) *suitable* triple such that

$$S \cup \left\{ sa \;\middle|\; (s,a,u) \in \mathit{Tran} \right\} \subseteq S_0; \quad \left\{ (sa, u) \;\middle|\; (s,a,u) \in \mathit{Tran} \right\} \subseteq \equiv_0;$$

and

$$\left\{ (sa, s'b) \;\middle|\; (s,a,u)\ I\ (s',b,u') \right\} \subseteq I_0;$$

and, for $i > 0$, $(S_i, \equiv_i, I_i)$ is the least *suitable* triple such that

($\Im$) $\quad S_{i-1}\lceil A_{i-1} \subseteq S_i; \quad \equiv_{i-1}\lceil A_{i-1} \subseteq \equiv_i; \quad (I_{i-1} \setminus TA_{i-1})\lceil A_{i-1} \subseteq I_i;$

(D1) $\quad xa,\ xb \in S_{i-1}\lceil A_{i-1}$ and $xa\ (I_{i-1} \setminus TA_{i-1})\lceil A_{i-1}\ xb$
implies $\quad xab,\ xba \in S_i$ and $xab \equiv_i xba$;

(D2) $\quad xa,\ xab \in S_{i-1}\lceil A_{i-1}$ and $xa\ (I_{i-1} \setminus TA_{i-1})\lceil A_{i-1}\ xab$
implies $\quad xb,\ xba \in S_i$ and $xab \equiv_i xba$;

where $A_i = \{a \in L \mid xa\ I_i\ xa\}$ and $TA_i = \{(xa, yb) \in I_i \mid a \in A_i \text{ or } b \in A_i\}$.

The inductive step extends a triple towards a transition system with independence by means of the rules *(D1)* and *(D2)*, whose intuitive meaning is clearly that of closing possibly incomplete diamonds. The process could create *autoindependent* transitions which must be eliminated. This is done by *($\Im$)* which drops them and adjusts $\equiv_i$ and I_i.

A simple inspection of the rules shows that if $a \in A_i$, then it will never appear again in the sequence. Thus, if x is removed from S_i, it will not be reintroduced, and the same applies to the pairs in $\equiv_i$ and I_i. Then, it is easy to identify the *limit* of the sequence as

$$\Big(S_\omega = \bigcup_{i\in\omega}\bigcap_{j\geq i} S_j, \quad \equiv_\omega = \bigcup_{i\in\omega}\bigcap_{j\geq i} \equiv_j, \quad I_\omega = \bigcup_{i\in\omega}\bigcap_{j\geq i} I_j\Big).$$

PROPOSITION 2.6.3
The triple $(S_\omega, \equiv_\omega, I_\omega)$ is suitable. Moreover, I_ω is irreflexive.

The following proposition gives an alternative characterization of $(S_\omega, \equiv_\omega, I_\omega)$ which will be useful later on. In the following let A_ω denote $\bigcup_{i\in\omega} A_i$ and let TA_ω be $\bigcup_{i\in\omega} TA_i$.

PROPOSITION 2.6.4
$$(S_\omega, \equiv_\omega, I_\omega) = \Big(\bigcup_{i\in\omega}(S_i{\restriction}A_\omega), \quad \bigcup_{i\in\omega}(\equiv_i{\restriction}A_\omega), \quad \bigcup_{i\in\omega}((I_i \setminus TA_\omega){\restriction}A_\omega)\Big).$$

Proof. If $x \in S_\omega$ then there exists i such that $x \in S_j$ for any $j \geq i$. It follows easily that $x{\restriction}A_j = x$ for any $j \in \omega$ and then $x = x{\restriction}A_\omega \in S_i{\restriction}A_\omega$. On the other hand, suppose that $x = \bar{x}{\restriction}A_\omega$, with $\bar{x} \in S_i$. Since $\bar{x}$ is a (finite) string, it must exist k such that $\bar{x}{\restriction}(\bigcup_{j=i,\ldots,k-1} A_j) = x$. Then $x \in S_k$, and for any $j \geq k$, since $x{\restriction}A_j = x$, $x \in S_j$. Thus, $x \in S_\omega$.

If $(x, y) \in \equiv_j$ for any $j \geq i$, then $x, y \in S_j$ for any $j \geq i$ and, reasoning as before, $x{\restriction}A_\omega = x$ and $y{\restriction}A_\omega = y$. Thus, $(x, y) = (x{\restriction}A_\omega, y{\restriction}A_\omega) \in \equiv_i{\restriction}A_\omega$. If instead $(x, y) = (\bar{x}{\restriction}A_\omega, \bar{y}{\restriction}A_\omega) \in \equiv_i{\restriction}A_\omega$, then there exists k such that $\bar{x}{\restriction}(\bigcup_{j=i,\ldots,k-1} A_j) = x$ and $\bar{y}{\restriction}(\bigcup_{j=i,\ldots,k-1} A_j) = y$. It follows easily that $(x, y) \in \equiv_i{\restriction}(\bigcup_{j=i,\ldots,k-1} A_j) \subseteq \equiv_k$, and then, for any $j \geq k$, $(x, y) \in \equiv_j$, i.e., $(x, y) \in \equiv_\omega$.

Finally, if $(xa, yb) \in I_j$ for any $j \geq i$, $a, b \notin A_\omega$ and then $(xa, yb) \in I_i \setminus TA_\omega$, whence it follows $(xa, yb) = (xa{\restriction}A_\omega, yb{\restriction}A_\omega) \in (I_i \setminus TA_\omega){\restriction}A_\omega$. On the contrary, if $(xa, yb) \in (I_i \setminus TA_\omega){\restriction}A_\omega$, it must be $xa = \bar{x}a{\restriction}A_\omega$ and $yb = \bar{y}b{\restriction}A_\omega$ with $(\bar{x}a, \bar{y}b) \in I_i$. Then, it can be proved as in the previous case that there exists k such that $(xa, yb) \in I_j$ for any $j \geq k$. ✓

Proposition 2.6.4 has the following immediate corollary.

COROLLARY 2.6.5
i) $x \in S_i$ *implies* $x{\restriction}A_\omega \in S_\omega$;
ii) $x \equiv_i y$ *implies* $(x{\restriction}A_\omega) \equiv_\omega (y{\restriction}A_\omega)$;
iii) $xa\ I_i\ yb$ *and* $a, b \notin A_\omega$ *implies* $(xa{\restriction}A_\omega)\ I_\omega\ (yb{\restriction}A_\omega)$.

In the following we shall refer to the sets obtained by applying rules *($\Im$)*, *(D1)* and *(D2)* to S_{i-1}, $\equiv_{i-1}$ and I_{i-1} as the *generators* of the suitable triple $(S_i, \equiv_i, I_i)$. Similarly, the sets $S \cup \{sa \mid (s, a, u) \in \mathit{Tran}\}$, $\{(sa, u) \mid (s, a, u) \in \mathit{Tran}\}$ and $\{(sa, s'b) \mid (s, a, u)\ I\ (s', b, u')\}$ are the generators of $(S_0, \equiv_0, I_0)$. We shall occasionally denote the generators of $(S_i, \equiv_i, I_i)$ by ${}_\gamma S_i$, ${}_\gamma{\equiv_i}$ and ${}_\gamma I_i$. Now, we take the occasion to show some general properties of the first step of the above construction.

Lemma 2.6.6
Let TI be a transition system with independence and $(S_i, \equiv_i, I_i)$ the corresponding sequence of triples. For $s,\ s' \in S,\ sa,\ s'b,\ s\alpha,\ s'\beta \in S_0$ we have that

i) $s\alpha \equiv_0 s'\beta$ *if there exists $u \in S$ and two sequences of transitions leading, respectively, from s to u with labels α and from s' to u with labels β;*

ii) $s' \equiv_0 sa$ *if $(s, a, s') \in Tran$.*

iii) $sa\ I_0\ s'b$ *if there exist transitions $(s, a, u)\ I\ (s', b, u')$ in TI.*

Proof. Observe that point *(ii)* is an easy corollary of point *(i)*.

Consider $X \subseteq S \cdot L^*$ such that $s\alpha \in X$ if and only if $s \in S$ and there is a sequence of transitions $(s, a_0, s_0) \cdots (s_{n-1}, a_n, s_n)$ in *DPT*, where $a_0 \cdots a_n$ is α. Then, consider the relations $\equiv\ \subseteq X \times X$ and $\bar{I} \subseteq X \times X$ such that $s\alpha \equiv s'\beta$ if and only if there are two corresponding sequences of transitions leading to the same state of *TI* and $s\alpha\ \bar{I}\ s'\beta$ if and only if there are two corresponding sequences whose last transitions are in the relation I of *TI*.

In order to show *(i)* and *(iii)* it suffices to show that $(X, \equiv, \bar{I}) \subseteq (S_0, \equiv_0, I_0)$. We do it by induction on the structure of the elements of X.

Clearly, by definition, $s \in X$ implies $s \in S_0$ and $sa\ \bar{I}\ s'b$ implies $sa\ I_0\ s'b$. Moreover, $s \equiv s'$ implies $s = s'$ and, therefore, $s \equiv_0 s'$. Now suppose that $s\alpha a \equiv s'\beta$. Then we have the sequences $(s, a_0, s_0) \cdots (s_{n-1}, a_n, s_n)(s_n, a, u)$ with $\alpha = a_0 \cdots a_n$, and $(s', b_0, s'_0) \cdots (s'_{m-1}, b_m, u)$, with $\beta = b_0 \cdots b_m$. Then, we have $s_n a \equiv_0 u \equiv_0 s'_{m-1} b_m$ and, by induction, since $s\alpha \equiv s_n$ and $s'\beta \equiv u$, it is $s\alpha \equiv_0 s_n$ and $s'\beta \equiv_0 u$. Then, by *(Cl1)*, we have $s\alpha a \in S_0$ and $s\alpha a \equiv_0 s_n a \equiv_0 u \equiv_0 s'_{m-1} b_m \equiv_0 s'\beta$.

Finally, if $s\alpha a\ \ \bar{I}\ \ s'\beta b$, we have the two sequences $(s, a_0, s_0) \cdots (s_n, a, u)$ and $(s', b_0, s'_0) \cdots (s'_m, b, u')$, where $(s_n, a, u)\ I\ (s'_m, b_m, u')$. Therefore, it is $s_n a\ I_0\ s'_m b$, and, by induction, $s\alpha \equiv_0 s_n$ and $s'\beta \equiv_0 s'_m$. Then, since $s_n a\ I_0\ s'_m b$, by *(Cl2)* we have $s\alpha a\ I_0\ s'_m b$ and since $s'\beta \equiv_0 s'_m$, again by *(Cl2)*, $s\alpha a\ I_0\ s'\beta b$. ✓

If *TI* is deterministic then there is a neat characterization of $(S_0, \equiv_0, I_0)$.

Lemma 2.6.7
Let TI be a deterministic transition system with independence. Then

i) $s\alpha \equiv_0 s'\beta$ *if and only if there is $u \in S$ and two sequences of transitions leading from s to u with labels α and from s' to u with labels β;*

ii) $s' \equiv_0 sa$ *if and only if $(s, a, s') \in Tran$.*

iii) $sa\ I_0\ s'b$ *if and only if there exist $(s, a, u)\ I\ (s', b, u')$ in TI.*

Proof. By exploiting the determinism of TI, point *(ii)* is again easily obtained from *(i)*.

Therefore, to show the thesis, it suffices to show that $(X, \equiv, \bar{I}) = (S_0, \equiv_0, I_0)$, where $(X, \equiv, \bar{I})$ is as defined in the proof of Lemma 2.6.6. Since we know that

$$({}_\gamma S_0, {}_\gamma{\equiv_0}, {}_\gamma I_0) \subseteq (X, \equiv, \bar{I}) \subseteq (S_0, \equiv_0, I_0)$$

respectively by definition and by Lemma 2.6.6, and since $(S_0, \equiv_0, I_0)$ is the least *suitable* triple which contains ${}_\gamma S_0$, ${}_\gamma{\equiv_0}$ and ${}_\gamma I_0$, it is enough to show that $(X, \equiv, \bar{I})$ is *suitable.*

Relations $\equiv$ and $\bar{I}$ are, respectively, an equivalence and a symmetric relation. Thus, we need to prove that $(X, \equiv, \bar{I})$ is closed with respect to *(Cl1)*, *(Cl2)* and *(Cl3)*.

(Cl1) By definition, if $s\alpha \equiv s'\beta$ then there is a state u and two sequences leading, respectively, from s to u with observation α and from s' to u with labels β. Moreover, if $s'\beta a \in X$, there is a transition $(u, a, u') \in \mathit{Tran}$ and therefore a sequence leading from s to u' with labels αa, whence $s\alpha a \in X$ and $s\alpha a \equiv s'\beta a$.

(Cl2) If $s\alpha \equiv s'\beta$, then, provided that they exist, $s\alpha a$ and $s'\beta a$ represent the same transition. Therefore, $s'\beta a \; \bar{I} \; yc$ implies $s\alpha a \; \bar{I} \; yc$.

(Cl3) Consider $s\alpha ab \equiv u \equiv s\alpha ba$ and $s\alpha a \; \bar{I} \; s\alpha b$ or $s\alpha a \; \bar{I} \; s\alpha ab$. Then, by definition we have $(\bar{s}, a, s') \; I \; (\bar{s}, b, s'')$ or $(\bar{s}, a, s') \; I \; (s', b, u)$, for $\bar{s} \equiv s\alpha$, $s' \equiv s\alpha a$ and $s'' \equiv s\alpha b$. In both cases we have $(s'', a, u) \sim (\bar{s}, a, s')$ and, therefore, for any $(\underline{s}, c, \underline{s}')$, we have $(\bar{s}, a, s') \; I \; (\underline{s}, c, \underline{s}')$ if and only if $(s'', a, u) \; I \; (\underline{s}, c, \underline{s}')$, i.e., $s\alpha a \; \bar{I} \; yc$ if and only if $s\alpha ba \; \bar{I} \; yc$. ✓

This result admits the following immediate corollary.

COROLLARY 2.6.8
If TI is deterministic, for any $x \in S_0$ there is exactly one $s \in S$ such that $x \equiv_0 s$.

As anticipated before, $(S_i, \equiv_i, I_i)$ encodes a deterministic pre transition system with independence which contains a deterministic version of the original TI we started from (apart from the autoindependent transitions). Formally, for each $\kappa \in \omega \cup \{\omega\}$, define

$$\mathit{TSys}_\kappa = \Big(S_\kappa/\equiv_\kappa, [s^I]_{\equiv_\kappa}, L_\kappa, \mathit{Tran}_{\equiv_\kappa}, I_{\equiv_\kappa}\Big),$$

where

- $([x]_{\equiv_\kappa}, a, [x']_{\equiv_\kappa}) \in \mathit{Tran}_{\equiv_\kappa}$ if and only if $x' \equiv_\kappa xa$;
- $([x]_{\equiv_\kappa}, a, [x']_{\equiv_\kappa}) \; I_{\equiv_\kappa} \; ([\bar{x}]_{\equiv_\kappa}, b, [\bar{x}']_{\equiv_\kappa})$ if and only if $xa \; I_\kappa \; \bar{x}b$;
- $L_\kappa = L \setminus \bigcup_{j<\kappa} A_j$.

Observe that the above definitions are well given. In fact, concerning $Tran_{\equiv_\kappa}$, since $xa \in S_i$ if and only if $\underline{x}a \in S_i$ for any $\underline{x} \equiv_i x$, and since $x' \equiv_i xa$ if and only if $\underline{x}' \equiv_i \underline{x}a$ for any $\underline{x} \equiv_i x$ and $\underline{x}' \equiv_i x'$, its definition is irrespective of the representive chosen. The same holds for the definition of $I_{\equiv_\kappa}$, since $xa\ I_i\ x'b$ if and only if $\underline{x}a\ I_i\ \underline{x}'b$ for any $\underline{x} \equiv_i x$ and $\underline{x}' \equiv_i x'$.

Proposition 2.6.9

$TSys_\kappa$ is a deterministic pre-transition system with independence.

Proof. $TSys_\kappa$ is certainly a transition system and since $(S_\kappa, \equiv_\kappa, I_\kappa)$ is *suitable*, $I_{\equiv_\kappa}$ is symmetric. Moreover, since $[x]_{\equiv_\kappa} \xrightarrow{a} [x']_{\equiv_\kappa}$ if and only if $x' \equiv_\kappa xa$, then if $[x]_{\equiv_\kappa} \xrightarrow{a} [x'']_{\equiv_\kappa}$, we have $[x'']_{\equiv_\kappa} = [x']_{\equiv_\kappa}$. Therefore, $TSys_\kappa$ is deterministic. ✓

Lemma 2.6.7, its Corollary 2.6.8 and the previous proposition show the similarity of $TSys_0$ with the construction of the deterministic version of a transition system as given in Section 2.2. Actually, starting from them, it is not difficult to see that, when applied to a transition system TS, i.e., a transition system with independence whose indepecence relation is empty, $TSys_0$ is a deterministic transition system isomorphic to $ts.dts(TS)$. This fact supports our claim that the construction we are about to give builds on $ts.dts$. However, in Section 2.2 a simpler construction was enough, because we did not need to manipulate transitions but only states.

Proposition 2.6.10

The pair (in, id), where $in: S \to S_0/{\equiv_0}$ is the function which sends s to its equivalence class $[s]_{\equiv_0}$ and id is the identity of L, is a morphism of pre-transition systems with independence from TI to $TSys_0$. Moreover, if TI is deterministic, then (in, id) is an isomorphism.

Proof. Since $(s, a, s') \in Tran$ implies that $s' \equiv_0 sa$ which in turn implies that $([s]_{\equiv_0}, a, [s']_{\equiv_0}) \in Tran_{\equiv_0}$, we have that (in, id) is a morphism of transition systems. If TI is deterministic then from Corollary 2.6.8 and from Lemma 2.6.7 *(ii)*, $(s, a, s') \in Tran$ if and only if $([s]_{\equiv_0}, a, [s']_{\equiv_0}) \in Tran_{\equiv_0}$, and thus (in, id) is an isomorphism of transition systems. Moreover, since $(s, a, s')\ I\ (\bar{s}, b, \bar{s}')$ implies $sa\ I_0\ \bar{s}b$, which in turn implies $([s]_{\equiv_0}, a, [s']_{\equiv_0})\ I_{\equiv_0}\ ([\bar{s}]_{\equiv_0}, b, [\bar{s}']_{\equiv_0})$, it follows that (in, id) is a morphism of pre-transition systems with independence. Finally, from Lemma 2.6.7 *(iii)*, if TI is deterministic, then $(s, a, s')\ I\ (\bar{s}, b, \bar{s}')$ if and only if $([s]_{\equiv_0}, a, [s']_{\equiv_0})\ I_{\equiv_0}\ ([\bar{s}]_{\equiv_0}, b, [\bar{s}']_{\equiv_0})$, i.e., (in, id) is an isomorphism of (pre) transition systems with independence. ✓

For $i \in \omega \setminus \{0\}$, consider the pair (in_i, id_i), where $in_i: S_{i-1}/{\equiv_{i-1}} \to S_i/{\equiv_i}$ is the function such that $in_i([x]_{\equiv_{i-1}}) = [x{\restriction}A_{i-1}]_{\equiv_i}$ and $id_i: L_{i-1} \rightharpoonup L_i$ is given by $id_i(a) = a$ if $a \notin A_{i-1}$ and $id_i{\uparrow}a$ otherwise. Then, we have the following.

Lemma 2.6.11

The pair (in_i, id_i): $TSys_{i-1} \to TSys_i$ is a morphism of pre-transition systems with independence.

Proof. Observe that since $x \equiv_{i-1} y$ implies that $x\lceil A_{i-1} \equiv_i y\lceil A_{i-1}$, in_i is well-defined.

i) $in_i([s^I]_{\equiv_{i-1}}) = [s^I\lceil A_{i-1}]_{\equiv_i} = [s^I]_{\equiv_i}$.

ii) Consider a transition $[x]_{\equiv_{i-1}} \xrightarrow{a} [xa]_{\equiv_{i-1}}$ in $TSys_{i-1}$. Now, if $a \in A_{i-1}$, then $in_i([x]_{\equiv_{i-1}}) = [x\lceil A_{i-1}]_{\equiv_i} = [xa\lceil A_{i-1}]_{\equiv_i} = in_i([xa]_{\equiv_{i-1}})$. Otherwise, $xa\lceil A_{i-1} = (x\lceil A_{i-1})a$, and then

$$in_i\Big([x]_{\equiv_{i-1}}\Big) = [x\lceil A_{i-1}]_{\equiv_i} \xrightarrow{a} [(x\lceil A_{i-1})a]_{\equiv_i} = in_i\Big([xa]_{\equiv_{i-1}}\Big).$$

iii) If $([x]_{\equiv_{i-1}}, a, [xa]_{\equiv_{i-1}})\ I_{\equiv_{i-1}}\ ([y]_{\equiv_{i-1}}, b, [yb]_{\equiv_{i-1}})$ and $a, b \notin A_{i-1}$, then we have $xa\ I_{i-1}\ yb$ and $(x\lceil A_{i-1})a\ I_i\ (y\lceil A_{i-1})b$, i.e.,

$$\Big([x\lceil A_{i-1}]_{\equiv_i} \xrightarrow{a} [(x\lceil A_{i-1})a]_{\equiv_i}\Big)\ I_{\equiv_i}\ \Big([y\lceil A_{i-1}]_{\equiv_i} \xrightarrow{b} [(y\lceil A_{i-1})b]_{\equiv_i}\Big),$$

i.e., $\Big(in_i([x]_{\equiv_{i-1}}) \xrightarrow{a} in_i([xa]_{\equiv_{i-1}})\Big)\ I_{\equiv_i}\ \Big(in_i([y]_{\equiv_{i-1}}) \xrightarrow{b} in_i([yb]_{\equiv_{i-1}})\Big)$. ✓

It is interesting to notice that $TSys_\omega$ is a colimit in the category $\underline{\mathsf{pTSI}}$.

PROPOSITION 2.6.12
$TSys_\omega$ is the colimit in $\underline{\mathsf{pTSI}}$ of the $\boldsymbol{\omega}$-diagram

$$\mathcal{D} = TSys_0 \stackrel{(in_1,id_1)}{\longrightarrow} TSys_1 \stackrel{(in_2,id_2)}{\longrightarrow} \cdots \stackrel{(in_i,id_i)}{\longrightarrow} TSys_i \stackrel{(in_{i+1},id_{i+1})}{\longrightarrow} \cdots$$

Proof. For any $i \in \omega$, consider the function $in_i^\omega\colon S_i/\!\equiv_i \to S_\omega/\!\equiv_\omega$ such that $in_i^\omega([x]_{\equiv_i}) = [x\lceil A_\omega]_{\equiv_\omega}$ and let $id_i^\omega\colon L_i \rightharpoonup L_\omega$ denote the function such that $id_i^\omega(a) = a$ if $a \notin A_\omega$ and $id_i^\omega\uparrow a$ otherwise. As for Lemma 2.6.11, it is easy to see that $(in_i^\omega, id_i^\omega)$ is a morphism of pre-transition systems with independence from $TSys_i$ to $TSys_\omega$.

Since for any i we have $in_{i+1}^\omega \circ in_{i+1} = in_i^\omega$ and $id_{i+1}^\omega \circ id_{i+1} = id_i^\omega$, then $TSys_\omega$ and the morphisms $\{(in_i^\omega, id_i^\omega) \mid i \in \omega\}$ form a cocone in $\underline{\mathsf{pTSI}}$ whose base is $\mathcal{D}$. Now, consider any cocone $\{(\sigma_i, \lambda_i)\colon TSys_i \to PT \mid i \in \omega\}$, for PT any pre-transition system with independence. Then, by definition of cocone, it must be $\sigma_i = \sigma_{i+1} \circ in_{i+1}$ for each $i \in \omega$, i.e., $\sigma_i([x]_{\equiv_i}) = \sigma_{i+1}([x\lceil A_i]_{\equiv_{i+1}})$, whence it follows easily that for any $x \in S_i$ and $y \in S_j$ such that $x\lceil A_\omega = y\lceil A_\omega$ it must be $\sigma_i([x]_{\equiv_i}) = \sigma_j([y]_{\equiv_j})$. Moreover, again by definition of cocone, it must be $\lambda_i = \lambda_{i+1} \circ id_{i+1}$. This implies that for $a \in L \setminus A_\omega$ we have $\lambda_i(a) = \lambda_{i+1}(a)$ for any $i \in \omega$, while for $a \in A_j$ it must be $\lambda_i\uparrow a$ for any $i \leq j$. In fact, if $a \notin A_\omega$, since $id_{i+1}(a) = a$, it must be $\lambda_i(a) = \lambda_{i+1}(a)$. Suppose instead that $a \in A_j$. Then, $id_{j+1}\uparrow a$ and thus $\lambda_j\uparrow a$. Now, since $id_i(a) = a$ if $i \leq j$, it follows that $\lambda_i\uparrow a$ for any $i \leq j$.

Now, define $(\bar\sigma, \bar\lambda)\colon TSys_\omega \to PT$, where $\bar\sigma([x]_{\equiv_\omega}) = \sigma_i([\bar x]_{\equiv_i})$ for any i and $\bar x \in S_i$ such that $\bar x\lceil A_\omega = x$, and take $\bar\lambda$ to be the restriction of λ_0 to L_w. Exploiting the features of the morphisms (σ_i, λ_i), it is easy to see that $(\sigma_i, \lambda_i) = (\bar\sigma, \bar\lambda) \circ (in_i^\omega, id_i^\omega)$ for each i, and that $(\bar\sigma, \bar\lambda)$ is the unique morphism which enjoys this property. Observe that, in view of Proposition 2.6.4, $\bar\sigma$ could be equivalently defined by saying that $\bar\sigma([x]_{\equiv_\omega}) = \sigma_i([x]_{\equiv_i})$ for any x such that $x \in S_i$. ✓

Besides enjoying a (co)universal property, $TSys_\omega$ has another property which the reader would have already guessed: it is actually a deterministic transition system with independence.

PROPOSITION 2.6.13
$TSys_\omega$ is a deterministic transition system with independence.

Proof. Proposition 2.6.9 shows that $Tsys_\omega$ is a deterministic pre-transition system with independence, while it follows immediately from Proposition 2.6.3 that $I_{\equiv_\omega}$ is irreflexive. Let us check that the axioms of transition systems with independence hold.

i) Trivial, since $TSys_\omega$ is deterministic.

ii) Suppose that $([x]_{\equiv_\omega}, a, [x']_{\equiv_\omega})\ I_{\equiv_\omega}\ ([x]_{\equiv_\omega}, b, [x'']_{\equiv_\omega})$. Then, $xa\ I_\omega\ xb$ and, therefore, there exists an index i such that $xa\ I_{i-1}\ xb$, which, in turn, implies that there exist $xab \equiv_i xba \in S_i$. Then, by *(Cl3)*, $xa\ I_i\ xb$ implies $xba\ I_i\ xb$ and $xb\ I_i\ xa$ implies $xab\ I_i\ xa$. Since $a, b \notin A_\omega$ and $x\lceil A_\omega = x$, then it is $xab \equiv_\omega xba$, and $xa\ I_\omega\ xab$ and $xb\ I_\omega\ xba$, which implies that there exists $[xab]_{\equiv_\omega} = [u]_{\equiv_\omega} = [xba]_{\equiv_\omega}$ in $S_\omega/\equiv_\omega$ such that $([x]_{\equiv_\omega}, a, [x']_{\equiv_\omega})\ I_{\equiv_\omega}\ ([x']_{\equiv_\omega}, b, [u]_{\equiv_\omega})$, and $([x]_{\equiv_\omega}, b, [x'']_{\equiv_\omega})\ I_{\equiv_\omega}\ ([x'']_{\equiv_\omega}, a, [u]_{\equiv_\omega})$.

iii) Similar to the previous point.

iv) It is enough to show that

$$([x]_{\equiv_\omega}, a, [x']_{\equiv_\omega})\ (\prec \cup \succ)\ ([x'']_{\equiv_\omega}, a, [u]_{\equiv_\omega})\ I_{\equiv_\omega}\ ([\bar{x}]_{\equiv_\omega}, b, [\bar{x}']_{\equiv_\omega}) \\ \text{implies } ([x]_{\equiv_\omega}, a, [x']_{\equiv_\omega})\ I_{\equiv_\omega}\ ([\bar{x}]_{\equiv_\omega}, b, [\bar{x}']_{\equiv_\omega}).$$

Suppose that the '$\prec$' case holds. Then, there exists an index i such that $x' \equiv_i xa$, $x'' \equiv_i xb$, $xa\ I_i\ xb$, $xab \equiv_i u \equiv_i xba$, and $xba\ I_i\ \bar{x}b$. Then, by *(Cl3)*, we have $xa\ I_i\ \bar{x}b$. Then, it is $xa\ I_\omega\ \bar{x}b$, whence it follows that $([x]_{\equiv_\omega}, a, [x']_{\equiv_\omega})\ I_{\equiv_\omega}\ ([\bar{x}]_{\equiv_\omega}, b, [\bar{x}']_{\equiv_\omega})$.

A similar proof shows the case in which '$\succ$' holds. ✓

Thus, $TSys_\omega$ is the deterministic transition system with independence we will associate to the transition system with independence TI. Formally, define the map $dtsi$ from the objects of <u>TSI</u> to the ojects of <u>dTSI</u> as $dtsi(TI) = TSys_\omega$. Figure 2.3 exemplifies the construction in an easy, yet interesting, case.

Let $TI = (S, s^I, L, Tran, I)$ and $TI' = (S', s'^I, L', Tran', I')$ together with a morphism $(\sigma, \lambda)\colon TI \to TI'$ in <u>TSI</u>. In the sequel, let $(S_\kappa, \equiv_\kappa, I_\kappa)$ and $(S'_\kappa, \equiv'_\kappa, I'_\kappa)$, $\kappa \in \omega \cup \{\omega\}$, be the sequences of suitable triples corresponding, respectively, to TI and TI'. Moreover, we shall write A_κ, TA_κ, L_κ, $TSys_\kappa$, A'_κ, TA'_κ, L'_κ and $TSys'_\kappa$ to denote the sets and the transition systems determined respectively by the sequences $(S_\kappa, \equiv_\kappa, I_\kappa)$ and $(S'_\kappa, \equiv'_\kappa, I'_\kappa)$. In the following, we shall construct a sequence of morphisms $(\bar{\sigma}_i, \lambda_i)\colon TSys_i \to TSys'_i$, which will determine a morphism $(\bar{\sigma}_\omega, \lambda_\omega)\colon TSys_\omega \to TSys'_\omega$, i.e., $dtsi\Big((\sigma, \lambda)\Big)$.

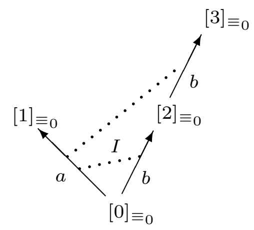
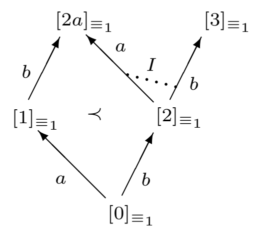
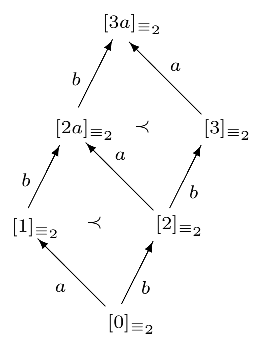
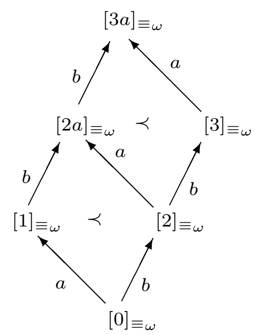
	$TSys_\kappa$	$\equiv_\kappa$	I_κ
$\kappa = 0$		$[1]_{\equiv_0} = \{1, 0a\}$ $[2]_{\equiv_0} = \{2, 0b\}$ $[3]_{\equiv_0} = \{3, 2b, 0bb\}$	$[0a]_{\equiv_0}\ I_{\equiv_0}\ [0b]_{\equiv_0}$ $[0a]_{\equiv_0}\ I_{\equiv_0}\ [2b]_{\equiv_0}$
$\kappa = 1$		$[1]_{\equiv_1} = \{1, 0a\}$ $[2]_{\equiv_1} = \{2, 0b\}$ $[3]_{\equiv_1} = \{3, 2b, 0bb\}$ $[2a]_{\equiv_1} = \{2a, 1b, 0ab, 0ba\}$	$[0a]_{\equiv_1}\ I_{\equiv_1}\ [0b]_{\equiv_1}$ $[0a]_{\equiv_1}\ I_{\equiv_1}$ $[0ab]_{\equiv_1}$ $[0b]_{\equiv_1}\ I_{\equiv_1}\ [0ba]_{\equiv_1}$ $[1b]_{\equiv_1}\ I_{\equiv_1}\ [2a]_{\equiv_1}$ $[0a]_{\equiv_1}\ I_{\equiv_1}\ [2b]_{\equiv_1}$ $[0ba]_{\equiv_1}\ I_{\equiv_1}\ [0bb]_{\equiv_1}$
$\kappa = 2$		$[0]_{\equiv_2} = \{0\}$ $[1]_{\equiv_2} = \{1, 0b\}$ $[2]_{\equiv_2} = \{2, 0a\}$ $[3]_{\equiv_2} = \{3, 2a, 0aa\}$ $[2b]_{\equiv_2} = \{1a, 2b, 0ba, 0ab\}$ $[3b]_{\equiv_2} = \{3b, 2ab, 2ba, 1aa,$ $0abb, 0baa, 0aba\}$	$[0a]_{\equiv_2}\ I_{\equiv_2}\ [0b]_{\equiv_2}$ $[0a]_{\equiv_2}\ I_{\equiv_2}\ [1b]_{\equiv_2}$ $[0b]_{\equiv_2}\ I_{\equiv_2}\ [2a]_{\equiv_2}$ $[1b]_{\equiv_2}\ I_{\equiv_2}\ [2a]_{\equiv_2}$ $[2a]_{\equiv_2}\ I_{\equiv_2}\ [2b]_{\equiv_2}$ $[2a]_{\equiv_2}\ I_{\equiv_2}\ [2bb]_{\equiv_2}$ $[2b]_{\equiv_2}\ I_{\equiv_2}\ [3a]_{\equiv_2}$ $[2bb]_{\equiv_2}\ I_{\equiv_2}\ [3a]_{\equiv_2}$
$\kappa = \omega$		COMMENTS. The transition system we start from gets us to $TSys_0$, where the dotted lines indicate relation I. $TSys_0$ fails to a be transition system with independence because there is no diamond for the transitions sticking out $[0]_{\equiv_0}$. In $TSys_1$, this problem has been solved by use of *(D1)*. However, now there is no diamond for the transitions leaving from $[2]_{\equiv_1}$, which are independent because of the closure *(Cl3)*. The problem is fixed in $TSys_2$ which is a transition system with independence and coincides with $TSys_\omega$.	

Figure 2.3: An example of the construction of $TSys_\omega$.

For $i \in \omega$, let σ_i be the function such that

$$\sigma_i(x) = \sigma(x) \quad \text{for } x \in S;$$

and

$$\sigma_i(xa) = \begin{cases} \sigma_i(x)\lambda_i(a) & \text{if } \lambda_i{\downarrow}a \\ \sigma_i(x) & \text{otherwise;} \end{cases}$$

where

$$\lambda_i(a) = \begin{cases} \lambda(a) & \text{if } \lambda(a) \notin \bigcup_{j<i} A'_j \\ \uparrow & \text{otherwise.} \end{cases}$$

The next step is to show that σ_i is well-defined, i.e., it is actually a function from S_i to S'_i, and that it respects both $\equiv_i$ and I_i. In order to do that, we need the following lemmas.

Lemma 2.6.14
If $x \in S_{i-1}\lceil A_{i-1}$ then $\sigma_i(x) = \sigma_{i-1}(x)\lceil A'_{i-1}$.

Proof. We proceed by induction on the structure of x. Observe that the claim trivially holds for the base case $x \in S$.

(Inductive step). Consider $xa \in S_{i-1}\lceil A_{i-1}$. Then, $x \in S_{i-1}\lceil A_{i-1}$ and by inductive hypothesis $\sigma_i(x) = \sigma_{i-1}(x)\lceil A_{i-1}$. Now, if $\lambda_i{\downarrow}a$ then $\lambda_{i-1}{\downarrow}a$ and $\lambda(a) \notin A'_{i-1}$. Thus, $\sigma_i(xa) = \sigma_i(x)\lambda(a) = (\sigma_{i-1}(x)\lceil A'_{i-1})\lambda(a) = (\sigma_{i-1}(x)\lambda(a)\lceil A'_{i-1}) = \sigma_{i-1}(xa)\lceil A'_{i-1}$. Suppose instead that $\lambda_i{\uparrow}a$. In case $\lambda_{i-1}{\uparrow}a$, it is $\sigma_i(xa) = \sigma_i(x) = \sigma_{i-1}(x)\lceil A'_{i-1} = \sigma_{i-1}(xa)\lceil A'_{i-1}$. Then, suppose that $\lambda_{i-1}{\downarrow}a$. This means that we have $\lambda(a) \in A'_{i-1}$. Then, $\sigma_i(xa) = \sigma_i(x) = \sigma_{i-1}(x)\lceil A'_{i-1} = (\sigma_{i-1}(x)\lambda(a))\lceil A'_{i-1} = \sigma_{i-1}(xa)\lceil A'_{i-1}$. ✓

Lemma 2.6.15
Suppose that $xa\ I_i\ yb$ and $\lambda_i{\downarrow}a$, $\lambda_i{\downarrow}b$ implies $\sigma_i(xa)\ I'_i\ \sigma_i(yb)$. Then, if $\bar{x}\lceil A_i = x$, it is $\sigma_i(\bar{x})\lceil A'_i = \sigma_i(x)\lceil A'_i$.

Proof. By induction on the structure of $\bar{x}$, the thesis being obvious for the base case $\bar{x} \in S$.

(Inductive step). Consider $\bar{x}a$. If $a \in A_i$, then $(\bar{x}a)\lceil A_i = \bar{x}\lceil A_i = x$ and then, by induction, we have $\sigma_i(\bar{x})\lceil A'_i = \sigma_i(x)\lceil A'_i$. Now, if $\lambda_i{\uparrow}a$, we have $\sigma_i(\bar{x}a)\lceil A'_i = \sigma_i(\bar{x})\lceil A'_i = \sigma_i(x)\lceil A'_i$. Otherwise, there are the following two cases.

1. If $\lambda(a) \in \bigcup_{j<i} A'_j$, then again $\sigma_i(\bar{x}a) = \sigma_i(\bar{x})$ and the induction is mantained.
2. If $\lambda(a) \notin \bigcup_{j<i} A'_j$, then $\sigma_i(\bar{x}a) = \sigma_i(\bar{x})\lambda(a)$. Since $a \in A_i$, there exists $ya \in S_i$ such that $ya\ I_i\ ya$, and by hypothesis we have that $\sigma_i(y)\lambda(a)\ I'_i\ \sigma_i(y)\lambda(a)$. Then, we have $\lambda(a) \in A'_i$. It follows again that $\sigma_i(\bar{x}a)\lceil A'_i = \sigma_i(\bar{x})\lceil A'_i = \sigma_i(x)\lceil A'_i$.

If otherwise $a \notin A_i$, then it is $x = x'a$ and $\bar{x}a\lceil A'_i = (\bar{x}\lceil A'_i)a = x'a$, with $\bar{x}\lceil a = x'$. Then, by inductive hypothesis we have $\sigma_i(\bar{x})\lceil A'_i = \sigma_i(x')\lceil A'_i$. Now, if $\lambda_i{\uparrow}a$, the thesis follows easily. Otherwise, if $\lambda_i{\downarrow}a$, there are again the two cases above which can be treated analogously. ✓

Now, we can prove that σ_i is well-defined and respects $\equiv_i$ and I_i.

LEMMA 2.6.16
For any $i \in \omega$, we have that

i) $x \in S_i$ *implies* $\sigma_i(x) \in S'_i$;

ii) $x \equiv_i y$ *implies* $\sigma_i(x) \equiv'_i \sigma_i(y)$;

iii) $xa\ I_i\ yb$ *and* $\lambda_i{\downarrow}a$, $\lambda_i{\downarrow}b$ *implies* $\sigma_i(xa)\ I'_i\ \sigma_i(yb)$.

Proof. We show the three points at the same time, by induction on i. The base case for $i = 0$ follows directly from the definition of σ_0 and from the fact that (σ, λ) is a morphism.

(Inductive step). We start by proving that the generators of $(S_i, \equiv_i, I_i)$ are mapped by σ_i to the generators of $(S'_i, \equiv'_i, I'_i)$. Let us start with the elements generated by *($\Im$)*. Let x belong to $S_{i-1}\lceil A_{i-1}$. Then, by Lemma 2.6.14, we have that $\sigma_i(x) = \sigma_{i-1}(x)\lceil A'_{i-1}$. Taken $\bar{x} \in S_{i-1}$ such that $\bar{x}\lceil A_{i-1} = x$, we can apply Lemma 2.6.15 and conclude that $\sigma_{i-1}(x)\lceil A'_{i-1} = \sigma_{i-1}(\bar{x})\lceil A'_{i-1}$. Since by inductive hypothesis $\sigma_{i-1}(\bar{x}) \in S'_i$, it follows that $\sigma_i(x) \in S'_i\lceil A'_i$. Suppose now that $(x, y) \in\ \equiv_{i-1}\lceil A_{i-1}$. Then there exists $(\bar{x}, \bar{y}) \in\ \equiv_{i-1}$ such that $\bar{x}\lceil A_{i-1} = x$ and $\bar{y}\lceil A_{i-1} = y$. By induction we have $\sigma_{i-1}(\bar{x}) \equiv'_{i-1} \sigma_{i-1}(\bar{y})$, and then $(\sigma_{i-1}(\bar{x})\lceil A'_{i-1}) \equiv'_{i-1} \lceil A'_{i-1}\ (\sigma_{i-1}(\bar{y})\lceil A'_{i-1})$. Then, again by Lemma 2.6.14 and Lemma 2.6.15, we have $\sigma_i(x) = \sigma_{i-1}(\bar{x})\lceil A'_{i-1}$ and $\sigma_i(y) = \sigma_{i-1}(\bar{y})\lceil A'_{i-1}$. It follows that $\sigma_i(x) \equiv'_{i-1} \lceil A'_{i-1}\ \sigma_i(y)$. Finally, suppose that $(xa, yb) \in (I_{i-1} \setminus TA_{i-1})\lceil A_{i-1}$ and that $\lambda_i{\downarrow}a$ and $\lambda_i{\downarrow}b$, which implies that $\lambda_{i-1}{\downarrow}a$ and $\lambda_{i-1}{\downarrow}b$. Reasoning as above, we can find $(\bar{x}a, \bar{y}a) \in I_{i-1}$ such that $\bar{x}\lceil A_{i-1} = x$ and $\bar{y}\lceil A_{i-1} = y$ and $\sigma_{i-1}(\bar{x}a)\ I'_{i-1}\ \sigma_{i-1}(\bar{y}b)$. Then, $(\sigma_{i-1}(\bar{x}a)\lceil A'_{i-1})\ (I'_{i-1} \setminus TA'_{i-1})\lceil A'_{i-1}\ (\sigma_{i-1}(\bar{y}b)\lceil A'_{i-1})$, which clearly implies $\sigma_i(xa)\ (I'_{i-1} \setminus TA'_{i-1})\lceil A'_{i-1}\ \sigma_i(yb)$.

Then, we have shown that σ_i maps $S_{i-1}\lceil A_{i-1}$ to $S'_{i-1}\lceil A'_{i-1}$, $\equiv_{i-1}\lceil A_{i-1}$ to $\equiv'_{i-1}\lceil A'_{i-1}$ and $(I_{i-1} \setminus TA_{i-1})\lceil A_{i-1}$ to $(I'_{i-1} \setminus TA'_{i-1})\lceil A'_{i-1}$. Then, the induction hypothesis ensures that *(i)*, *(ii)* and *(iii)* hold for elements generated by *($\Im$)*. Now, we need to prove the claim for the generators introduced in the *i-th* inductive step.

Consider $x \in {}_{\gamma}S_i$. Let us show that $\sigma_i(x) \in {}_{\gamma}S'_i$. The following two cases are possible.

1. x is generated by *(D1)*, i.e., x is yba for $yb\ (I_{i-1} \setminus TA_{i-1})\lceil A_{i-1}\ ya$ in $S_{i-1}\lceil A_{i-1}$. Then, if $\lambda_i{\uparrow}a$ then $\sigma_i(x) = \sigma_i(yb) \in S'_{i-1}\lceil A'_{i-1} \subseteq {}_{\gamma}S'_i$. Otherwise, if $\lambda_i{\uparrow}b$, then $\sigma_i(x) = \sigma_i(y)\lambda(a) = \sigma_i(ya) \in {}_{\gamma}S'_i$. Finally, if λ_i is defined on both a and b, then $\sigma_i(y)\lambda(a)\ (I'_{i-1} \setminus TA'_{i-1})\lceil A'_{i-1}\ \sigma_i(y)\lambda(b)$ and therefore $\sigma_i(y)\lambda(b)\lambda(a) = \sigma_i(x)$ is in ${}_{\gamma}S'_i$.
2. x is generated by *(D2)*, i.e., x is ya or yab for $yb\ (I_{i-1} \setminus TA_{i-1})\lceil A_{i-1}\ yba$ in $S_{i-1}\lceil A_{i-1}$. Then, the proof goes as in the previous case.

Now, suppose $x\ {}_{\gamma}{\equiv_i}\ y$ and let us show that $\sigma_i(x)\ {}_{\gamma}{\equiv'_i}\ \sigma_i(y)$. Again, there are two cases.

1. $x \;{}_\gamma{\equiv_i}\; y$ is generated by *(D1)*, which means that x is of the kind zab and y is zba for $za\ (I_{i-1} \setminus TA_{i-1}){\restriction}A_{i-1}\ zb$ in $S_{i-1}{\restriction}A_{i-1}$. Then, if $\lambda_i{\uparrow}a$ or $\lambda_i{\uparrow}b$, we have that $\sigma_i(x) = \sigma_i(y)$ and we are done. Otherwise, if λ_i is defined on both a and b, since $\sigma_i(z)\lambda(a)\ (I'_{i-1} \setminus TA'_{i-1}){\restriction}A'_{i-1}\ \sigma_i(z)\lambda(b)$, it is $\sigma_i(x) = \sigma_i(z)\lambda(a)\lambda(b)\ {}_\gamma{\equiv'_i}\ \sigma_i(z)\lambda(b)\lambda(a) = \sigma_i(y)$.
2. $x \equiv_i y$ is generated by *(D2)*, which means that x is of the kind zab and y is zba for $za\ (I_{i-1} \setminus TA_{i-1}){\restriction}A_{i-1}\ zab$ in $S_{i-1}{\restriction}A_{i-1}$. This case is analogous to the previous one.

Then, we have proved that $x \in {}_\gamma S_i$ implies $\sigma_i(x) \in {}_\gamma S'_i$, $x\ {}_\gamma{\equiv_i}\ y$ implies $\sigma_i(x)\ {}_\gamma{\equiv'_i}\ \sigma_i(y)$, and $xa\ {}_\gamma I_i\ yb$ and $\lambda_i{\downarrow}a$, $\lambda_i{\downarrow}b$ implies $\sigma_i(xa)\ {}_\gamma I'_i\ \sigma_i(yb)$. To conclude the proof, now it is enough to check that the closure rules preserve points *(i)*, *(ii)* and *(iii)*.

(Cl1) & (Cl2) Suppose that $x \equiv_i z$ and $za \in S_i$, where $\sigma_i(x) \equiv'_i \sigma_i(z)$ and $\sigma_i(za) \in S'_i$. Now, if $\lambda_i{\uparrow}a$, then $\sigma_i(xa) \equiv'_i \sigma_i(za) \in S'_i$, and we are done. Otherwise, since $\sigma_i(x) \equiv'_i \sigma_i(z)$ and $\sigma_i(z)\lambda(a) \in S'_i$, we have $\sigma_i(xa) = \sigma_i(x)\lambda(a) \equiv'_i \sigma_i(z)\lambda(a) = \sigma_i(za)$ in S'_i, since $(S'_i, \equiv'_i, I'_i)$ is *suitable*.

Now suppose that $za\ I_i\ yc$, $\lambda_i{\downarrow}a$ and $\lambda_i{\downarrow}c$, and assume $\sigma_i(z)\lambda(a)\ I'_i\ \sigma_i(y)\lambda(c)$. Then, $\sigma_i(x)\lambda(a) \equiv'_i \sigma_i(z)\lambda(a)$ and, therefore, $\sigma_i(xa) = \sigma_i(x)\lambda(a)\ I'_i\ \sigma_i(y)\lambda(c) = \sigma_i(yc)$, since $(S'_i, \equiv'_i, I'_i)$ is *suitable*.

(Cl3) Suppose $xab \equiv_i xba$ and $xa\ I_i\ xb$ or $xa\ I_i\ xab$. Moreover suppose that $\lambda_i{\downarrow}a$ and $\lambda_i{\downarrow}c$. Then if $\sigma_i(x)\lambda(a)\lambda(b) \equiv'_i \sigma_i(x)\lambda(b)\lambda(a)$ and $\sigma_i(x)\lambda(a)\ I'_i\ \sigma_i(x)\lambda(b)$ or $\sigma_i(x)\lambda(a)\ I'_i\ \sigma(x)\lambda(a)\lambda(b)$, we are in the condition of *(Cl3)*, and we conclude that $\sigma(xa)\ I'_i\ \sigma(yc)$ if and only if $\sigma(xba)\ I'_i\ \sigma(yc)$. ✓

It follows immediately from Lemma 2.6.16 that for $i \in \omega$, $\bar\sigma_i$, defined to be the map which sends $[x]_{\equiv_i}$ to $[\sigma_i(x)]_{\equiv'_i}$ is a well-defined function from $S_i/{\equiv_i}$ to $S'_i/{\equiv'_i}$. Then, the following lemma follows easily.

Lemma 2.6.17

For $i \in \omega$, the map $(\bar\sigma_i, \lambda_i)\colon TSys_i \to TSys'_i$ is a morphism of pre-transition systems with independence.

For any $i \in \omega$, consider the morphism of pre-transition systems with independence $(in'^{\omega}_i, id'^{\omega}_i) \circ (\bar\sigma_i, \lambda_i)\colon TSys_i \to TSys'_\omega$. Recall that for $x \in S_i$, we have $\sigma_{i+1}(x{\restriction}A_i) = \sigma_i(x{\restriction}A_i){\restriction}A'_i = \sigma_i(x){\restriction}A'_i$, whence $\sigma_{i+1}(x{\restriction}A_i){\restriction}A'_\omega = \sigma_i(x){\restriction}A'_\omega$. Then

$$\begin{aligned} in'^{\omega}_i \circ \bar\sigma_i([x]_{\equiv_i}) &= in'^{\omega}_i([\sigma_i(x)]_{\equiv'_i}) = [\sigma_i(x){\restriction}A'_\omega]_{\equiv'_\omega} \\ &= [\sigma_{i+1}(x{\restriction}A_i){\restriction}A'_\omega]_{\equiv'_\omega} = in'^{\omega}_{i+1}([\sigma_{i+1}(x{\restriction}A_i)]_{\equiv'_{i+1}}) \\ &= in'^{\omega}_{i+1} \circ \bar\sigma_{i+1}([x{\restriction}A_i]_{\equiv_{i+1}}) = in'^{\omega}_{i+1} \circ \bar\sigma_{i+1} \circ in_{i+1}([x]_{\equiv_i}), \end{aligned}$$

i.e., $in'^{\omega}_i \circ \bar\sigma_i = in'^{\omega}_{i+1} \circ \bar\sigma_{i+1} \circ in_{i+1}$ for any $i \in \omega$. Moreover, since $a \in A_i$ implies $\lambda(a) \in A'_i$, it is easy to see that $id'^{\omega}_i \circ \lambda_i = id'^{\omega}_{i+1} \circ \lambda_{i+1} \circ id_{i+1}$ for any $i \in \omega$. Thus, we have that

$$\left\{ (in'^{\omega}_i, id'^{\omega}_i) \circ (\bar\sigma_i, \lambda_i)\colon TSys_i \to TSys'_\omega \;\middle|\; i \in \omega \right\}$$

is a cocone for the $\boldsymbol{\omega}$-diagram $\mathcal{D}$ given in Proposition 2.6.12. Then, there exists a unique $(\bar{\sigma}_\omega, \lambda_\omega): TSys_\omega \to TSys'_\omega$ induced by the colimit construction, which is the morphism of transition systems with independence we associate to (σ, λ), i.e., $dtsi\Big((\sigma,\lambda)\Big) = (\bar{\sigma}_\omega, \lambda_\omega)$. From Proposition 2.6.12, it is immediate to see that $\bar{\sigma}_\omega([x]_{\equiv_\omega}) = [\sigma_i(\bar{x})\restriction A'_\omega]_{\equiv'_\omega}$ for $\bar{x} \in S_i$ such that $\bar{x}\restriction A_\omega = x$, or, equivalently, $\bar{\sigma}_\omega([x]_{\equiv_\omega}) = [\sigma_i(x)\restriction A'_\omega]_{\equiv'_\omega}$ for any i such that $x \in S_i$, and that

$$\lambda_\omega(a) = \begin{cases} \lambda(a) & \text{if } \lambda(a) \notin A'_\omega \\ \uparrow & \text{otherwise.} \end{cases}$$

The following proposition follows easily from the universal properties of colimits.

Proposition 2.6.18 *($dtsi$: $\underline{\mathsf{TSI}} \to \underline{\mathsf{dTSI}}$ is a functor)*
The map $dtsi$ is a functor from $\underline{\mathsf{TSI}}$ to $\underline{\mathsf{dTSI}}$.

The question we address next concerns what we get when we apply $dtsi$ to a deterministic transition system with independence DTI. We shall see that in this case the inductive construction of $TSys_\omega$ gives a transition system which is isomorphic to DTI. More precisely, each $\equiv_\omega$-equivalence class of $(S_{DTI})_\omega$ contains exactly *one* state of the original transition system, and the transition system with independence morphism $(in_0^\omega \circ in, id_0^\omega): DTI \to dtsi(DTI)$—whose transition component sends $s \in S_{DTI}$ to $[s]_{\equiv_\omega}$—is actually an isomorphism. Moreover, we shall see that its inverse (ε, id), where $\varepsilon([x]_{\equiv_\omega})$ is the unique $s \in S_{DTI}$ such that $s \equiv_\omega x$, is the counit of the adjunction.

Lemma 2.6.19
Let $DTI = (S, s^I, L, Tran, I)$ be a deterministic transition system with independence. Then, $(S_1, \equiv_1, I_1)$ coincides with $(S_0, \equiv_0, I_0)$. Therefore, $(in_0^\omega \circ in, id_0^\omega)$ is an isomorphism whose inverse is (ε, id).

Proof. We know from Proposition 2.6.10 that (in, id) is an isomorphism if DTI is deterministic. Thus, $(in_0^\omega \circ in, id_0^\omega)$ is an isomorphism if and only if $(in_0^\omega, id_0^\omega): TSys_0 \to TSys_\omega$ is so, which, in turn, is a consequence of the first part of the claim.

Observe that $A_0 = \varnothing$ and, therefore, $TA_0 = \varnothing$. In fact, since DTI and $TSys_0$ are isomorphic, if there were $xa\ I_0\ xa$, then I_{DTI} would not be irreflexive. Then, in order to show that $(S_1, \equiv_1, I_1) = (S_0, \equiv_0, I_0)$, it is enough to see that no new elements are introduced by *(D1)* and *(D2)*. In fact, in this case, $(S_1, \equiv_1, I_1)$ would be the least *suitable* triple which contains $(S_0, \equiv_0, I_0)$ which is clearly $(S_0, \equiv_0, I_0)$ itself.

(D1) Suppose $xa\ I_0\ xb$. Then, by Corollary 2.6.8, there exist s, s', $s'' \in S$ such that $s \equiv_0 x$, $s' \equiv_0 xa$ and $s'' \equiv_0 xb$. Therefore, by Lemma 2.6.7, we have $(s, a, s')\ I\ (s, b, s'')$ in $Tran$. Since DTI is a transition system with independence, there exists u such that

$$Diam\Big((s,a,s'), (s,b,s''), (s',b,u), (s'',a,u)\Big),$$

and then we have $sab \equiv_0 u \equiv_0 sba$ and, therefore, by *(Cl1)*, we already have $xab \equiv_0 xba$ in $(S_0, \equiv_0, I_0)$.

(D2) Analogous to the previous case. ✓

Thus, we have proved the following corollary.

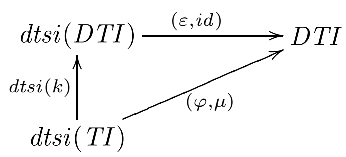
Corollary 2.6.20
$(\varepsilon, id)\colon dtsi(DTI) \to DTI$ *is a transition system with independence isomorphism.*

Before showing that (ε, id) is the counit of the reflection of dTSI in TSI, we need the following lemma which characterizes the behaviour of transition system with independence morphisms whose target is deterministic.

Lemma 2.6.21
Let DTI *be a deterministic transition system with independence and consider a morphism* $(\sigma, \lambda)\colon TI \to DTI$ *in* TSI. *Let* $TSys_\kappa$, $\kappa \in \omega \cup \{\omega\}$ *be the sequence of pre-transition systems with independence associated to* TI. *Consider* $a \in L_{TI}$ *and suppose that* $a \in A_i$. *Then* $\lambda\!\uparrow\! a$.

We are ready now to show that (ε, id) is couniversal.

Proposition 2.6.22 *($(\varepsilon, id)\colon dtsi(DTI) \to DTI$ is couniversal)*
For any transition system with independence TI, *deterministic transition system with independence* DTI *and morphism* $(\varphi, \mu)\colon dtsi(TI) \to DTI$, *there exists a unique* $k\colon TI \to DTI$ *such that* $(\varepsilon, id) \circ dtsi(k) = (\varphi, \mu)$.

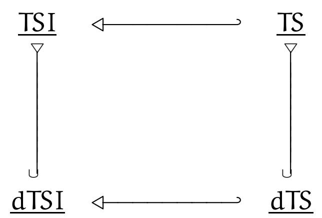


Proof. Let us consider $k = (\sigma, \lambda)$, where $\sigma(s) = \varphi([s]_{\equiv_\omega})$ and λ is the function which coincides with μ on $(L_{TI})_\omega$ and is undefined elsewhere. Observe that this is the only possible choice for k. In fact, any $k'\colon TI \to DTI$ which has to make the diagram commute must be of the kind (σ', λ') with $\lambda'(a) = \mu(a) = \lambda(a)$ for $a \in (L_{TI})_\omega$. Moreover, by Lemma 2.6.21, if $a \in A_\omega$, it must be $\lambda'\!\uparrow\! a$, i.e., $\lambda' = \lambda$. Furthermore, $\sigma'(s)$ must be an $\bar{s}$ in S_{DTI} such that $\varepsilon([\bar{s}]_{\equiv_\omega}) = \bar{s}$ coincides with $\varphi([s]_{\equiv_\omega})$, i.e., σ' is the σ we have chosen.

In order to show that (σ, λ) is a morphism of pre-transition systems with independence, it is enough to observe that (σ, λ) can be expressed as the composition of the transition system with independence morphisms $(\varphi, \mu) \circ (in_0^\omega \circ in, id_0^\omega)\colon TI \to dtsi(TI) \to DTI$. This makes easy to conclude the proof. ✓

COROLLARY 2.6.23 *(dtsi $\dashv \hookleftarrow$)*
Functor dtsi is left adjoint to the inclusion functor $\underline{\mathsf{dTSI}} \hookrightarrow \underline{\mathsf{TSI}}$. *Therefore, the adjunction* $\langle dtsi, \hookleftarrow\rangle\colon \underline{\mathsf{dTSI}} \rightharpoonup \underline{\mathsf{TSI}}$ *is a reflection.*

The adjunction $\underline{\mathsf{dTSI}} \hookrightarrow\!\!\lhd \underline{\mathsf{TSI}}$ that we have so established closes another face of the cube. In particular, we have obtained the following square, which matches the one presented in Section 2.2.



2.7 Deterministic Labelled Event Structures

In this section we prove that there exists a reflection from the category of deterministic labelled event structures to labelled event structures. A reflection $\underline{\mathsf{dLES}} \hookrightarrow\!\!\lhd \underline{\mathsf{LES}}$ does exist, for it follows from the reflections we have presented in the previous sections. In fact, the results in Section 2.5 and 2.6 show that there exist adjunctions

$$\underline{\mathsf{dLES}} \hookrightarrow\!\!\rhd \underline{\mathsf{dTSI}} \hookrightarrow\!\!\lhd \underline{\mathsf{TSI}} \lhd\!\!\hookleftarrow \underline{\mathsf{LES}}.$$

Now, in order to show that there is a coreflection from $\underline{\mathsf{dLES}}$ to $\underline{\mathsf{LES}}$, since $\underline{\mathsf{dLES}} \cong \underline{\mathsf{doTSI}_\mathsf{E}}$ and $\underline{\mathsf{LES}} \cong \underline{\mathsf{oTSI}_\mathsf{E}}$, it is enough to show that $\underline{\mathsf{dTSI}} \hookrightarrow\!\!\lhd \underline{\mathsf{TSI}}$ cuts down to a reflection $\underline{\mathsf{doTSI}_\mathsf{E}} \hookrightarrow\!\!\lhd \underline{\mathsf{oTSI}_\mathsf{E}}$. In this case, we would have an adjunction

$$\underline{\mathsf{dLES}} \cong \underline{\mathsf{doTSI}_\mathsf{E}} \hookrightarrow\!\!\lhd \underline{\mathsf{oTSI}_\mathsf{E}} \cong \underline{\mathsf{LES}},$$

whose right adjoint is isomorphic to the inclusion functor $\underline{\mathsf{dLES}} \hookrightarrow \underline{\mathsf{LES}}$. As usual, to establish that $\underline{\mathsf{doTSI}_\mathsf{E}} \hookrightarrow\!\!\lhd \underline{\mathsf{oTSI}_\mathsf{E}}$, it is enough to show that if OTI in $\underline{\mathsf{oTSI}}$ satisfies axiom (E), then $dtsi(OTI)$ is a deterministic occurrence transition system with independence which satisfies (E).

However, since this task is rather boring, we prefer to introduce the reflection $\underline{\mathsf{dLES}} \hookrightarrow\!\!\lhd \underline{\mathsf{LES}}$ as a construction given directly on labelled event structures. In order to simplify the exposition, we factorize $\underline{\mathsf{dLES}} \hookrightarrow\!\!\lhd \underline{\mathsf{LES}}$ in two parts: $\underline{\mathsf{dLES}} \hookrightarrow\!\!\lhd \underline{\mathsf{LES}_\mathsf{I}} \hookrightarrow\!\!\lhd \underline{\mathsf{LES}}$, where $\underline{\mathsf{LES}_\mathsf{I}}$ is the category of labelled event structures without autoconcurrency, i.e., those labelled event structures in which all the concurrent events carry distinct labels.

Labelled Event Structures without Autoconcurrency

As already remarked in Section 2.6, the only way to cope with *autoconcurrent* events is by eliminating them. However, the reader will notice that the task is now much easier than in the case of transition systems with independence. Once again, this is due to the difference between independence and concurrency and it gives a "measure" of how this difference can play a role when dealing with transition systems with independence.

Let $ES = (E, \#, \leq, \ell, L)$ be a labelled event structure. Consider the sets $A(ES) = \{a \in L \mid \exists e, e' \in E, e \; co \; e' \text{ and } \ell(e) = a = \ell(e')\}$ and $TA(ES) = \{e \in E \mid \ell(e) \in A(ES)\}$. Then define

$$lesi(ES) = \Big(\bar{E},\, \# \cap (\bar{E} \times \bar{E}),\, \leq \cap (\bar{E} \times \bar{E}),\, \bar{\ell},\, \bar{L}\Big)$$

where $\bar{E} = E \setminus TA(ES)$, $\bar{L} = L \setminus A(ES)$ and $\bar{\ell} \colon \bar{E} \to \bar{L}$ is ℓ restricted to $\bar{E}$.

Of course $lesi(ES)$ is a labelled event structure without autoconcurrency. As a candidate for the unit of the adjunction, consider the map $(\bar{in}, \bar{id}) \colon ES \to lesi(ES)$ where

$$\bar{in}(e) \;=\; \begin{cases} e & \text{if } e \in \bar{E} \\ \uparrow & \text{otherwise;} \end{cases}$$

and

$$\bar{id}(a) \;=\; \begin{cases} a & \text{if } a \in \bar{L} \\ \uparrow & \text{otherwise} \end{cases}$$

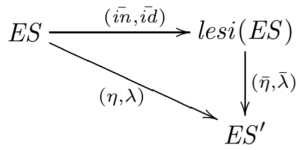
It is extremely easy to verify that this definition gives a morphism in $\underline{\mathsf{LES}}$.

Lemma 2.7.1
Let $(\eta, \lambda) \colon ES \to ES'$ be a morphism of labelled event structures and suppose that ES' has no autoconcurrency. Then, $\eta{\uparrow}e$ for any $e \in TA(ES)$.

Proof. Let $e \in TA(ES)$. Then $\ell_{ES}(e) \in A(ES)$ and therefore there exist $e', e'' \in E_{ES}$ such that $e' \; co \; e''$ and $\ell_{ES}(e') = \ell_{ES}(e) = \ell_{ES}(e'')$. Suppose now that $\eta{\downarrow}e'$ and $\eta{\downarrow}e''$. Then, by general properties of event structures morphisms, it must be $\eta(e') \; co \; \eta(e'')$ in ES'. But $\ell_{ES'}(\eta(e')) = \lambda(\ell_{ES}(e')) = \lambda(\ell_{ES}(e'')) = \ell_{ES'}(\eta(e''))$, which is impossible, since ES' has no autoconcurrency. It follows that $\eta{\uparrow}e'$ or $\eta{\uparrow}e''$. Without loss of generality suppose that $\eta{\uparrow}e'$. Then, $\lambda{\uparrow}\ell(e')$ and thus $\lambda{\uparrow}\ell(e)$. Therefore, it must be $\eta{\uparrow}e$ (and, of course, $\eta{\uparrow}e''$). ✓

It is now easy to show that *lesi* extends to a functor from $\underline{\mathsf{LES}}$ to $\underline{\mathsf{LES}}_{\mathrm{I}}$ which is left adjoint to the inclusion $\underline{\mathsf{LES}}_{\mathrm{I}} \hookrightarrow \underline{\mathsf{LES}}$.

PROPOSITION 2.7.2 *($(\bar{in}, \bar{id})\colon ES \to lesi(ES)$ is universal)*
For any labelled event structure ES, any labelled event structure without autoconcurrency ES' and any morphism $(\eta, \lambda)\colon ES \to ES'$, there exists a unique $(\bar{\eta}, \bar{\lambda})\colon lesi(ES) \to ES'$ in $\underline{\mathsf{LES}}_{\mathrm{I}}$ such that $(\bar{\eta}, \bar{\lambda}) \circ (\bar{in}, \bar{id}) = (\eta, \lambda)$.



Proof. Consider $\bar{\eta}\colon \bar{E}_{ES} \to E_{ES'}$ and $\bar{\lambda}\colon \bar{L}_{ES} \to L_{ES'}$ to be, respectively, η restricted to $\bar{E}_{ES}$ and λ restricted to $\bar{L}_{ES}$. Exploiting Lemma 2.7.1 it is not difficult to conclude the proof. ✓

Therefore, we have the following corollary.

COROLLARY 2.7.3 *($lesi \dashv \hookrightarrow$)*
The map lesi extends to a functor from $\underline{\mathsf{LES}}$ to $\underline{\mathsf{LES}}_{\mathrm{I}}$ which is left adjoint to the inclusion functor $\underline{\mathsf{LES}}_{\mathrm{I}} \hookrightarrow \underline{\mathsf{LES}}$. Therefore, the adjunction $\langle lesi, \hookrightarrow\rangle\colon \underline{\mathsf{LES}} \rightharpoonup \underline{\mathsf{LES}}_{\mathrm{I}}$ is a reflection.

DETERMINISTIC LABELLED EVENT STRUCTURES

Let us now turn our attention to $\underline{\mathsf{dLES}} \hookrightarrow\!\!\lhd\ \underline{\mathsf{LES}}_{\mathrm{I}}$. Given a labelled event structure without autoconcurrency $ES = (E, \leq, \#, \ell, L)$, consider the sequence of relations $(\sim_i, \leq_i, \#_i)$, for $i \in \omega$, where

- $\sim_0 = \left\{(e, e) \;\middle|\; e \in E\right\}$; $\quad \leq_0 = \leq$; $\quad \#_0 = \#$;

and, for $i > 0$,

- $\sim_i$ is the least equivalence on E such that

  i) $\sim_{i-1} \subseteq \sim_i$;

  ii) $e \not\leq_{i-1} e',\ e' \not\leq_{i-1} e,\ \ell(e) = \ell(e')$
  $\lfloor e \rfloor_{\leq_{i-1}} \not\#_{i-1} \lfloor e' \rfloor_{\leq_{i-1}} \setminus \{e'\}$ and
  $\lfloor e \rfloor_{\leq_{i-1}} \not\#_{i-1} \lfloor e' \rfloor_{\leq_{i-1}} \setminus \{e'\}$
  implies $e \sim_i e'$,

  where $\lfloor e \rfloor_{\leq_i}$ is a shorthand for $\{e' \in E \mid e' \leq_i e\}$ and, for $x, y \subseteq E$, $x \not\#_i y$ stands for $\forall\, e \in x,\ \forall\, e' \in y,$ not $e \#_i e'$.

- $e \leq_i e'$ if and only if $\forall\, \bar{e}' \sim_i e'\; \exists\, \bar{e} \sim_i e.\ \bar{e} \leq_{i-1} \bar{e}'$;
- $e \#_i e'$ if and only if $\forall \bar{e}' \sim_i e'\; \forall \bar{e} \sim_i e.\ e \#_{i-1} e'$;

Observe from the previous definitions that, while $\sim_i \subseteq \sim_{i+1}$ and $\#_i \supseteq \#_{i+1}$, it is $\leq_i \not\subseteq \leq_{i+1}$ and $\leq_i \not\supseteq \leq_{i+1}$. Each triple $(\sim_i, \leq_i, \#_i)$ represents a *quotient* of the original labelled event structure in which—informally speaking—the *"degree"* of non-determinism has decreased. This will be delevoped in the following.

LEMMA 2.7.4
For any $i \in \omega$, $\leq_i$ is a preorder on E such that

$$e \leq_i e' \text{ and } e' \leq_i e \quad \text{if and only if} \quad e \sim_i e'.$$

Proof. Reflexivity and transitivity of $\leq_i$ are obvious. Concernig the rest of the claim, for $i = 0$ there is nothing to show, while for any i the '$\Leftarrow$' implication follows immediately by taking the $\bar{e}'$ in the definition of $\leq_i$ to be $\bar{e}$ itself. To conclude, we show the '$\Rightarrow$' implication by induction on i, the base case for $i = 0$ being already discussed.

Suppose $e \leq_i e'$ and $e' \leq_i e$. Then, since $e \leq_i e'$, there exists $\bar{e} \sim_i e$ such that $\bar{e} \leq_{i-1} e'$, and since $e' \leq_i e$, there exists $\bar{e}' \sim_i e'$ such that $\bar{e}' \leq_{i-1} \bar{e}$. Thus, $\bar{e}' \leq_{i-1} \bar{e} \leq_{i-1} e'$. Now, since $\bar{e}' \sim_i e'$, we have $\bar{e}' \sim_{i-1} e'$ and thus, exploiting the '$\Leftarrow$' implication, we also have $e' \leq_{i-1} \bar{e}'$. Then we have $e' \leq_{i-1} \bar{e}' \leq_{i-1} \bar{e} \leq_{i-1} e'$, whence $e' \leq_{i-1} \bar{e}$ and $\bar{e} \leq_{i-1} e'$. So, by induction, $\bar{e} \sim_{i-1} e'$ and, therefore, $\bar{e} \sim_i e'$. Finally, since $\bar{e} \sim_i e$, we conclude that $e \sim_i e'$. ✓

LEMMA 2.7.5
For any $i \in \omega$, $\#_i$ is a symmetric, irreflexive relation on E which satisfies

$$e \#_i e' \leq_i e'' \quad \text{implies} \quad e \#_i e''.$$

Proof. The fact that $\#_i$ is symmetric and irreflexive follows by straightforward induction from the fact that $\#_0$ is such. We show the other property by induction on i.

The claim is immediate for $i = 0$. For a generic i, since $e' \leq_i e''$, $\forall\, \bar{e}'' \sim_i e''$, $\exists\, \bar{e}' \sim_i e'$ such that $\bar{e}' \leq_{i-1} \bar{e}''$. Moreover, since $e \#_i e'$, $\forall\, \bar{e} \sim_i e$, we have $\bar{e} \#_{i-1} \bar{e}'$, i.e., $\bar{e} \#_{i-1} \bar{e}' \leq_i \bar{e}''$. Then, by induction hypothesis, $\bar{e} \#_{i-1} \bar{e}''$. Summarizing, $\forall\, \bar{e} \sim_i e$, $\forall\, \bar{e}'' \sim_i e''$ we have $\bar{e} \#_{i-1} \bar{e}''$, i.e., $e \#_i e''$. ✓

Observe that a direct consequence of the previous two lemmas is that $e \sim_i e'$ implies $e \not\#_i e'$.

Lemma 2.7.6
For any $i \in \omega$, for any $e \in E$, the set $\left\{ [e']_{\sim_i} \;\middle|\; e' \leq_i e,\ e' \in E \right\}$ is finite.

Proof. The proof is straightforward, by induction on i. For $i = 0$ the thesis is trivially true, since *ES* is a labelled event structure. For a general i the existence of an infinite set of predecessors of $[e]_{\sim_i}$ for $\leq_i$ would imply, by definition of $\leq_i$ and since $\sim_{i-1} \subseteq \sim_i$, the existence of an infinite set of predecessors of $[e]_{\sim_{i-1}}$ for $\leq_{i-1}$. ✓

It follows immediately from the previous lemmas that, for any $i \in \omega$,

$$Ev_i = (E/\sim_i, \leq_{\sim_i}, \#_{\sim_i}, \ell_{\sim_i}, L),$$

where

- $E/\sim_i$ is the set of $\sim_i$-classes of E;
- $[e]_{\sim_i} \leq_{\sim_i} [e']_{\sim_i}$ if and only if $e \leq_i e'$;
- $[e]_{\sim_i} \#_{\sim_i} [e']_{\sim_i}$ if and only if $e \#_i e'$;
- $\ell_{\sim_i}([e]_{\sim_i}) = \ell(e)$;

is a labelled event structure. Observe that Ev_0 is (isomorphic to) the labelled event structure *ES* we started from. Using the same notation as in Section 2.6, we denote by $(in, id)\colon ES \to Ev_0$ the isomorphism which sends e to $[e]_{\sim_0}$.

The interesting fact about the labelled event structures Ev_i is that they have no autoconcurrency. This fact plays a crucial role in establishing the adjunction we are seeking. We shall prove it by means of the following lemmas.

Lemma 2.7.7
Suppose that Ev_i has no autoconcurrency and that $e \sim_{i+1} e'$ and $e \not\sim_i e'$. Then, $\forall\, \bar{e} \in E$ such that $\ell(\bar{e}) = \ell(e) = \ell(e')$ we have $\bar{e} \leq_i e \Leftrightarrow \bar{e} \leq_i e'$.

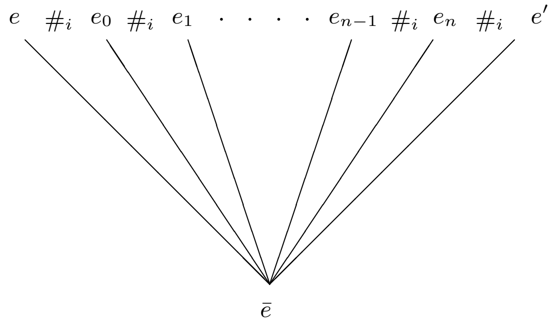
Proof. Since $e \not\sim_i e'$ and $e \sim_{i+1} e'$, we must have a chain

$$e \gtrless_i e_0 \gtrless_i e_1 \gtrless \cdots \gtrless_i e_{n-1} \gtrless_i e_n \gtrless_i e',$$

where the events adjacent to each other satisfy condition *(ii)* of the definition of $\sim_{i+1}$. Now, since Ev_i has no autoconcurrency, and since all the events in the chain have the same label, it must be

$$e \#_i e_0 \#_i e_1 \#_i \cdots \#_i e_{n-1} \#_i e_n \#_i e'.$$

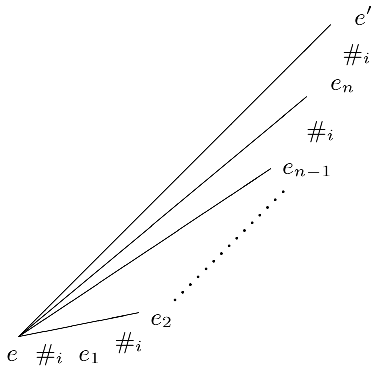
Now suppose that $\bar{e} \leq_i e'$. Then, it cannot be $e_n \leq_i \bar{e}$, otherwise we would have $e' \#_i e_n \leq_i \bar{e} \leq_i e'$ and so $e' \#_i e'$, which is impossible. Moreover, it cannot be $\bar{e} \#_i e_n$, otherwise, since $\bar{e} \in \lfloor e' \rfloor_{\leq_i} \setminus \{e'\}$ and $e_n \in \lfloor e_n \rfloor_{\leq_i}$, e' and e_n would not satisfy condition *(ii)*, as supposed. Finally, $\bar{e}$ and e_n cannot be concurrent, because they have the same label. Then, necessarily, it is $\bar{e} \leq_i e_n$. Therefore, repeating inductively the same argument for the other elements of the chain, we have $\bar{e} \leq_i e'$, $\bar{e} \leq_i e_n, \ldots, \bar{e} \leq_i e_0$, $\bar{e} \leq_i e$, as illustrated in the following picture.



Thus $\bar{e} \leq_i e$. A symmetric argument shows that if $\bar{e} \leq_i e$, then $\bar{e} \leq_i e'$. ✓

Lemma 2.7.8
Suppose that $e \sim_{i+1} e'$ and $e \not\sim_i e'$. Then, if Ev_i has no autoconcurrency, we have $e \not\leq_i e'$ and $e' \not\leq_i e$, and therefore, since $e \sim_{i+1} e'$ implies that $\ell(e) = \ell(e')$, and since Ev_i is in $\underline{\mathsf{LES}}_{\mathrm{I}}$, it follows that $e \#_i e'$.

Proof. Since $\ell(e) = \ell(e')$ we are in the hypothesis of the previous lemma. Then, if it were $e \leq_i e'$, it would be $e' \leq_i e$ and so $e \sim_i e'$, which is a contradiction. The same argument shows that it cannot be $e' \leq_i e$.



The above picture illustrates how, starting from the hypothesis $e \leq_i e'$ and applying the same argument exploited in the proof of Lemma 2.7.7, a contradiction would be reached, since it would be $e_1 \#_i e \leq_i e_1$. ✓

Lemma 2.7.9
For any $i \in \omega$, Ev_i is in $\underline{\mathsf{LES}}_{\mathrm{I}}$.

Proof. Since Ev_0 is (isomorphic to) ES, by hypothesis it has no autoconcurrency. Then, in order to show the claim, it is enough to show that if Ev_{i-1} has no autoconcurrency, then so does Ev_i.

Suppose that $[e]_{\sim_i} \not\leq_{\sim_i} [e']_{\sim_i}$ and $[e]_{\sim_i} \not\geq_{\sim_i} [e']_{\sim_i}$, and that $\ell(e) = \ell(e')$. Then, by definition, there exist $\underline{e} \in [e]_{\sim_i}$ and $\underline{e}' \in [e']_{\sim_i}$ such that

$$\begin{array}{lll} \forall\, \bar{e} \sim_i e & \bar{e} \not\leq_{i-1} \underline{e}', & \text{i.e.} \quad [\bar{e}]_{\sim_{i-1}} \not\leq_{\sim_{i-1}} [\underline{e}']_{\sim_{i-1}}; \\ \forall\, \bar{e}' \sim_i e' & \bar{e}' \not\leq_{i-1} \underline{e}, & \text{i.e.} \quad [\bar{e}']_{\sim_{i-1}} \not\leq_{\sim_{i-1}} [\underline{e}]_{\sim_{i-1}}. \end{array}$$

In particular, we have $[\underline{e}]_{\sim_{i-1}} \not\leq_{\sim_{i-1}} [\underline{e}']_{\sim_{i-1}}$ and $[\underline{e}]_{\sim_{i-1}} \not\geq_{\sim_{i-1}} [\underline{e}']_{\sim_{i-1}}$. Then, since $\ell(\underline{e}) = \ell(\underline{e}')$ and Ev_{i-1} has no autoconcurrency, it must be $[\underline{e}]_{\sim_{i-1}} \#_{\sim_{i-1}} [\underline{e}']_{\sim_{i-1}}$.

Consider now any $\bar{e} \sim_i e$ and $\bar{e}' \sim_i e'$. It must be either $\bar{e} \leq_{i-1} \bar{e}'$ or $\bar{e} \geq_{i-1} \bar{e}'$ or $\bar{e} \#_{i-1} \bar{e}'$. If $\bar{e} \sim_{i-1} \underline{e}$ and $\bar{e}' \sim_{i-1} \underline{e}'$ we obviously have $\bar{e} \#_{i-1} \bar{e}'$. Then, suppose $\bar{e} \sim_{i-1} \underline{e}$ and $\bar{e}' \not\sim_{i-1} \underline{e}'$. The first hypothesis prevents $\bar{e}$ to be below $\bar{e}'$, otherwise we would have $\bar{e} \leq_{i-1} \bar{e}' \leq_{i-1} \underline{e}'$, which is a contradiction. On the other hand, if it were $\bar{e}' \leq_{i-1} \bar{e}$, then, by Lemma 2.7.7, it would be $\bar{e}' \leq_{i-1} \underline{e}$, which again is impossible. Therefore, it must be $\bar{e} \#_{i-1} \bar{e}'$.

A symmetric argument shows that it must necessarily be $\bar{e} \#_{i-1} \bar{e}'$ whenever $\bar{e} \not\sim_{i-1} \underline{e}$ and $\bar{e}' \sim_{i-1} \underline{e}'$. Finally, two applications of Lemma 2.7.7 show that, if $\bar{e} \not\sim_{i-1} \underline{e}$ and $\bar{e}' \not\sim_{i-1} \underline{e}'$, once again we have $\bar{e} \#_{i-1} \bar{e}'$.

Thus, $\forall\, \bar{e} \sim_i e\ \forall\, \bar{e}' \sim_i e'$ it is $\bar{e} \#_{i-1} \bar{e}'$, i.e., $e \#_i e'$. Then, $[e]_{\sim_i} \#_{\sim_i} [e']_{\sim_i}$ and Ev_i is in $\underline{\mathsf{LES}}_{\mathrm{I}}$. ✓

As a consequence of the results just presented, we have that if ES has no autoconcurrency, then all the elements in any $\sim_i$ class are pairwise conflicting, i.e., if $e \neq e'$ and $e \sim_i e'$ for some $i \in \omega$, then $e \# e'$. In fact, if $e \sim_i e'$ and $e \neq e'$, then there exists j such that $e \not\sim_j e'$ and $e \sim_{j+1} e'$. Then, since Ev_j is in $\underline{\mathsf{LES}}_{\mathrm{I}}$, by Lemma 2.7.8, it is $e \# e'$.

Useful alternative characterizations of $\leq_i$ and $\#_i$ are given by the following crucial lemma.

LEMMA 2.7.10
Let $(\sim_i, \leq_i, \#_i)$, $i \in \omega$, be the sequence constructed from a labelled event structure without autoconcurrency ES as given above. Then

i) for any $i \in \omega$ and for any $j \leq i$, $\quad e \leq_i e' \Leftrightarrow \forall\, \bar{e}' \sim_i e'\ \exists\, \bar{e} \sim_i e.\ \bar{e} \leq_j \bar{e}'$;

ii) for any $i \in \omega$ and for any $j \leq i$, $\quad e \#_i e' \Leftrightarrow \forall\, \bar{e} \sim_i e\ \forall\, \bar{e}' \sim_i e'.\ \bar{e} \#_j \bar{e}'$.

Proof. We start by proving *(ii)* by induction on $i - j$. For $j = i$ the statement follows immediately from Lemmas 2.7.4 and 2.7.5. Otherwise, consider $\bar{e} \sim_i e$ and $\bar{e}' \sim_i e'$. Then, by induction hypothesis, we have $\bar{e} \#_{i-1} \bar{e}'$ if and only if $\forall\, \bar{\bar{e}} \sim_{i-1} \bar{e}$ and $\forall\, \bar{\bar{e}}' \sim_{i-1} \bar{e}'$ it is $\bar{\bar{e}} \#_j \bar{\bar{e}}'$. Then, the thesis follows immediately, since, by definition, $e \#_i e'$ if and only if $\forall\, \bar{e} \sim_i e$ and $\forall\, \bar{e}' \sim_i e'$ it is $\bar{e} \#_{i-1} \bar{e}'$, i.e., by induction, if and only if $\forall\, \bar{e} \sim_i e$ and $\forall\, \bar{e}' \sim_i e'$ we have that $\forall\, \bar{\bar{e}} \sim_{i-1} \bar{e}$ and $\forall\, \bar{\bar{e}}' \sim_{i-1} \bar{e}'$ it is $\bar{\bar{e}} \#_j \bar{\bar{e}}'$, which, in turn, happens if and only if $\forall\, \bar{\bar{e}} \sim_i e$ and $\forall\, \bar{\bar{e}}' \sim_i e'$ it is $\bar{\bar{e}} \#_j \bar{\bar{e}}'$.

Concerning point *(i)*, we first show the '$\Rightarrow$' implication by induction on $i-j$. If $j = i$, then the results follows directly from Lemma 2.7.4. Otherwise, suppose that $e \leq_i e'$. Then, by definition, $\forall\, \bar{e}' \sim_i e'$, $\exists\, \bar{e} \sim_i e$ such that $\bar{e} \leq_{i-1} \bar{e}'$. Thus, by induction hypotesis, we have that $\exists\, \bar{\bar{e}} \sim_{i-1} \bar{e}$ such that $\bar{\bar{e}} \leq_j \bar{e}'$. Clearly, we have $\bar{\bar{e}} \sim_i e$, whence the thesis.

Let us show the '$\Leftarrow$' implication, again by induction on $i - j$. Observe that the base case for $j = i$ has already been proved. Then, suppose that $\forall\, \bar{e}' \sim_i e'$, $\exists\, \bar{e} \sim_i e$ such that $\bar{e} \leq_j \bar{e}'$. We must show that $e \leq_i e'$. Exploiting the induction hypothesis, it is enough to show that $\forall\, \bar{e}' \sim_i e'$, $\exists\, \bar{e} \sim_i e$ such that $\bar{e} \leq_{j+1} \bar{e}'$, i.e., such that $\forall\, \bar{\bar{e}}' \sim_{j+1} \bar{e}'$, $\exists\, \bar{\bar{e}} \sim_{j+1} \bar{e}$ such that $\bar{\bar{e}} \leq_j \bar{\bar{e}}'$.

Thus, given $\bar{e}' \sim_i e'$, consider any $\bar{e} \sim_i e$ such that $\bar{e} \leq_j \bar{e}'$. By hypothesis, such $\bar{e}$ exists. Again by hypothesis, since $\bar{\bar{e}}' \sim_{j+1} \bar{e}'$ implies $\bar{\bar{e}}' \sim_i e'$, we know that $\exists\, \bar{\bar{e}} \sim_i e$, and thus $\bar{\bar{e}} \sim_i \bar{e}$, such that $\bar{\bar{e}} \leq_j \bar{\bar{e}}'$. Then, it is enough to show that, actually, $\bar{\bar{e}} \sim_{j+1} \bar{e}$. To this aim, suppose that $\bar{\bar{e}} \not\sim_{j+1} \bar{e}$. Then, by Lemma 2.7.8, $\bar{\bar{e}} \#_{j+1} \bar{e}$. The situation is summarized in the picture on the side.

$$\begin{array}{ccc} \bar{\bar{e}}' & \sim_{j+1} & \bar{e}' \\ \leq_j \Big| & & \Big| \leq_j \\ \bar{\bar{e}} & \#_{j+1} & \bar{e} \end{array}$$

Clearly, it must be $\bar{\bar{e}}' \not\sim_j \bar{e}'$. In fact, if it were $\bar{\bar{e}}' \sim_j \bar{e}'$, it would also be, by Lemma 2.7.4, $\bar{\bar{e}}' \leq_j \bar{e}'$, and the situation would be the one indicated in the figure on the side. Now, since $\bar{e} \#_j \bar{\bar{e}} \leq_j \bar{\bar{e}}' \leq_j \bar{e}'$, we have $\bar{e} \#_j \bar{e}'$ and then, since $\bar{e}' \#_j \bar{e} \leq_j \bar{e}'$, it would be $\bar{e}' \#_j \bar{e}'$, which is absurd.

$$\begin{array}{ccc} \bar{\bar{e}}' & \leq_j & \bar{e}' \\ \leq_j \Big| & & \Big| \leq_j \\ \bar{\bar{e}} & \#_j & \bar{e} \end{array}$$

Then, we must have a chain

$$\bar{\bar{e}}' \#_j e_0' \#_j e_1' \cdots \#_j e_{n-1}' \#_j e_n' \#_j \bar{e}'$$

whose elements are directly related by clause *(ii)* of the definition of $\sim_{j+1}$. Since $\bar{\bar{e}}' \sim_{j+1} e_0' \sim_{j+1} \cdots \sim_{j+1} e_n' \sim_{j+1} \bar{e}'$, then, by hypothesis, we can find $e_0, \ldots, e_n$ such that $e_k \leq_j e_k'$, for $k = 0, \ldots, n$ and such that $\bar{\bar{e}} \sim_i e_0 \sim_i \cdots \sim_i e_n \sim_i \bar{e}$, as shown in the following picture.

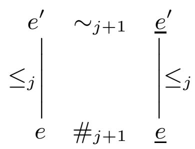
$$\begin{array}{ccccccc} \bar{\bar{e}}' & \#_j & e_0' & \cdots & e_n' & \#_j & \bar{e}' \\ \leq_j \Big| & & \Big| \leq_j & & \leq_j \Big| & & \Big| \leq_j \\ \bar{\bar{e}} & \sim_i & e_0 & \cdots & e_n & \sim_i & \bar{e} \end{array}$$

Now, it must be $\bar{\bar{e}} \sim_j e_0$, otherwise it would be $\bar{\bar{e}} \#_j e_0$ and thus $\lfloor \bar{\bar{e}} \rfloor_{\leq_j}$ and $\lfloor e_0 \rfloor_{\leq_j} \setminus \{e_0\}$ would not be conflict free. Then, inductively, for $k = 0, \ldots, n-1$ it is $e_k \sim_j e_{k+1}$ and finally $e_n \sim_j \bar{e}$. This contradicts the hypothesis that $\bar{\bar{e}} \not\sim_{j+1} \bar{e}$, and, therefore, concludes the proof. ✓

The next lemma shows that, although neither $\leq_i \subseteq \leq_{i+1}$ nor $\leq_i \supseteq \leq_{i+1}$, the "behaviour" of the sequence of preorders $\leq_i$, is not so bad as it could seem. This lemma is the last step we miss in order to see that the sequence $(\sim_i, \leq_i, \#_i)$ admits a limit.

LEMMA 2.7.11
If $e \leq_j e'$ and $e \not\leq_{j+1} e'$, then $\forall\, i > j.\ e \not\leq_i e'$.

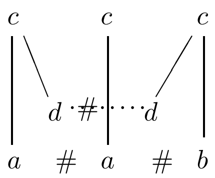
Proof. Since $e \not\leq_{j+1} e'$, there exists $\underline{e}' \sim_{j+1} e'$ such that for any $\bar{e} \sim_{j+1} e$ it is $\bar{e} \not\leq_j \underline{e}'$. Now, suppose that there exists $i > j$ such that $e \leq_i e'$. Then, there exists $\underline{e} \sim_i e$ such that $\underline{e} \leq_j \underline{e}'$, using the characterization of $\leq_i$ given in Lemma 2.7.10. Since $\underline{e} \leq_j \underline{e}'$, $\underline{e} \not\sim_{j+1} e$ and, therefore, $\underline{e} \#_{j+1} e$, we are in the following situation.



But, since $e \leq_i e'$, this is exactly the situation we already came across in the proof of Lemma 2.7.10 and which led us to a contradiction. Thus, i cannot exist. ✓

Observe that the analogous property for $\not\leq_i$ does *not* hold, i.e., it is possible that $e \not\leq_j e'$, $e \leq_{j+1} e'$ and $e \not\leq_i e'$ for $j+1 < i$. This is illustrated by the following example.

EXAMPLE 2.7.12
Consider the event structure ES in the picture.



As a convention, assume that events carrying the same label are distinguished by numbering them starting from the left. We have that $a_2 \not\leq_0 c_1$. Then, since $[a_2]_{\sim_1} = \{a_1, a_2\}$ and $[c_1]_{\sim_1} = \{c_1\}$, we have that $a_2 \leq_1 c_1$. However, since $[a_2]_{\sim_2} = [a_2]_{\sim_1}$ while $[c_1]_{\sim_2} = \{c_1, c_2, c_3\}$, it is $a_2 \not\leq_2 c_1$.

Now, consider the triple of relations $(\sim_\omega, \leq_\omega, \#_\omega)$, where

$$\sim_\omega = \bigcup_{i\in\omega} \sim_i, \qquad \leq_\omega = \bigcup_{i\in\omega} \bigcap_{j>i} \leq_i, \qquad \#_\omega = \bigcap_{i\in\omega} \#_i$$

or, equivalenty, $\leq_\omega$ is defined by

$$e \leq_\omega e' \quad \text{if and only if} \quad \exists\, k\ \forall\, i > k \quad e \leq_i e'.$$

Thanks to Lemma 2.7.11, it is immediate to show that $\leq_\omega$ enjoys the following relevant property.

Lemma 2.7.13
$e \not\leq_\omega e'$ if and only if $\exists\, k.\ \forall\, i > k \quad e \not\leq_i e'$.

Proof. By definition, $e \not\leq_\omega e'$ if and only if $\forall\, k\ \exists\, i > k.\ \ e \not\leq_i e'$. Then, the '$\Leftarrow$' implication is obviously true. Suppose that $e \not\leq_\omega e'$ and for $k = 0$ let i_0 be the first index such that $e \not\leq_{i_0} e'$. Now, if $\forall\, i > i_0$ it is $e \not\leq_i e'$ we are done. Otherwise let $k_0 > i_0$ be an index such that $e \leq_{k_0} e'$. Then, we can find $i_1 > k_0$ such that $e \not\leq_{i_1} e'$ and, from Lemma 2.7.11, we know that $\forall\, i > i_1$ it must be $e \not\leq_i e'$. ✓

The following characterization of $\leq_\omega$ derives easily from Lemma 2.7.10 and Lemma 2.7.13.

Lemma 2.7.14
For any $j \in \omega$, $e \leq_\omega e'$ if and only if $\forall\, \bar{e}' \sim_\omega e'\ \exists\, \bar{e} \sim_\omega e.\ \bar{e} \leq_j \bar{e}'$.

Proof. Suppose that $e \leq_\omega e'$. Then, there exists k such that for any $i > k$ it is $e \leq_i e'$. Then, chosen any j and exploiting Lemma 2.7.10, $\forall\, i > \max\{j, k\}\ \forall\, \bar{e}' \sim_i e'\ \exists\, \bar{e} \sim_i e.\ \bar{e} \leq_j \bar{e}'$, whence it immediately follows that $\forall\, \bar{e}' \sim_\omega e'\ \exists\, \bar{e} \sim_\omega e.\ \bar{e} \leq_j \bar{e}'$.

In order to show the inverse implication, we start by showing that, if $\forall\, \bar{e}' \sim_\omega e'\ \exists\, \bar{e} \sim_\omega e$ such that $\bar{e} \leq_j \bar{e}'$ holds for a particular j, then it holds for all $j \in \omega$. It is immediately clear from the definitions that

$$\forall\, \bar{e}' \sim_\omega e'\ \exists\, \bar{e} \sim_\omega e.\ \ \bar{e} \leq_j \bar{e}' \quad \text{implies} \quad \forall\, \bar{e}' \sim_\omega e'\ \exists\, \bar{e} \sim_\omega e.\ \ \bar{e} \leq_{j-1} \bar{e}'.$$

Therefore, it is enough to show—with a proof similar to the one of Lemma 2.7.10—that

$$\forall\, \bar{e}' \sim_\omega e'\ \exists\, \bar{e} \sim_\omega e.\ \ \bar{e} \leq_j \bar{e}' \quad \text{implies} \quad \forall\, \bar{e}' \sim_\omega e'\ \exists\, \bar{e} \sim_\omega e.\ \ \bar{e} \leq_{j+1} \bar{e}'.$$

Consider $\bar{e}' \sim_\omega \bar{e}$. Then, we know that $\exists\, \bar{e} \sim_\omega e$ such that $\bar{e} \leq_j \bar{e}'$. Now, for any $\bar{\bar{e}}' \sim_{j+1} \bar{e}'$, again from the hypothesis, we can find $\bar{\bar{e}} \sim_\omega e \sim_\omega \bar{e}$ such that $\bar{\bar{e}} \leq_j \bar{\bar{e}}'$. Now, assuming that $\bar{\bar{e}}\ \#_{j+1}\ \bar{e}$ and reasoning as in the proof of Lemma 2.7.10 would lead to a contradiction. Therefore, it must be $\bar{\bar{e}} \sim_{j+1} \bar{e}$. Then, we have shown that $\forall\, \bar{\bar{e}}' \sim_{j+1} \bar{e}'\ \exists\, \bar{\bar{e}} \sim_{j-1} \bar{e}$ such that $\bar{\bar{e}} \leq_j \bar{\bar{e}}'$, i.e., $\bar{e} \leq_{j+1} \bar{e}'$, and thus we have that $\forall\, \bar{e}' \sim_\omega e'\ \exists\, \bar{e} \sim_\omega e.\ \ \bar{e} \leq_{j+1} \bar{e}'$.

To conclude the proof suppose that $\forall\, \bar{e}' \sim_\omega e'\ \exists\, \bar{e} \sim_\omega e$ such that $\bar{e} \leq_j \bar{e}'$. Then, in particular $\forall\, j \in \omega\ \exists\, \bar{e}_j \sim_\omega e$ such that $\bar{e}_j \leq_j e'$. We claim that the sequence of the $\bar{e}_j$'s cannot be such that $\forall\, k\ \exists\, j > k.\ \ \bar{e}_k \not\sim_j \bar{e}_j$. In fact, if this were the case, then we could build the following subsequence $\bar{e}_{j_n}$, $n \in \omega$, of the $\bar{e}_j$'s.

Let j_0 and k_0 be 0, and let j_1 be the least index greater than k_0 such that $\bar{e}_{j_1} \not\sim_{j_1} \bar{e}_{j_0}$. Observe that j_1 exists by hypothesis. Moreover, let k_1 be the least integer (greater that j_1) such that $\bar{e}_{j_1} \sim_{k_1} \bar{e}_{j_0}$, which exists since $\bar{e}_{j_0} \sim_\omega e \sim_\omega \bar{e}_{j_1}$.

Inductively, $\bar{e}_{j_n}$ is chosen to be the event in $\{\bar{e}_j \mid j \in \omega\}$ whose index j_n is the least index greater than k_{n-1} such that $\bar{e}_{j_n} \not\sim_{j_n} \bar{e}_{j_{n-1}}$. Then, k_n is the least integer (greater than j_{n-1}) such that $\bar{e}_{j_n} \sim_{k_n} \bar{e}_{j_{n-1}}$.

It is immediate to understand that the sequence $\bar{e}_{j_n}$, $n \in \omega$, is built in such a way that

$$\begin{array}{rll} i) & \forall n\, \forall m, m' < n, & \bar{e}_{j_m} \sim_{j_n} \bar{e}_{j_{m'}}; \\ ii) & \forall n\, \forall m < n, & \bar{e}_{j_n} \not\sim_{j_n} \bar{e}_{j_m}. \end{array}$$

In fact, since j_n is greater than k_{n-1}, we have that $\bar{e}_{j_{n-1}} \sim_{j_n} \bar{e}_{j_{n-2}}$ and thus, inductively, $\forall\, m, m' < n$ it is $\bar{e}_{j_m} \sim_{j_n} \bar{e}_{j_{m'}}$. Therefore, if it were $\bar{e}_{j_n} \sim_{j_n} \bar{e}_{j_m}$ for some $m < n$, it would be $\bar{e}_{j_n} \sim_{j_n} \bar{e}_{j_{n-1}}$, which is impossible by construction.

Thus, we are in the situation illustrated by the following picture.

$$\begin{array}{ccccccccccc} e' & & e' & & e' & & e' & & e' & & \cdots\cdots \\ \Big| \leq_{j_0} & & \Big| \leq_{j_1} & & \Big| \leq_{j_2} & & \Big| \leq_{j_3} & & \Big| \leq_{j_4} & & \\ \bar{e}_{j_0} & \not\sim_{j_1} & \bar{e}_{j_1} & \not\sim_{j_2} & \bar{e}_{j_2} & \not\sim_{j_3} & \bar{e}_{j_3} & \not\sim_{j_4} & \bar{e}_{j_4} & & \cdots\cdots \end{array}$$

It is now easy to show that this situation implies that e' has infinitely many pre-events in the event structure ES or, equivalenty, in Ev_0. In fact, by Lemma 2.7.10, since $\bar{e}_{j_0} \leq_{j_0} e'$, there exists $\bar{\bar{e}}_{j_0} \sim_{j_0} \bar{e}_{j_0}$ such that $\bar{\bar{e}}_{j_0} \leq e'$. (Here we are confusing $\leq$ and $\leq_0$.) For the same reason, there exists $\bar{\bar{e}}_{j_1} \sim_{j_1} \bar{e}_{j_1}$ such that $\bar{\bar{e}}_{j_1} \leq e'$. Of course, it must be $\bar{\bar{e}}_{j_0} \not\sim_{j_0} \bar{\bar{e}}_{j_1}$, otherwise it would be $\bar{e}_{j_1} \sim_{j_1} \bar{e}_{j_0}$, which is impossible. It follows that $\bar{\bar{e}}_{j_0} \neq \bar{\bar{e}}_{j_1}$. In general, for any $n \in \omega$, it must exist $\bar{\bar{e}}_{j_n} \sim_{j_n} \bar{e}_{j_n}$ such that $\bar{\bar{e}}_{j_n} \leq e'$, and such that, for any $m < n$, $\bar{\bar{e}}_{j_n} \not\sim_{j_{n-1}} \bar{\bar{e}}_{j_m}$, since this last condition would imply $\bar{e}_{j_n} \sim_{j_n} \bar{e}_{j_m}$. We conclude that, therefore, for any $m < n$, $\bar{\bar{e}}_{j_n} \neq \bar{\bar{e}}_{j_m}$.

It follows from the previous discussion that, if $\forall\, k\ \exists\, j > k.\ \ \bar{e}_k \not\sim_j \bar{e}_j$, then we have that $\{e \leq e' \mid e' \in E_{ES}\} \supseteq \{\bar{\bar{e}}_{j_n} \mid n \in \omega\}$ is infinite. But this is absurd, since ES is an event structure.

Thus, we can assume that $\exists\, k.\ \ \forall\, i > k\ \bar{e}_k \sim_i \bar{e}_i$. But this means that $\forall\, i > k$ we have $\bar{e}_k \leq_i \bar{e}_i \leq_i e'$. Then, since $\exists\, k'$ such that $\bar{e}_k \sim_{k'} e$, we have that $\forall\, i > \max\{k, k'\}$ $e \leq_i \bar{e}_k \leq_i e'$, i.e., $e \leq_\omega e'$. ✓

The next three lemmas are the equivalent of Lemma 2.7.4, Lemma 2.7.5 and Lemma 2.7.6 for the triple $(\sim_\omega, \leq_\omega, \#_\omega)$.

LEMMA 2.7.15
Relation $\leq_\omega$ is a preorder such that

$$e \leq_\omega e' \text{ and } e' \leq_\omega e \quad \text{if and only if} \quad e \sim_\omega e'.$$

Proof. That $\leq_\omega$ is a preorder is easily shown from the definition, exploiting the fact that, for any $i \in \omega$, $\leq_i$ is a preorder.

Now suppose that $e \leq_\omega e'$ and $e' \leq_\omega e$. Then, there exists i such that $e \leq_i e'$ and $e' \leq_i e$. Thus, by Lemma 2.7.4, $e \sim_i e'$ whence $e \sim_\omega e'$. On the contrary, if $e \sim_\omega e'$, then there exists k such that $\forall\, i > k\ e \sim_i e'$. Then, again by the same lemma, for any $i > k$, $e \leq_i e'$ and $e' \leq_i e$, i.e., $e \leq_\omega e'$ and $e' \leq_\omega e$. ✓

Lemma 2.7.16
Relation $\#_\omega$ is symmetric, irreflexive and such that

$$e \;\#_\omega\; e' \leq_\omega e'' \quad \textit{implies} \quad e \;\#_\omega\; e''.$$

Proof. Since for any $i \in \omega$ $\#_i$ is symmetric and irreflexive and since $\#_\omega = \bigcap_{i\in\omega} \#_i$, the first part of the claim is immediate.

Suppose that $e \;\#_\omega\; e'$ and $e' \leq_\omega e''$. By definition of $\#_\omega$, for any $i \in \omega$ we have $e \;\#_i\; e'$. Moreover, there exists k such that $\forall\, i > k$ it is $e' \leq_i e''$. Thus, $\forall\, i > k$ we have $e \;\#_i\; e'$ and $e' \leq_i e''$, i.e., by Lemma 2.7.5, $e \;\#_i\; e''$. Then, since $\#_i \subseteq \#_{i-1}$ we have $e \;\#_i\; e''$ for any $i \in \omega$, i.e., $e \;\#_\omega\; e''$. ✓

Lemma 2.7.17
For any $e \in E$, the set $\left\{ [e']_{\sim_\omega} \;\middle|\; e' \leq_\omega e,\ e' \in E \right\}$ is finite.

Proof. If $[e']_{\sim_\omega} \leq_{\sim_\omega} [e]_{\sim_\omega}$, then there exists $\bar{e}' \sim_\omega e'$ such that $\bar{e}' \leq_0 e$. Moreover, if $[e']_{\sim_\omega} \neq [e]_{\sim_\omega}$, then clearly $\bar{e}' \neq e$. Hence, if we have infinitely many events in $\left\{ [e']_{\sim_\omega} \;\middle|\; e' \leq_\omega e \right\}$, we also have infinitely many events in $\left\{ [e']_{\sim_0} \;\middle|\; e' \leq_0 e \right\}$, which, by Lemma 2.7.6, is impossible. ✓

Then, we can define a labelled event structure Ev_ω out of the triple $(\sim_\omega, \leq_\omega, \#_\omega)$ exactly as for the Ev_i's: Ev_ω is the event structure $(E/{\sim_\omega}, \leq_{\sim_\omega}, \#_{\sim_\omega}, \ell_{\sim_\omega}, L)$, where

- $E/{\sim_\omega}$ is the set of $\sim_\omega$-classes of E;
- $[e]_{\sim_\omega} \leq_{\sim_\omega} [e']_{\sim_\omega}$ if and only if $e \leq_\omega e'$;
- $[e]_{\sim_\omega} \;\#_{\sim_\omega}\; [e']_{\sim_\omega}$ if and only if $e \;\#_\omega\; e'$;
- $\ell_{\sim_\omega}([e]_{\sim_\omega}) = \ell(e)$.

Similarly to the case of the sequence $TSys_i$, $i \in \omega$, presented in Section 2.6, event structures Ev_i are related to each other by *inclusion* morphisms. For $i \in \omega \setminus \{0\}$, let $in_i \colon E/{\sim_{i-1}} \to E/{\sim_i}$ be the function such that $in_i\Big([e]_{\sim_{i-1}}\Big) = [e]_{\sim_i}$. Then we have the following.

LEMMA 2.7.18
For any $i \in \omega \setminus \{0\}$, $(in_i, id)\colon Ev_{i-1} \to Ev_i$ is a labelled event structure morphism.

Proof. Axiom *(iii)* of Definition 2.3.1 of labelled event structure morphisms trivially holds for (in_i, id_i). Let us check axioms *(i)* and *(ii)*.

i) $in_i\Big(\Big\lfloor [e]_{\sim_{i-1}} \Big\rfloor_{\leq_{i-1}}\Big) \subseteq \Big\lfloor in_i\Big([e]_{\sim_{i-1}}\Big)\Big\rfloor_{\leq_i}$, i.e., for any $[e']_{\sim_i} \leq_{\sim_i} [e]_{\sim_i}$ there exists $[\bar{e}']_{\sim_{i-1}} \leq_{\sim_{i-1}} [e]_{\sim_{i-1}}$ such that $[\bar{e}']_{\sim_i} = [e']_{\sim_i}$. But this is immediate because $e' \leq_i e$ implies, by definition of $\leq_i$, that there exists $\bar{e}' \sim_i e'$ such that $\bar{e}' \leq_{i-1} e$.

ii) $in_i\Big([e]_{\sim_{i-1}}\Big) \curlyvee in_i\Big([e']_{\sim_{i-1}}\Big)$ implies $[e]_{\sim_{i-1}} \curlyvee [e']_{\sim_{i-1}}$, i.e., $[e]_{\sim_i} \curlyvee [e']_{\sim_i}$ implies $[e]_{\sim_{i-1}} \curlyvee [e']_{\sim_{i-1}}$. Clearly, by definition, $[e]_{\sim_i} \#_{\sim_i} [e']_{\sim_i}$ implies $[e]_{\sim_{i-1}} \#_{\sim_{i-1}} [e']_{\sim_{i-1}}$.
Now suppose that $[e]_{\sim_i} = [e']_{\sim_i}$ but $[e]_{\sim_{i-1}} \neq [e']_{\sim_{i-1}}$. Then, we have $e \sim_i e'$ but $e \not\sim_{i-1} e'$. Then, by Lemma 2.7.8, we have $e \#_{i-1} e'$, i.e., $[e]_{\sim_{i-1}} \#_{\sim_{i-1}} [e']_{\sim_{i-1}}$. ✓

Next, we shall show that Ev_ω is the colimit of the $\boldsymbol{\omega}$-diagram formed by the Ev_i's. For any $i \in \omega$, consider the mapping $in_i^\omega\colon E/{\sim_i} \to E/{\sim_\omega}$ which, for any $e \in E$, sends $[e]_{\sim_i}$ to $[e]_{\sim_\omega}$.

LEMMA 2.7.19
For any $i \in \omega$, $(in_i^\omega, id)\colon Ev_i \to Ev_\omega$ is a labelled event structure morphism

Proof. The proof is formally identical to that of Lemma 2.7.18. With respect to Definition 2.3.1, property *(i)*, i.e., $in_i^\omega\Big(\Big\lfloor [e]_{\sim_i} \Big\rfloor_{\leq_i}\Big) \subseteq \Big\lfloor in_i^\omega\Big([e]_{\sim_i}\Big)\Big\rfloor_{\leq_\omega}$, derives immediately from Lemma 2.7.10; property *(ii)*, i.e., $in_i^\omega\Big([e]_{\sim_i}\Big) \curlyvee in_i^\omega\Big([e']_{\sim_i}\Big)$ implies $[e]_{\sim_i} \curlyvee [e']_{\sim_i}$ derives immediately from Lemma 2.7.8; and property *(iii)* is trivially true. ✓

PROPOSITION 2.7.20
Ev_ω is the colimit in $\underline{\mathsf{LES}}_{\mathrm{I}}$ of the $\boldsymbol{\omega}$-diagram

$$\mathcal{D} = Ev_0 \stackrel{(in_1,id)}{\longrightarrow} Ev_1 \stackrel{(in_2,id)}{\longrightarrow} \cdots \stackrel{(in_{i-1},id)}{\longrightarrow} Ev_i \stackrel{(in_i,id)}{\longrightarrow} \cdots$$

Proof. Since for any $1 < j < i$ it is $in_j^\omega = in_i^\omega \circ in_i \circ \cdots \circ in_{j+1}$, then $\{(in_i^\omega, id)\colon Ev_i \to Ev_\omega \mid i \in \omega\}$ is a cocone with base $\mathcal{D}$.

Consider now any other cocone $\{(\eta_i, \lambda_i)\colon Ev_i \to ES \mid i \in \omega\}$ for $\mathcal{D}$, ES being any object in $\underline{\mathsf{LES}}_{\mathrm{I}}$. Since for any i it is $(\eta_i, \lambda_i) = (\eta_{i+1}, \lambda_{i+1}) \circ (in_{i+1}, id)$, it must necessarily be $\lambda_i = \lambda_0 = \lambda$ and $\eta_i\Big([e]_{\sim_i}\Big) = \eta_{i+1}\Big([e]_{\sim_{i+1}}\Big)$, for any i. Thus, we can define $\bar{\eta}: E/\sim_\omega \to E_{ES}$ by $\bar{\eta}\Big([e]_{\sim_\omega}\Big) = \eta_0\Big([e]_{\sim_0}\Big)$. Clearly, we have that

$$\bar{\eta}\Big([e]_{\sim_\omega}\Big) = \eta_0\Big([e]_{\sim_0}\Big) = \eta_i\Big([e]_{\sim_i}\Big),$$

i.e., for any i, $\eta_i = \bar{\eta} \circ in_i^\omega$, and, moreover, $\bar{\eta}$ is clearly the unique mapping for which that happens. Thus, to conclude the proof, we only miss to show that $(\bar{\eta}, \lambda): Ev_\omega \to ES$ is a labelled event structure morphism.

Again, axiom *(iii)* of Definition 2.3.1 trivially holds by definition. Let us show axioms *(i)* and *(ii)*.

i) Consider $\bar{e} \in E_{ES}$ such that $\bar{e} \le \bar{\eta}\Big([e]_{\sim_\omega}\Big)$. Then, by definition, for each i, $\bar{e} \le \eta_i\Big([e]_{\sim_i}\Big)$ and, since (η_i, λ_i) is a morphism, for any i there exists $\bar{\bar{e}}_i \in E$ such that $\bar{\bar{e}}_i \le_i e$ and $\eta_i\Big([\bar{\bar{e}}_i]_{\sim_i}\Big) = \bar{e}$. Thus, we are in the same situation we came across while proving Lemma 2.7.14. And in fact, reasoning as in that proof, we have that there can be only finitely many $\bar{\bar{e}}_i$ which are not in the relation $\sim_\omega$, i.e., there must be a k such that $\bar{\bar{e}}_k \le_i e$ for any $i > k$. Then, $\bar{\bar{e}}_k \le_\omega e$ and $\bar{\eta}\Big([\bar{\bar{e}}_k]_{\sim_\omega}\Big) = \eta_k\Big([\bar{\bar{e}}_k]_{\sim_k}\Big) = \bar{e}$.

ii) Suppose that $\bar{\eta}\Big([e]_{\sim_\omega}\Big) \veebar \bar{\eta}\Big([e']_{\sim_\omega}\Big)$. Then, clearly, for any $i \in \omega$ we have that $\eta_i\Big([e]_{\sim_i}\Big) \veebar \eta_i\Big([e']_{\sim_i}\Big)$. Now, if there exists i such that $[e]_{\sim_i} = [e']_{\sim_i}$, then $[e]_{\sim_\omega} = [e']_{\sim_\omega}$. Otherwise, $[e]_{\sim_i} \#_{\sim_i} [e']_{\sim_i}$ for any $i \in \omega$, i.e., $[e]_{\sim_\omega} \#_{\sim_\omega} [e']_{\sim_\omega}$. In both cases, $[e]_{\sim_\omega} \veebar [e']_{\sim_\omega}$ and we are done. ✓

Exploiting the characterizations of $\le_\omega$ previously given, it is not difficult to show the following.

Lemma 2.7.21
Ev_ω is deterministic.

Proof. Consider a configuration c_ω of Ev_ω and two events $[e]_{\sim_\omega} \neq [e']_{\sim_\omega}$ enabled at c_ω, i.e., such that $c_\omega \vdash [e]_{\sim_\omega}$ and $c_\omega \vdash [e']_{\sim_\omega}$. We shall show that there exist i and a configuration c_i of Ev_i in which $[e]_{\sim_i}$ and $[e']_{\sim_i}$ are enabled. Since $[e]_{\sim_{i+1}} \neq [e']_{\sim_{i+1}}$, which follows from the fact that $[e]_{\sim_\omega} \neq [e']_{\sim_\omega}$, and since $[e]_{\sim_i} \lneqq\!\!\!\!\gneqq_i [e']_{\sim_i}$, which follows from the fact that both are enabled at c_i, it must necessarily be

$$\ell_{\sim_\omega}\Big([e]_{\sim_\omega}\Big) = \ell_{\sim_i}\Big([e]_{\sim_i}\Big) \neq \ell_{\sim_i}\Big([e']_{\sim_i}\Big) = \ell_{\sim_\omega}\Big([e']_{\sim_\omega}\Big),$$

which shows that Ev_ω is deterministic.

Suppose that $[\bar{e}]_{\sim_\omega} \le_{\sim_\omega} [e]_{\sim_\omega}$. Then, there exists k such that $\forall\, j > k\ [\bar{e}]_{\sim_j} \le_{\sim_j} [e]_{\sim_j}$. Since the set $\Big\lfloor [e]_{\sim_\omega} \Big\rfloor_{\le_\omega}$ is finite, we can find $\bar{k}'$ such that for any $j > \bar{k}'$ $[\bar{e}]_{\sim_\omega} \le_{\sim_\omega}$

$[e]_{\sim_\omega}$ implies $[\bar{e}]_{\sim_j} \leq_{\sim_j} [e]_{\sim_j}$. If otherwise $[\bar{e}]_{\sim_\omega} \not\leq_{\sim_\omega} [e]_{\sim_\omega}$ and $[\bar{e}]_{\sim_{\bar{k}'}} \leq_{\sim_{\bar{k}'}} [e]_{\sim_{\bar{k}'}}$, there exists $j > k$ such that $[\bar{e}]_{\sim_j} \not\leq_{\sim_j} [e]_{\sim_j}$. Now, observe that there can be only finitely many such $[\bar{e}]_{\sim_k}$. In fact, if it were not, reasoning as done in previous proofs, and exploiting Lemmas 2.7.10 and 2.7.13, we would easily derive a contradiction showing that e has infinitely many pre-events in Ev_0. Then, we can find $\bar{k}''$ such that for any $j > \bar{k}''$

$$[\bar{e}]_{\sim_\omega} \leq_{\sim_\omega} [e]_{\sim_\omega} \quad \Leftrightarrow \quad [\bar{e}]_{\sim_j} \leq_{\sim_j} [e]_{\sim_j}.$$

In the same way, we can find $\bar{\bar{k}}''$ such that for any $j > \bar{\bar{k}}''$, $[\bar{e}]_{\sim_\omega} \leq_{\sim_\omega} [e']_{\sim_\omega}$ if and only if $[\bar{e}]_{\sim_j} \leq_{\sim_j} [e']_{\sim_j}$. Thus, considering $\bar{k} = \max\{\bar{k}'', \bar{\bar{k}}''\}$ we have that for any $i > \bar{k}$

$$[\bar{e}]_{\sim_\omega} \in \Big\lfloor [e]_{\sim_\omega} \Big\rfloor_{\leq_\omega} \quad \Leftrightarrow \quad [\bar{e}]_{\sim_i} \in \Big\lfloor [e]_{\sim_i} \Big\rfloor_{\leq_i}$$

and

$$[\bar{e}]_{\sim_\omega} \in \Big\lfloor [e']_{\sim_\omega} \Big\rfloor_{\leq_\omega} \quad \Leftrightarrow \quad [\bar{e}]_{\sim_i} \in \Big\lfloor [e']_{\sim_i} \Big\rfloor_{\leq_i}.$$

Now, consider $[\bar{e}]_{\sim_{\bar{k}}} \leq_{\bar{k}} [e]_{\sim_{\bar{k}}}$ and $[\bar{e}']_{\sim_{\bar{k}}} \leq_{\bar{k}} [e']_{\sim_{\bar{k}}}$. Clearly, it is still possible that $[\bar{e}]_{\sim_{\bar{k}}} \#_{\sim_{\bar{k}}} [\bar{e}']_{\sim_{\bar{k}}}$. However, since $[\bar{e}]_{\sim_\omega} \not\#_{\sim_\omega} [\bar{e}']_{\sim_\omega}$, it must exist k such that for any $i > k$ $[\bar{e}]_{\sim_i} \not\#_{\sim_i} [\bar{e}']_{\sim_i}$. Then, since there can be only finitely many such pairs, we can find an integer i (greater than $\bar{k}$) such that the set $\Big\lfloor [e]_{\sim_i} \Big\rfloor_{\leq_i} \cup \Big\lfloor [e']_{\sim_i} \Big\rfloor_{\leq_i}$ is conflict free (wrt. to $\#_{\sim_i}$).

It is immediate now to see that

$$c_i = \left(\Big\lfloor [e]_{\sim_i} \Big\rfloor_{\leq_i} \setminus \Big\{ [e]_{\sim_i} \Big\} \right) \cup \left(\Big\lfloor [e']_{\sim_i} \Big\rfloor_{\leq_i} \setminus \Big\{ [e']_{\sim_i} \Big\} \right)$$

is a configuration of Ev_i which enables $[e]_{\sim_i}$ and $[e']_{\sim_i}$. ✓

As it was probably clear from a while, the object component of the functor $dles\colon \underline{\mathsf{LES}}_{\mathrm{I}} \to \underline{\mathsf{dLES}}$ is the function which maps a labelled event structure without autoconcurrency ES to the deterministic event structure Ev_ω limit of the sequence of event structures Ev_i built from it.

An example of the construction is given in Figure 2.4, in which we also show how the construction of the deterministic transition system with independence works on the transition system of configurations of ES. Now, the last step we miss is to show that $dles$ can be extended to a functor which is left adjoint to the inclusion functor $\underline{\mathsf{dLES}} \hookrightarrow \underline{\mathsf{LES}}_{\mathrm{I}}$. This is done in the following proposition.

PROPOSITION 2.7.22 *($(in_0^\omega \circ in, id)\colon ES \to dles(ES)$ is universal)*
For any labelled event structure without autoconcurrency ES, any deterministic labelled event structure DES and any $(\eta, \lambda)\colon ES \to DES$, there exists a unique

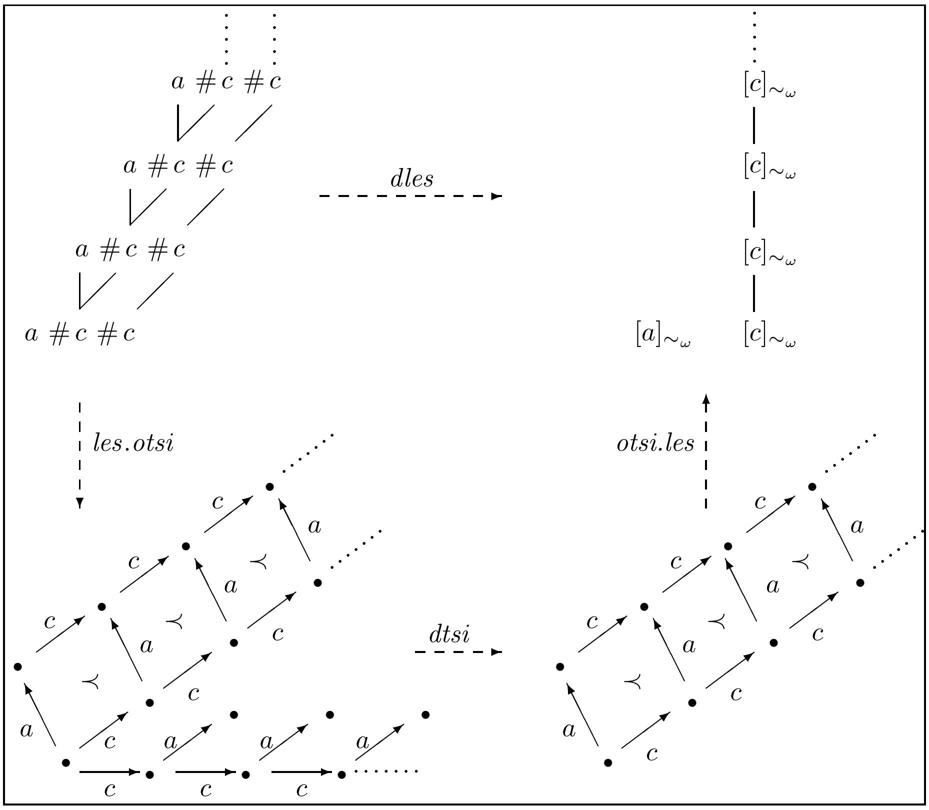
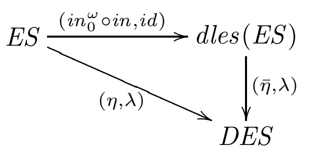


Figure 2.4: An event structure ES and $dles(ES)$

$(\bar{\eta}, \lambda)$: $dles(ES) \to DES$ *in* $\underline{\mathsf{dLES}}$ *such that* $(\bar{\eta}, \lambda) \circ (in_0^\omega \circ in, id) = (\eta, \lambda)$.



Proof. Suppose for a while that we are able to show that (η, λ): $ES \to DES$ gives rise to a cocone $\{(\eta_i, \lambda): Ev_i \to DES \mid i \in \omega\}$ with base $\mathcal{D}$ such that $\eta = \eta_0 \circ in$, (in, id) being the isomorphism of ES and Ev_0. Then, by Proposition 2.7.20, there exists a unique $(\bar{\eta}, \lambda)$: $dles(ES) \to DES$ such that, for any i, $\eta_i = \bar{\eta} \circ in_i^\omega$. Then, in particular,

$\eta_0 = \bar{\eta} \circ in_0^\omega$, and thus

$$(\eta, \lambda) = (\eta_0 \circ in, \lambda) = (\bar{\eta}, \lambda) \circ (in_0^\omega \circ in, id).$$

In other words, $\bar{\eta}: E_{ES}/\!\sim_\omega \to E_{DES}$ given, as in the proof of Proposition 2.7.20, by $\bar{\eta}\Big([e]_{\sim_\omega}\Big) = \eta(e)$ would be such that $(\bar{\eta}, \lambda)$ makes the diagram commute.

The hypothesis above is sufficient also to show the uniqueness of $(\bar{\eta}, \lambda)$. Suppose in fact that there exists $\bar{\bar{\eta}}$ such that $\eta = \bar{\bar{\eta}} \circ (in_0^\omega \circ in)$, then it is $\eta_0 = \bar{\bar{\eta}} \circ in_0^\omega$. It follows that, for any i,

$$\begin{aligned} \eta_i\Big([e]_{\sim_i}\Big) &= (\bar{\eta} \circ in_i^\omega)\Big([e]_{\sim_i}\Big) = (\bar{\eta} \circ in_0^\omega)\Big([e]_{\sim_0}\Big) \\ &= (\bar{\bar{\eta}} \circ in_0^\omega)\Big([e]_{\sim_0}\Big) = (\bar{\bar{\eta}} \circ in_i^\omega)\Big([e]_{\sim_i}\Big), \end{aligned}$$

i.e., for any i it is $\eta_i = \bar{\bar{\eta}} \circ in_i^\omega$. Then, by definition of colimit, it is $\bar{\bar{\eta}} = \bar{\eta}$.

Thus, we only need to show that the cocone $\{(\eta_i, \lambda): Ev_i \to DES \mid i \in \omega\}$ actually exists. However, by defining for any integer i the mapping $\eta_i: E_{ES}/\!\sim_i \to E_{DES}$ by $\eta_i\Big([e]_{\sim_i}\Big) = \eta(e)$, we obviously have a cocone $\{(\eta_i, \lambda) \mid i \in \omega\}$ as required. So, we just need to prove that (η_i, λ) is a well-defined labelled event structure morphism from Ev_i to DES. We shall do it by induction on i.

For $i = 0$ everything is fine, since (η_0, λ) is the composition of two morphisms, namely $(\eta, \lambda) \circ (in^{-1}, id)$. So, assuming that (η_i, λ) is a well-defined morphism from Ev_i to DES, let us show that $(\eta_{i+1}, \lambda): Ev_{i+1} \to DES$ is such.

To show that η_{i+1} is well-defined, it is enough to show that $\eta(e) = \eta(e')$ whenever e and e' are related directly from condition *(ii)* of the definition of $\sim_{i+1}$. Since $\sim_{i+1}$ is an equivalence generated by that unique rule starting from $\sim_i$, the other cases follow immediately using the inductive hypothesis. Thus, it is $e \lneqq\!\!\!\!\gneqq_i e'$, $\ell(e) = \ell(e')$ and

$$\lfloor e \rfloor_{\leq_i} \setminus \{e\} \;\#_i\; \lfloor e' \rfloor_{\leq_i} \quad \text{and} \quad \lfloor e \rfloor_{\leq_i} \;\#_i\; \lfloor e' \rfloor_{\leq_i} \setminus \{e'\}.$$

First observe that $\eta\!\downarrow\! e$ if and only if $\lambda\!\downarrow\!\ell(e)$ if and only if $\lambda\!\downarrow\!\ell(e')$ if and only if $\eta\!\downarrow\! e'$. Thus, η_{i+1} is defined or not on a class $[e]_{\sim_{i+1}}$ irrespectively of the representative chosen. Suppose then that η is defined on e. Thus η_i is defined on both $[e]_{\sim_i}$ and $[e']_{\sim_i}$. By general properties of event structure morphisms,

$$\eta_i\Big(\Big(\lfloor e \rfloor_{\leq_i} \setminus \{e\}\Big) \cup \Big(\lfloor e' \rfloor_{\leq_i} \setminus \{e'\}\Big)\Big)$$

is a configuration of DES which enables both $\eta_i\Big([e]_{\sim_i}\Big)$ and $\eta_i\Big([e']_{\sim_i}\Big)$. Then, since those two events have the same label in DES, which is deterministic, it follows that $\eta_i\Big([e]_{\sim_i}\Big) = \eta_i\Big([e']_{\sim_i}\Big)$.

Let us show now that (η_{i+1}, λ) is a morphism.

i) $\Big\lfloor \eta_{i+1}\Big([e]_{\sim_{i+1}}\Big)\Big\rfloor_{\leq} \subseteq \eta_{i+1}\Big(\Big\lfloor [e]_{\sim_{i+1}} \Big\rfloor_{\leq_{\sim_{i+1}}}\Big)$.

Consider $\underline{e} \leq \eta_{i+1}\Big([e]_{\sim_{i+1}}\Big)$. Then, for any $e' \sim_{i+1} e$, we have that $\eta_i\Big([e']_{\sim_i}\Big) = \eta_{i+1}\Big([e]_{\sim_{i+1}}\Big)$. Thus, we have $\underline{e} \leq \eta_i\Big([e']_{\sim_i}\Big)$ and, by induction, there exists $\bar{e}' \leq_i e'$ such that $\eta_i\Big([\bar{e}]_{\sim_i}\Big) = \underline{e}$. In the following, let $\bar{x}$ denote the event identified in such a way in correspondence of a generic event x. Observe that, since by induction (η_i, λ) is a morphism, for any $e' \sim_{i+1} e$ it is $[\bar{e}]_{\sim_i} \vee\!\!\vee [\bar{e}']_{\sim_i}$. Consider now $e' \sim_{i+1} e$. Then, there exists a chain

$$e' \#_i e_0 \#_i \cdots \#_i e_n \#_i e,$$

where the elements adjacent to each other satisfies condition *(ii)* of the definition of $\sim_{i+1}$. Thus we are led to the following situation.

$$\begin{array}{ccccccccc}
e' & \#_i & e_0 & \#_i & \cdots & \#_i & e_n & \#_i & e \\
\Big|\leq_i & & \Big|\leq_i & & & & \Big|\leq_i & & \Big|\leq_i \\
\bar{e}' & \vee\!\!\vee & \bar{e}_0 & \vee\!\!\vee & \cdots & \vee\!\!\vee & \bar{e}_n & \vee\!\!\vee & \bar{e}
\end{array}$$

However, since by hypothesis it cannot be $[\bar{e}']_{\sim_i} \#_i [\bar{e}_0]_{\sim_i}$, we must conclude that $[\bar{e}']_{\sim_i} = [\bar{e}_0]_{\sim_i}$ and, inductively, that $\bar{e}' \sim_i \bar{e}$. Summing up, there exists $\bar{e}$ such that $\forall\, e' \sim_{i+1} e \;\exists\, \bar{e}' \sim_{i+1} \bar{e}.\; \bar{e}' \leq_i e'$, i.e., $\bar{e} \leq_{i+1} e$, and such that $\eta_{i+1}\Big([\bar{e}]_{\sim_{i+1}}\Big) = \underline{e}$.

ii) $\eta_{i+1}\Big([e]_{\sim_{i+1}}\Big) \vee\!\!\vee \eta_{i+1}\Big([e]_{\sim_{i+1}}\Big)$ implies $[e]_{\sim_{i+1}} \vee\!\!\vee [e]_{\sim_{i+1}}$.

For any $[\bar{e}]_{\sim_i} \subseteq [e]_{\sim_{i+1}}$ and any $[\bar{e}']_{\sim_i} \subseteq [e']_{\sim_{i+1}}$ we have

$$\eta_i\Big([\bar{e}]_{\sim_i}\Big) = \eta_{i+1}\Big([e]_{\sim_{i+1}}\Big) \vee\!\!\vee \eta_{i+1}\Big([e']_{\sim_{i+1}}\Big) = \eta_i\Big([\bar{e}']_{\sim_i}\Big)$$

Then, by induction, $[\bar{e}]_{\sim_i} \vee\!\!\vee [\bar{e}']_{\sim_i}$. Thus, if $[\bar{e}]_{\sim_i} = [\bar{e}']_{\sim_i}$ for any such $\bar{e}$ and $\bar{e}'$, it is $[\bar{e}]_{\sim_{i+1}} = [\bar{e}']_{\sim_{i+1}}$, i.e., $[e]_{\sim_{i+1}} = [e]_{\sim_{i+1}}$. Otherwise, $\forall\, \bar{e} \sim_{i+1} e \;\forall\, \bar{e}' \sim_{i+1} e'$ it is $\bar{e} \#_i \bar{e}'$. Then, by Lemma 2.7.10, we have $e \#_{i+1} e'$. In both cases, $[e]_{\sim_{i+1}} \vee\!\!\vee [e]_{\sim_{i+1}}$.

iii) $\ell_{DES}\Big(\eta_i\Big([e]_{\sim_{i+1}}\Big)\Big) = \lambda\Big(\ell_{\sim_{i+1}}\Big([e]_{\sim_{i+1}}\Big)\Big)$.

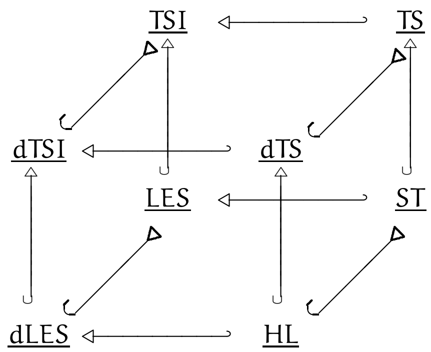
Trivial, since $\ell_{DES}\Big(\eta(e)\Big) = \lambda\Big(\ell_{ES}(e)\Big)$. ✓

As usual, the universality of $(in_0^\omega \circ in, id)$ allows us to conclude what follows.

Corollary 2.7.23 *(dles $\dashv \hookrightarrow$)*
The mapping dles extends to a functor which is left adjoint of the inclusion of <u>dLES</u> *in* $\underline{\mathsf{LES}}_{\mathrm{I}}$. *Then,* $\langle dles, \hookrightarrow\rangle$ *is a reflection.*

The coreflection <u>dLES</u> $\hookrightarrow\!\!\lhd$ <u>LES</u> closes the last two faces of the cube. So, our results may be summed up in the following cube of relationships among models.

THEOREM 2.7.24 *(The Cube)*



Conclusions

This chapter established a complete "cube" of formal relationships between well-known and (a few) new models for concurrency. Thus, we have a complete picture of how to translate between these models via adjunctions along the axes of "interleaving/noninterleaving", "linear/branching" and "behaviuor/system". Notice also the pleasant conformity in the picture, with *coreflections* along the "interleaving/noninterleaving" and "behaviour/system" axes, and *reflections* along "linear/branching".

It should be mentioned that not all squares (surfaces) of the "cube" commute. Of course, they do with directions along those of the embeddings.

It is worth remarking that all the adjunctions in this chapter would still hold if we modified uniformly the morphisms of the involved categories by eliminating the label component. However, if we considered only total morphisms, the reflections $\underline{\mathsf{dTSI}} \hookrightarrow\!\!\!\lhd \underline{\mathsf{TSI}}$ and $\underline{\mathsf{dLES}} \hookrightarrow\!\!\!\lhd \underline{\mathsf{LES}}$ would not exist.

Chapter 3

Infinite Computations

ABSTRACT. For any cardinal $\aleph$, there exists a KZ-doctrine on the 2-category of the locally small categories whose algebras are exactly the categories which admits all the colimits indexed by chains of cardinality not greater than $\aleph$. The chapter presents a wide survey of this topic.

In addition, we show that this inductive completion KZ-doctrine lifts smoothly to KZ-doctrines on (many variations of) the 2-categories of monoidal and symmetric monoidal categories, thus yielding, in particular, a universal construction of colimits of ω-chains in those categories. Since the processes of Petri nets may be axiomatized in terms of symmetric monoidal categories this result provides a universal construction of the algebra of infinite processes of a Petri net.

So that we may say that the door is now opened . . .
to a new method . . . which in future years
will command the attention of other minds.

Galileo Galilei

Tra questa immensità s'annega il pensier mio:
e il naufragar mi è dolce in questo mare.

Giacomo Leopardi

L'infinito è drasticamente diverso
da tutte le cose finite.

Antonino Zichichi, L'infinito

This chapter is based on joint work with José Meseguer and Ugo Montanari [123].

Introduction

The idea of completing a mathematical stucture by adding to it some desirable limit “points” is indeed a very natural one and it arises in many different fields of mathematics, and in particular topology and partial order theory. Since categories are a generalization of the notion of partial orders, the issue of completing categories for a given class of limits or colimits arose rather early in the development of the theory (see [79] and references therein).

As far as computer science is concerned, the theory of complete partial orders and associated completion techniques have assumed great relevance since the pioneering work on semantics by D. Scott [128, 133]. In the last few years, however, many computing systems have been given a semantics through the medium of category theory, the general pattern being to look at objects as representing states and at arrows as representing computations. It is therefore natural to expect that the theory of *cocompletion of categories* may play an interesting role in this kind of semantics. The main purpose of this chapter is to illustrate how this theory fits well with the issue of infinite computations and, therefore, to make it more easily available to the computer science community. In a sense, by viewing categories as generalized posets, this view of infinite computations is very natural and indeed generalizes to categories similar constructions for adding limits to posets. We motivate this further in terms of processes of Petri nets in Section 3.1.

Petri nets [109] are probably the most clear examplification of the categorical semantics pattern discussed above. They are unanimously considered among the most representative *models for concurrency*, since they are a fairly simple and natural model of *concurrent* and *distributed* computation. Recent works [97, 16, 121, 122] have shown that the semantics of Petri nets can be understood in terms of *symmetric monoidal categories*—where objects are states, arrows processes, and the tensor product and the arrow composition model respectively the operations of parallel and sequential composition of processes. This yields an axiomatization of the causal behaviour of nets as an *essentially algebraic theory*. However, when modeling perpetual systems, describing finite processes is not enough: we need to consider also *infinite behaviours*. We remark that our interest here resides on *processes* of Petri nets, i.e., on structures able to describe concurrent computations taking into account causality. More precisely, we aim at defining an *algebra* of net computations which includes *infinite processes* as well. To the best of our knowledge, this issue is still completely unexplored.

Although we are mainly interested in considering colimits of ω-chains, we shall present the theory of the cocompletion for filtered colimits of cardinality not greater than $\aleph$, where $\aleph$ is an infinite cardinal. In particular, since $\aleph$-filtered cocompleteness is equivalent to the cocompleteness by colimits taken over chains of cardinality not greater that $\aleph$, for $\aleph = \omega$ we have the ω-cocompletion as instantiation of the general theory. More precisely, for any infinite cardinal $\aleph$, we define a *Kock-Zöberlain (KZ-)doctrine* $\mathtt{Ind}(\_)_\aleph$ on the 2-category of locally small categories such that the categories admitting filtered colimits of cardinality $\aleph$ are the *algebras* [76, 149] for the doctrine, while the $\mathtt{Ind}$–homomorphisms are the functors which preserve those colimits. This result has already appeared in several different forms in the literature, e.g. [71, 43] and the related [76, 79, 28, 149, 150, 137, 56, 58]. However, we aim at giving a simple and *complete* tutorial, which integrates the best features of the existing approaches and explores the application of these ideas to computer science.

Concerning the organization of this chapter, in Section 3.2 we recall that, for any *small* category $\underline{\mathsf{C}}$, the category $\underline{\mathsf{Set}}^{\underline{\mathsf{C}}^{\mathrm{op}}}$ of all presheaves on $\underline{\mathsf{C}}$ may be considered the "free" cocompletion of $\underline{\mathsf{C}}$ under all small colimits. This suggests immediately a strategy for identifying the cocompletion of $\underline{\mathsf{C}}$ for $\aleph$-filtered colimits, i.e., to look for an appropriate subcategory of $\underline{\mathsf{Set}}^{\underline{\mathsf{C}}^{\mathrm{op}}}$. Moreover, this approach generalizes also to *locally small* categories. This part of the theory is illustrated in Section 3.3. Then Section 3.5 gives the functors $\mathtt{Ind}(\_)_\aleph$ bulding on the theory of KZ-doctrines, whose definitions and basic results are given in Section 3.4.

Getting back to our starting motivations about the process semantics of Petri nets, in Section 3.6 we study the application of the doctrine to symmetric monoidal categories. Precisely, we show that the KZ-doctrine $\mathtt{Ind}(\_)_\aleph$ lifts to KZ-doctrines respectively on any of the 2-categories of monoidal categories appearing in Table 3.1. This, from the technical point of view, is the main original contribution of the chapter. Finally, in Section 3.7, we discuss how this result generalizes the algebraic approach to the process semantics of Petri nets discussed in detail in Chapter 1 to the case in which infinite processes and composition operations on them are considered. In particular, the infinite processes of a Petri net can in this way be given an algebraic presentation which combines the *essentially algebraic* presentation of monoidal categories with the *monadic* presentation of their completion in terms of KZ-doctrines. A further link with Chapter 1, and in particular with the results of Section 1.9 about the relationships between process, unfolding, and algebraic views of net semantics is provided by showing that the arrows of the cocompletion of the category of decorated concatenable processes of N coincides with the configurations of the event structure associated to N by the unfolding semantics.

However, the correspondence between infinite net processes obtained via the cocompletion doctrine and the algebraic theory of net processes is not as "precise" as one would like. In fact, in addition to the symmetric monoidality, the categories of processes of Petri nets satisfy further axioms which are, in general, not preserved by $\mathtt{Ind}(\_)_\aleph$. Therefore, although the arrows of the cocomplete category correspond precisely to infinite computations, they do not enjoy the global structural properties of finite net processes, i.e., $\mathtt{Ind}(\_)_\aleph$ does not restrict to an endofunctor on the category of categories of net processes. It is still an open problem whether a more satisfactory solution to this problem can be found.

		small		locally small	
		monoidal	strict monoidal	monoidal	strict monoidal
STRICT	non symmetric	$\underline{\underline{\mathsf{MonCat}}}$	$\underline{\underline{\mathsf{sMonCat}}}$	$\underline{\underline{\mathsf{MonCAT}}}$	$\underline{\underline{\mathsf{sMonCAT}}}$
	symmetric	$\underline{\underline{\mathsf{SMonCat}}}$	$\underline{\underline{\mathsf{SsMonCat}}}$	$\underline{\underline{\mathsf{SMonCAT}}}$	$\underline{\underline{\mathsf{SsMonCAT}}}$
	strictly symmetric	$\underline{\underline{\mathsf{sSMonCat}}}$	$\underline{\underline{\mathsf{sSsMonCat}}}$	$\underline{\underline{\mathsf{sSMonCAT}}}$	$\underline{\underline{\mathsf{sSsMonCAT}}}$
STRONG	non symmetric	$\underline{\underline{\mathsf{MonCat}}}^{\star}$	$\underline{\underline{\mathsf{sMonCat}}}^{\star}$	$\underline{\underline{\mathsf{MonCAT}}}^{\star}$	$\underline{\underline{\mathsf{sMonCAT}}}^{\star}$
	symmetric	$\underline{\underline{\mathsf{SMonCat}}}^{\star}$	$\underline{\underline{\mathsf{SsMonCat}}}^{\star}$	$\underline{\underline{\mathsf{SMonCAT}}}^{\star}$	$\underline{\underline{\mathsf{SsMonCAT}}}^{\star}$
	strictly symmetric	$\underline{\underline{\mathsf{sSMonCat}}}^{\star}$	$\underline{\underline{\mathsf{sSsMonCat}}}^{\star}$	$\underline{\underline{\mathsf{sSMonCAT}}}^{\star}$	$\underline{\underline{\mathsf{sSsMonCAT}}}^{\star}$
MONOIDAL	non symmetric	$\underline{\underline{\mathsf{MonCat}}}^{\star\star}$	$\underline{\underline{\mathsf{sMonCat}}}^{\star\star}$	$\underline{\underline{\mathsf{MonCAT}}}^{\star\star}$	$\underline{\underline{\mathsf{sMonCAT}}}^{\star\star}$
	symmetric	$\underline{\underline{\mathsf{SMonCat}}}^{\star\star}$	$\underline{\underline{\mathsf{SsMonCat}}}^{\star\star}$	$\underline{\underline{\mathsf{SMonCAT}}}^{\star\star}$	$\underline{\underline{\mathsf{SsMonCAT}}}^{\star\star}$
	strictly symmetric	$\underline{\underline{\mathsf{sSMonCat}}}^{\star\star}$	$\underline{\underline{\mathsf{sSsMonCat}}}^{\star\star}$	$\underline{\underline{\mathsf{sSMonCAT}}}^{\star\star}$	$\underline{\underline{\mathsf{sSsMonCAT}}}^{\star\star}$

LEGENDA: The data in the definition of monoidal categories and functors (see Section A.2 for the relevant definitions) give rise to many combinations according to whether the monoidality and the symmetry are strict or not and so on. To fix notation, we propose the nomenclature above. The idea is that, since we consider the categories with *strict* monoidal functors as the "normal" categories, we explicitly indicate with simple and double superscripted $\star$'s the categories with, respectively, *strong* monoidal functors and simply *monoidal* functors. This is indicated by the leftmost column in the table. Clearly, the categories of symmetric monoidal categories consists always of *symmetric* monoidal functors. Moreover, sS means *strictly symmetric* while sMon means *monoidal strict*. We distinguish between categories of locally small and of small categories by using uppercase letters in the first case. Of course, there is an analogous table for the categories above considered as one-dimensional categories. We use a single underline in order to distinguish the two situations.

Table 3.1: A nomenclature for categories of monoidal categories

3.1 Motivations from Net Theory

Building on the formalization of the processes of a net N as a monoidal category, it looks conceptually very simple to describe the infinite computations of N. However, in order to appreciate the discussion in this section, it is not necessary to know in detail what Petri nets are. Indeed, the relevant facts are, as already stressed in the introduction, that we have a category $\underline{\mathsf{C}}$ whose objects represent the states and whose arrows represent the finite processes of a computational device, say a net N. To make the situation more interesting, we assume that, in addition, there is a notion of parallel composition of transitions, expressed by the fact that $\underline{\mathsf{C}}$ is a *monoidal category*. Then, since arrows in the category $\underline{\mathsf{C}}$, are *finite* processes, and since we understand infinite computations as "limits" of countable sequences of finite processes, we can think of them as sequences of arrows in $\underline{\mathsf{C}}$, i.e., $\underline{\mathsf{C}}$-valued, ω-shaped diagrams

$$c_0 \xrightarrow{f_0} c_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} c_n \xrightarrow{f_n} c_{n+1} \xrightarrow{f_{n+1}} \cdots$$

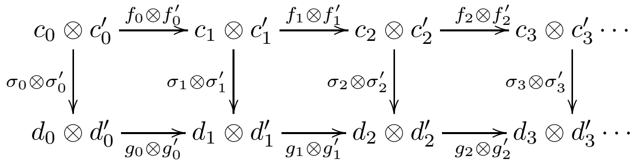
which are exactly the functors from the partial ordered category $\underline{\omega} = \{0 < 1 < 2 < 3 < \cdots\}$ to $\underline{\mathsf{C}}$.

However, this is only part of the story, actually the easiest. First of all, we are not interested in a set-theoretic treatment of infinite processes, but rather in their categorical description. In other terms, we aim at extending our category $\underline{\mathsf{C}}$ to a larger category whose objects represent states and whose arrows represent the processes of N, including the infinite ones. Secondly, but not less importantly, we want to preserve for infinite computations the view already available for categories of finite net computations as models of (essentially) algebraic theories.

Changing the viewpoint, one may look at an ω-diagram F in $\underline{\mathsf{C}}$ as a "formal state", rather than as a computation, namely the state reached by (the computation represented by) F. Then, a tentative solution which immediately arises is provided by $\underline{\mathsf{C}}^{\underline{\omega}}$, the category of functors from $\underline{\omega}$ to $\underline{\mathsf{C}}$ and natural transformations. The tensor product $\otimes$ on $\underline{\mathsf{C}}$ is easily lifted to $\underline{\mathsf{C}}^{\underline{\omega}}$ by defining

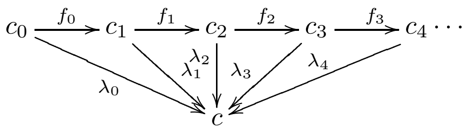
$$\begin{array}{ccc}
\underline{\mathsf{C}}^{\underline{\omega}} \times \underline{\mathsf{C}}^{\underline{\omega}} & \xrightarrow{\quad\tilde{\otimes}\quad} & \underline{\mathsf{C}}^{\underline{\omega}} \\
(F, F') & \longmapsto & \otimes \circ \langle F, F' \rangle \\
{\scriptstyle (\sigma,\sigma')}\big\downarrow & & \big\downarrow{\scriptstyle \sigma \tilde{\otimes} \sigma'} \\
(G, G') & \longmapsto & \otimes \circ \langle G, G' \rangle
\end{array}$$

where $\langle \_, \_ \rangle$ is the pairing of functors induced by the product $\underline{\mathsf{C}} \times \underline{\mathsf{C}}$ and $\tilde{\otimes}$ acts on the natural transformations σ and σ' componentwise. The tensor $\tilde{\otimes}$ is exemplified in the diagram below.



However, it is easily realized that the functor category $\underline{C}^{\underline{\omega}}$ is quite removed from the category we are looking for. For example, for any non-identity arrow f in $\underline{C}$, there are infinitely many $F \in \underline{C}^{\underline{\omega}}$ such that $F(j < j+1) = f$, for some $j \in \omega$, while for any $i \neq j$, $F(i < i+1)$ is an identity arrow. Although in our intended interpretation all these functors clearly represent the same computation, viz. f, they are distinct in $\underline{C}^{\underline{\omega}}$ and, even worse, they are not necessarily isomorphic to each other, which is the very least one would desire. Of course, a way out of this problem could be to construct a suitable quotient of $\underline{C}^{\underline{\omega}}$, or, more in the spirit of category theory, to make some appropriate arrows be isomorphisms; otherwise said, the notion of morphism for the category we search for is not at all self-evident.

Another conceptual approach to the issue of infinite computations which lies fully in the categorical framework is to exploit the notion of colimit. Suppose that we can "complete" $\underline{C}$ by adding suitable objects and arrows so that we can ensure that every ω-diagram in the completed category has a colimit. Then, in particular, for every sequence $c_0 \xrightarrow{f_0} c_1 \xrightarrow{f_1} \cdots$ of processes of N, we have a *unique* (up to isomorphism) object c and a cocone



where, by definition, $f_i; \lambda_{i+1} = \lambda_i$ for any $i \in \omega$. Then, it follows immediately that the arrow $\lambda_0 : c_0 \to c$ represents the infinite computation $c_0 \xrightarrow{f_0} c_1 \xrightarrow{f_1} \cdots$.

In the following, we shall give (some different representations of) the *"free"* (in a lax sense to be explained later) completion of $\underline{C}$ by $\underline{\omega}$-colimits. Moreover, Section 3.6 will clarify how the two, seemingly different, approaches discussed in this section can be reconciled. Although we shall not achieve a representation of the category of infinite computations of N (or equivalently a free cocompletion of $\underline{C}$) where "just the needed points" and "nothing else" is added, nevertheless, all the desired infinite computations will be represented faithfully, and all the computations which are "intuitively" the same will have isomorphic colimits in the completed category. Also, $\underline{C}$ will be embedded (fully and faithfully) by means of a *strict* monoidal functor in it. We start working by analogy with the well know case of the completion of $\underline{C}$

by arbitrary small colimits; the next section recalls this result, while the following ones exploit the same techniques to our case.

<u>REMARK</u>. Concerning foundational issues, we assume as usual the existence of a fixed *universe* $\mathfrak{U}$ of *small* sets [55, 43] upon which *small* and *locally small* categories are built [89]. A category is small if the collection of its arrows form a small set, i.e., it belongs to $\mathfrak{U}$; it is locally small if the collection of arrows between any two objects of the category is a small set (see [24, 84] for smoother choices about foundations of category theory; a brief discussion of the topic is provided in Appendix A.2).

<u>NOTATION</u>. In the following, $\underline{\mathsf{Set}}$, $\underline{\mathsf{Cat}}$ and $\underline{\mathsf{CAT}}$ are, respectively, the category of small sets and functions, the category of small categories and functors and the category of locally small categories and functors. Concerning notation, we shall use a double underlying to denote a 2-category. Thus, $\underline{\underline{\mathsf{Cat}}}$ and $\underline{\underline{\mathsf{CAT}}}$ are the 2-categories corresponding to $\underline{\mathsf{Cat}}$ and $\underline{\mathsf{CAT}}$. Apart from the *large* categories above, and unless differently specified, in the following $\underline{\mathsf{C}}$ stands for a generic locally small category. We denote indifferently by juxtaposition and by $\_ \circ \_$ the composition of functors, while the composition of arrows is always written as $\_\circ\_$, except in the categories of net processes where, in order to emphasize the fact that it represents sequentialization, we write composition as $\_;\_$ and we use the (left to right) diagrammatic order. Finally, we shall preferably denote homsets in $\underline{\mathsf{C}}$ by $\mathsf{Hom}_{\underline{\mathsf{C}}}(a,b)$. However, since this notation can easily become heavy, we shall occasionally write $\underline{\mathsf{C}}[a,b]$.

3.2 Presheaf Categories as Free Cocompletions

Given a locally small $\underline{\mathsf{C}}$, a *presheaf* on $\underline{\mathsf{C}}$ is a contravariant functor $P\colon \underline{\mathsf{C}}^{\mathrm{op}} \to \underline{\mathsf{Set}}$. The (not necessarily locally small) category $\underline{\mathsf{Set}}^{\underline{\mathsf{C}}^{\mathrm{op}}}$ is the category of all presheaves on $\underline{\mathsf{C}}$.

We remind the reader that $\underline{\mathsf{C}}$ is embedded fully and faithfully in $\underline{\mathsf{Set}}^{\underline{\mathsf{C}}^{\mathrm{op}}}$ via the *Yoneda embedding* Y defined as follows. To any $c \in \underline{\mathsf{C}}$ we associate the presheaf $\mathsf{Y}(c) = \mathsf{Hom}_{\underline{\mathsf{C}}}(\_, c)$, often denoted by $\mathfrak{h}_c$. This is the presheaf which associates to $d \in \underline{\mathsf{C}}$ the set $\mathsf{Hom}_{\underline{\mathsf{C}}}(d,c)$ and to $f\colon d' \to d$ in $\underline{\mathsf{C}}$ the function $(\_ \circ f)\colon \mathsf{Hom}_{\underline{\mathsf{C}}}(d,c) \to \mathsf{Hom}_{\underline{\mathsf{C}}}(d',c)$, as in the diagram below.

$$
\begin{array}{ccc}
\underline{\mathsf{C}}^{\mathrm{op}} & \xrightarrow{\ \mathsf{Y}(c)\ } & \underline{\mathsf{Set}} \\
d & \longmapsto & \mathsf{Hom}_{\underline{\mathsf{C}}}(d,c) \\
{\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle (\_\circ f)} \\
d' & \longmapsto & \mathsf{Hom}_{\underline{\mathsf{C}}}(d',c)
\end{array}
$$

Now, Y can be extended to the arrows of $\underline{\mathsf{C}}$ by mapping $f\colon c \to c'$ to the (constant) natural transformation $(f \circ \_)\colon \mathsf{Y}(c) \stackrel{\cdot}{\to} \mathsf{Y}(c')$. It is very easy to see that this definition makes Y into a (covariant) functor from $\underline{\mathsf{C}}$ to $\underline{\mathsf{Set}}^{\underline{\mathsf{C}}^{\mathrm{op}}}$.

$$\begin{array}{ccc}
\underline{\mathsf{C}} & \xrightarrow{\quad \mathsf{Y} \quad} & \underline{\mathsf{Set}}^{\underline{\mathsf{C}}^{\mathrm{op}}} \\
c & \longmapsto & \mathsf{Hom}_{\underline{\mathsf{C}}}(_, c) = \mathsf{Y}(c) \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle (f\circ_)} \\
c' & \longmapsto & \mathsf{Hom}_{\underline{\mathsf{C}}}(_, c') = \mathsf{Y}(c')
\end{array}$$

Functors of the form $\mathsf{Y}(c)$, i.e., those set-valued contravariant functors on $\underline{\mathsf{C}}$ isomorphic to $\mathsf{Y}(c)$ for some $c \in \underline{\mathsf{C}}$, are very important for at least two reasons. Firstly, they represent faithfully the category $\underline{\mathsf{C}}$, and secondly, if $\underline{\mathsf{C}}$ is small, they *generate* via small colimits all the others presheaves in $\underline{\mathsf{Set}}^{\underline{\mathsf{C}}^{\mathrm{op}}}$. They are called *representable functors*.

Lemma 3.2.1 *(Yoneda Lemma)*
For any $P \in \underline{\mathsf{Set}}^{\underline{\mathsf{C}}^{\mathrm{op}}}$ we have that $\mathsf{Hom}_{\underline{\mathsf{Set}}^{\underline{\mathsf{C}}^{\mathrm{op}}}}(\mathsf{Y}(c), P) \cong P(c)$ via the natural isomorphism θ which sends $\sigma\colon \mathsf{Y}(c) \stackrel{\cdot}{\to} P$ to $\sigma_c(id_c)$.

In other words, there is a natural isomorphism between the natural transformations from $\mathsf{Y}(c)$ to P and the elements of $P(c)$.

Corollary 3.2.2 *(Yoneda Embedding)*
The functor $\mathsf{Y}\colon \underline{\mathsf{C}} \to \underline{\mathsf{Set}}^{\underline{\mathsf{C}}^{\mathrm{op}}}$ is full and faithful. Thus, Y determines an equivalence between $\underline{\mathsf{C}}$ and its replete image in $\underline{\mathsf{Set}}^{\underline{\mathsf{C}}^{\mathrm{op}}}$, i.e., between $\underline{\mathsf{C}}$ and the full subcategory of $\underline{\mathsf{Set}}^{\underline{\mathsf{C}}^{\mathrm{op}}}$ consisting of the representable functors.

Proof. Immediate: $\underline{\mathsf{Set}}^{\underline{\mathsf{C}}^{\mathrm{op}}}[\mathsf{Y}(c), \mathsf{Y}(c')] \cong \mathsf{Y}(c')(c) = \underline{\mathsf{C}}[c, c']$. ✓

There is also a contravariant version of Yoneda's embedding $\mathsf{Y}'\colon \underline{\mathsf{C}}^{op} \to \underline{\mathsf{Set}}^{\underline{\mathsf{C}}}$ defined as follows:

$$\begin{array}{ccc}
\underline{\mathsf{C}}^{op} & \xrightarrow{\quad \mathsf{Y}' \quad} & \underline{\mathsf{Set}}^{\underline{\mathsf{C}}} \\
c & \longmapsto & \mathsf{Hom}_{\underline{\mathsf{C}}}(c, _) = \mathsf{Y}'(c) \\
{\scriptstyle f}\downarrow & & \uparrow{\scriptstyle (_\circ f)} \\
c' & \longmapsto & \mathsf{Hom}_{\underline{\mathsf{C}}}(c', _) = \mathsf{Y}'(c')
\end{array}$$

Y' is dual to Y; in particular there is a version of Yoneda's Lemma which says that, for each $c \in \underline{\mathsf{C}}$ and for each $P \in \underline{\mathsf{Set}}^{\underline{\mathsf{C}}}$, it is

$$\underline{\mathsf{Set}}^{\underline{\mathsf{C}}}[\mathsf{Hom}_{\underline{\mathsf{C}}}(c, \_), P] \cong P(c).$$

It is worthwhile to recall that Y preserves limits and Y' preserves colimits, i.e., for any $\mathsf{F}\colon \mathsf{J} \to \underline{\mathsf{C}}$, if F has a limit, one has $\mathsf{Y}(\varprojlim \mathsf{F}) \cong \varprojlim(\mathsf{YF})$, and if F has a colimit one has $\mathsf{Y}'(\varinjlim \mathsf{F}) \cong \varprojlim(\mathsf{Y}'\mathsf{F})$. By rewriting these formulas in terms of the explicit definitions of Y and Y', we have

$$\mathsf{Hom}_{\underline{\mathsf{C}}}(\_, \varprojlim \mathsf{F}) \cong \varprojlim_{\mathsf{J}} \mathsf{Hom}_{\underline{\mathsf{C}}}(\_, \mathsf{F}j) \quad \text{and} \quad \mathsf{Hom}_{\underline{\mathsf{C}}}(\varinjlim \mathsf{F}, \_) \cong \varprojlim_{\mathsf{J}} \mathsf{Hom}_{\underline{\mathsf{C}}}(\mathsf{F}j, \_),$$

which say that the functor $\mathsf{Hom}_{\underline{\mathsf{C}}}(\_,\_)\colon \underline{\mathsf{C}}^{op} \times \underline{\mathsf{C}} \to \underline{\mathsf{Set}}$ is "continuous" in the second argument and "cocontinuous" in the first.

Definition 3.2.3 *(The Category of Elements of P)*
Let $\underline{\mathsf{C}}$ be a locally small category. Given $P \in \underline{\mathsf{Set}}^{\underline{\mathsf{C}}^{\mathrm{op}}}$, the category of elements of P, $\int_{\underline{\mathsf{C}}} P$, has objects the pairs (c,p) with $c \in \underline{\mathsf{C}}$ and $p \in P(c)$, and arrows $u\colon (c',p') \to (c,p)$ if $u\colon c \to c'$ in $\underline{\mathsf{C}}$ and $P(u)(p) = p'$.

Since, thanks to Yoneda's lemma, we can identify c and $\mathsf{Y}(c)$, an equivalent description of $\int_{\underline{\mathsf{C}}} P$ is obtained by taking the objects to be pairs $(\mathsf{Y}(c), p)$, where p is a natural transformation from $\mathsf{Y}(c)$ to P, and the arrows from $(\mathsf{Y}(c'), p') \to (\mathsf{Y}(c), p)$ to be natural transformations $u\colon \mathsf{Y}(c') \stackrel{.}{\to} \mathsf{Y}(c)$, i.e., an arrow $f\colon c' \to c$, such that $p \circ u = p'$. The reader will have already recognized this as being the (comma) category of the representable functors over P in $\underline{\mathsf{Set}}^{\underline{\mathsf{C}}^{\mathrm{op}}}$. Throughout the chapter we shall often switch between these two descriptions. Notice that the category of elements of P is locally small, respectively small, when $\underline{\mathsf{C}}$ locally small, respectively small.

Observe that the projection on the first component $\pi_P\colon \int_{\underline{\mathsf{C}}} P \to \underline{\mathsf{C}}$ defined by the diagram below is a functor.

$$\begin{array}{ccc}
\int_{\underline{\mathsf{C}}} P & \xrightarrow{\pi_P} & \underline{\mathsf{C}} \\
(c',p') & \longmapsto & c' \\
\Big\downarrow u & & \Big\downarrow u \\
(c,p) & \longmapsto & c
\end{array}$$

Let $\underline{\mathsf{C}}$ be a small category. In the following we shall see that every presheaf on $\underline{\mathsf{C}}$ is a colimit of representables in a canonical way. In other words, there is a canonical way to associate to $P \in \underline{\mathsf{Set}}^{\underline{\mathsf{C}}^{\mathrm{op}}}$ a small diagram $D\colon \mathsf{J} \to \underline{\mathsf{C}}$ such that $P \cong \varinjlim(\mathsf{Y} \circ D)$. In more complex, but equivalent terms, in view of Corollary 3.2.2, one could say that $\underline{\mathsf{C}}$ is *dense* in $\underline{\mathsf{Set}}^{\underline{\mathsf{C}}^{\mathrm{op}}}$ or that the identity on $\underline{\mathsf{Set}}^{\underline{\mathsf{C}}^{\mathrm{op}}}$ is the left

Kan extension of the Y along Y itself. The proof here follows along the lines of [92]. Recall from general results in category theory (see also Appendix A.1) that functor categories are as cocomplete (and complete) as their target categories, the colimits being computed *"pointwise"*. Thus, $\underline{\mathsf{Set}}^{\underline{\mathsf{C}}^{op}}$ is cocomplete, and so we can consider the colimit of any small diagram of presheaves. Consider now a cocomplete category $\underline{\mathcal{E}}$ and suppose that there is a functor $\mathsf{A}\colon \underline{\mathsf{C}} \to \underline{\mathcal{E}}$. Then define $\mathsf{R}\colon \underline{\mathcal{E}} \to \underline{\mathsf{Set}}^{\underline{\mathsf{C}}^{op}}$ as follows:

$$\begin{array}{ccc}
\underline{\mathcal{E}} & \xrightarrow{\ \mathsf{R}\ } & \underline{\mathsf{Set}}^{\underline{\mathsf{C}}^{op}} \\[1em]
e & \longmapsto & \mathsf{Hom}_{\underline{\mathcal{E}}}(\mathsf{A}(_), e) \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle (f\circ_)} \\
e' & \longmapsto & \mathsf{Hom}_{\underline{\mathcal{E}}}(\mathsf{A}(_), e')
\end{array}$$

<u>REMARK</u>. The restriction to a small category $\underline{\mathsf{C}}$ is needed in order to define L properly. In fact, colimits indexed on locally small categories, in general, do *not* exist in $\underline{\mathsf{Set}}$. Of course, another possible way out of the problem consists of choosing a "superlarge" category of large sets which, therefore, admits large colimits. However, we prefer to stick to the standard definition of $\underline{\mathsf{Set}}$.

THEOREM 3.2.4
R *has a left adjoint* $\mathsf{L}\colon \underline{\mathsf{Set}}^{\underline{\mathsf{C}}^{op}} \to \underline{\mathcal{E}}$ *which sends* P *to* $\varinjlim\left(\int_{\underline{\mathsf{C}}} P \xrightarrow{\pi_P} \underline{\mathsf{C}} \xrightarrow{\mathsf{A}} \underline{\mathcal{E}}\right)$.

Proof. An arrow $\tau\colon P \stackrel{\cdot}{\to} \mathsf{R}(e)$ is a family of arrows $\{\tau_c\colon P(c) \to \underline{\mathcal{E}}[\mathsf{A}(c), e]\}$ indexed by the objects of $\underline{\mathsf{C}}$. The naturality condition for τ says that for any $u\colon c' \to c$ in $\underline{\mathsf{C}}$, the following diagram commutes

$$\begin{array}{ccc}
P(c) & \xrightarrow{\ \tau_c\ } & \underline{\mathcal{E}}[\mathsf{A}(c), e] \\
{\scriptstyle P(u)}\downarrow & & \downarrow{\scriptstyle (_\circ \mathsf{A}(u))} \\
P(c') & \xrightarrow[\ \tau_{c'}\ ]{} & \underline{\mathcal{E}}[\mathsf{A}(c'), e]
\end{array}$$

We can safely consider it a family of arrows $\{\tau_c(p)\colon \mathsf{A}(c) \to e\}$ indexed by the object of the category of elements of P. Then, the diagram above becomes $\tau_c(p) \circ \mathsf{A}(u) = \tau_{c'}(P(u)(p))$. In other words, for any $u\colon c' \to c$, putting $p' = P(u)(p)$, we have the commutativity of the following diagram.

$$\begin{array}{ccc}
\mathsf{A}(c) = \mathsf{A}(\pi_P(c,p)) & & \\
 & \searrow^{\tau_c(p)} & \\
{\scriptstyle A(u)}\uparrow & & e \\
 & \nearrow_{\tau_{c'}(p')} & \\
\mathsf{A}(c') = \mathsf{A}(\pi_P(c',p')) & &
\end{array}$$

In other words, $\left\{\tau_c(p) \mid (c,p) \in \int_{\underline{C}} P\right\}$ is a cocone for $\mathsf{A} \circ \pi_P$. Since we have this correspondence between natural transformations from P to $\mathsf{R}(e)$ and cocones for $\mathsf{A}\circ\pi_P$ with vertex e, it follows easily from the universal property of colimits that there is a natural bijection between natural transformations from P to $\mathsf{R}(e)$ and arrows from $\varinjlim(\mathsf{A} \circ \pi_P)$ to e, i.e., $\underline{\mathsf{Set}}^{\underline{\mathsf{C}}^{\mathrm{op}}}[P, \mathsf{R}(e)] \cong \underline{\mathcal{E}}[\mathsf{L}(P), e]$, which means that $\mathsf{L} \dashv \mathsf{R}$. ✓

Corollary 3.2.5
Every presheaf is a colimit of representable functors.

Proof. Apply the previous theorem with $\underline{\mathsf{Set}}^{\underline{\mathsf{C}}^{\mathrm{op}}}$ in place of $\underline{\mathcal{E}}$ and Y in place of A. Observe that then we have $\mathsf{L}, \mathsf{R}\colon \underline{\mathsf{Set}}^{\underline{\mathsf{C}}^{\mathrm{op}}} \to \underline{\mathsf{Set}}^{\underline{\mathsf{C}}^{\mathrm{op}}}$. Now, by definition, $\mathsf{R}(P)(c) = \underline{\mathsf{Set}}^{\underline{\mathsf{C}}^{\mathrm{op}}}[\mathsf{Y}(c), P]$ and then, by Yoneda's lemma, $\mathsf{R}(P)(c) \cong P(c)$ for any $c \in \underline{\mathsf{C}}$ and any $P \in \underline{\mathsf{Set}}^{\underline{\mathsf{C}}^{\mathrm{op}}}$. If follows that $\mathsf{R} \cong Id_{\underline{\mathsf{Set}}^{\underline{\mathsf{C}}^{\mathrm{op}}}}$. Now, the identity is left adjoint to itself and since adjoints are unique up to isomorphisms and since we know that L is left adjoint to R and, therefore, to the identity, we conclude that $\mathsf{L} \cong Id_{\underline{\mathsf{Set}}^{\underline{\mathsf{C}}^{\mathrm{op}}}}$. Therefore

$$P \cong \mathsf{L}(P) = \varinjlim\left(\int_{\underline{\mathsf{C}}} P \xrightarrow{\pi_P} \underline{\mathsf{C}} \xrightarrow{\mathsf{Y}} \underline{\mathsf{Set}}^{\underline{\mathsf{C}}^{\mathrm{op}}}\right) = \varinjlim\left(\mathsf{Y} \circ \pi_P\right).$$

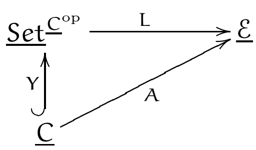
✓

An application of this result is that the morphisms in $\underline{\mathsf{Set}}^{\underline{\mathsf{C}}^{\mathrm{op}}}$ can be characterized as follows:

$$\begin{aligned}\underline{\mathsf{Set}}^{\underline{\mathsf{C}}^{\mathrm{op}}}[P,Q] &\cong \underline{\mathsf{Set}}^{\underline{\mathsf{C}}^{\mathrm{op}}}\left[\varinjlim\left(\int_{\underline{\mathsf{C}}} P \xrightarrow{\mathsf{Y}\circ\pi_P} \underline{\mathsf{Set}}^{\underline{\mathsf{C}}^{\mathrm{op}}}\right), Q\right] \\ &\cong \varprojlim_{c\in\int P} \underline{\mathsf{Set}}^{\underline{\mathsf{C}}^{\mathrm{op}}}[\mathsf{h}_c, Q] \cong \varprojlim_{c\in\int P} Q(c).\end{aligned}$$

We have also the following corollary that states that $\underline{\mathsf{Set}}^{\underline{\mathsf{C}}^{\mathrm{op}}}$ is "universal" up to isomorphism and bring us back to the issue of cocompletion of categories.

Corollary 3.2.6
Let $\underline{\mathsf{C}}$ be a small category. Then, for any cocomplete category $\underline{\mathcal{E}}$ and any functor $\mathsf{A}\colon \underline{\mathsf{C}} \to \underline{\mathcal{E}}$, there is a functor $\mathsf{L}\colon \underline{\mathsf{Set}}^{\underline{\mathsf{C}}^{\mathrm{op}}} \to \underline{\mathcal{E}}$ which preserves the colimits and such that the following diagram commutes.



Moreover, L is unique up to isomorphisms.

Proof. Let us consider $\mathsf{L}(P) = \varinjlim\Big(\int_{\underline{\mathsf{C}}} P \xrightarrow{\pi_P} \underline{\mathsf{C}} \xrightarrow{\mathsf{A}} \underline{\mathcal{E}}\Big)$. Since L is a left adjoint, it preserves colimits. Now, let us show that the diagram commutes. If P is representable, then $P \cong \mathsf{Y}(c)$ for some $c \in \underline{\mathsf{C}}$. It is then easy to realize that the category of elements of P has a terminal object, namely the object (c, id_c). Thus, without loss of generality, may assume that

$$\mathsf{L}(\mathsf{Y}(c)) = \varinjlim(\mathsf{A} \circ \pi_{\mathsf{Y}(c)}) = \mathsf{A}(\pi_{\mathsf{Y}(c)}(c, id_c)) = \mathsf{A}(c),$$

i.e., the diagram commutes.

Now, suppose that $\mathsf{K}\colon \underline{\mathsf{Set}}^{\underline{\mathsf{C}}^{\mathrm{op}}} \to \underline{\mathcal{E}}$ renders the diagram commutative and preserves colimits. Then, since for any presheaf P we have $P \cong \varinjlim(\mathsf{Y} \circ \pi_P)$, it is

$$\begin{aligned} \mathsf{L}(P) \cong \mathsf{L}(\varinjlim(\mathsf{Y} \circ \pi_P)) &\cong \varinjlim(\mathsf{L} \circ \mathsf{Y} \circ \pi_P) = \varinjlim(\mathsf{A} \circ \pi_P) \\ &= \varinjlim(\mathsf{K} \circ \mathsf{Y} \circ \pi_P) \cong \mathsf{K}(\varinjlim(\mathsf{Y} \circ \pi_P)) \cong \mathsf{K}(P), \end{aligned}$$

and thus $\mathsf{K} \cong \mathsf{L}$. ✓

Observe that the above construction does not give rise to an adjunction from $\underline{\mathsf{Cat}}$ to some "superlarge" category of large cocomplete categories for at least two reasons. First, there are problems of cardinality, since $\underline{\mathsf{Set}}^{\underline{\mathsf{C}}^{\mathrm{op}}}$ is not necessarily small and thus there is no evident forgetful functor which could make it so. Equally important, and more relevant from our point of view, is that the free construction above is defined only up to equivalences. We shall get back to this point later.

3.3 ℵ-Filtered Cocompletion

In this section we apply the techniques of Section 3.2 to the case of the ℵ-chain cocompletion. The understanding of $\underline{\mathsf{Set}}^{\underline{\mathsf{C}}^{\mathrm{op}}}$ as the "free" completion of a small category $\underline{\mathsf{C}}$ by arbitrary small colimits gives a strong hint on what the completion of $\underline{\mathsf{C}}$ by ℵ-chains, say $\widehat{\underline{\mathsf{C}}}_\aleph$, should be: the *full* subcategory of $\underline{\mathsf{Set}}^{\underline{\mathsf{C}}^{\mathrm{op}}}$ which contains the *representables* and their ℵ-chain colimits in $\underline{\mathsf{Set}}^{\underline{\mathsf{C}}^{\mathrm{op}}}$. We shall see that this is indeed the case, i.e., that $\widehat{\underline{\mathsf{C}}}_\aleph$ is closed for colimits of ℵ-chains and it is *"universal"* (in the weak sense above) among the ℵ-chain cocomplete categories which extends $\underline{\mathsf{C}}$. Moreover, since the definition of $\widehat{\underline{\mathsf{C}}}_\aleph$ makes sense for any locally small category, the following theory applies to any such $\underline{\mathsf{C}}$.

Although chains are pretty simple structures, they are rather uncomfortable to manage, at least in our context, since, as categories, they are very strongly restricted. For instance, one cannot form products and coproducts of chains. This situation, however, is common to other fields of mathematics, like partial order theory and topology, where one uses the—equivalent in many contexts—notion of directed set [93, 52, 61, 133]. Thus, as a first step, we abandon ℵ-chains and we broaden the class of index categories we use for the colimits, but, as we shall see, this will not change the kind of cocompleteness $\widehat{\underline{C}}_{\aleph}$ will have. Of course, as in many examples in the literature, e.g. [43, 58], we use filtered categories, which are the categorical correspondent of directed sets.

CHAINS VERSUS DIRECTED PARTIAL ORDERS

In this subsection we recall the relevant results about the equivalence of the notions of chain-completeness and direct-completeness for posets. By applying to categories, one sees that a category admits all the colimits indexed over chains if and only if it admits all colimits indexed by directed sets. The development here follows [52, 130, 93]. Our reference for set theory is [55].

DEFINITION 3.3.1
A non-empty subset D of a partial order P is directed if any pair of elements in D has an upper bound in D. Equivalently, D is directed if it contains an upper bound of any of its finite subsets. A chain is a non-empty partial order which is totally ordered.

P is $\aleph$-directed complete if P has the least upper bound of any directed subset of cardinality not greater than $\aleph$. It is directed complete if it has the least upper bound of any directed subset. P is $\aleph$-chain complete if P has the least upper bound of any chain of cardinality not greater than $\aleph$. It is chain complete if it has the least upper bound of any chain.

Directed sets and chains are, as usual, required not to be empty. This is because the least upper bound of an empty set, if it exists, is the *least* element of the poset and in many applications—and in particular here—one desires to consider limits without requiring the existence of such an element.

LEMMA 3.3.2
Let D be an infinite directed set. Then, there exists a transfinite sequence $\{D_\alpha\}_\alpha$ of directed subsets of D, with $\alpha < |D|$, such that

i) For any α
- *If α is finite so is D_α;*
- *If α is infinite, then $|D_\alpha| = |\alpha|$ (and therefore $|D_\alpha| < |D|$).*

ii) For any ordinals $\alpha < \beta < |D|$, $D_\alpha \subset D_\beta$.

iii) $D = \bigcup_\alpha D_\alpha$.

Proof. Let γ be the cardinality of D. Remember that the cardinals correspond to the least ordinal of a given cardinality. Let $\{x_\alpha\}_{\alpha<\gamma}$ be a well-ordering of D and for each finite subset F of D, let u_F denote an upper bound of F in D. Then consider the countable sequence

$$\begin{aligned} D_0 &= \{x_0\}; \\ D_{i+1} &= D_i \cup \left\{ y_{i+1}, u_{D_i \cup \{y_{i+1}\}} \right\}; \end{aligned}$$

where y_{i+1} is the least element of $D \setminus D_i$ (wrt. to the order $\{x_\alpha\}_{\alpha<\gamma}$). Of course, for any $i \in \omega$, D_i is finite of cardinality greater or equal to i, is directed, and for any $j > i$ we have $D_i \subset D_j$. Let D_ω be $\bigcup_{i<\omega} D_i$. Now, if $\gamma = \omega$, we are done. In fact, $D_\omega \subseteq D$ by construction, while if $x \in D$, then $x = x_j$ for some j and therefore $x \in D_j \subseteq D_\omega$, whence it follows $D_\omega = D$.

Suppose instead that $\gamma > \omega$. Consider now $\beta < \gamma$ and suppose that the sequence $\{D_\alpha\}_{\alpha<\beta}$ enjoys properties *(i)* and *(ii)* above. We show that the sequence can be extended to β. There are the following two cases.

(a) If β is a limit ordinal, then define $D_\beta = \bigcup_{\alpha<\beta} D_\alpha$. Then, for any $\alpha < \beta$ it is $D_\alpha \subset D_\beta$. Moreover, since $|D_\alpha| = |\alpha|$ and since $\beta = \bigcup_{\alpha<\beta} \alpha$, we have $|D_\beta| = \Sigma_{\alpha<\beta}|\alpha| = |\beta| \times \sup|\alpha| = |\beta| \times |\beta| = |\beta|$.

(b) $\beta = \delta + 1$. Then consider the countable sequence

$$\begin{aligned} D_{\beta,0} &= D_\delta \cup \{y'_\beta\},\ y'_\beta \text{ being the least element in } D \setminus D_\delta; \\ D_{\beta,i+1} &= D_{\beta,i} \cup \left\{ u_F \mid F \subset D_{\beta,i},\ F \text{ finite} \right\}; \end{aligned}$$

and take $D_\beta = \bigcup_{i<\omega} D_{\beta,i}$. Let S be a finite subset of D_β. Then, $S \subset D_{\beta,i}$ for some i and, therefore, $u_F \in D_{\beta,i+1}$. It follows that D_β is directed. Moreover, $|D_{\beta,i}| = |D_\delta|$, since the set of the finite subsets of X has the same cardinality as X. Then $|D_\beta| = \omega \times |D_\delta| = |D_\delta| = |\delta| = |\delta + 1| = |\beta|$. Finally, we of course have $D_\delta \subset D_\beta$.

Now, the thesis follows by transfinite induction. ✓

The relevance of the previous lemma shows in the proof of the following.

Corollary 3.3.3
P is ℵ-directed complete if and only if P is ℵ-chain complete.

Proof. One implication is obvious, since chains are directed sets. Let us show the other implication.

Suppose that the conclusion is false, i.e., that there exists a directed $D \subseteq P$ with $|D| \leq \aleph$ such that $\sup D$ does not exist. Observe that D cannot be finite, since a finite directed set always has a least upper bound: its greatest element. Clearly, we can assume that every $D' \subset D$ with $|D'| < |D|$ has a least upper bound. Then we are in the hypothesis of Lemma 3.3.2. Let $\{D_\alpha\}$, with $\alpha < |D|$, be the sequence obtained by applying it to D. Since $|D_\alpha| < |D|$, $\sup D_\alpha$ exists in P. Therefore we have a chain $\{\sup D_\alpha\}_{\alpha<|D|}$ of cardinality $|D|$. Then, $\sup\{\sup D_\alpha\}_{\alpha<|D|}$ exists and it is clearly the least upper bound of D. ✓

The previous proof depends heavily on the *axiom of choice*, because of the well-ordering chosen for D. Observe that Corollary 3.3.3 does *not* say that from any directed set we can extract a *cofinal subchain*, which is in fact *false*. (For a counterexample see e.g. [130].)

COROLLARY 3.3.4
P is directed complete if and only if P is chain complete.

Proof. Immediate from Corollary 3.3.3. ✓

Next, we apply Lemma 3.3.2 to the notion of cocompleteness in categories. Since an ordinal number α is an ordered set, which in turn is a category, we can consider functors $F\colon \alpha \to \underline{C}$. We shall refer to such functors as *chain* functors, or simply chains. We call F a $\aleph$-chain functor, or simply $\aleph$-*chain*, if $\alpha = \aleph$, where $|\_|$ gives the cardinality of sets. Similarly, for D a directed set, a functor $F\colon D \to \underline{C}$ is called *directed* functor. If $|D| \le \aleph$ then F is called $\aleph$-*directed*.

DEFINITION 3.3.5
A category $\underline{C}$ is ($\aleph$-)directed cocomplete if it admits colimits of all ($\aleph$-)directed functors. It is ($\aleph$-)chain cocomplete if it has colimits of all ($\aleph$-)chain functors.

Of course, as in the case of posets, for $\aleph$ finite, the $\aleph$-directed and $\aleph$-chain cocompleteness are trivial notions, since they are enjoyed by any category.

Remarkably, the notion of colimits indexed over directed *posets* is a pretty powerful one—indeed it was the original definition of colimits in category theory. In fact, if $\underline{C}$ has all finite coproducts and colimits of directed sets, then it is cocomplete, i.e., it has all small colimits [90, chap. IX, pp. 208].

PROPOSITION 3.3.6
$\underline{C}$ is $\aleph$-directed cocomplete if and only if it is $\aleph$-chain cocomplete.

Proof. Once again one implication is trivial. Let us show that when $\underline{C}$ admits colimits of all $\aleph$-chains it has colimits of all $\aleph$-directed functors. If $\aleph$ is finite the thesis is trivial; so suppose $\aleph$ infinite. Let $F\colon D \to \underline{C}$ be a $\aleph$-directed functor. If F is n-directed then D has a greatest element, say d, and, therefore, F has a colimit, namely Fd with the obvious limit cocone.

Now suppose that a colimit exists for any β-directed functor with $\beta < \gamma \le \aleph$, γ an infinite cardinal, and let $|D|$ be γ in $F\colon D \to \underline{C}$. Applying Lemma 3.3.2, we get a sequence $\{D_\alpha\}_{\alpha<\gamma}$ with $D_\alpha \subset D$, $|D_\alpha| = |\alpha| < \gamma$, D_α directed and $\bigcup_{\alpha<\gamma} D_\alpha = D$. Let $in_\alpha\colon D_\alpha \to D$ denote the injection of D_α in D. For any $\alpha < \gamma$, let $F_\alpha\colon D_\alpha \overset{in_\alpha}{\hookrightarrow} D \overset{F}{\longrightarrow} \underline{C}$ be the restriction of F to D_α and let $\sigma_\alpha\colon F_\alpha \stackrel{\cdot}{\to} c_\alpha$ be a colimit for F_α. Now, observe that for any $\alpha \le \beta$, since $D_\alpha \subseteq D_\beta$, we have $(\sigma_\beta)_\alpha\colon F_\alpha \stackrel{\cdot}{\to} c_\beta$, where $(\sigma_\beta)_\alpha$ is the restriction of σ_β to D_α. Then, by definition of colimit, there exists a unique induced arrow $f_{\alpha,\beta}\colon c_\alpha \to c_\beta$ such that

$$f_{\alpha,\beta} \circ (\sigma_\alpha)_d = (\sigma_\beta)_d \quad \text{for any } d \in D_\alpha. \tag{3.1}$$

It follows, again from the universal property of colimits, that the following definition defines a functor $G: \gamma \to \underline{C}$.

$$G\alpha = c_\alpha \quad \text{for any } \alpha \in \gamma; \qquad G(\alpha \leq \beta) = f_{\alpha,\beta}.$$

Since G is a γ-chain we can consider $\varinjlim G$ in $\underline{C}$. Let $\lambda: G \stackrel{\bullet}{\to} c$ be the limit cocone. We claim that c is $\varinjlim_D F$.

Next we define a cocone $\sigma: F \stackrel{\bullet}{\to} c$ for F and we show that it is the limit cocone. Since for any $d \in D$ there exists α such that $d \in D_\alpha$, the obvious choice is $\sigma_d = \lambda_\alpha \circ (\sigma_\alpha)_d$. Observe that σ_d is irrespective of the choice of α. In fact, suppose $d \in D_\beta$ and let σ'_d be $\lambda_\beta \circ (\sigma_\beta)_d$. Without loss of generality suppose $\alpha \leq \beta$. Then, since we have $\lambda: G \stackrel{\bullet}{\to} c$, it is $\lambda_\alpha = \lambda_\beta \circ f_{\alpha,\beta}$. Moreover, by construction, $(\sigma_\beta)_d = f_{\alpha,\beta} \circ (\sigma_\alpha)_d$. Then, $\lambda_\beta \circ (\sigma_\beta)_d = \lambda_\beta \circ f_{\alpha,\beta} \circ (\sigma_\alpha)_d = \lambda_\alpha \circ (\sigma_\alpha)_d = \sigma_d$. Thus, σ is well-given. Let us see that it is a cocone. Suppose $d \leq e$ in D. Let α be large enough so that $d, e \in D_\alpha$. Now, since we have $\sigma_\alpha: F_\alpha \to c_\alpha$, it is $(\sigma_\alpha)_d = (\sigma_\alpha)_e \circ F_\alpha(d \leq e)$ and then, taking into account also that $F_\alpha(d \leq e) = F(d \leq e)$, it is $\lambda_\alpha \circ (\sigma_\alpha)_d = \lambda_\alpha \circ (\sigma_\alpha)_e \circ F(d \leq e)$ which is $\sigma_d = \sigma_e \circ F(d \leq e)$.

Then, we only miss to show that σ is universal. Let $\eta: F \stackrel{\bullet}{\to} c'$ be a cocone with vertex c' for F. In particular, for any $\alpha \in \gamma$, η restricts on D_α to $(\eta)_\alpha: D_\alpha \stackrel{\bullet}{\to} c'$, and thus it exists a unique $\nu_\alpha: c_\alpha \to c$ induced by the colimit, i.e., such that $\nu_\alpha \circ (\sigma_\alpha)_d = \eta_d$ for any $d \in D_\alpha$. Let us show that the collection of the ν's form a cocone with vertex c' for G. We must show that, for any $\alpha \leq \beta$, $\nu_\alpha = \nu_\beta \circ f_{\alpha,\beta}$ which comes as follows: for any $\alpha \leq \beta$, and for any $d \in D_\alpha$, we have by (3.1) that $\nu_\beta \circ f_{\alpha,\beta} \circ (\sigma_\alpha)_d = \nu_\beta \circ (\sigma_\beta)_d = \eta_d$, which is enough since ν_α is the unique arrow with this property. Thus, we have shown that a cocone $\eta: F \stackrel{\bullet}{\to} c'$ determines a unique $\nu: G \stackrel{\bullet}{\to} c'$. Then, by universality of λ, there exists a unique $k: c \to c'$ such that $\nu_\alpha = k \circ \lambda_\alpha$ for any $\alpha \in \gamma$, which implies $\nu_\alpha \circ (\sigma_\alpha)_d = k \circ \lambda_\alpha \circ (\sigma_\alpha)_d$ for any $\alpha \in \gamma$ and for any $d \in D_\alpha$, which implies $\eta_d = k \circ \lambda_\alpha \circ (\sigma_\alpha)_d$ for any $\alpha \in \gamma$ and for any $d \in D_\alpha$, which is $\eta_d = k \circ \sigma_d$ for any $d \in D$. The construction itself shows that k is the unique morphism from c to c' enjoying this property, and thus the proof is concluded. ✓

A particular case which will interest us in the following is the one where $\aleph$ is ω. Of course, we have the following corollary.

Corollary 3.3.7
$\underline{C}$ is directed cocomplete if and only if $\underline{C}$ is chain cocomplete.

Proof. Immediate from Proposition 3.3.6. ✓

Filtered Categories and Cofinal Functors

In this subsection we broaden further the kind of index categories over which colimits are considered. In particular, we recall the basic facts about *filtered categories* and *cofinal functors*. Moreover, we state results which show that requiring the existence of colimits indexed by filtered categories is not stronger than requiring the existence of colimits indexed by directed sets and, therefore, of chain colimits.

Definition 3.3.8 *(Filtered Categories)*
A category J *is filtered if it is not empty and*

i) for all $j, j' \in \mathsf{J}$ *there exists* k *and* $u: j \to k$, $v: j' \to k$, *i.e.,* $j \xrightarrow{u} k \xleftarrow{v} j'$.

ii) for all $i \overset{u}{\underset{v}{\rightrightarrows}} j$ *in* J, *there exists* $w: j \to k$ *such that* $w \circ u = w \circ v$, *i.e.,* $i \overset{u}{\underset{v}{\rightrightarrows}} j \xrightarrow{w} k$ *is commutative.*

A functor $\mathsf{F}: \mathsf{J} \to \underline{\mathsf{C}}$ *is filtered if* J *is filtered. By filtered colimits we mean colimits of filtered functors.*

The following propositions list some of the good properties enjoyed by filtered categories.

Proposition 3.3.9
The cartesian product of filtered categories is filtered.

Proof. Immediate. ✓

A good point is that filtered colimits in $\underline{\mathsf{Set}}$ are easily characterized.

Proposition 3.3.10
Let $\mathsf{F}: \mathsf{J} \to \underline{\mathsf{Set}}$ *be filtered and suppose that* J *is small. Consider the set* $\coprod_{j\in\mathsf{J}} \mathsf{F}j$ *and the binary relation* $\mathcal{R}$ *on it defined as follows*

$$in_i(x)\ \mathcal{R}\ in_j(y) \quad \Leftrightarrow \quad \exists k \in \mathsf{J} \text{ and } i \xrightarrow{u} k \xleftarrow{v} j \text{ such that } \mathsf{F}(u)(x) = \mathsf{F}(v)(y).$$

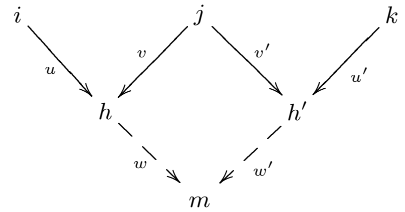
Then

i) $\mathcal{R}$ *is an equivalence relation;*

ii) $\left(\coprod_{j\in\mathsf{J}} \mathsf{F}j\right) \big/ \mathcal{R} \cong \varinjlim_{\mathsf{J}} \mathsf{F}$;

iii) Given $x, y \in \mathsf{F}k$, *then* $in_k(x)\ \mathcal{R}\ in_k(y)$ *if and only if there exists* $u: k \to h$ *such that* $\mathsf{F}(u)(x) = \mathsf{F}(u)(y)$.

Proof. *(i).* $\mathcal{R}$ is manifestly reflexive and symmetric. Let us show that it is symmetric. Suppose $in_i(x)\ \mathcal{R}\ in_j(y)$ via $u\colon i \to h$ and $v\colon j \to h$, and suppose that $in_j(y)\ \mathcal{R}\ in_k(z)$ via $v'\colon j \to h'$ and $u'\colon k \to h'$. Using jointly the properties *(i)* and *(ii)* of filtered categories, we find $w\colon h \to m$ and $w'\colon h' \to m$ such that $w \circ v = w' \circ v$. Now we have $\mathsf{F}(w \circ u)(x) = \mathsf{F}(w)(\mathsf{F}(u)(x)) = \mathsf{F}(w)(\mathsf{F}(v)(y)) = \mathsf{F}(w \circ v)(y) = \mathsf{F}(w' \circ v')(y) = \mathsf{F}(w')(\mathsf{F}(v')(y)) = \mathsf{F}(w')(\mathsf{F}(u')(z)) = \mathsf{F}(w' \circ u')(z)$, i.e., $in_i(x)\ \mathcal{R}\ in_k(z)$ via w and w'. This is summarized by the following picture.



(ii). For any $j \in \mathsf{J}$, let σ_j be the composition of the injection of $\mathsf{F}j$ in $\coprod_{j\in\mathsf{J}} \mathsf{F}j$ with the quotient map $\coprod_{j\in\mathsf{J}} \mathsf{F}j \mapsto \left(\coprod_{j\in\mathsf{J}} \mathsf{F}j\right) \big/ \mathcal{R}$. Since for any $u\colon i \to j$ in J, it is obviously $in_i(x)\ \mathcal{R}\ in_j(\mathsf{F}(u)(x))$ via u and id_j, we have $\sigma_i = \sigma_j \circ \mathsf{F}(u)$. Thus, the collection of the σ's is a cocone $\sigma\colon \mathsf{F} \stackrel{\cdot}{\to} \left(\coprod_{j\in\mathsf{J}} \mathsf{F}j\right) \big/ \mathcal{R}$ which is easily shown to be universal. In fact, if $\eta\colon \mathsf{F} \stackrel{\cdot}{\to} X$ is another cocone, by definition it must be $\eta_i(x) = \eta_j(y)$ whenever $in_i(x)\ \mathcal{R}\ in_j(y)$ whence the result follows at once.

(iii). Obvious from point *(i)* and condition *(ii)* of Definition 3.3.8. ✓

Proposition 3.3.11
The category of small filtered categories has all small filtered colimits.

Proof. It is easy to see that the characterization of indexed colimits in $\underline{\mathsf{Set}}$ given by Proposition 3.3.10 can be extended to $\underline{\mathsf{Cat}}$-valued filtered functors. Then, it is easy to show that the category resulting as colimit of a filtered diagram of filtered categories is itself filtered. ✓

Cofinal subcategories are the categorical generalization of the set-theoretic notion of cofinal chains. Intuitively, a subcategory I of J is cofinal in J if the colimit of any J-indexed diagram coincides with the colimit of the same diagram restricted to I. Of course, there is no conceptual need to limit oneself to subcategories, and that is why one introduces *cofinal functors.*

Definition 3.3.12 *(Cofinal Functors)*
A functor $\phi\colon \mathsf{I} \to \mathsf{J}$ is cofinal if for any functor $\mathsf{F}\colon \mathsf{J} \to \underline{\mathsf{C}}$

$$\varinjlim_{\mathsf{I}}(\mathsf{F}\circ\phi) \text{ exists} \quad\Rightarrow\quad \varinjlim_{\mathsf{J}} \mathsf{F} \text{ exists} \quad\text{and}\quad \varinjlim_{\mathsf{I}}(\mathsf{F}\circ\phi) \cong \varinjlim_{\mathsf{J}} \mathsf{F},$$

the isomorphism being via the canonical comparison map $\varinjlim_{\mathsf{I}}(\mathsf{F} \circ \phi) \to \varinjlim_{\mathsf{J}} \mathsf{F}$ *induced by the colimit.*

A subcategory I *of* J *is cofinal if the inclusion functor is cofinal.*

Of course the name is inherited from the corresponding notion in set theory and the "co" prefix has nothing to do with duality in categories. For this reason, MacLane [90, chap. IX] and others use the term *"final"* to name the concept. However, once the reader has been warned about this mismatch, we prefer to keep using the classical terminology.

The proof of the following proposition is immediate.

PROPOSITION 3.3.13
The composition of cofinal functors is a cofinal functor.

The following key lemma gives a characterization of cofinal functors between filtered categories.

LEMMA 3.3.14
For a functor $\phi: \mathsf{I} \to \mathsf{J}$ *the following properties can be stated:*

$\mathbf{F_1}$: *for any* $j \in \mathsf{J}$*, there exists* $i \in \mathsf{I}$ *such that* $\mathsf{Hom}_{\mathsf{J}}(j, \phi i) \neq \varnothing$.

$\mathbf{F_2}$: *for any* $i \in \mathsf{I}$ *and for any* $j \underset{g}{\overset{f}{\rightrightarrows}} \phi i$ *in* J*, there exists* $h: i \to k$ *in* I *such that* $\phi h \circ f = \phi h \circ g$.

Then, we have the following facts:

i) if ϕ *is cofinal, then* $\mathbf{F_1}$ *holds;*

ii) if I *is filtered, then* ϕ *is cofinal if and only if* $\mathbf{F_1}$ *and* $\mathbf{F_2}$ *hold and, in this case,* J *is also filtered;*

iii) if J *is filtered and* ϕ *is full and faithful, then* ϕ *is cofinal if and only if* $\mathbf{F_1}$ *holds, and, in this case,* I *is also filtered.*

Proof. Let us consider the Yoneda's embedding $\mathsf{Y}: \mathsf{J} \to \underline{\mathsf{Set}}^{\mathsf{J}^{op}}$. Without loss of generality, we can assume that $\underline{\mathsf{Set}}$ is based on a universe $\mathfrak{U}$ such that I and J are small with respect to $\mathfrak{U}$. In fact, if this were not the case we could replace $\underline{\mathsf{Set}}$ with a larger category and the following arguments would apply unchanged. Then, we can consider the $\varinjlim_{\mathsf{I}}(\mathsf{Y} \circ \phi)$, which exists by the hypothesis above. Since ϕ is cofinal, this implies that $\varinjlim_{\mathsf{J}} \mathsf{Y}$ exists.

Now, in order to show that $\mathbf{F_1}$ is a necessary condition, suppose that there exists $\bar{\jmath} \in \mathsf{J}$ such that, for any $i \in \mathsf{I}$, $\mathsf{Hom}_{\mathsf{J}}(\bar{\jmath}, \phi i) = \varnothing$. Then, again by cofinality of ϕ, it must be

$$\varinjlim_{\mathsf{I}}(\mathsf{Y} \circ \phi) = \varinjlim_{i \in \mathsf{I}} \mathsf{Hom}_{\mathsf{J}}(\_, \phi i) \cong \varinjlim_{j \in \mathsf{J}} \mathsf{Hom}_{\mathsf{J}}(\_, j) = \varinjlim_{\mathsf{J}} \mathsf{Y}.$$

But this is impossible, since $\left(\varinjlim_{i\in \mathsf{I}} \mathsf{Hom}_{\mathsf{J}}(\_,\phi i)\right)\bar{\jmath} \cong \varinjlim_{\imath\in \mathsf{I}} \mathsf{Hom}_{\mathsf{J}}(\bar{\jmath},\phi i) = \varnothing$ by hypothesis on $\bar{\jmath}$, while $\left(\varinjlim_{j\in \mathsf{J}} \mathsf{Hom}_{\mathsf{J}}(\_,j)\right)\bar{\jmath} \cong \varinjlim_{j\in \mathsf{J}} \mathsf{Hom}_{\mathsf{J}}(\bar{\jmath},j) \neq \varnothing$, since it must at least contain an element to match $id_{\bar{\jmath}} \in \mathsf{Hom}_{\mathsf{J}}(\bar{\jmath},\bar{\jmath})$.

Assume now that I be filtered. Let us see that J is filtered. Consider $j, j' \in \mathsf{J}$. Then, by $\mathbf{F_1}$, we can find $j \xrightarrow{u} \phi i$ and $j' \xrightarrow{u'} \phi i'$ in J, and since I is filtered we have $i \xrightarrow{w'} k$ and $i' \xrightarrow{w'} k$ in I whose image via ϕ gives $j \xrightarrow{\phi w\circ u} \phi k \xleftarrow{\phi w'\circ u'} j'$. Thus, property *(i)* of the definition of filtered categories is shown.

In order to show the other condition, we need the following lemma: for any $k \in \mathsf{I}$ it is $\varinjlim_{\mathsf{I}} \mathsf{Hom}_{\mathsf{I}}(k,\_) = \{*\}$. Observe that this is an easy consequence of the (contravariant) Yoneda's Lemma: let $\Delta(S)\colon \mathsf{I} \to \underline{\mathsf{Set}}$ be the constant functor which returns the set S; then

$$\underline{\mathsf{Set}}^{\mathsf{I}}[\mathsf{Hom}_{\mathsf{I}}(k,\_),\Delta(S)] \cong S \cong \underline{\mathsf{Set}}[\{*\},S],$$

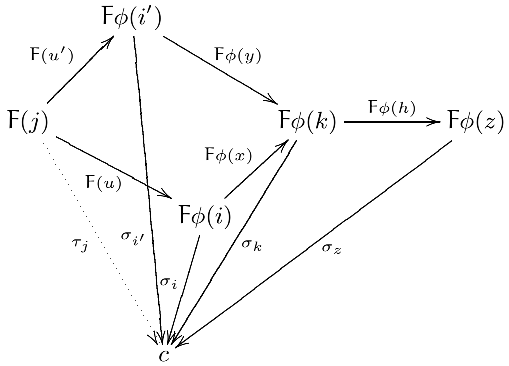
which is the required adjointness condition for colimits. Consider now $\bar{\jmath} \underset{g}{\overset{f}{\rightrightarrows}} \bar{\jmath}'$ and take $x\colon \bar{\jmath}' \to \phi\bar{\imath}$, which exists by $\mathbf{F_1}$. Then, $\varinjlim_{i\in \mathsf{I}} \mathsf{Hom}_{\mathsf{I}}(\bar{\jmath},\phi i) = \varinjlim_{j\in \mathsf{J}} \mathsf{Hom}_{\mathsf{J}}(\bar{\jmath},j)$ by cofinality of ϕ, which is $\{*\}$ by the previous argument. Then, by point *(iii)* of Proposition 3.3.10, since $x\circ f, x\circ g \in \mathsf{Hom}_{\mathsf{I}}(\bar{\jmath},\phi\bar{\imath})$ and $x\circ f \;\mathcal{R}\; x\circ g$, it must exist $h\colon \bar{\imath} \to k$ in I such that $\phi h\circ x\circ f = \phi h\circ x\circ g$. Thus, $\phi h\circ x$ equalizes[1] f and g and the second condition of filteredness is shown. Now, if we consider $\bar{\jmath} \underset{g}{\overset{f}{\rightrightarrows}} \phi\bar{\imath}$, reasoning as above, we find $h\colon \bar{\imath} \to k$ such that $\phi h\circ f = \phi h\circ g$. Thus, the necessity of $\mathbf{F_2}$ is shown.

Next, we see that $\mathbf{F_1}$ and $\mathbf{F_2}$ are sufficient to ensure that ϕ is cofinal. Let $\mathsf{F}\colon \mathsf{J} \to \underline{\mathsf{C}}$ be a functor and suppose that $c = \varinjlim(\mathsf{F}\circ\phi)$. It is possible to show that there is a one-to-one correspondence between cocones with base $\mathsf{F}\circ\phi$ and vertex c and cocones with base F and vertex c. Let $\sigma\colon \mathsf{F}\circ\phi \xrightarrow{\cdot} c$. Now, for $j\in \mathsf{J}$, chosen $u\colon j \to \phi i$, define $\tau_j = \sigma_i \circ \mathsf{F}u$. First, we have to show that τ_j does not depend on the choice of $u\colon j\to\phi i$. In fact, if $u'\colon j \to \phi i'$, since I is filtered, we can find $x\colon i\to k$ and $y\colon i'\to k$. Moreover, by $\mathbf{F_2}$ we can find $h\colon k\to z$ such that $\phi h\circ\phi x\circ u = \phi h\circ \phi y\circ u'$. Then

$$\begin{aligned} \tau_j' = \sigma_{i'}\circ \mathsf{F}u' &= \sigma_k \circ \mathsf{F}\phi y\circ \mathsf{F}u' \\ &= \sigma_z\circ \mathsf{F}\phi h\circ \mathsf{F}\phi y\circ \mathsf{F}u' \\ &= \sigma_z\circ \mathsf{F}\phi h\circ \mathsf{F}\phi x\circ \mathsf{F}u \\ &= \sigma_k\circ \mathsf{F}\phi x\circ \mathsf{F}u \\ &= \sigma_i\circ \mathsf{F}u = \tau_j \end{aligned}$$

[1] Here and in the following the term equalizer is used in its English sense, not in the categorical one.

The situation is summarized by the following picture.



Thus, the τ's are well-defined. Now consider a morphism $w: j \to j'$ in J. By $\mathbf{F_1}$ and $\mathbf{F_2}$, exploiting that I is filtered, we can find $i \in \mathsf{I}$, $u: j \to \phi i$ and $v: j' \to \phi i$ such that $u = v \circ w$. Then, $\tau_j = \sigma_i \circ \mathsf{F}u = \sigma_i \circ \mathsf{F}v \circ \mathsf{F}w = \tau_{j'} \circ \mathsf{F}w$. Therefore, $\tau: \mathsf{F} \stackrel{\cdot}{\to} c$ is a cocone. Observe finally that, since I is filtered, by definition of cocone, any $\tau: \mathsf{F} \to c$ is completely determined through the construction just illustrated by the morphisms of the form $\tau_{\phi i}$. It follows that the restriction of τ to the elements $\tau_{\phi i}$ is the inverse of constructing τ from σ. Therefore, point *(ii)* is proved.

Lastly, concerning point *(iii)*, if J is filtered and ϕ is fully faithful it is immediate to realize that $\mathbf{F_1}$ implies that I is filtered. Consider now $j \overset{f}{\underset{g}{\rightrightarrows}} \phi i'$ in J. By filteredness of J we find $h: \phi i' \to j'$ which equalizes f and g, and by $\mathbf{F_1}$ we find $k: j' \to \phi i''$. Thus, by hypothesis on ϕ, there exists $x: i' \to i''$ in I such that $\phi x = k \circ h$ and which, therefore, equalizes f and g. This means that $\mathbf{F_2}$ follows from the hypothesis and, exploiting the previous points, concludes the proof. ✓

DEFINITION 3.3.15 *(Filt and* $\mathrm{Filt}_{\aleph}$*)*
Given a category J*, by the cardinality of* J*, in symbols* $|\mathsf{J}|$*, we mean, as usual, the cardinality of the underlying set of arrows of* J*. If* J *is filtered and* $|\mathsf{J}| \leq \aleph$*, we say that* J *is* $\aleph$*-filtered.*

Let $\underline{\mathrm{Filt}}$ *denote the full subcategory of* $\underline{\mathrm{Cat}}$ *consisting of filtered categories and let* $\underline{\mathrm{Filt}}_{\aleph}$ *be the full subcategory of the* $\aleph$*-filtered categories.*

The next proposition shows that requiring the existence of filtered colimits is not more than requiring the existence of directed colimits.

PROPOSITION 3.3.16
Let J *be small and filtered. Then, there exists a directed set* D *and* $\phi: D \to \mathsf{J}$ *which is cofinal. Moreover, if* $|\mathsf{J}|$ *is infinite, then* $|\mathsf{J}| = |D|$*.*

Proof. First suppose that J has no greatest object, i.e., that for any $j \in \mathsf{J}$ it is possible to find $j' \in \mathsf{J}$, $j' \neq j$, such that $\mathsf{Hom}_{\mathsf{J}}(j, j') \neq \varnothing$. D consists of all the finite subcategories A of J which have a *unique* terminal object ordered by inclusion. Of course, one defines $\phi(\mathsf{A})$ to be its terminal object. Moreover, if $\mathsf{A} \subseteq \mathsf{B}$, there is a unique arrow f in B such that $f\colon \phi(\mathsf{A}) \to \phi(\mathsf{B})$, and one defines $\phi(\mathsf{A} \subseteq \mathsf{B})$ to be such an arrow.

For A and B in D an upper bound is identified as follows. Let a and b be the terminal objects of, respectively, A and B. Since J is filtered one can find $u\colon a \to j$ and $v\colon b \to j$. Then consider the category E whose objects are the union of those of A and B plus j and whose arrows are the union of those of A and B plus u and v. Clearly, E is a finite subcategory of J with a unique terminal object and is an upper bound for A and B. Thus, D is directed.

In order to show that ϕ is cofinal we use point *(ii)* of Lemma 3.3.14. It is obvious that $\mathbf{F_1}$ holds: for any $j \in \mathsf{J}$ consider the subcategory of J consisting of the only j with its identity arrow. Verifying that $\mathbf{F_2}$ holds requires a bit more of work. Let $j \underset{g}{\overset{f}{\rightrightarrows}} \phi(\mathsf{A})$ be in J. Since J is filtered, we can find $u\colon \phi(\mathsf{A}) \to j'$ which equalizes them. Since we assume that J has no greatest object, we can suppose that j' is different from $\phi(\mathsf{A})$. It follows that j' cannot be an object of A and that the subcategory B obtained from A by adding the new object j' and the arrow u belongs to D. Moreover, $\phi(\mathsf{A} \subseteq \mathsf{B}) = u$ equalizes f and g as required.

We miss the case in which J has a greatest element. In this case one considers the category $\underline{\omega} \times \mathsf{J}$, which is filtered and does not have a greatest object. Then one finds $D \xrightarrow{\phi} \underline{\omega} \times \mathsf{J}$, with ϕ cofinal and compose it with the projection $\underline{\omega} \times \mathsf{J} \xrightarrow{\pi} \mathsf{J}$, which is manifestly cofinal because of Lemma 3.3.14, point *(iii)*. Thus, $D \xrightarrow{\pi \circ \phi} \mathsf{J}$ is cofinal. The statement about cardinalities follows at once, since the set of finite subcategories of J has the same cardinality as J if this is infinite. ✓

Recall that we say that a functor $\mathsf{F}\colon \mathsf{J} \to \underline{\mathsf{C}}$ is *filtered* if $\mathsf{J} \in \underline{\mathtt{Filt}}$. Moreover, if $\mathsf{J} \in \underline{\mathtt{Filt}}_{\aleph}$, then F is $\aleph$-*filtered*. A category $\underline{\mathsf{C}}$ is $\aleph$-filtered cocomplete if it admits colimits of all $\aleph$-filtered functors, and it filtered cocomplete if it has colimits of all filtered functors.

Corollary 3.3.17

A category $\underline{\mathsf{C}}$ is $\aleph$-filtered cocomplete if and only if it is $\aleph$-directed cocomplete, if and only if it is $\aleph$-chain cocomplete.

A category $\underline{\mathsf{C}}$ is filtered cocomplete if and only if is directed cocomplete, if and only if it is chain cocomplete.

Proof. Concerning the first claim, the second double implication is Proposition 3.3.6 and one direction of the first one is trivial. The other direction follows immediately by Proposition 3.3.16 as follows. Let $\underline{\mathsf{C}}$ be directed complete and consider $\mathsf{F}\colon \mathsf{J} \to \underline{\mathsf{C}}$. Suppose that $\aleph$ is infinite. Then, we can find a directed D and $\phi\colon D \to \mathsf{J}$ such that $|D| = |\mathsf{J}|$ and ϕ is cofinal. Then, $\mathsf{F}\phi\colon D \to \underline{\mathsf{C}}$ has a colimit and therefore, by cofinality, $\varinjlim \mathsf{F}$ exists in $\underline{\mathsf{C}}$. Suppose instead that $\aleph$ is finite. If J has no greatest object, the

D obtained from Proposition 3.3.16 is finite and the argument above can be applied, since the colimits indexed by finite D's always exist in $\underline{\mathsf{C}}$. Finally, if J has a greatest object, take ϕ to be the functor which maps the singleton set to such an object. Of course ϕ is cofinal, and thus we can apply again the argument above. (Observe that this just proves that, as in the case of directed colimits, finite filtered colimits are a trivial notion, since they exist in any category $\underline{\mathsf{C}}$.)

The second statement is proved similarly. ✓

Exploiting further the notion of cofinal functor, we can still broaden further the class of index categories over which we shall consider colimits. We shall do so by being as lax as possible with respect to cardinalities.

DEFINITION 3.3.18 *(Essentially $\aleph$-Filtered Categories)*
A category J is essentially $\aleph$-filtered if it is locally small and if there exists $\mathsf{I} \in \underline{\mathsf{Filt}}_\aleph$ together with $\phi: \mathsf{I} \to \mathsf{J}$ which is cofinal. J is essentially filtered if it is essentially $\aleph$-filtered for some cardinal $\aleph$.

Let $\underline{\mathcal{E}\text{-}\mathsf{Filt}}$ be the full subcategory of $\underline{\mathsf{CAT}}$ consisting of the essentially filtered categories. $\underline{\mathcal{E}\text{-}\mathsf{Filt}}_\aleph$ denotes the full subcategory of $\underline{\mathcal{E}\text{-}\mathsf{Filt}}$ consisting of the essentially $\aleph$-filtered categories.

We say that $\underline{\mathsf{C}}$ is *essentially ($\aleph$-)filtered cocomplete* if it admits colimits of all functors indexed over essentially ($\aleph$-)filtered categories. Then, we have the following immediate corollary which closes this subsection and allows us in the rest of the chapter to use the quite liberal essentially $\aleph$-filtered categories in place of $\aleph$-chains without changing the kind of cocompleteness of the category involved.

COROLLARY 3.3.19
A category $\underline{\mathsf{C}}$ is essentially ($\aleph$-)filtered cocomplete if and only if it is ($\aleph$-)filtered cocomplete if and only if it is ($\aleph$-)directed cocomplete if and only if it is ($\aleph$-)chain cocomplete.

Proof. The missing step is to prove that if $\underline{\mathsf{C}}$ is ($\aleph$-)filtered cocomplete then it is essentially ($\aleph$)-filtered cocomplete. But this follows immediately from Corollary 3.3.17 and from the definition of cofinal functor. ✓

Thus, although we shall preferably use the ($\aleph$-)filtered alternative, the previous corollary allows us to use the terms above interchangeably.

REMARK. We have already observed that the notion of cocompleteness coincide (trivially) for all the finite cardinals. However, this is just a limit case of a general situation: if $\aleph$ and $\aleph'$ are infinite cardinals having the same *cofinality*, then $\underline{\mathsf{C}}$ is $\aleph$-filtered cocomplete if and only if it is $\aleph'$-filtered cocomplete. Clearly, different *regular* cardinals give rise to different notions of cocompleteness.

ℵ-Ind-Representable Functors

We are ready now to give the results about the chain cocompletion, or equivalently the essentially filtered cocompletion, of $\underline{\mathsf{C}}$.

Definition 3.3.20 *(ℵ-Ind-Representable Functors)*
Given a locally small category $\underline{\mathsf{C}}$*, let* $\widehat{\underline{\mathsf{C}}}_{\aleph}$ *denote the full subcategory of* $\underline{\mathsf{Set}}^{\underline{\mathsf{C}}^{\mathrm{op}}}$ *which contains the representables* h_c*,* $c \in \underline{\mathsf{C}}$*, and* $\varinjlim \mathsf{YF}$ *for any ℵ-chain functor* $\mathsf{F}\colon \alpha \to \underline{\mathsf{C}}$*,* Y *being the Yoneda's embedding.*

Similarly, $\widehat{\underline{\mathsf{C}}}$ *denotes the full subcategory of* $\underline{\mathsf{Set}}^{\underline{\mathsf{C}}^{\mathrm{op}}}$ *which contains the representables* h_c *and* $\varinjlim \mathsf{YF}$ *for any chain functor* F*.*

The presheaves in $\widehat{\underline{\mathsf{C}}}_{\aleph}$ *are called ℵ-ind-representable (ind standing for inductively), those in* $\widehat{\underline{\mathsf{C}}}$ *are called the ind-representable functors.*

Observe that, in view of the development in the previous sections, $\widehat{\underline{\mathsf{C}}}$ ($\widehat{\underline{\mathsf{C}}}_{\aleph}$) can be equivalently defined as the full subcategory of $\underline{\mathsf{Set}}^{\underline{\mathsf{C}}^{\mathrm{op}}}$ which contains the representables and $\varinjlim(\mathsf{YF})$ for any $\mathsf{F}\colon \mathsf{J} \to \underline{\mathsf{C}}$, where $\mathsf{J} \in \underline{\mathcal{E}\text{-}\mathsf{Filt}}$ ($\mathsf{J} \in \underline{\mathcal{E}\text{-}\mathsf{Filt}}_{\aleph}$). In the following, we shall use often this equivalent description.

Proposition 3.3.21
The categories $\widehat{\underline{\mathsf{C}}}_{\aleph}$ *and* $\widehat{\underline{\mathsf{C}}}$ *are locally small.*

Proof. Since $\widehat{\underline{\mathsf{C}}}_{\aleph}$ is a full subcategory of $\widehat{\underline{\mathsf{C}}}$, it is enough to prove the result for $\widehat{\underline{\mathsf{C}}}$. Given P and Q in $\widehat{\underline{\mathsf{C}}}$ we can assume they are of the form $P \cong \varinjlim_{\mathsf{I}} \mathsf{h}_{c_i}$ and $Q \cong \varinjlim_{\mathsf{J}} \mathsf{h}_{c_j}$, for $\mathsf{I}, \mathsf{J} \in \underline{\mathsf{Filt}}$. Then,

$$\begin{aligned}
\widehat{\underline{\mathsf{C}}}[P,Q] &= \underline{\mathsf{Set}}^{\underline{\mathsf{C}}^{\mathrm{op}}}[P,Q] \\
&\cong \underline{\mathsf{Set}}^{\underline{\mathsf{C}}^{\mathrm{op}}}[\varinjlim\nolimits_{\mathsf{I}} \mathsf{h}_{c_i}, \varinjlim\nolimits_{\mathsf{J}} \mathsf{h}_{c_j}] \\
&\cong \varprojlim\nolimits_{\mathsf{I}} \underline{\mathsf{Set}}^{\underline{\mathsf{C}}^{\mathrm{op}}}[\mathsf{h}_{c_i}, \varinjlim\nolimits_{\mathsf{J}} \mathsf{h}_{c_j}] \text{ (since } \mathsf{Y}' \text{ preserves colimits)} \\
&\cong \varprojlim\nolimits_{\mathsf{I}} \Big(\varinjlim\nolimits_{\mathsf{J}} \mathsf{h}_{c_j}\Big) c_i \text{ (by the Yoneda's lemma)} \\
&\cong \varprojlim\nolimits_{\mathsf{I}} \varinjlim\nolimits_{\mathsf{J}} \mathsf{Hom}_{\underline{\mathsf{C}}}(c_i, c_j).
\end{aligned}$$

Thus, as a $\varprojlim \varinjlim$ construction in $\underline{\mathsf{Set}}$ indexed by small categories, $\mathsf{Hom}_{\widehat{\underline{\mathsf{C}}}}(P,Q)$ is a small set. ✓

The following is an interesting property of ind-representables.

Proposition 3.3.22
ℵ-ind-representable functors preserve finite colimits.

Proof. Let I be finite and consider $\mathsf{F}\colon \mathsf{I} \to \underline{\mathsf{C}}$ such that $\varinjlim \mathsf{F}$ exists. Let us show that for every P be in $\widehat{\underline{\mathsf{C}}}_{\aleph}$ it is $\varprojlim(P \circ \mathsf{F}) \cong P(\varinjlim \mathsf{F})$.

First observe that this is true for the representable functors.

$$\begin{array}{rcl} \mathsf{h}_c(\varinjlim \mathsf{F}) & = & \mathsf{Hom}_{\underline{\mathsf{C}}}(\varinjlim \mathsf{F}, c) \\ & \cong & \varprojlim \mathsf{Hom}_{\underline{\mathsf{C}}}(\mathsf{F}, c) \text{ (since } \mathsf{Y}' \text{ preserves colimits)} \\ & \cong & \varprojlim(\mathsf{h}_c \circ \mathsf{F}) \end{array}$$

Now, if P is $\aleph$-ind-representable, then $P = \varinjlim_{\mathsf{J}} \mathsf{h}_{c_j}$. Therefore,

$$\begin{array}{rcl} (\varinjlim_{\mathsf{J}} \mathsf{h}_{c_j})(\varinjlim_{\mathsf{I}} \mathsf{F}) = \varinjlim_{\mathsf{J}}(\mathsf{h}_{c_j}(\varinjlim_{\mathsf{I}} \mathsf{F})) & = & \varinjlim_{\mathsf{J}} \varprojlim_{\mathsf{I}}(\mathsf{h}_{c_j} \circ \mathsf{F}) \\ & \cong & \varprojlim_{\mathsf{I}} \varinjlim_{\mathsf{J}}(\mathsf{h}_{c_j} \circ \mathsf{F}) \\ & = & \varprojlim_{\mathsf{I}}((\varinjlim_{\mathsf{J}} \mathsf{h}_{c_j}) \circ \mathsf{F}) = \varprojlim_{\mathsf{I}}(P \circ \mathsf{F}), \end{array}$$

the key passage being because finite limits and filtered colimits commute in $\underline{\mathsf{Set}}$. ✓

We have seen in Section 3.2 that the Yoneda's embedding Y preserves arbitrary small limits. If we look at Y as a functor from $\underline{\mathsf{C}}$ to the category of ($\aleph$-)ind-representable functors, we can show that Y preserves finite colimits. However, it is false that Y preserves filtered colimits which exist in $\underline{\mathsf{C}}$, i.e., $c = \varinjlim \mathsf{F}$ in $\underline{\mathsf{C}}$ does not imply $\mathsf{h}_c \cong \varinjlim(\mathsf{YF})$.

PROPOSITION 3.3.23
$\mathsf{Y}\colon \underline{\mathsf{C}} \to \widehat{\underline{\mathsf{C}}}$ ($\mathsf{Y}\colon \underline{\mathsf{C}} \to \widehat{\underline{\mathsf{C}}}_{\aleph}$) preserves finite colimits

Proof. Let $c = \varinjlim_{\mathsf{I}} c_i$. Then, for any $P \in \widehat{\underline{\mathsf{C}}}$, supposing that $P \cong \varinjlim_{\mathsf{K}} \mathsf{h}_{x_k}$, we have

$$\begin{array}{rcl} \underline{\mathsf{Set}}^{\underline{\mathsf{C}}^{\mathrm{op}}}[\mathsf{h}_c, P] & \cong & P(c) \cong (\varinjlim_{\mathsf{K}} \mathsf{h}_{x_k})c \cong \varinjlim_{\mathsf{K}} \mathsf{Hom}_{\underline{\mathsf{C}}}(c, x_k) \\ & \cong & \varinjlim_{\mathsf{K}} \mathsf{Hom}_{\underline{\mathsf{C}}}(\varinjlim_{\mathsf{I}} c_i, x_k) \cong \varinjlim_{\mathsf{K}} \varprojlim_{\mathsf{I}} \mathsf{Hom}_{\underline{\mathsf{C}}}(c_i, x_k) \\ & \cong & \varprojlim_{\mathsf{I}} \varinjlim_{\mathsf{K}} \mathsf{Hom}_{\underline{\mathsf{C}}}(c_i, x_k) \cong \varprojlim_{\mathsf{I}} \underline{\mathsf{Set}}^{\underline{\mathsf{C}}^{\mathrm{op}}}[\mathsf{h}_{c_i}, \varinjlim_{\mathsf{K}} \mathsf{h}_{x_k}] \\ & \cong & \underline{\mathsf{Set}}^{\underline{\mathsf{C}}^{\mathrm{op}}}[\varinjlim_{\mathsf{I}} \mathsf{h}_{c_i}, P]. \end{array}$$

It follows that $\mathsf{Y}(\varinjlim_{\mathsf{I}} c_i) \cong \varinjlim_{\mathsf{I}} \mathsf{Y}(c_i)$. ✓

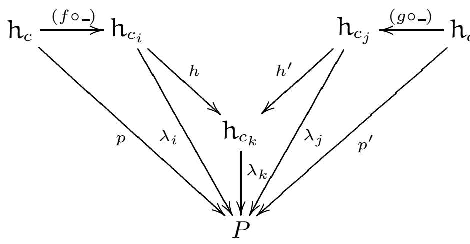
Now, we see that the results about $\underline{\mathsf{Set}}^{\underline{\mathsf{C}}^{\mathrm{op}}}$ in Section 3.2 can be mimicked for $\widehat{\underline{\mathsf{C}}}_{\aleph}$. The key fact is the following proposition, which gives us a *canonical* way to see every $\aleph$-ind-representable functor as an essentially $\aleph$-filtered diagram in $\underline{\mathsf{C}}$.

PROPOSITION 3.3.24
A presheaf P on $\underline{\mathsf{C}}$ is $\aleph$-ind-representable if and only if $\int_{\underline{\mathsf{C}}} P$ is essentially $\aleph$-filtered.

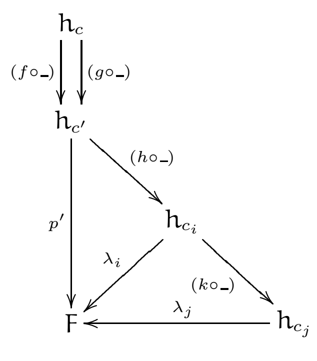
Proof. Since for any presheaf P we have $P \cong \varinjlim\left(\int_{\underline{\mathsf{C}}} P \xrightarrow{\pi_P} \underline{\mathsf{C}} \xrightarrow{\mathsf{Y}} \underline{\mathsf{Set}}^{\underline{\mathsf{C}}^{\mathrm{op}}}\right)$, one implication is obvious.

Therefore, if the category of elements of P is essentially $\aleph$-filtered then P is in $\widehat{\underline{\mathsf{C}}}_{\aleph}$.

Suppose now that $P \cong \varinjlim_{\mathsf{I}} h_{c_i}$, where I is essentially ℵ-filtered and let $\lambda_i : \mathsf{h}_{c_i} \to P$ be the components of the limit cocone. Let (c, p) and (c', p') be objects of $\int_{\underline{\mathsf{C}}} P$. Recall that p and p' can be thought of as morphisms $p : \mathsf{h}_c \xrightarrow{\cdot} P$ and $p' : \mathsf{h}_{c'} \xrightarrow{\cdot} P$, and since $\underline{\mathsf{Set}}^{\underline{\mathsf{C}}^{\mathrm{op}}}[\mathsf{h}_c, P] \cong \varinjlim_{\mathsf{I}} \mathsf{Hom}_{\underline{\mathsf{C}}}(c, c_i)$, from the characterization of set-valued filtered colimits of Proposition 3.3.10, these transformations must come from morphisms $f : c \to c_i$ and $g : c' \to c_j$ in $\underline{\mathsf{C}}$, for $i, j \in \mathsf{I}$, such that $p = \lambda_i \circ (f \circ \_)$ and $p' = \lambda_j \circ (g \circ \_)$. Thus we have $(c, p) \xrightarrow{f} (c_i, \lambda_i)$ and $(c', p') \xrightarrow{g} (c_j, \lambda_j)$ in $\int_{\underline{\mathsf{C}}} P$. Now, since I is filtered, we can find k in I and $h : c_j \to c_k$ and $h' : c'_j \to c_k$ in $\underline{\mathsf{C}}$. Consider (c_k, λ_k). Of course, we have $\lambda_i = \lambda_k \circ (h \circ \_)$ and $\lambda_j = \lambda_k \circ (h' \circ \_)$ and thus we have $(c, p) \xrightarrow{f} (c_i, \lambda_i) \xrightarrow{h} (c_k, \lambda_k)$ and $(c', p') \xrightarrow{g} (c_j, \lambda_j) \xrightarrow{h'} (c_k, \lambda_k)$ in $\int_{\underline{\mathsf{C}}} P$. This is summarized by the following picture.



Consider now $(c, p) \underset{g}{\overset{f}{\rightrightarrows}} (c', p')$. Recall that by definition of morphism in the category of elements of P, $p' \circ (f \circ \_) = p = p' \circ (g \circ \_)$. Reasoning as before we get that p' corresponds to $h : c' \to c_i$ such that $(c', p') \xrightarrow{h} (c_i, \lambda_i)$. Then we have $\lambda_i \circ (h \circ f \circ \_) = \lambda_i \circ (h \circ g \circ \_)$ By the characterization of filtered colimits in $\underline{\mathsf{Set}}$ of Proposition 3.3.10 this means that it must exist an arrow $i \to j$ in the index category I whose image in $\underline{\mathsf{C}}$, say k, equalizes $h \circ f$ and $h \circ g$. It follows that $(\mathsf{h}_{c_i}, \lambda_i) \xrightarrow{k} (\mathsf{h}_{c_j}, \lambda_j)$ equalizes f and g in $\int_{\underline{\mathsf{C}}} P$. This is illustrated below.



Finally, we need to see that $\int_{\underline{\mathsf{C}}} P$ is essentially $\aleph$-filtered. To this aim consider the full subcategory J of $\int_{\underline{\mathsf{C}}} P$ consisting of objects of the form (c_i, λ_i), which is isomorphic to I and, therefore, $\aleph$-filtered. Then, to conclude the proof it is enough to show that the inclusion of J in $\int_{\underline{\mathsf{C}}} P$ is cofinal. However, reasoning as before, this follows immediately from Lemma 3.3.14 *(iii)*. ✓

Of course, it follows from the proof above that P is ind-representable if and only if its category of elements is essentially filtered. Proposition 3.3.24 allows us to show the following.

PROPOSITION 3.3.25
The category $\widehat{\underline{\mathsf{C}}}_{\aleph}$ is $\aleph$-filtered cocomplete.

Proof. Consider a diagram $\mathsf{F}\colon \mathsf{J} \to \widehat{\underline{\mathsf{C}}}_{\aleph}$, where J belongs to $\underline{\mathcal{E}}\text{-}\underline{\mathsf{Filt}}_{\aleph}$. Consider $\mathsf{T} = \varinjlim_{\mathsf{J}} \mathsf{F}$ in $\underline{\mathsf{Set}}^{\underline{\mathsf{C}}^{\mathrm{op}}}$. Let us construct $\mathsf{K} = \int_{\underline{\mathsf{C}}} \mathsf{T}$. Now, reasoning as in the proof of Proposition 3.3.24, it is not difficult to see that K is essentially $\aleph$-filtered.

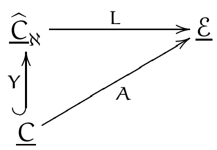
In the following let P_j denote $\mathsf{F}j$. Consider $(c, p) \in \mathsf{K}$. Then $p\colon \mathsf{h}_c \to \varinjlim_{\mathsf{J}} P_j$ corresponds to an element of $(\varinjlim_{\mathsf{J}} P_j)c = \varinjlim_{\mathsf{J}}(P_j(c))$ which by Yoneda's lemma is $\varinjlim_{\mathsf{J}} \mathsf{Hom}(\mathsf{h}_c, P_j)$. Therefore, p factorizes through an object P_j and the component λ_j of the limit cocone for T. Then, we can proceed as in Proposition 3.3.24 to conclude the proof. It follows that $\widehat{\underline{\mathsf{C}}}_{\aleph}$ in $\aleph$-filtered cocomplete. ✓

Clearly, there is the corresponding result for the ind-representables.

PROPOSITION 3.3.26
The category $\widehat{\underline{\mathsf{C}}}$ is filtered cocomplete.

Finally, we have the expected 2-universal property for $\widehat{\underline{\mathsf{C}}}_{\aleph}$ and $\widehat{\underline{\mathsf{C}}}$.

COROLLARY 3.3.27
For any $\aleph$-filtered cocomplete $\underline{\mathcal{E}}$, any locally small $\underline{\mathsf{C}}$ and any functor $\mathsf{A}\colon \underline{\mathsf{C}} \to \underline{\mathcal{E}}$, there is a functor $\mathsf{L}\colon \widehat{\underline{\mathsf{C}}}_{\aleph} \to \underline{\mathcal{E}}$ which preserves $\aleph$-filtered colimits and such that the following diagram commutes.



Morevover, L is unique up to isomorphisms.

Proof. Consider $\mathsf{L}(P) = \varinjlim\Big(\int_{\underline{\mathsf{C}}} P \xrightarrow{\pi_P} \underline{\mathsf{C}} \xrightarrow{\mathsf{Y}} \widehat{\underline{\mathsf{C}}}_{\aleph}\Big)$. Observe that this is just the restriction to $\widehat{\underline{\mathsf{C}}}_{\aleph}$ of the functor used for the general case of $\underline{\mathsf{Set}}^{\underline{\mathsf{C}}^{\mathrm{op}}}$ in Corollary 3.2.6, and that this restriction can be used because of the particular shape the category of elements of P takes when P is $\aleph$-ind-representable. It follows at once that it preserves the colimits which exist in $\widehat{\underline{\mathsf{C}}}_{\aleph}$ and that it renders the diagram commutative. The uniqueness up to isomorphisms of L comes from these two facts following the same formal development of Corollary 3.2.6. ✓

Corollary 3.3.28
For any filtered cocomplete $\underline{\mathcal{E}}$, any locally small $\underline{\mathsf{C}}$ and any functor $\mathsf{A}\colon \underline{\mathsf{C}} \to \underline{\mathcal{E}}$, there is a functor, unique up to isomorphisms, $\mathsf{L}\colon \widehat{\underline{\mathsf{C}}} \to \underline{\mathcal{E}}$ which preserves filtered colimits and such that $\mathsf{LY} = \mathsf{A}$.

Observe that the cardinality problem we met at the end of Section 3.2 disappears in the case of the $\aleph$-filtered cocompletion. Therefore, the construction above gives a 2-categorical (pseudo) adjunction [40]. However, it does not look very easy to define such a construction functorially on $\underline{\underline{\mathsf{CAT}}}$. Moreover, the fact that the universal property holds only up to isomorphisms, brings us away from the classical world of free constructions. Next section is devoted to a wide discussion of this fact.

3.4 KZ-Doctrines and Pseudo-Monads

Free constructions are defined up to isomorphism, which reflects the fact that the stress is on the essential structure to be added, irrespective of the actual representation chosen for such a structure. Correspondingly, free constructions, as left adjoints to forgetful functors, give rise to *monads* [22] (see also Appendix A.1), i.e., to algebraic constructions.

In the formulation above, however, the freeness condition is verified only up to isomorphism of the functors involved. Thus, the free object is identified up to equivalence of categories, in the precise sense that the (infinitely many) categories which enjoy the universal property are "only" equivalent—as opposed to isomorphic—to each other. Correspondingly, the cocompletion functors do not give rise to cocompletion functors do not give rise to an adjunction or, equivalently, to a monad (however see [71]). We would like to stress that, since the notion of colimit is defined only up to isomorphism, it is *not* a strictly algebraic operation, and therefore it would not be reasonable to expect a stronger form of universality.

Situations like this arise often in the everyday practice in mathematics, and a lot of work has been done in order to formalize them in category theory, e.g. [5, 42, 76, 39, 40, 135, 137, 10, 149], where equality is replaced by equivalence of morphisms or, even weaker, by the mere existence of a 2-cell between two morphisms. Needless to say, it is very often the case that this 2-cells have to be related by coherence isomorphisms themselves. So these constructions make sense in 3-categories, like $\underline{\underline{\underline{\mathsf{Cat}}}}$. There are many natural examples of situations where the *pseudo* version (i.e. up to coherent isomorphisms) or the *lax* version (i.e. up to coherent 2-cells) of algebraic laws seems to be the natural requirement. Perhaps the most evident example is the case of monoidal categories [21] where the standard notion of *monoidal functor* is not required to commute with the monoidal structure "on the nose", but only up to isomorphisms or up to canonical 2-cells.

Of course another such example is, in our opinion, the cocompletion construction we are interested in. Thus, instead of trying to "fish out" some peculiar representatives in order to make the colimit notion behave strictly algebraically, we prefer to adopt a viewpoint also taken by other authors [71, 76, 149] who recognize its 2-categorical "lax" nature and formalize it as a pseudo-adjunction, or equivalently as a pseudo-monad. However, the problem with this approach is that the needed coherence conditions may look quite overwhelming sometimes. For instance, Zöberlein's *2-doctrines* [149, 150], i.e., 2-functors on 2-categories with unit and multiplication natural only up to isomorphism for which the laws for monads, algebras and homomorphisms hold up to isomorphisms, must be provided with 19 coherence axioms. Fortunately enough, in the nice case of *coquasi-idempotent doctrines* [149, 150], which are what is needed for the cocompletion construction, most of them disappear. In the following we shall recall the basics of *KZ-doctrines* or *KZ-monads* [76, 137] (KZ standing for Kock-Zöberlein), which are a simpler representation of the cited notion. In particular, the most relevant feature of KZ-doctrines is that all we need about coherent isomorphisms of 1-cells is contained in a single piece of information, namely a family of 2-cells.

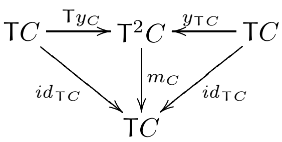
Remark. In the following we shall be dealing with 2-categories (see Appendix A.3). As a matter of notation, we shall denote by $\_ * \_$ the horizontal composition (Godement [35]) and by $\_ \bullet \_$ the vertical composition of 2-cells, while we stick to the classical $\_ \circ \_$ for the horizontal composition of 1-cells. Identity 1-cells are written as id_C, or simply id, while for the identity 2-cell of a 1-cell f we use f itself, since confusion is never possible. Moreover, when the 1-cell involved is not relevant, we write $\mathbf{1}$ to indicate a generic identity 2-cell. We tend to avoid parenthesis around the arguments of 2-functors.

Definition 3.4.1 *(KZ-Doctrines)*
A KZ-doctrine on a 2-category $\underline{\underline{\mathsf{C}}}$ *is a tuple* $(\mathsf{T}, y, m, \lambda)$, *where*

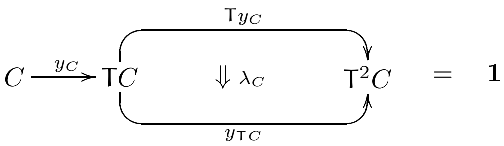
- $\mathsf{T} \colon \underline{\underline{\mathsf{C}}} \to \underline{\underline{\mathsf{C}}}$ *is a 2-endofunctor;*
- $y \colon Id \stackrel{\cdot}{\to} \mathsf{T}$ *and* $m \colon \mathsf{T}^2 \stackrel{\cdot}{\to} \mathsf{T}$ *are 2-natural transformations;*
- λ *is a family of 2-cells* $\{\lambda_C \colon \mathsf{T}y_C \Rightarrow y_{\mathsf{T}C} \colon \mathsf{T}C \to \mathsf{T}^2 C\}_{C \in \underline{\underline{\mathsf{C}}}}$ *indexed by the objects of* $\underline{\underline{\mathsf{C}}}$*;*

satisfying the following axioms.

T0: $m_C \circ \mathsf{T}y_C = m_C \circ y_{\mathsf{T}C} = id_{\mathsf{T}C}$;

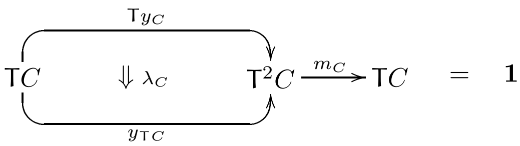


T1: $\lambda_C * y_C = \mathbf{1}$;



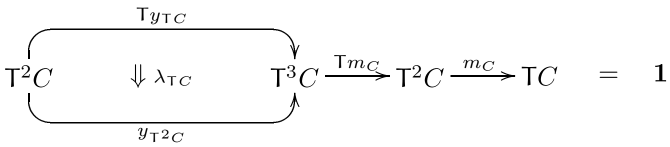
Observe that $\mathsf{T}y_C \circ y_C = y_{\mathsf{T}C} \circ y_C$ *follows by naturality of* y*, so the equation between 2-cells makes sense.*

T2: $m_C * \lambda_C = \mathbf{1}$;



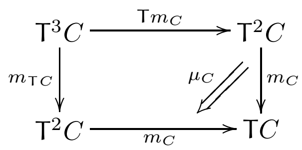
Observe that by **T0** *we have* $m_C \circ \mathsf{T}y_C = m_C \circ y_{\mathsf{T}C} = id_{\mathsf{T}C}$.

T3: $m_C * \mathsf{T}m_C * \lambda_{\mathsf{T}C} = \mathbf{1}$;



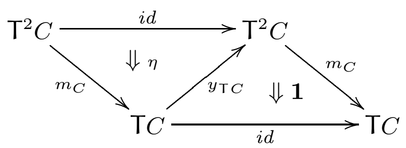
Here $m_C \circ \mathsf{T}m_C \circ \mathsf{T}y_{\mathsf{T}C} = m_C \circ \mathsf{T}m_C \circ y_{\mathsf{T}^2C} = m_C$ *comes as follows. We have* $\mathsf{T}m_C \circ \mathsf{T}y_{\mathsf{T}C} = \mathsf{T}(m_C \circ y_{\mathsf{T}C}) = id_{\mathsf{T}^2C}$*, by* **T0**. *By naturality of* y*, we have* $\mathsf{T}m_C \circ y_{\mathsf{T}^2C} = y_{\mathsf{T}C} \circ m_C$*, and thus* $m_C \circ \mathsf{T}m_C \circ y_{\mathsf{T}^2C} = m_C \circ y_{\mathsf{T}C} \circ m_C = m_C$*, the last equality by* **T0**.

Thus, T, y and m play the role of the *functor*, *unit* and *multiplication* of an ordinary 2-monad [10]. In particular, y and m are *actual* (not pseudo) 2-natural transformations. As anticipated, the only additional 2-dimensional information around is λ and every coherence isomorphism is obtained from it. Axiom $\mathbf{T_0}$ corresponds to the *unit law* of monads, that therefore holds strictly also in KZ-doctrines. Axioms $\mathbf{T_1}$, $\mathbf{T_2}$ and $\mathbf{T_3}$ express the coherence of λ with the unit and the multiplication. Observe that there is no explicit mention of a pseudo form of the *multiplication law*. However, we shall see later that this is indeed the case and, for any $C \in \underline{\underline{C}}$, there exists an isomorphism $\mu_C \colon m_C \circ \mathsf{T}m_C \Rightarrow m_C \circ m_{\mathsf{T}C} \colon \mathsf{T}^3C \to \mathsf{T}C$.

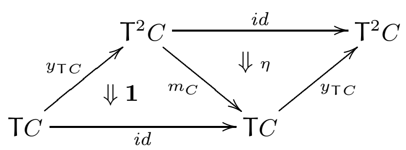


PROPOSITION 3.4.2
For any $C \in \underline{\underline{C}}$ we have a reflection $m_C \dashv y_{\mathsf{T}C} \colon \mathsf{T}^2C \rightharpoonup \mathsf{T}C$, the unit of the adjunction being $\mathsf{T}m_C * \lambda_{\mathsf{T}C} \colon id_{\mathsf{T}^2C} \Rightarrow y_{\mathsf{T}C} \circ m_C$.

Proof. Here and in the following, we denote by η and ε respectively the unit and counit of a generic adjunction. Let us check that the triangular identities hold. Remember that for a reflection the counit is the identity 2-cell.



$= \quad m_C * \mathsf{T}m_C * \lambda_C = \mathbf{1}, \quad$ by $\mathbf{T_3}$.



$= \quad \mathsf{T}m_C * \lambda_{\mathsf{T}C} * y_{\mathsf{T}C} = \mathbf{1}, \quad$ by $\mathbf{T_1}$.

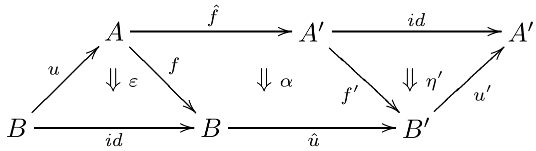
✓

DEFINITION 3.4.3 *(T-Algebras)*
An algebra for T is an object $A \in \underline{\underline{C}}$ together with a structure map $\mathfrak{a} \colon \mathsf{T}A \to A$ which is a reflection left adjoint for $y_A \colon A \to \mathsf{T}A$.

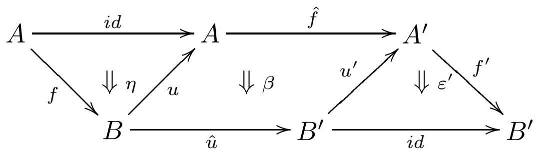
Thus, *structures* are *adjoints* to *units* [76]. Observe that, since $\mathfrak{a} \dashv y_A$ is a reflection, we have $\mathfrak{a} \circ y_A = id$. Therefore, as in the case of the KZ-doctrine itself, the *unit law* for the structure of an algebra holds strictly. Since we have $m_C \dashv y_{\mathsf{T}C}$, for any $C \in \underline{\underline{C}}$ there is a "free" algebra on C, namely $(\mathsf{T}C, m_C)$.

In order to define T-homomorphisms, we need to recall the well known concept of 2-cells *mates under adjunctions* (see, e.g., [69]). It is based on the operation of *diagram pasting* [5, 69] (see also Appendix A.3). Given the adjunctions $f \dashv u\colon A \to B$ and $f' \dashv u'\colon A' \to B'$ whose respective units and counits are η, ε and η', ε' and given the 1-cells $\hat{f}\colon A \to A'$ and $\hat{u}\colon B \to B'$, there is a bijection between the 2-cells $\alpha\colon f' \circ \hat{f} \Rightarrow \hat{u} \circ f$ and $\beta\colon \hat{f} \circ u \Rightarrow u' \circ \hat{u}$ given by the following correspondence:

$\alpha \quad \mapsto$



$\beta \quad \mapsto$



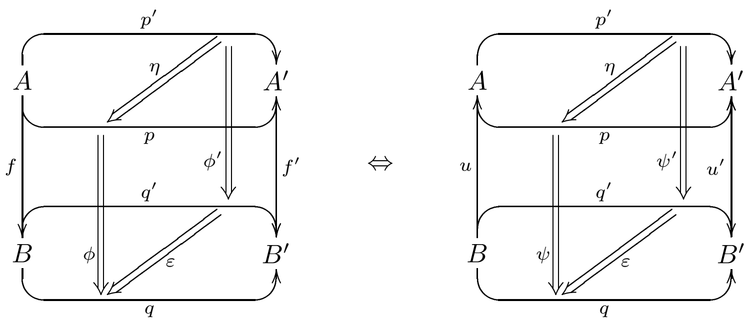
In other words, the mate of a 2-cell is obtained by pasting the appropriate unit and counit at its ends. Although the mate of an identity 2-cell is not necessarily an identity, the bijection above respects composition, both horizontal and vertical. Corresponding α and β are said to be *mates* under the adjunctions $f \dashv u$ and $f' \dashv u'$ (wrt. $\hat{f}$ and $\hat{u}$).

Using the fact that *mating* respects vertical composition, it is easy to show the following easy lemma [76], which will be useful later on.

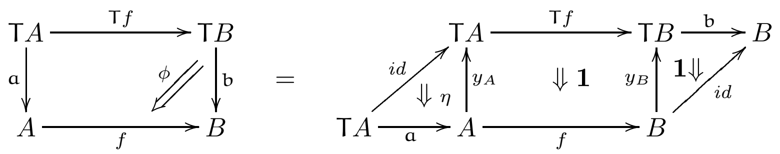
Lemma 3.4.4
Let $f \dashv u\colon A \rightharpoonup B$ and $f' \dashv u'\colon A' \rightharpoonup B'$ be adjunctions, and let $q, q'\colon A \to A'$ and $p, p'\colon B \to B'$ together with 2-cells $\eta\colon p' \Rightarrow p$ and $\varepsilon\colon q' \Rightarrow q$ such that $\varepsilon * f = f' * \eta$. Now consider the 2-cells $\phi\colon f' \circ p \Rightarrow q \circ f$ and $\phi'\colon f' \circ p' \Rightarrow q' \circ f$ and let ψ and ψ' be the respective mates under the given adjunctions (wrt. p and q and to p' and q', respectively). Then

$$\phi \centerdot (f' * \eta) = (\varepsilon * f) \centerdot \phi' \quad \Leftrightarrow \quad \psi \centerdot (\eta * u) = (u' * \varepsilon) \centerdot \psi'$$

The situation is summarized in the picture below.



Given T-algebras $(A, \mathfrak{a})$ and $(B, \mathfrak{b})$, consider a morphism $f: A \to B$ in $\underline{\underline{C}}$. By naturality of y, we have that $\mathsf{T}f \circ y_A = y_B \circ f$. Thus, we can consider the identity $\mathbf{1}: \mathsf{T}f \circ y_A \Rightarrow y_B \circ f$ and its mate $\phi: \mathfrak{b} \circ \mathsf{T}f \Rightarrow f \circ \mathfrak{a}$ under the adjunctions $\mathfrak{a} \dashv y_A$ and $\mathfrak{b} \dashv y_B$ wrt. f and $\mathsf{T}f$.



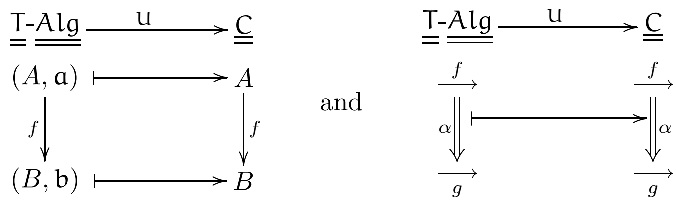
We shall refer to ϕ as the *canonical 2-cell* associated to f.

DEFINITION 3.4.5 *(T-homomorphisms)*
A T-homomorphism f from the T-algebra $(A, \mathfrak{a})$ to the T-algebra $(B, \mathfrak{b})$ is a morphism $f: A \to B$ whose canonical 2-cell is invertible.

Since the calculus of mates preserves composition, given the algebras $(A, \mathfrak{a})$ and $(B, \mathfrak{b})$, we have that if ϕ_f and ϕ_g are the canonical 2-cells of $f: A \to B$ and $g: B \to C$, then the canonical 2-cell $\phi_{g \circ f}$ associated to $g \circ f$ is $\phi_g * \phi_f$. Moreover, a simple shot of pasting shows that the canonical 2-cell associated to id_A is the identity 2-cell. Therefore, we have the following.

PROPOSITION 3.4.6
T-algebras and T-homomorphisms form a category $\underline{\mathsf{T}\text{-}\mathsf{Alg}}$ which is lifted to a 2-category $\underline{\underline{\mathsf{T}\text{-}\mathsf{Alg}}}$ by enriching it with all the 2-cells in $\underline{\underline{C}}$.

It follows immediately from the definitions that the forgetful functor



is *faithful* and *locally fully faithful*, i.e., $\underline{\underline{\mathsf{T}\text{-}\mathsf{Alg}}}[f, g] = \underline{\underline{C}}[f, g]$.

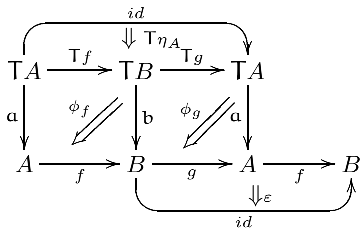
Next, we state two important cases in which one can conclude that a morphism is a T-homomorphism.

PROPOSITION 3.4.7
Let $(A, \mathfrak{a})$, $(B, \mathfrak{b})$ be T-algebras. If $f: A \to B$ is invertible, then f is a T-homomorphism.

Proof. The canonical 2-cell associated to $f^{-1} \circ f = id = f \circ f^{-1}$ is the identity 2-cell, but it is also the composition $\phi_{f^{-1}} * \phi_f$ and the composition $\phi_f * \phi_{f^{-1}}$. This means that ϕ_f is invertible. ✓

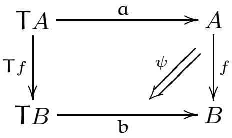
PROPOSITION 3.4.8
Let $(A, \mathfrak{a})$, $(B, \mathfrak{b})$ be T-algebras. If $f: A \to B$ is a left adjoint, then f is a T-homomorphism.

Proof. Let $\eta, \varepsilon: f \dashv g: A \rightharpoonup B$ be an adjunction. Observe that, since T is a 2-functor, we also have an adjunction $\mathsf{T}\eta, \mathsf{T}\varepsilon: \mathsf{T}f \dashv \mathsf{T}g: \mathsf{T}A \rightharpoonup \mathsf{T}B$. Now consider the following diagram built on the canonical 2-cells ϕ_f and ϕ_g respectively for f and g.

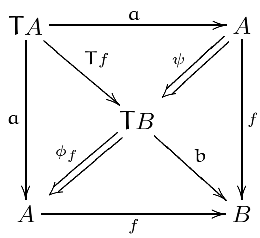


(3.2)

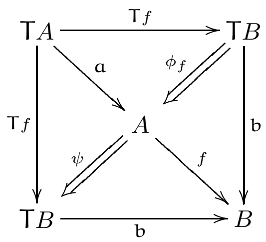
Now, dropping from (3.2) the square corresponding to ϕ_f and pasting the rest of it we obtain a 2-cell $\psi: f \circ \mathfrak{a} \Rightarrow \mathfrak{b} \circ \mathsf{T}f$, which is the mate of ϕ_g under the adjunctions $f \dashv g$ and $\mathsf{T}f \dashv \mathsf{T}g$.



Now, we claim that



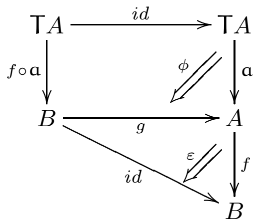
$= \mathbf{1}$ and



$= \mathbf{1}$,

i.e., ψ and ϕ_f are each other's inverse, which shows the proposition. In order to show the first of the two equations, one has to show that pasting (3.2) yields the identity 2-cell. By 2-naturality of y, it is $\mathsf{T}\eta_A * y_A = y_A * \eta_A$, which can be written as $\mathbf{1} \centerdot (\mathsf{T}\eta_A * y_A) = (y_A * \eta_A) \centerdot \mathbf{1}$.

Now, by applying Lemma 3.4.4 to the equality above, we get $(\eta_A * \mathfrak{a}) \centerdot \bar{\phi} = \bar{\phi}' \centerdot (\mathfrak{a} * \mathsf{T}\eta_A)$. But $\bar{\phi}$ is the mate of $\mathbf{1}_{y_A}$, which is easily seen to be an identity 2-cell, while $\bar{\psi}'$ reduces to $\phi_g * \psi_f$. Therefore, we have $\eta_A * \mathfrak{a} = (\phi_g * \psi_f) \centerdot (\mathfrak{a} * \mathsf{T}\eta_A)$. Therefore, diagram (3.2) reduces to



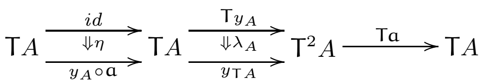
which is $(\varepsilon * f * \mathfrak{a}) \centerdot (f * \eta * \mathfrak{a}) = ((\varepsilon * f) \centerdot (f * \eta)) * \mathfrak{a} = f * a = \mathbf{1}$, the last but one equality being because the first subterm is a triangular identity for $f \dashv g$.

Thus, the first equality is shown. The other one is obtained in an similar way working on the diagram obtained from (3.2) by moving the part corresponding to ϕ_f on the right hand side of (3.2). ✓

T-algebras are characterized by structure maps which are adjoint to a given morphism. Therefore, the structure on a given algebra is unique up to isomorphisms. Moreover, fixed a structure map the unit of the adjunction is uniquely determined via λ.

PROPOSITION 3.4.9
Let $(A, \mathfrak{a})$ be an algebra and suppose that η is the unit of $\mathfrak{a} \dashv y_A$. Then, $\eta = \mathsf{T}\mathfrak{a} * \lambda_A$. Moreover, a morphism $\mathfrak{a}: \mathsf{T}A \to A$ such that $\mathfrak{a} \circ y_A = id$ is a structure map if and only if $\mathfrak{a} * \mathsf{T}\mathfrak{a} * \lambda_A = \mathbf{1}$.

Proof. Consider



We can calculate the pasting in two different ways as follows.

(i) $(\mathsf{T}\mathfrak{a} * y_{\mathsf{T}A} * \eta) \centerdot (\mathsf{T}\mathfrak{a} * \lambda_A)$ which, by naturality of y, is $(y_A * \mathfrak{a} * \eta) \centerdot (\mathsf{T}\mathfrak{a} * \lambda_A) = \mathsf{T}\mathfrak{a} * \lambda_A$, since $\mathfrak{a} * \eta = \mathbf{1}$ is one of the triangular identities.

(ii) $(\mathsf{T}\mathfrak{a} * \lambda_A * y_A * \mathfrak{a}) \centerdot (\mathsf{T}\mathfrak{a} * \mathsf{T}y_A * \eta)$, which, by $\mathbf{T_1}$, reduces to $(\mathsf{T}\mathfrak{a} * \mathsf{T}y_A * \eta) = \mathsf{T}(\mathfrak{a} \circ y_A) * \eta = \eta$, since $\mathfrak{a} \circ y_A = id$ by hypothesis.

Therefore, $\eta = \mathsf{T}\mathfrak{a} * \lambda_A$. Take now any $\mathfrak{a}: \mathsf{T}A \to A$ such that $\mathfrak{a} \circ y_A = id$. Suppose that $\mathfrak{a} \dashv y_A$. Then, by definition of adjunction, $\mathfrak{a} * \eta = \mathbf{1}$, and since $\eta = \mathsf{T}\mathfrak{a} * \lambda_A$, we are done. On the other hand, suppose that $\mathsf{T}\mathfrak{a} * \lambda_A * \mathfrak{a} = \mathbf{1}$. Then, choosing $\eta = \mathsf{T}\mathfrak{a} * \lambda_A$, it is easy to verify that the conditions for adjointness hold: $\mathfrak{a} * \eta = \mathbf{1}$ because of our choice and $\eta * y_A = \mathsf{T}\mathfrak{a} * \lambda_A * y_A = \mathbf{1}$ by $\mathbf{T_1}$. ✓

Since the canonical 2-cells of morphisms $f: A \to B$ are mates of an identity 2-cell under adjunctions whose units can be expressed through λ and whose counits are identities, the following result is vary natural.

Proposition 3.4.10
Let $(A, \mathfrak{a})$ and $(B, \mathfrak{b})$ be algebras and consider $f: A \to B$. Then, the canonical 2-cell associated to f is $\phi_f = \mathfrak{b} * \mathsf{T}f * \mathsf{T}a * \lambda_A$.

Proof. The mate under $\mathfrak{a} \dashv y_A$ and $\mathfrak{b} \dashv y_B$ of $\mathbf{1}: \mathsf{T}f \circ y_A \Rightarrow y_B \circ f$, is easily computed as $(f * \varepsilon_B) \centerdot (\mathfrak{b} * \mathbf{1} * \mathfrak{a}) \centerdot (\mathfrak{b} * \mathsf{T}f * \eta_A) = \mathfrak{b} * \mathsf{T}f * \eta_A = \mathfrak{b} * \mathsf{T}f * \mathsf{T}\mathfrak{a} * \lambda_A$. ✓

The following result, due to Street [135], is rather interesting.

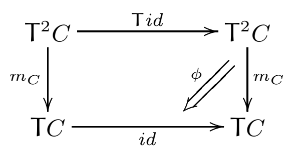
Proposition 3.4.11 *(Recognition Lemma)*
Let $(A, \mathfrak{a})$ and $(B, \mathfrak{b})$ be algebras and consider a morphism $f: A \to B$. Then, a 2-cell $\phi: \mathfrak{b} \circ \mathsf{T}f \Rightarrow f \circ \mathfrak{a}$ is the canonical 2-cell of f if and only if $\phi * y_A = \mathbf{1}$.

Proof. If ϕ is the canonical 2-cell associated to f, then, by the previous proposition, $\phi = \mathfrak{b} * \mathsf{T}f * \mathsf{T}\mathfrak{a} * \lambda_A$ and thus $\phi * y_A = \mathbf{1}$ by $\mathbf{T_1}$. Suppose instead that $\phi * y_A = \mathbf{1}$. The mate of ϕ under the adjunctions $\mathfrak{a} \dashv y_A$ and $\mathfrak{b} \dashv y_B$ is $\psi = (y_B * \phi * y_A) \centerdot (\eta * y_B * f) = (y_B * \phi * y_A)$, since η being the unit of $\mathfrak{b} \dashv y_B$ is annihilated by y_B. Therefore, if $\phi * y_A = \mathbf{1}$ then $\psi = \mathbf{1}$. Since the calculus of mates provides a bijection, it follows that ϕ is the mate of the appropriate identity, i.e., the canonical 2-cell of f. ✓

Next, we recall some properties of the multiplication m. The following proposition identifies an equation "dual" to $\mathbf{T_3}$.

Proposition 3.4.12
In a KZ-doctrine we have $\mathbf{T'_3}$: $m_C * m_{\mathsf{T}C} * \mathsf{T}\lambda_C = \mathbf{1}$.

Proof. Observe that the source and the target of the 2-cell above are both m_C. In fact, $m_C * m_{\mathsf{T}C} * \mathsf{T}\lambda_C: m_C \circ m_{\mathsf{T}C} \circ \mathsf{T}^2 y_C \Rightarrow m_C \circ m_{\mathsf{T}C} \circ \mathsf{T}y_{\mathsf{T}C}$. Now, $m_{\mathsf{T}C} \circ \mathsf{T}y_{\mathsf{T}C} = id$, by $\mathbf{T_0}$, and thus the target is m_C. On the other hand, by naturality of m, we have that $m_{\mathsf{T}C} \circ \mathsf{T}^2 y_C = \mathsf{T}y_C \circ m_C$, whence, again by $\mathbf{T_0}$, it follows $m_C \circ m_{\mathsf{T}C} \circ \mathsf{T}^2 y_C = m_C$. Therefore, putting $\phi = m_C * m_{\mathsf{T}C} * \mathsf{T}\lambda_C$, we have



and then, to conclude that $\phi = \mathbf{1}$, it is enough to see that it is the canonical 2-cell associated to $id: \mathsf{T}C \to \mathsf{T}C$. From Proposition 3.4.11, this follows if $\phi * y_{\mathsf{T}C} = \mathbf{1}$.

$$\begin{array}{rcll} m_C * m_{\mathsf{T}C} * \mathsf{T}\lambda_C * y_{\mathsf{T}C} & = & m_C * m_{\mathsf{T}C} * y_{\mathsf{T}^2C} * \lambda_C & \text{by 2-naturality of } y \\ & = & m_C * \lambda_C & \text{by } \mathbf{T_0} \\ & = & \mathbf{1} & \text{by } \mathbf{T_2}, \end{array}$$

which proves the proposition. ✓

The definition of KZ-doctrine implies that for any $C \in \underline{\underline{C}}$ there is a reflection $m_C \dashv y_{\mathsf{T}C}$. It follows from $\mathbf{T'_3}$ that m_C has also a left adjoint.

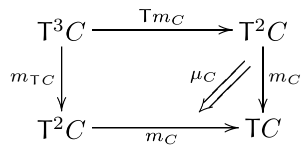
PROPOSITION 3.4.13
For any $C \in \underline{\underline{C}}$ there is a coreflection $\mathsf{T}y_C \dashv m_C$.

Proof. We choose $\varepsilon = m_{\mathsf{T}C} * \mathsf{T}\lambda_C\colon \mathsf{T}y_C \circ m_C \Rightarrow id$ as a candidate for the counit. Let us verify the triangular identities.

(i) $(\varepsilon * \mathsf{T}y_C) \centerdot (\mathsf{T}y_C * \mathbf{1}) = m_{\mathsf{T}C} * \mathsf{T}\lambda_C * \mathsf{T}y_C = m_{\mathsf{T}C} * \mathsf{T}(\lambda_C * y_C)$, which collapse to $\mathbf{1}$ because of $\mathbf{T_1}$.

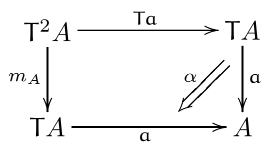
(ii) $(m_C * \varepsilon) \centerdot (\mathbf{1} * m_C) = m_C * \varepsilon = m_C * m_{\mathsf{T}C} * \mathsf{T}\lambda_C$, which, by $\mathbf{T_3'}$, is $\mathbf{1}$. ✓

We complete this recapitulation section about KZ-doctrines by stating more precisely in what sense they are pseudo-monads. First of all, notice that the associativity of m holds up to a canonical isomorphism. In fact, since $m_C\colon \mathsf{T}^2C \to \mathsf{T}C$ is left adjoint to $y_{\mathsf{T}C}$, by Proposition 3.4.8, it is a T-homomorphism with canonical 2-cell $\mu_C\colon m_C \circ \mathsf{T}m_C \Rightarrow m_C \circ m_{\mathsf{T}C}$, i.e.,



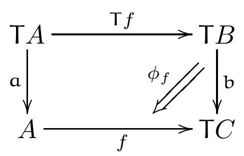
Of course, as canonical 2-cell associated to m_C, we have by Proposition 3.4.10 that $\mu_C = m_C * \mathsf{T}m_C * \mathsf{T}m_{\mathsf{T}C} * \lambda_{\mathsf{T}^2C}$. Moreover, Proposition 3.4.11 provides *coherence conditions* which link μ with y, namely $\mu_C * y_{\mathsf{T}^2C} = \mathbf{1}$ and $\mu_C * \mathsf{T}y_{\mathsf{T}C} = \mathbf{1}$. While the first equation comes directly from the recognition lemma, the second one can be proved as follows. $\mu_C * \mathsf{T}y_{\mathsf{T}C}\colon m_C \circ \mathsf{T}m_C \circ \mathsf{T}y_{\mathsf{T}C} \Rightarrow m_C \circ m_{\mathsf{T}C} \circ \mathsf{T}y_{\mathsf{T}C}$, i.e., from $\mathbf{T_0}$ $\mu_C * \mathsf{T}y_{\mathsf{T}C}\colon m_C \Rightarrow m_C$, which is of the form $m_C \circ \mathsf{T}id_{\mathsf{T}C} \Rightarrow id_{\mathsf{T}C} \circ m_C$. Then, by Proposition 3.4.11, if $\mu_C * \mathsf{T}y_{\mathsf{T}C} * y_{\mathsf{T}C} = \mathbf{1}$, we have that $\mu_C * \mathsf{T}y_{\mathsf{T}C}$ is the canonical 2-cell associated to $id_{\mathsf{T}C}$, i.e., $\mathbf{1}$. Now, $\mu_C * \mathsf{T}y_{\mathsf{T}C} * y_{\mathsf{T}C} = \mu_C * y_{\mathsf{T}^2C} * y_{\mathsf{T}C}$, by naturality of y, which is the identity 2-cell by $\mathbf{T_2}$. Recall from Definition 3.4.1 $\mathbf{T_0}$ that the unit law holds strictly.

The same things can be said for any algebra $(A, \mathfrak{a})$: since $\mathfrak{a}\colon \mathsf{T}A \to A$ is a left adjoint we have an isomorphism $\alpha\colon \mathfrak{a} \circ \mathsf{T}\mathfrak{a} \Rightarrow \mathfrak{a} \circ m_A$, i.e., the associativity law for $\mathfrak{a}$ holds up to isomorphisms.



Moreover, α is coherent with y in the sense that $\alpha * y_{\mathsf{T}A} = \mathbf{1}$ and $\alpha * \mathsf{T}y_A = \mathbf{1}$, which can be shown as in the case of μ. Also in this case the fact that $\mathfrak{a} \dashv y_A$ is a *reflection* implies that the unit law holds strictly for T-algebras.

Finally, the fact that here the monad-theoretic definition of homomorphisms is matched up to isomorphism is just the definition of T-homomorphism.



Once more, the isomorphism $\phi_f\colon \mathfrak{b} \circ \mathsf{T}f \Rightarrow f \circ \mathfrak{a}$ satisfies the coherence condition that $\phi_f * y_A = \mathbf{1}$.

Summing up, KZ-doctrines $(\mathsf{T}, y, m, \lambda)$ can be thought as pseudo-monads (T, y, m, μ) where the unit law holds strictly and the associativity law holds up to the family of isomorphisms μ, which satisfies the coherence conditions $\mu * y\mathsf{T}^2 = \mu * \mathsf{T}y\mathsf{T} = \mathbf{1}$. For such pseudo-monads one defines a pseudo-algebra to be a triple $(A, \mathfrak{a}, \alpha)$, where $\mathfrak{a}\colon \mathsf{T}A \to A$ satisfies the unit law strictly and the associativity up to the isomorphism α which satisfies the coherence axioms $\alpha * y_{\mathsf{T}A} = \alpha * \mathsf{T}y_A = \mathbf{1}$. Finally, one defines the homomorphisms of pseudo-algebras to be pair (f, ϕ_f) where $f\colon A \to B$ is a morphism which satisfies the condition $b \circ \mathsf{T}f = f \circ \mathfrak{a}$ only up to the isomorphism ϕ_f. Furthermore, ϕ_f is required to satisfy the coherence $\phi_f * y_A = \mathbf{1}$.

Let us denote by $\underline{\underline{\mathsf{Pseudo}}}\text{-}\underline{\underline{\mathsf{Alg}}}$ the 2-category of pseudo-algebras and their morphisms, the 2-cells being those of $\underline{\underline{\mathsf{C}}}$. Then, it is not difficult to show the following.

Proposition 3.4.14
$\underline{\underline{\mathsf{T}}}\text{-}\underline{\underline{\mathsf{Alg}}} \cong \underline{\underline{\mathsf{Pseudo}}}\text{-}\underline{\underline{\mathsf{Alg}}}$

In the following section, we shall see how this notion is perfectly suited to describe the cocompletion construction. Of course, this is not surprising since KZ-monads arose from Kock's work on completion of categories [71].

3.5 A KZ-Doctrine for the Ind Completion

Although the construction of Section 3.3 is perfectly satisfactory from the theoretical viewpoint, it does not lend itself easily to a functorial treatment, which is indeed a desirable property. Moreover, in many occasions a more concrete description of the objects of $\underline{\widehat{\mathsf{C}}}$ may be useful. In particular, we think of something very close to the description of infinite processes we have sketched in Section 3.1. Close to this issue, there is the fact that, as already noticed at the beginning of Section 3.4, one would often like a more algebraic description of colimits in terms of pseudo monads. In this section we study a KZ-doctrine for the $\aleph$-filtered cocompletion. In other words, we study alternative representations for the ind-representable presheaves.

<u>REMARK</u>. Since the case of $\aleph$ finite is trivial, in the following we implicitly assume that $\aleph$ is infinite. Moreover, since everything which follows is parametric with respect to $\aleph$, we shall mention it explicitly only when unavoidable. However, the reader should keep in mind that every statement below about "filtered" objects can safely be restated in terms of "$\aleph$-filtered" objects.

IND OBJECTS

The following definition of ind-object follows the same simple idea about representation of infinite computations we discussed in Section 3.1.

DEFINITION 3.5.1 *(Ind-objects)*
A functor $X\colon \mathsf{J} \to \underline{\mathsf{C}}$ is an ind-object if J is small and filtered. If $|\mathsf{J}| \leq \aleph$, then X is a $\aleph$-ind-object.

We shall identify ind-objects $X\colon \mathsf{I} \to \underline{\mathsf{C}}$ and $Y\colon \mathsf{J} \to \underline{\mathsf{C}}$ if there exists a cofinal $\phi\colon \mathsf{I} \to \mathsf{J}$ whose object component is an isomorphism and such that $Y \circ \phi = X$.

Thus, ind-objects are nothing but filtered diagrams in $\underline{\mathsf{C}}$. We can think of ind-objects as "syntactic" representations of ind-representable functors. In particular, we shall say that the ind-object $X\colon \mathsf{I} \to \underline{\mathsf{C}}$ represents the ind-representable functor

$$\mathsf{L}(X) = \varinjlim(\mathsf{I} \xrightarrow{X} \underline{\mathsf{C}} \xrightarrow{\mathsf{Y}} \widehat{\underline{\mathsf{C}}}) \cong \varinjlim_{i\in\mathsf{I}} \mathsf{Hom}_{\underline{\mathsf{C}}}(\_, X(i)).$$

Observe then that the imposed equalities are perfectly harmless because they identify objects which represent isomorphic presheaves.

In order to simplify notation, we shall often use the so-called *indexed* notation for ind-objects. We write $(X_i)_{i\in\mathsf{I}}$ for $X\colon \mathsf{I} \to \underline{\mathsf{C}}$ with $X(i) = X_i$. Admittedly this notation is rather poor but it will not be misleading, and therefore it will be acceptable, provided one never forgets that we are not handling sequences or chains of objects, but filtered diagrams in $\underline{\mathsf{C}}$. Of course, given an ind-object $(X_i)_{i\in\mathsf{I}}$, we reserve the right to use X also in every context in which a functor is expected.

As already observed in Section 3.1, the key point is the definition of morphisms for ind-objects. The right notion should be such that it identifies, i.e., it makes isomorphic, ind-objects which intuitively should be the same. Moreover, it has to make the category of ind-objects filtered cocomplete. Clearly, the theory exposed in Section 3.3 allows us to identify such notion of morphism immediately.

DEFINITION 3.5.2
The category $\mathtt{Ind}(\underline{\mathsf{C}})$ is the category whose objects are the ind-objects of $\underline{\mathsf{C}}$, and whose homsets are defined by

$$\mathsf{Hom}_{\mathtt{Ind}(\underline{\mathsf{C}})}(X, Y) = \mathsf{Hom}_{\widehat{\underline{\mathsf{C}}}}(\mathsf{L}(X), \mathsf{L}(Y)).$$

The category $\mathtt{Ind}(\underline{\mathsf{C}})_\aleph$ is the full subcategory of $\mathtt{Ind}(\underline{\mathsf{C}})$ whose object are $\aleph$-filtered functors.

This makes L into a full and faithful functor from $\mathtt{Ind}(\underline{\mathsf{C}})$ to $\widehat{\underline{\mathsf{C}}}$. However, observe that it is far from being injective on the objects. Nevertheless, we have that $\mathtt{Ind}(\underline{\mathsf{C}})$ and $\widehat{\underline{\mathsf{C}}}$ are *(weakly)* equivalent.

Proposition 3.5.3
$\mathtt{Ind}(\underline{\mathsf{C}}) \dot{\cong} \widehat{\underline{\mathsf{C}}}$.

Proof. Exploiting Corollary 3.3.17 and Proposition 3.3.6, it is immediate to see that L is a weak equivalence, i.e., a full and faithful functor whose replete image is the whole target category, i.e., such that every object P in $\widehat{\underline{\mathsf{C}}}$ is isomorphic to some $\mathsf{L}(X)$ for X in $\mathtt{Ind}(C)$. Freyd and Scedrov [26] show that the hypothesis that every weak equivalence is a strict (classical) one is equivalent to the axiom of choice. Since in our context we have largely used such an axiom, we can also assume that L is an equivalence. ✓

Proposition 3.5.4
$\mathtt{Ind}(\underline{\mathsf{C}})_{\aleph} \dot{\cong} \widehat{\underline{\mathsf{C}}}_{\aleph}$.

An interesting fact about $\mathtt{Ind}(\underline{\mathsf{C}})$ is the following.

Proposition 3.5.5
If $\underline{\mathsf{C}}$ *is small, then so are* $\mathtt{Ind}(C)_{\aleph}$ *and* $\mathtt{Ind}(\underline{\mathsf{C}})$.

Proof. There is "only" a small set of ($\aleph$-)filtered diagrams in $\underline{\mathsf{C}}$ and, therefore, the objects of $\mathtt{Ind}(\underline{\mathsf{C}})$ ($\mathtt{Ind}(\underline{\mathsf{C}})_{\aleph}$) form a small set. Regarding morphisms, by Proposition 3.3.21, the morphisms of $\mathtt{Ind}(\underline{\mathsf{C}})$ ($\mathtt{Ind}(\underline{\mathsf{C}})_{\aleph}$) are a family of small sets indexed by a small set, and therefore a small set [89]. ✓

Another immediate consequence of the definition of $\mathtt{Ind}(\underline{\mathsf{C}})$ is that the first requirement we made on the morphisms is satisfied. Of course, the same holds for $\mathtt{Ind}(\underline{\mathsf{C}})_{\aleph}$.

Proposition 3.5.6
Consider the ind-objects $(X_i)_{i\in \mathsf{I}}$ *and* $(Y_j)_{j\in \mathsf{J}}$ *and suppose that there exists a cofinal* $\phi\colon \mathsf{I} \to \mathsf{J}$ *such that* $Y \circ \phi = X$. *Then,* $(X_i)_{i\in \mathsf{I}} \cong (Y_j)_{j\in \mathsf{J}}$ *in* $\mathtt{Ind}(\underline{\mathsf{C}})$.

Proof. Since ϕ is cofinal $\mathsf{L}(X) = \mathsf{L}(Y \circ \phi) \cong \mathsf{L}(Y)$. ✓

Next, we give more explicit representations of morphisms of ind-objects. In particular, we describe three more different ways of understanding such morphisms.

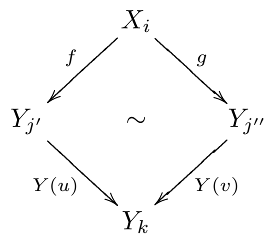
Recalling a computation we have done in the previous section, we have that

$$\begin{aligned}
\mathsf{Hom}_{\mathtt{Ind}(\underline{\mathsf{C}})}((X_i)_{i\in \mathsf{I}}, (Y_j)_{j\in \mathsf{J}}) &= \underline{\mathsf{Set}}^{\underline{\mathsf{C}}^{\mathrm{op}}}[\mathsf{L}(X), \mathsf{L}(Y)] \\
&\cong \underline{\mathsf{Set}}^{\underline{\mathsf{C}}^{\mathrm{op}}}[\varinjlim_{\mathsf{I}} \mathsf{h}_{X_i}, \varinjlim_{\mathsf{J}} \mathsf{h}_{Y_j}] \\
&\cong \varprojlim_{\mathsf{I}} \underline{\mathsf{Set}}^{\underline{\mathsf{C}}^{\mathrm{op}}}[\mathsf{h}_{X_i}, \varinjlim_{\mathsf{J}} \mathsf{h}_{Y_j}] \quad \mathsf{Y}' \text{ preserves colimits} \\
&\cong \varprojlim_{\mathsf{I}} \varinjlim_{\mathsf{J}} \underline{\mathsf{Set}}^{\underline{\mathsf{C}}^{\mathrm{op}}}[\mathsf{h}_{X_i}, \mathsf{h}_{Y_j}] \quad \text{Yoneda's lemma} \\
&\cong \varprojlim_{\mathsf{I}} \varinjlim_{\mathsf{J}} \mathsf{Hom}_{\underline{\mathsf{C}}}(X_i, Y_j)
\end{aligned}$$

From Proposition 3.3.10, we know how to compute filtered colimits in $\underline{\mathsf{Set}}$. An element $x \in \varinjlim_{\mathsf{J}} \mathsf{Hom}_{\underline{\mathsf{C}}}(X_i, Y_j)$ is an equivalence class of arrows $[f]_\sim$ each representative of which is an arrow $f\colon X_i \to Y_j$ of $\underline{\mathsf{C}}$ for some $j \in \mathsf{J}$ and where

$$\Big(f\colon X_i \to Y_{j'}\Big) \sim \Big(g\colon X_i \to Y_{j''}\Big) \quad \Leftrightarrow \quad \exists \begin{array}{c} j' \xrightarrow{u} \\ j'' \xrightarrow{v} \end{array} k \text{ in } \mathsf{J} \text{ st. } Y(u) \circ f = Y(v) \circ g,$$

as summarized below



Concerning limits, their calculus in $\underline{\mathsf{Set}}$ is much simpler than that of colimits. Let us remind it.

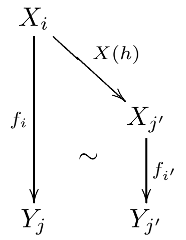
PROPOSITION 3.5.7
Let I be a small category and $\mathsf{F}\colon \mathsf{I} \to \underline{\mathsf{C}}$. By $\langle f \rangle$ we denote a function $f\colon Obj(\mathsf{I}) \to \bigcup_{i \in Obj(\mathsf{I})} \mathsf{F}(i)$ such that, for all $i \in Obj(\mathsf{I})$, it is $f(i) \in \mathsf{F}(i)$. Then,

$$\varprojlim_{\mathsf{I}} \mathsf{F} = \Big\{ \langle f \rangle \;\Big|\; \mathsf{F}(h)(f(i)) = f(j),\ \forall h\colon i \to j \text{ in } \mathsf{I} \Big\}.$$

Proof. Let S denote the set indicated above. For any $i \in \mathsf{I}$ define the projection $\pi_i\colon S \to \mathsf{F}(i)$ such that $\pi_i(\langle f \rangle) = f(i)$. Since for any $h\colon i \to j$ in I it is $\mathsf{F}(h) \circ \pi_i(\langle f \rangle) = \mathsf{F}(h)(f(i)) = f(j) = \pi_j(\langle f \rangle)$, i.e., $\mathsf{F}(h) \circ \pi_i = p_j$, we have that the family $\{\pi_i\}$ is a cone with base F and vertex S.

Suppose now that there is a cone $\beta\colon X \to \mathsf{F}$. For any $x \in X$ we can consider the function $f_x\colon Obj(\mathsf{I}) \to \bigcup_{i \in Obj(\mathsf{I})} \mathsf{F}(i)$ such that $f_x(i) = \beta_i(x)$ for any $i \in \mathsf{I}$. Since β is a cone, we have that f_x is of the kind $\langle f \rangle$. Thus, $f_x \in S$. Therefore, $\alpha\colon X \to S$ which sends x to f_x is well defined. Moreover, for any $i \in \mathsf{I}$, it is $\pi_i \circ \alpha(x) = \pi_i(f_x) = f_x(i) = \beta_i(x)$ for any $x \in X$ and of course α is the unique function with such a property. Therefore, $S = \varprojlim \mathsf{F}$. ✓

Getting back to our problem, the elements x in $\varprojlim_{\mathsf{I}} \varinjlim_{\mathsf{J}} \mathsf{Hom}_{\underline{\mathsf{C}}}(X_i, Y_j)$ are therefore a *collection of equivalence classes* $[f_i]_\sim$ indexed by the objects of I which are *compatible* in the precise sense that for any $h\colon i \to i'$ in I it is $[f_i]_\sim = [f_{i'} \circ X(h)]_\sim$, as shown below.



We shall denote this kind of families of equivalence classes with a notation similar to the one used for ind-objects, namely $([f_i])_{i\in\mathsf{I}} \colon (X_i)_{i\in\mathsf{I}} \to (Y_j)_{j\in\mathsf{J}}$, where f_i is an arrow from X_i to some Y_j. The square brackets remind us that each component is an equivalence class and the index i means that f_i is a representative for the i-th class. We nearly always avoid explicit mention of $\sim$. However it should be taken into account that $\sim$, and thus the elements of $[f_i]_\sim$, of course depends on the actual J, while the compatibility of the various components depends also on I.

The composition of ind-morphisms from $(X_i)_{i\in\mathsf{I}}$ to $(Y_j)_{j\in\mathsf{J}}$ and from $(Y_j)_{j\in\mathsf{J}}$ to $(Z_k)_{k\in\mathsf{K}}$ can be of course defined explicitly through the canonical function induced by the limit

$$\varprojlim_{\mathsf{J}} \varinjlim_{\mathsf{K}} \mathsf{Hom}_{\underline{\mathsf{C}}}(Y_j, Z_k) \times \varprojlim_{\mathsf{I}} \varinjlim_{\mathsf{J}} \mathsf{Hom}_{\underline{\mathsf{C}}}(X_i, Y_j) \to \varprojlim_{\mathsf{I}} \varinjlim_{\mathsf{K}} \mathsf{Hom}_{\underline{\mathsf{C}}}(X_i, Z_k).$$

However, the equivalent description in terms of families of equivalence classes above is simpler: given

$$([f_i])_{i\in\mathsf{I}} \colon (X_i)_{i\in\mathsf{I}} \to (Y_j)_{j\in\mathsf{J}} \quad \text{and} \quad ([g_j])_{j\in\mathsf{J}} \colon (Y_j)_{j\in\mathsf{J}} \to (Z_k)_{k\in\mathsf{K}},$$

their composition is the I-indexed family whose i-th component is $[g \circ f]_{\sim'}$ for $(f \colon X_i \to Y_j) \in [f_i]_\sim$ and $(g \colon Y_j \to Z_k) \in [g_j]_{\sim'}$, with the equivalence $\sim$ being relative to J and the equivalence $\sim'$ to K. In other words, the i-th class of the composition is obtained by considering the class (wrt. K) of the composition of one representative of the i-th component of $([f_i])_{i\in\mathsf{I}}$ and one representative of the j_i-th component of $([g_j])_{j\in\mathsf{J}}$, where j_i is determined by the chosen representative of f_i.

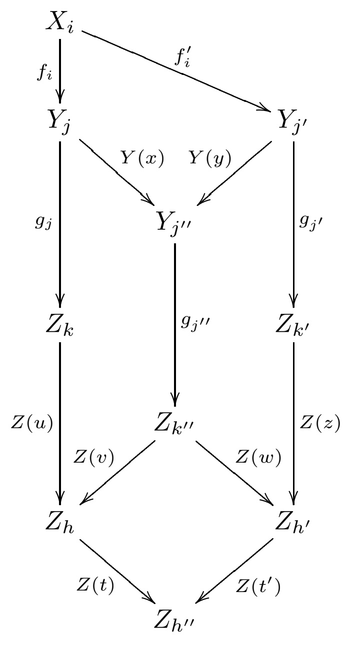
Of course, one needs to show that this is well-defined, i.e., that the definition does not depend on the choice of f and g above and that it gives an ind-morphism from $(X_i)_{i\in\mathsf{I}}$ to $(Z_k)_{k\in\mathsf{K}}$.

Concerning the first issue, for $f_i \colon X_i \to Y_j$, let us see that the composite does not depend on the choice of $g \in [g_j]_\sim$. Consider $(g_j \colon Y_j \to Z_k) \sim (g'_j \colon Y_j \to Z_{k'})$. Of course, since there exist $u \colon k \to k''$ and $v \colon k' \to k''$ such that $Z(u) \circ g_i = Z(v) \circ g'_i$, we have $Z(u) \circ g_j \circ f_i = Z(v) \circ g'_j \circ f_i$, i.e., $[g_j \circ f_i]_\sim = [g'_j \circ f_i]_\sim$.

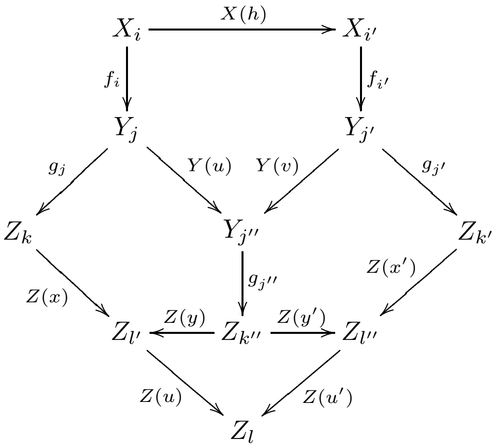
Now, let us verify that the composite does not depend on the choice of $f \in [f_i]_\sim$. To this aim, take $(f_i \colon X_i \to Y_j) \sim (f'_i \colon X_i \to Y_{j'})$. By definition of $\sim$, we have

$Y(x)\circ f_i = Y(y)\circ f'_i$ for $x\colon j \to j''$ and $y\colon j' \to j''$ in J. Consider $(g_j\colon Y_j \to Z_k) \in [g_j]_\sim$ and $(g_{j''}\colon Y_{j''} \to Z_{k''}) \in [g_{j''}]_\sim$. Then, again by definition of $\sim$, there are $u\colon k \to h$ and $v\colon k'' \to h$ in K such that $Z(u) \circ g_j = Z(v) \circ g_{j''} \circ Y(x)$. In the same way, we have $Z(x) \circ g_{j'} = Z(w) \circ g_{j''} \circ Y(y)$, for $z\colon k' \to h'$ and $w\colon k'' \to h'$ in K. Finally, since K is filtered, we find $t\colon h \to h''$ and $t'\colon h' \to h''$ such that $t \circ v = t' \circ w$. Then, $t\circ u\colon k \to h''$ and $t'\circ z\colon k' \to h''$ show that $(g_j \circ f_i) \sim (g_{j'} \circ f'_i)$. (See the commutative diagram below.) In fact,

$$
\begin{aligned}
Z(t \circ u) \circ g_j \circ f_i &= Z(t) \circ Z(v) \circ g_{j''} \circ Y(x) \circ f_i \\
&= Z(t') \circ Z(w) \circ g_{j''} \circ Y(x) \circ f_i \\
&= Z(t') \circ Z(w) \circ g_{j''} \circ Y(y) \circ f'_i \\
&= Z(t') \circ Z(z) \circ g_{j'} \circ f_i \\
&= Z(t' \circ z) \circ g_{j'} \circ f_i
\end{aligned}
$$

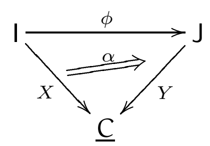


Thus, the definition does not depend on the choice of the representatives. The point we miss in order to conclude that $([g_j])_{j\in\mathsf{J}} \circ ([f_i])_{i\in\mathsf{I}}$ is well-defined, is to see that it is an ind-morphism. Let $h\colon i \to i'$ be in I. We have to see that $(g_j \circ f_i) \sim (g_{j'} \circ f_{i'})$. Of course, since $([f_i])_{i\in\mathsf{I}}$ is an ind-morphism, we have $u\colon j \to j''$ and $v\colon j' \to j''$ in J such that $Y(u) \circ f_i = Y(v) \circ f_{i'} \circ X(h)$. Then, the proof follows very much on the same line as before, as shown by the following commutative diagram (which in fact, apart from the first level, is isomorphic to the previous one).

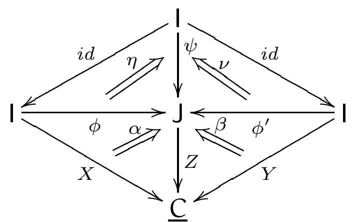


In order to remember that in the composition $([g_j])_{j\in \mathsf{J}} \circ ([f_i])_{i\in \mathsf{I}}$ the i-th equivalence class can be obtained by composing any representative of $[f_i]_\sim$ with any representative of an appropriate $[g_{j_i}]_\sim$, we shall denote it $(g_j * f_i)_{i\in \mathsf{I}}$. Sometimes, we shall write explicitly $([g_{j_i} \circ f_i])_{i\in \mathsf{I}}$, meaning that j_i is the index of the target of f_i.

An alternative description of ind-morphisms can be obtained using equivalence classes of families of arrows, rather than families of equivalence classes of arrows. At a first attempt, one *would* say that a morphism from the ind-object $(X_i)_{i\in \mathsf{I}}$ to the ind-object $(Y_j)_{j\in \mathsf{J}}$ is a pair (α, ϕ) where $\phi\colon \mathsf{I} \to \mathsf{J}$ is a functor and $\alpha\colon X \stackrel{.}{\to} Y\phi$ is a natural transformation



subject to the equivalence $\simeq$ defined as follows: $(X \stackrel{\alpha}{\Longrightarrow} Y\phi) \simeq (X \stackrel{\beta}{\Longrightarrow} Y\phi')$ if and only if $\exists\, \psi\colon \mathsf{I} \to \mathsf{J}$ and $\eta\colon \phi \stackrel{.}{\to} \psi$, $\nu\colon \phi' \stackrel{.}{\to} \psi$ such that $Z\eta \circ \alpha = Z\nu \circ \beta$.



However, although this definition can be easily proved equivalent to that of ind-morphisms when J is a chain, we currently do *not* know whether this is true in general. More precisely, every transformation $\alpha: X \to Y\phi$ is clearly an ind-morphism, but it is not clear whether, given $([f_i])_{i\in\mathsf{I}}: (X_i)_{i\in\mathsf{I}} \to (Y_j)_{j\in\mathsf{J}}$, it is possible to choose an element $f_{i,j_i}: X_i \to Y_{j_i}$ from $[f_i]_\sim$ for any $i \in \mathsf{I}$ in such a way that the mapping $i \rightsquigarrow j_i$ could be extended to a functor $\phi: \mathsf{I} \to \mathsf{J}$. When J is a chain, a morphism $([f_i])_{i\in\mathsf{I}}$ has an easy description. In fact, since for any pair of objects in $j, k \in \mathsf{J}$ there is exactly one morphism $j \to k$ or one morphism $k \to j$, the equivalence $\sim$ assumes a simpler form, namely $(f: X_i \to Y_j) \sim (g: X_i \to Y_k)$ if and only if $Y(j \le k) \circ f = g$ and $j \le k$ or $Y(k \le j) \circ g = f$ and $k \le j$, which makes very easy to define the wanted functor ϕ.

Therefore, while maintaining the notation and terminology of functors and natural transformations, for the time being, we work with the following definition, which consists merely of taking a representative for each class of an ind-morphism.

A morphism from the ind-object $(X_i)_{i\in\mathsf{I}}$ to the ind-object $(Y_j)_{j\in\mathsf{J}}$ is a pair (α, ϕ), where $\phi: \mathit{Obj}(\mathsf{I}) \to \mathit{Obj}(\mathsf{J})$ is a function and $\{\alpha_i: X_i \to Y_{\phi(i)}\}_{i\in\mathsf{I}}$ is a family of arrows of $\underline{\mathsf{C}}$ such that for any $h: i \to i'$ in I, there exists $k: \phi(i) \to j$ and $k': \phi(i') \to j$ in J such that $Y(k) \circ \alpha_i = Y(k') \circ \alpha_{i'} \circ X(h)$. These transformations are subject to the following equivalence: $(X \xrightarrow{\alpha} Y\phi) \simeq (X \xrightarrow{\beta} Y\phi')$ if there exists a function $\psi: \mathit{Obj}(\mathsf{I}) \to \mathit{Obj}(\mathsf{J})$ and two families of arrows $\{\eta_i: \phi(i) \to \psi(i)\}_{i\in\mathsf{I}}$ and $\{\nu_i: \phi'(i) \to \psi(i)\}_{i\in\mathsf{I}}$ such that $Y\eta \circ \alpha = Y\nu \circ \beta$, the last equation meaning, as in the case of actual natural transformations, that, for all $i \in \mathsf{I}$, $Y(\eta_i) \circ \alpha_i = Y(\nu_i) \circ \beta_i$.

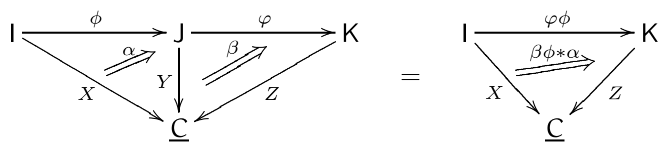
It is now immediate to see that this is nothing but an alternative description of ind-morphisms.

Proof. Given the ind-morphism $([f_i])_{i\in\mathsf{I}}$, choose any representative $\alpha_i = f_{i,j_i}: X_i \to Y_{j_i}$ from each equivalence class. Then, take $\phi: \mathit{Obj}(\mathsf{I}) \to \mathit{Obj}(\mathsf{J})$ which sends i to j_i.

Let $\alpha'_i = f_{i,j'_i}$ be the transformation obtained with a different choice of the f's in the equivalence classes. Of course, α_i and α'_i are equivalent:in fact, by definition of ind-morphism, for any i there exist $\psi(i)$, $\eta_i: j_i \to \psi(i)$ and $\nu_i: j'_i \to \psi(i)$ such that $Y(\eta_i) \circ f_{i,j_i} = Y(\nu_i) \circ f_{i,j'_i}$.

On the contrary, given a transformation α, just take $([\alpha_i])_{i\in\mathsf{I}}$. Of course, this translations are inverse to each other. ✓

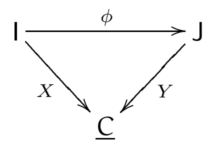
This description allows a simple formula for the composition of ind-morphisms



where, by (ab)using the notation of natural transformations, the i-th component of $\beta\phi * \alpha$ is $\beta_{\phi(i)} \circ \alpha_i$.

A third description of ind morphisms can be found via a *category of fractions* construction [29, 127] (see also Appendix A.4). The interest of this approach, introduced in [149, 150], resides in the fact that it explains the cocompletion construction "just" by making invertible a class of arrows in a universal way. Moreover, such class of arrows is as simple as possible and, therefore, the approach gives insights on the subject by making intuitively clear how the arrows chosen similarly to Section 3.1 should be enriched in order to get cocompleteness.

Let $\underline{\mathsf{Filt}}/\underline{\mathsf{C}}$ be the category of filtered categories over $\underline{\mathsf{C}}$ in $\underline{\mathsf{CAT}}$, i.e., the comma category $\langle \underline{\mathsf{Filt}} \downarrow \underline{\mathsf{C}} \rangle$ in $\underline{\mathsf{CAT}}$. The objects of this category are functors $X : \mathsf{I} \to \underline{\mathsf{C}}$, for $\mathsf{I} \in \underline{\mathsf{Filt}}$, i.e., they are exactly the ind-objects of $\mathsf{Ind}(\underline{\mathsf{C}})$. The arrows, however, are pretty simple: for $X : \mathsf{I} \to \underline{\mathsf{C}}$ and $Y : \mathsf{J} \to \underline{\mathsf{C}}$ a functor $\phi : \mathsf{I} \to \mathsf{J}$ is a arrow in $\underline{\mathsf{Filt}}/\underline{\mathsf{C}}$ from X to Y if it makes the following diagram commutative.



Consider now the class of arrows $\Sigma = \{\phi \text{ in } \underline{\mathsf{Filt}}/\underline{\mathsf{C}} \mid \phi \text{ is cofinal}\}$. Let us see that Σ admits a *calculus of left fractions* (see Appendix A.4).

Proposition 3.5.8

Σ admits a calculus of left fractions.

Proof. Let us check the condition of page 340. Of course Σ contains the identities and, by Proposition 3.3.13, is closed by composition.

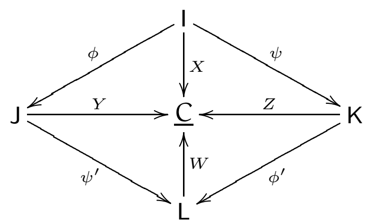
Concerning point *(iii)*, given $X : \mathsf{I} \to \underline{\mathsf{C}}$, $Y : \mathsf{J} \to \underline{\mathsf{C}}$ and $Z : \mathsf{K} \to \underline{\mathsf{C}}$, let $\phi : X \to Y$ belong to $\underline{\mathsf{Filt}}/\underline{\mathsf{C}}$ and $\psi : X \to Z$ to Σ. Then, consider the *pushout* of $\phi : \mathsf{I} \to \mathsf{J}$ and $\psi : \mathsf{I} \to \mathsf{K}$ in $\underline{\mathsf{Cat}}$. Let us denote it by L together with the functors $\psi' : \mathsf{J} \to \mathsf{L}$ and $\phi' : \mathsf{K} \to \mathsf{L}$. Remind that this can be described as the quotient of $\mathsf{J} + \mathsf{K}$ modulo the congruence generated by the rule

$$in_1(\phi(x)) \mathrel{\mathcal{R}} in_2(\psi(x)),$$

ψ' and ϕ' being the injections in the respective equivalence classes. Then, we have $W : \mathsf{L} \to \underline{\mathsf{C}}$ in $\underline{\mathsf{Filt}}/\underline{\mathsf{C}}$ defined by

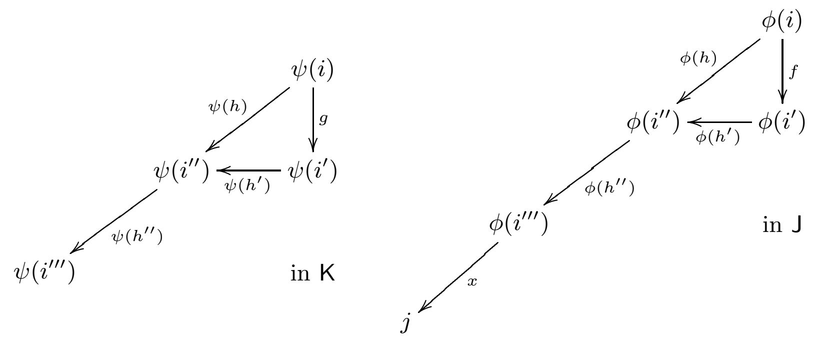
$$W([in_1(x)]_{\mathcal{R}}) = Y(x) \quad \text{and} \quad W([in_2(x)]_{\mathcal{R}}) = Z(x).$$

Observe that this definition is well given, since $Y\phi = X = Z\psi$ implies that whenever $[in_1(x)]_{\mathcal{R}} = [in_2(y)]_{\mathcal{R}}$ we have $Y(x) = Z(y)$. This is summarized by the picture below.



Let us see that L is filtered. Consider l and l' in L. If $l = [in_i(x)]_{\mathcal{R}}$ and $l' = [in_i(y)]_{\mathcal{R}}$, for $i \in \{1,2\}$, then we find an upper bound for l and l', since J and K are filtered. Suppose instead that $l = [in_1(j)]_{\mathcal{R}}$ and $l' = [in_2(k)]_{\mathcal{R}}$. Let us consider k in K. Since ψ is cofinal, we find $k \xrightarrow{u} \psi(k')$, and therefore an upper bound $j \xrightarrow{v} j'$, $\phi(k') \xrightarrow{w} j'$ for j and $\phi(k')$ in J. Then, we have in L the arrows $[in_1(v)]_{\mathcal{R}}: [in_1(j)]_{\mathcal{R}} \to [in_1(j')]_{\mathcal{R}}$ and $[in_1(w)]_{\mathcal{R}} \circ [in_2(u)]_{\mathcal{R}}: [in_2(k)]_{\mathcal{R}} \to [in_1(j')]_{\mathcal{R}}$.

Consider now $l \underset{v}{\overset{u}{\rightrightarrows}} l'$. Again, the only interesting case is when $u = [in_1(f)]_{\mathcal{R}}$ and $v = [in_2(g)]_{\mathcal{R}}$. In this hypothesis, it must necessarily be $f: \phi(i) \to \phi(i')$ and $g: \psi(i) \to \psi(i')$. Now, if f is of the kind $\phi(h)$ (g is of the kind $\psi(h)$), we have $\psi(i) \underset{g}{\overset{\phi(h)}{\rightrightarrows}} \psi(i')$ in K ($\psi(i) \underset{\psi(h)}{\overset{f}{\rightrightarrows}} \phi(i')$ in J), and we find an equalizer for u and v in L, since K is filtered (J is filtered). Thus, suppose that f and g are not in the image of ϕ and ψ, respectively. Since I is filtered we find $h: i \to i''$ and $h': i' \to i''$. Then, by cofinality of ψ, we can find h'' such that $\psi(h'' \circ h) = \psi(h'' \circ h') \circ g$. Now, by filteredness of J, we have an equalizer x for the parallel pair $\phi(h'' \circ h) = \phi(h'' \circ h') \circ f$.



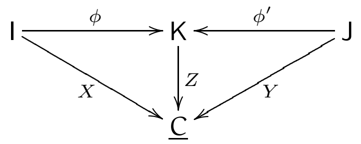
Then, we clearly have

$$\begin{aligned}
[x \circ \phi(h'' \circ h')]_{\mathcal{R}} \circ [f]_{\mathcal{R}} &= [x \circ \phi(h'' \circ h') \circ f]_{\mathcal{R}} = [x \circ \phi(h'' \circ h)]_{\mathcal{R}} \\
&= [x \circ \psi(h'' \circ h)]_{\mathcal{R}} = [x \circ \psi(h'' \circ h') \circ g]_{\mathcal{R}} \\
&= [x \circ \psi(h'' \circ h')]_{\mathcal{R}} \circ [g]_{\mathcal{R}} \\
&= [x \circ \phi(h'' \circ h')]_{\mathcal{R}} \circ [g]_{\mathcal{R}}
\end{aligned}$$

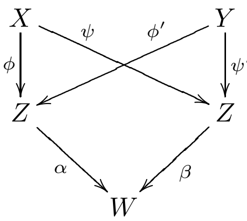
Thus, L is filtered. Now, using Lemma 3.3.14, it is now immediate to see that ψ' is cofinal and thus belongs to Σ, and thus condition *(iii)* is verified.

Consider now $\phi, \phi': X \to Y$ in $\underline{\mathsf{Filt}}/\underline{\mathsf{C}}$ and suppose that there exists $\psi: Z \to X$ in Σ which equalizes ϕ and ϕ', i.e., $\phi \circ \psi = \phi' \circ \psi$. Consider the *coequalizer* of $\phi, \phi': \mathsf{I} \to \mathsf{J}$ in $\underline{\mathsf{Cat}}$. This construction can be described as a quotient of J for the congruence generated by $\phi(x)\ \mathcal{R}\ \phi'(x)$. Then, if the coequalizer is $\psi': \mathsf{J} \to \mathsf{L}$, $W: \mathsf{L} \to \underline{\mathsf{C}}$ can be defined as above and, exploiting the existence of ψ, it can be proved in very much the same way of the previous case that L is filtered and ψ' is cofinal. ✓

Then, we can consider the category $\Sigma^{-1}\underline{\mathsf{Filt}}/\underline{\mathsf{C}}$ of left fractions of $\underline{\mathsf{Filt}}/\underline{\mathsf{C}}$ for Σ, i.e., the category obtained from $\underline{\mathsf{Filt}}/\underline{\mathsf{C}}$ by making formally invertible the arrows in Σ. By standard results of the theory of the categories of fractions [29], an arrow $\bar{\phi}: X \to Y$ in $\Sigma^{-1}\underline{\mathsf{Filt}}/\underline{\mathsf{C}}$ can be described as an equivalence class of pairs $(\phi: X \to Z, \phi': Y \to Z)$ of arrows of $\underline{\mathsf{Filt}}/\underline{\mathsf{C}}$ where ϕ' is cofinal



the equivalence given by $(\phi: X \to Z, \phi': Y \to Z) \sim (\psi: X \to Z', \psi': Y \to Z')$ if there exist $Z \xrightarrow{\alpha} W$ and $Z' \xrightarrow{\beta} W$ such that $\alpha \circ \phi'$ and $\beta \circ \psi'$ are cofinal and the following diagram commutes.



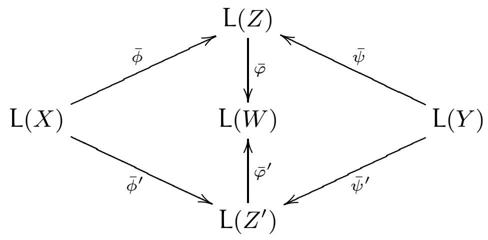
Now, it is not difficult to show that this yields ind-morphisms.

Proposition 3.5.9
$\Sigma^{-1}\underline{\mathsf{Filt}}/\underline{\mathsf{C}} \cong \mathsf{Ind}(\underline{\mathsf{C}})$.

Proof. Since $\mathsf{Ind}(\underline{\mathsf{C}})$ and $\Sigma^{-1}\underline{\mathsf{Filt}}/\underline{\mathsf{C}}$ have the same objects, it is enough to show that $\mathsf{Hom}_{\Sigma^{-1}\underline{\mathsf{Filt}}/\underline{\mathsf{C}}}(X, Y) \cong \mathsf{Hom}_{\mathsf{Ind}(\underline{\mathsf{C}})}(X, Y) = \underline{\mathsf{Set}}^{\underline{\mathsf{C}}^{\mathrm{op}}}[\mathsf{L}(X), \mathsf{L}(Y)]$, where L is the usual $\varinjlim \mathsf{Y}X$. Every morphism $\phi: X \to Y$ in $\underline{\mathsf{Filt}}/\underline{\mathsf{C}}$ induces a *canonical* morphism $\bar{\phi}: \varinjlim(\mathsf{Y}X) \to \varinjlim(\mathsf{Y}Y)$, namely the morphism identified by the universal property of colimits for the cocone $\lambda\phi: \mathsf{Y}X \stackrel{\cdot}{\to} \varinjlim(\mathsf{Y}Y)$, where $\lambda: \mathsf{Y}Y \stackrel{\cdot}{\to} \varinjlim(\mathsf{Y}Y)$ is the limit cocone for $\varinjlim(\mathsf{Y}Y)$. Of course, if ϕ is cofinal, then $\bar{\phi}$ is invertible.

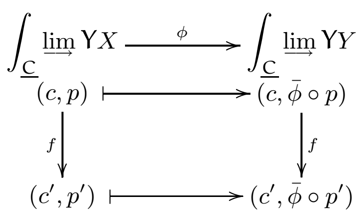
Now, fixed $X: \mathsf{I} \to \underline{\mathsf{C}}$ and $Y: \mathsf{J} \to \underline{\mathsf{C}}$, consider the mapping $\mathcal{L}: \Sigma^{-1}\underline{\mathsf{Filt}}/\underline{\mathsf{C}}[X, Y] \to \underline{\mathsf{Set}}^{\underline{\mathsf{C}}^{\mathrm{op}}}[\mathsf{L}(X), \mathsf{L}(Y)]$ which assigns to each $[(\phi, \psi)]_\sim$ the corresponding $\bar{\psi}^{-1} \circ \bar{\phi}$. We shall show that $\mathcal{L}$ is a set-isomorphism.

First, we see that $\mathcal{L}$ is well-defined. Suppose that $(\phi, \psi) \sim (\phi', \psi')$, where $\phi: X \to Z$, $\psi: Y \to Z$, $\phi': X \to Z'$, $\psi': Y \to Z'$, and consider $\bar{\psi}^{-1} \circ \bar{\phi}$ and $\bar{\psi}'^{-1} \circ \bar{\phi}'$. By definition of $\sim$, we find $\varphi: Z \to W$ and $\varphi': Z' \to W$ such that $\varphi \circ \phi = \varphi' \circ \phi'$ and $\varphi \circ \psi = \varphi' \circ \psi'$, the latter being cofinal. This gives us the following situation in $\underline{\mathsf{Set}}^{\underline{\mathsf{C}}^{\mathrm{op}}}$

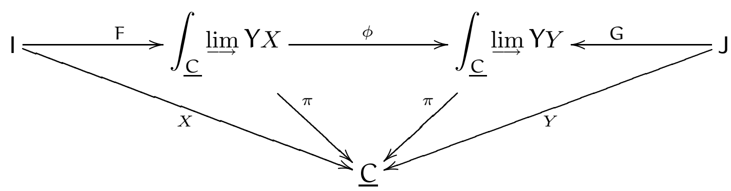


where $(\varphi \circ \psi)^{-1} \circ \varphi \circ \phi = (\varphi' \circ \psi')^{-1} \circ \varphi' \circ \phi'$. Now, since in $\underline{\mathsf{Set}}^{\underline{\mathsf{C}}^{\mathrm{op}}}$ the isomorphisms are the injective and surjective (on each component) transformations, and since ψ and ψ' are invertible, it follows that φ and φ' are invertible and it is $(\varphi \circ \psi)^{-1} = \psi^{-1} \circ \varphi^{-1}$ and $(\varphi' \circ \psi')^{-1} = \psi'^{-1} \circ \varphi'^{-1}$ and then, from the equation above, it follows $\psi_{-1} \circ \phi = \psi^{-1} \circ \phi$.

Next, we see that $\mathcal{L}$ is surjective. Consider any morphism $\bar{\phi}\colon \varinjlim(\mathsf{Y}X) \to \varinjlim(\mathsf{Y}Y)$. Then, consider the following functor ϕ between the categories of elements of $\mathsf{L}(X) = \varinjlim(\mathsf{Y}X)$ and $\mathsf{L}(Y) = \varinjlim(\mathsf{Y}Y)$.

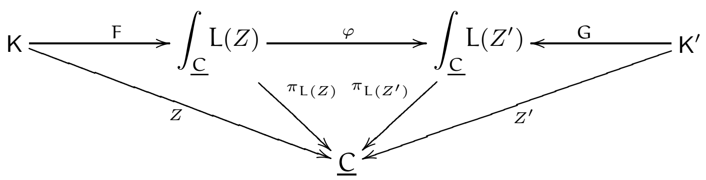


Observe that we are indentifying p with an arrow in $\underline{\mathsf{Set}}^{\underline{\mathsf{C}}^{\mathrm{op}}}$ $p\colon \mathsf{h}_c \stackrel{\cdot}{\to} \varinjlim(\mathsf{Y}X)$, and so the mapping above is well-defined on the objects. Concerning the morphisms, we have $f\colon (c, p) \to (c'p')$ if f is a morphism $c \to c'$ in $\underline{\mathsf{C}}$ such that $p' \circ (f \circ \_) = p$, which implies that $\bar{\phi} \circ p' \circ (f \circ \_) = \bar{\phi} \circ p$, i.e., $f\colon \phi(c, p) \to \phi(c', p')$. It follows at once that ϕ is well-defined and is a functor. We have already noticed in the proof of Proposition 3.3.24 that I is cofinal in the category of elements of $\varinjlim(\mathsf{Y}X)$ via the functor F which sends i to (X_i, σ_i), σ_i being the component at i of the limit cocone for $\varinjlim(\mathsf{Y}X)$, and which is the identity on the arrows. In the same way, we conclude that J is cofinal in the category of elements of $\varinjlim(\mathsf{Y}Y)$ via the analogous functor G. So we have the following diagram, which is immediately proved commutative.



Observe further that $\int_{\underline{\mathsf{C}}} \varinjlim \mathsf{Y}X$ and $\int_{\underline{\mathsf{C}}} \varinjlim \mathsf{Y}Y$ are not necessarily filtered, since they are "only" *essentially* filtered. However, this is not a problem, since we can just work with any filtered small part of them, e.g., the smallest filtered subcategory of $\int_{\underline{\mathsf{C}}} (\varinjlim \mathsf{Y}X)$ which contains the image of I through $\phi \circ \mathsf{F}$ and the image of J through G, which is clearly small. It is now evident that, $\overline{\phi \circ \mathsf{F}} = \bar{\phi}$, while $\overline{\mathsf{G}} = id$. Therefore, $\mathcal{L}([(\phi \circ \mathsf{F}, \mathsf{G})]_{\sim}) = \bar{\phi}$.

Finally, we need to show that $\mathcal{L}$ is injective. Consider $\phi: X \to Z$, $\psi: Y \to Z$, $\phi': X \to Z'$, $\psi': Y \to Z'$ and suppose that $\bar{\psi}^{-1} \circ \bar{\phi} = \bar{\psi}'^{-1} \circ \bar{\phi}'$. Then, $\mathsf{L}(Z)$ and $\mathsf{L}(Z')$ are isomorphic and, in particular, $\bar{\varphi} = \bar{\psi}' \circ \bar{\psi}^{-1}: \mathsf{L}(Z) \to \mathsf{L}(Z')$ is an isomorphism such that $\bar{\varphi} \circ \bar{\psi} = \bar{\psi}^{-1}$ and $\bar{\varphi} \circ \bar{\phi} = \bar{\phi}'$. Now, construct as above φ:



From the discussion above it is immediate to realize that $\varphi \circ \mathsf{F}$ and G make commutative the diagram which shows $(\phi, \psi) \sim (\phi', \psi)'$. Lastly, observe that since $\bar{\varphi}$ is invertible, then φ above is a isomorphism whose inverse sends (c, p) to $(c, \bar{\varphi}^{-1} \circ p)$. Therefore, φ is cofinal, which concludes the proof. ✓

Clearly, the same description of morphisms holds in $\mathcal{I}nd(\underline{\mathsf{C}})_{\aleph}$ for any cardinal $\aleph$.

When $\underline{\mathsf{C}}$ is a poset P, there is the following connection of $\mathcal{I}nd(\underline{\mathsf{C}})$ with the theory of complete posets.

Proposition 3.5.10
Let P be a small poset. Then $\mathcal{I}nd(P)$ is equivalent to the ideal completion of P viewed as a category.

Proof. An ideal in P is a *downward closed* directed subset $I \subseteq P$, i.e., a set such that $i \in I$ and $j \leq i$ implies $j \in I$. Observe that $\mathsf{Hom}_{\mathcal{I}nd(\mathrm{P})}(X, Y) = \varprojlim \varinjlim \mathsf{Hom}_P(X_i, Y_j)$ must be either a singleton or the empty set, since each $\mathsf{Hom}_P(X_i, Y_j)$ is such. Then $\mathcal{I}nd(\mathrm{P})$ is a preorder. Moreover, an ideal I of P is naturally an ind-object, namely the inclusion $I \hookrightarrow P$. Conversely, an ind-object $X: \mathsf{I} \to P$ can be thought as an ideal just by taking the *"downward" closure* of its image in P. Since X is cofinal in the ind-object corresponding to such a closure, this defines an equivalence. ✓

In the same way it can be shown that $\mathcal{I}nd(\mathrm{P})_{\aleph}$ is equivalent to the completion of P by all ideals of cardinality not greater than $\aleph$. A general treatment of the completion of posets in a categorical framework, namely via monads, has been given in [95].

$\mathrm{Ind}(\_)$ AS A 2-ENDOFUNCTOR ON $\underline{\underline{\mathsf{CAT}}}$

We have already seen that if $\underline{\mathsf{C}}$ is locally small, then so is $\widehat{\underline{\mathsf{C}}}$. It follows immediately from the definitions that the same holds for $\mathrm{Ind}(\underline{\mathsf{C}})$ and $\mathrm{Ind}(\underline{\mathsf{C}})_{\aleph}$. Therefore, it may look plausible that $\mathrm{Ind}(\_)$ is the object part of an endofunctor on $\underline{\mathsf{CAT}}$. In this subsection we show that this is the case. In particular, we show that $\mathrm{Ind}(\_)$ can be extended to the 2-cells of $\underline{\underline{\mathsf{CAT}}}$ obtaining in this way a 2-functor.

Let $\mathsf{F}\colon \underline{\mathsf{C}} \to \underline{\mathsf{D}}$ be a functor in $\underline{\underline{\mathsf{CAT}}}$. We define a functor $\mathrm{Ind}(\mathsf{F})\colon \mathrm{Ind}(\underline{\mathsf{C}}) \to \mathrm{Ind}(\underline{\mathsf{D}})$. Concerning the objects, the definition is evident: we map X to the composition of X with F.

$$\mathsf{I} \xrightarrow{X} \underline{\mathsf{C}} \xrightarrow{\mathsf{F}} \underline{\mathsf{D}}.$$

For the morphisms the situation is slightly more difficult.

Consider $X\colon \mathsf{I} \to \underline{\mathsf{C}}$ and $Y\colon \mathsf{J} \to \underline{\mathsf{D}}$. By definition, for any $i \in \mathsf{I}$ and any $j \in \mathsf{J}$, F induces a function $\mathsf{F}_{i,j}\colon \mathsf{Hom}_{\underline{\mathsf{C}}}(X_i, Y_j) \to \mathsf{Hom}_{\underline{\mathsf{D}}}(\mathsf{F}X_i, \mathsf{F}Y_j)$ which sends f to $\mathsf{F}f$. Therefore, by injecting $\mathsf{F}f$ in its equivalence class $[\mathsf{F}f]_{\sim}$, we get a cocone $\mathsf{F}_{X,Y}\colon \mathsf{Hom}_{\underline{\mathsf{C}}}(X_i, Y_j) \to \varinjlim_{\mathsf{J}} \mathsf{Hom}_{\underline{\mathsf{D}}}(\mathsf{F}X_i, \mathsf{F}Y_j)$, and thus an induced function $\mathsf{F}_i\colon \varinjlim_{\mathsf{J}} \mathsf{Hom}_{\underline{\mathsf{C}}}(X_i, Y_j) \to \varinjlim_{\mathsf{J}} \mathsf{Hom}_{\underline{\mathsf{D}}}(\mathsf{F}X_i, \mathsf{F}Y_j)$. Now, composing each F_i with the i-th component of the limit cone for $\varprojlim_{\mathsf{I}} \varinjlim_{\mathsf{J}} \mathsf{Hom}_{\underline{\mathsf{C}}}(X_i, Y_j)$, we get a cone with base $\mathsf{I} \to \varinjlim_{\mathsf{J}} \mathsf{Hom}_{\underline{\mathsf{D}}}(\mathsf{F}X_i, \mathsf{F}Y_j)$, which therefore gives

$$\mathsf{F}^*_{X,Y}\colon \varprojlim_{\mathsf{I}} \varinjlim_{\mathsf{J}} \mathsf{Hom}_{\underline{\mathsf{C}}}(X_i, Y_j) \to \varprojlim_{\mathsf{I}} \varinjlim_{\mathsf{J}} \mathsf{Hom}_{\underline{\mathsf{D}}}(\mathsf{F}X_i, \mathsf{F}Y_j).$$

By the universal properties of limits and colimits, it is easy to see that

i) $\mathsf{F}^*_{X,X}(id_X) = id_{\mathsf{F}X}$;

ii) $\mathsf{F}^*_{Y,Z}(g) \circ \mathsf{F}^*_{X,Y}(f) = \mathsf{F}^*_{X,Z}(g \circ f)$;

iii) $Id^*_{X,Y}(f) = f$;

iv) $\mathsf{G}^*_{\mathsf{F}X,\mathsf{F}Y} \circ \mathsf{F}^*_{X,Y} = (\mathsf{G} \circ \mathsf{F})^*_{X,Y}$.

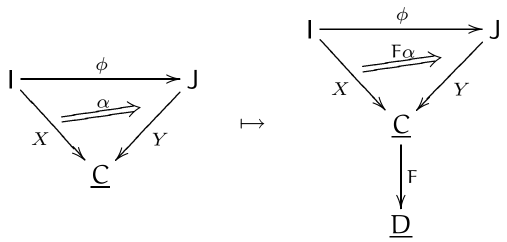
Points *(i)* and *(ii)* above make it clear that the following scheme defines a functor.

$$\begin{array}{ccc} \mathrm{Ind}(\underline{\mathsf{C}}) & \xrightarrow{\mathrm{Ind}(\mathsf{F})} & \mathrm{Ind}(\underline{\mathsf{D}}) \\ X & \longmapsto & \mathsf{F} \circ X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle F^*_{X,Y}(f)} \\ Y & \longmapsto & \mathsf{F} \circ Y \end{array}$$

In terms of the representation of ind-morphisms by families of equivalence classes, we have the following obvious situation:

$$([f_i])_{i\in\mathsf{I}}\colon (X_i)_{i\in\mathsf{I}} \to (Y_j)_{j\in\mathsf{J}} \mapsto \ ([\mathsf{F}f_i])_{i\in\mathsf{I}}\colon (\mathsf{F}X_i)_{i\in\mathsf{I}} \to (\mathsf{F}Y_j)_{j\in\mathsf{J}},$$

which can be also written in terms of equivalence classes of families as



Observe that, of course, $[\mathsf{F}f_i]_\sim$ is not necessarily the image of $[f_i]_\sim$, since, intuitively, $\underline{\mathsf{D}}$ could have "more" morphisms.

Thanks to points *(iii)* and *(iv)* above, we conclude that $\mathtt{Ind}(\_)$ is a functor from $\underline{\underline{\mathsf{CAT}}}$ to itself. Consider now $\mathsf{F}, \mathsf{G} \colon \underline{\mathsf{C}} \to \underline{\mathsf{D}}$. Given a natural transformation $\alpha \colon \mathsf{F} \stackrel{\cdot}{\to} \mathsf{G}$ there is then an obvious candidate for $\mathtt{Ind}(\alpha) \colon \mathtt{Ind}(\mathsf{F}) \stackrel{\cdot}{\to} \mathtt{Ind}(\mathsf{G})$, namely the family $\{\alpha_X\}_{X \in \mathtt{Ind}(\underline{\mathsf{C}})}$ where $\alpha_X \colon (\mathsf{F}X_i)_{i \in \mathsf{I}} \to (\mathsf{G}X_i)_{i \in \mathsf{I}}$ is the arrow of $\mathtt{Ind}(\underline{\mathsf{D}})$ whose i-th component is $[\alpha_{X_i}]_\sim$. In other words, $\mathtt{Ind}(\alpha)$ is determined by (taking the equivalence classes of the component arrows of) $\alpha_X \colon \mathsf{F}X \stackrel{\cdot}{\to} \mathsf{G}X$. Observe that the condition $[\alpha_{X_i}]_\sim = [\alpha_{X_j} \circ \mathsf{F}X(h)]_\sim$ for any $h \colon i \to j$ in I comes directly from the naturality in $\underline{\underline{\mathsf{CAT}}}$ of α.

$$
\begin{array}{ccc}
\mathsf{F}X_i & \xrightarrow{\alpha_{X_i}} & \mathsf{G}X_i \\
{\scriptstyle \mathsf{F}X(h)}\downarrow & & \downarrow{\scriptstyle \mathsf{G}X(h)} \\
\mathsf{F}Y_j & \xrightarrow[\alpha_{Y_j}]{} & \mathsf{G}Y_j
\end{array}
$$

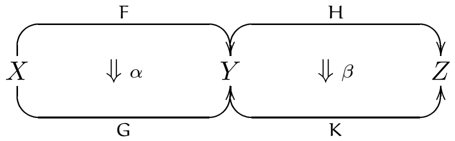
In order to show that $\{\alpha_X\}_{X \in \mathtt{Ind}(\underline{\mathsf{C}})}$ is natural, we have to see that, for any $f = ([f_i])_{i \in \mathsf{I}} \colon (X_i)_{i \in \mathsf{I}} \to (Y_j)_{j \in \mathsf{J}}$, the following diagram commutes.

$$
\begin{array}{ccc}
(\mathsf{F}X_i)_{i \in \mathsf{I}} & \xrightarrow{\alpha_X} & (\mathsf{G}X_i)_{i \in \mathsf{I}} \\
{\scriptstyle \mathtt{Ind}(\mathsf{F})(f)}\downarrow & & \downarrow{\scriptstyle \mathtt{Ind}(\mathsf{G})(f)} \\
(\mathsf{F}Y_j)_{j \in \mathsf{J}} & \xrightarrow[\alpha_Y]{} & (\mathsf{G}Y_j)_{j \in \mathsf{J}}
\end{array}
$$

Again, this comes easily from the naturality of α.

$$
\begin{aligned}
\alpha_Y \circ \mathtt{Ind}(\mathsf{F})(f) &= ([\alpha_{Y_j}]_\sim)_{j \in \mathsf{J}} \circ ([\mathsf{F}f_i])_{i \in \mathsf{I}} \\
&= ([\alpha_{Y_j} * \mathsf{F}f_i]_\sim)_{i \in \mathsf{I}} = ([\mathsf{G}f_i * \alpha_{X_i}]_\sim)_{i \in \mathsf{I}} \\
&= ([\mathsf{G}f_i])_{i \in \mathsf{I}} \circ ([\alpha_{X_i}]_\sim)_{i \in \mathsf{I}} = \mathtt{Ind}(\mathsf{G})(f) \circ \alpha_X .
\end{aligned}
$$

Next, we show that this definition makes $\mathsf{Ind}(\_)$ into a 2-functor. It comes directly from the definition that the identity natural transformation is sent to the identity and that $\mathsf{Ind}(\_)$ respects the *vertical composition* of natural transformations. Consider now $\alpha\colon \mathsf{F} \stackrel{\cdot}{\to} \mathsf{G}$ and $\beta\colon \mathsf{H} \stackrel{\cdot}{\to} \mathsf{K}$ as in the picture below.



The horizontal composition of α and β is $\gamma = \beta\mathsf{G} \circ \mathsf{H}\alpha$. Let us fix the attention on the X-th component of $\mathsf{Ind}(\gamma)$. We have

$$\begin{aligned}\mathsf{Ind}(\gamma)_X &= ([(\beta_{\mathsf{G}X_i} \circ \mathsf{H}\alpha_{X_i})]_\sim)_{i\in\mathsf{I}} = ([(\beta_{\mathsf{G}X_i}]_\sim)_{i\in\mathsf{I}} \circ ([\mathsf{H}\alpha_{X_i}]_\sim)_{i\in\mathsf{I}} \\ &= ([(\beta_{X_i}]_\sim)_{i\in\mathsf{I}}\mathsf{Ind}(\mathsf{G}) \circ \mathsf{Ind}(\mathsf{H})([\alpha_{X_i}]_\sim)_{i\in\mathsf{I}} \\ &= (\mathsf{Ind}(\beta)\mathsf{Ind}(\mathsf{G}) \circ \mathsf{Ind}(\mathsf{H})\mathsf{Ind}(\alpha))_X,\end{aligned}$$

which is the X-th component of the horizontal composition of $\mathsf{Ind}(\alpha)$ and $\mathsf{Ind}(\beta)$. It follows that $\mathsf{Ind}(\_)$ respects horizontal composition. Thus, we have proved the following.

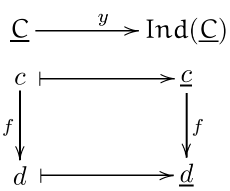
Proposition 3.5.11
$\mathsf{Ind}(\_)\colon \underline{\underline{\underline{\mathsf{CAT}}}} \to \underline{\underline{\underline{\mathsf{CAT}}}}$ *is a 2-functor.*

It follows immediately from the definitions that, for any cardinal $\aleph$, $\mathsf{Ind}(\_)$ restricts to a 2-endofunctor $\mathsf{Ind}(\_)_\aleph\colon \underline{\underline{\underline{\mathsf{CAT}}}} \to \underline{\underline{\underline{\mathsf{CAT}}}}$.

Constant Ind-objects: the 2-natural unit y

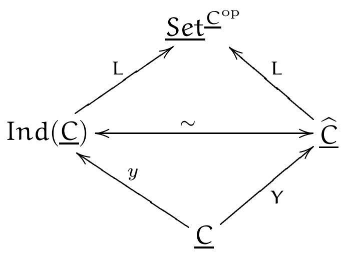
In this section we see that the Yoneda embedding $\mathsf{Y}\colon \underline{\mathsf{C}} \to \widehat{\underline{\mathsf{C}}}$ has an analogous embedding $y\colon \underline{\mathsf{C}} \to \mathsf{Ind}(\underline{\mathsf{C}})$. This shall provide us with a 2-natural transformation for the KZ-doctrine we are building.

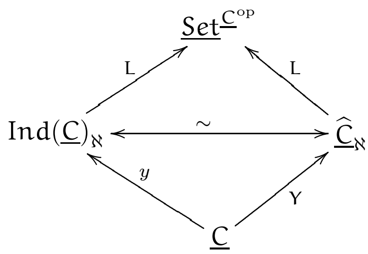
The category $\underline{1}$ consisting of a unique element and its identity arrow, i.e., the terminal object in $\underline{\mathsf{CAT}}$, is a filtered category. For any $c \in \underline{\mathsf{C}}$ we denote by $\underline{c}$ the ind-object $\underline{c}\colon \underline{1} \to \underline{\mathsf{C}}$ which picks up c. These kind of ind-objects are called *constant ind-objects* and provide a full and faithful image of $\underline{\mathsf{C}}$ in $\mathsf{Ind}(\underline{\mathsf{C}})$ via the functor y defined below.



Observe that, by definition $y(f)$, is $[f]_\sim$. However, since the index category for $\underline{d}$ is $\underline{1}$, in this case $\sim$ is trivial, i.e., $[f]_\sim$ consists of the unique element f.

Since $\mathsf{L}(\underline{c}) = \varinjlim_{\underline{1}}(\mathsf{Y} \circ \underline{c}) = h_c$, we have the following commuting diagrams.





Thus, y plays the role which Y played in the case of $\widehat{\underline{\mathsf{C}}}$. Of course there are many objects in $\mathrm{Ind}(\underline{\mathsf{C}})$ which can represent $\underline{\mathsf{C}}$ and, consequently, many possible embeddings y's. (For instance, in Section 3.6 we shall use another y.) If we consider a constant functor $\underline{\underline{c}}\colon \mathsf{I} \to \underline{\mathsf{C}}$, which always takes the value c, we have that $\mathsf{L}(\underline{\underline{c}}) = \varinjlim \mathfrak{h}_c = \mathfrak{h}_c$, i.e., $\underline{c}$ and $\underline{\underline{c}}$ are isomorphic in $\mathrm{Ind}(\underline{\mathsf{C}})$. The same happens if we consider a finite index category I and a functor $X\colon \mathsf{I} \to \underline{\mathsf{C}}$ which sends the greatest element of I to c. The ind-objects X such that $X \cong \underline{c}$ for some $c \in \underline{\mathsf{C}}$, or equivalently such that $\mathsf{L}(X) \cong \mathfrak{h}_c$, are called *essentially constant ind-objects*. Of course, y is an equivalence of $\underline{\mathsf{C}}$ and the full subcategory of the essentially constant ind-objects in $\mathrm{Ind}(\underline{\mathsf{C}})$.

A first connection with KZ-doctrines is the following proposition, where the reader will recognize the similarity with the definition of algebra for a KZ-doctrine.

Proposition 3.5.12
A locally small category $\underline{\mathsf{C}}$ is $\aleph$-filtered cocomplete if and only if $y\colon \underline{\mathsf{C}} \to \mathrm{Ind}(\underline{\mathsf{C}})_\aleph$ has a left adjoint.

Proof. $\underline{\mathsf{C}}$ has $\aleph$-filtered colimits if and only if for every $(X_i)_{i\in\mathsf{I}}$ in $\mathrm{Ind}(\underline{\mathsf{C}})_\aleph$ and c in $\underline{\mathsf{C}}$ there is a natural isomorphism

$$\mathsf{Hom}_{\underline{\mathsf{C}}}(\varinjlim X, c) \cong \underline{\mathsf{C}}^{\mathsf{I}}[X, \Delta_c],$$

where $\Delta_c\colon \mathsf{I} \to \underline{\mathsf{C}}$ is the constant functor which selects c. By definition of $\underline{c}$, it is immediate to see that such cocones $X \stackrel{\cdot}{\to} \Delta_c$ are in one-to-one correspondence with ind-morphisms $X \to \underline{c}$. It follows that $\underline{\mathsf{C}}$ has all $\aleph$-filtered colimits if and only if there is a natural isomorphism

$$\mathsf{Hom}_{\underline{\mathsf{C}}}(\varinjlim X, c) \cong \mathsf{Hom}_{\mathrm{Ind}(\underline{\mathsf{C}})_\aleph}(X, y(c)),$$

which is the isomorphism for the adjointness $\varinjlim \dashv y$. Observe that, since y is full and faithful, the adjunction is a reflection. ✓

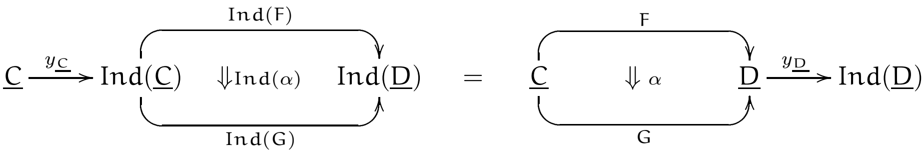
Of course, the above result holds also for $\mathsf{Ind}(\underline{\mathsf{C}})$

PROPOSITION 3.5.13
A locally small category $\underline{\mathsf{C}}$ is filtered cocomplete if and only if $y\colon \underline{\mathsf{C}} \to \mathsf{Ind}(\underline{\mathsf{C}})$ has a left adjoint.

Now, we see that the family $\{y_{\underline{\mathsf{C}}}\}_{\underline{\mathsf{C}}\in\underline{\underline{\mathsf{CAT}}}}$ is a 2-natural transformation $Id \stackrel{\cdot}{\to} \mathsf{Ind}(\_)$. The task is fairly easy. Concerning naturality, we have to prove that the following diagram commutes for every $\underline{\mathsf{C}}$, $\underline{\mathsf{D}}$ and $\mathsf{F}\colon \underline{\mathsf{C}} \to \underline{\mathsf{D}}$, which is really immediate.

$$\begin{array}{ccc} \underline{\mathsf{C}} & \xrightarrow{y_{\underline{\mathsf{C}}}} & \mathsf{Ind}(\underline{\mathsf{C}}) \\ {\scriptstyle \mathsf{F}}\downarrow & & \downarrow{\scriptstyle \mathsf{Ind}(\mathsf{F})} \\ \underline{\mathsf{D}} & \xrightarrow[y_{\underline{\mathsf{D}}}]{} & \mathsf{Ind}(\underline{\mathsf{D}}) \end{array}$$

For any $\alpha\colon \mathsf{F} \to \mathsf{G}$ in $\underline{\underline{\mathsf{CAT}}}$, the equation for 2-naturality is $\mathsf{Ind}(\alpha)y_{\underline{\mathsf{C}}} = y_{\underline{\mathsf{D}}}\alpha$, i.e.,



Now, the X-th component of $\mathsf{Ind}(\alpha)$ is $([\alpha_{X_i}]_\sim)_{i\in\mathsf{I}}\colon (\mathsf{F}X_i)_{i\in\mathsf{I}} \to (\mathsf{G}X_i)_{i\in\mathsf{I}}$, and therefore the c-th component of $\mathsf{Ind}(\alpha)y_{\underline{\mathsf{C}}}$ is $[\alpha_c]_\sim$. On the other hand, the c-th component of $y_{\underline{\mathsf{D}}}\alpha$ is $y_{\underline{\mathsf{D}}}(\alpha_c)$ which is again $[\alpha_c]_\sim$. Thus, we can conclude this subsection with the following propositions.

PROPOSITION 3.5.14
$y\colon Id \stackrel{\cdot}{\to} \mathsf{Ind}(\_)$ is a 2-natural transformation.

PROPOSITION 3.5.15
$y\colon Id \stackrel{\cdot}{\to} \mathsf{Ind}(\_)_\aleph$ is a 2-natural transformation.

FILTERED COLIMITS IN $\mathsf{Ind}(\underline{\mathsf{C}})$: THE 2-NATURAL MULTIPLICATION

In this subsection we show that $\mathsf{Ind}(\underline{\mathsf{C}})$ ($\mathsf{Ind}(\underline{\mathsf{C}})_\aleph$) is ($\aleph$-)filtered cocomplete. This is not surprising, since we have already seen in Section 3.3 that $\widehat{\underline{\mathsf{C}}}$ ($\widehat{\underline{\mathsf{C}}}_\aleph$) is the "free" cocompletion of $\underline{\mathsf{C}}$ by ($\aleph$-)filtered colimits and we have shown that $\mathsf{Ind}(\underline{\mathsf{C}})$ and $\widehat{\underline{\mathsf{C}}}$ ($\mathsf{Ind}(\underline{\mathsf{C}})_\aleph$ and $\widehat{\underline{\mathsf{C}}}_\aleph$) are equivalent (Proposition 3.5.3). However, we shall see that the calculus of colimits in $\mathsf{Ind}(\underline{\mathsf{C}})$ ($\mathsf{Ind}(\underline{\mathsf{C}})_\aleph$) may be expressed "naturally" in $\underline{\mathsf{C}}$, i.e., that it gives rise to a 2-natural transformation on $\underline{\underline{\mathsf{CAT}}}$.

Consider a filtered diagram T of ind-objects, i.e., an ind-object in $\mathtt{Ind}^2(\underline{\mathsf{C}}) = \mathtt{Ind}(\mathtt{Ind}(\underline{\mathsf{C}}))$. Suppose that $T(i) = (X_{i,j})_{j \in \mathsf{J}_i}$, and define the functor $U\colon \mathsf{K} \to \underline{\mathsf{C}}$ as follows:

- the objects of K are the pairs (i, j) where $i \in \mathsf{I}$ and $j \in \mathsf{J}_i$;
- the arrows of K are pairs $(\alpha, f)\colon (i, j) \to (h, k)$ where $\alpha\colon i \to h$ in I and $f\colon X_{i,j} \to X_{h,k}$ is a representative of the j-th component of $T(\alpha)\colon T(i) \to T(h)$, i.e., $(X_{i,j})_{j\in \mathsf{J}_i} \to (X_{h,k})_{k \in \mathsf{J}_h}$;

the composition in K being obviously given by $(\beta, g) \circ (\alpha, f) = (\beta \circ \alpha, g \circ f)$. Now, U is defined by

$$\begin{array}{ccc}
\mathsf{K} & \xrightarrow{U} & \underline{\mathsf{C}} \\
(i,j) & \longmapsto & X_{i,j} \\
{\scriptstyle (\alpha,f)}\downarrow & & \downarrow{\scriptstyle f} \\
(h,k) & \longmapsto & X_{h,k}
\end{array}$$

Lemma 3.5.16
$U\colon \mathsf{K} \to \underline{\mathsf{C}}$ is a functor

Proof. Of course U is well given. Since the identities of K are the pairs $(id_i, id_{X_{i,j}})$, U respects them. It follows immediately from the definition of composition that U respects it. ✓

Moreover, we have the following.

Lemma 3.5.17
K is filtered.

Proof. Consider (i, j) and (h, k) in K. Since I is filtered, we find an upper bound $\alpha\colon i \to x$, $\beta\colon h \to x$. Then we have $(i, j) \xrightarrow{(\alpha,f)} (x, m)$ and $(h, k) \xrightarrow{(\beta,g)} (x, n)$ in K and since $T(x)$ is filtered we have $u\colon m \to l$ and $v\colon n \to l$ in $T(x)$, which have to be component of the identity $T(x) \to T(x)$. Then, we have an upper bound for (i, j) and (h, k).

$$\begin{array}{ccccc}
(i,j) & \xrightarrow{(\alpha,f)} & (x,m) & & \\
 & & & \searrow{\scriptstyle (id,u)} & \\
 & & & & (x,l) \\
 & & & \nearrow{\scriptstyle (id,v)} & \\
(h,k) & \xrightarrow[(\beta,g)]{} & (x,n) & &
\end{array}$$

Consider now $(i,j) \overset{(\alpha,f)}{\underset{(\alpha',f')}{\rightrightarrows}} (i',j')$. We can equalize α and α' in I via β. Consider then any arrow in the j'-th component of $T(\beta)$, say g. Then we are in the following situation

$$X_{i,j} \overset{f}{\underset{f'}{\rightrightarrows}} X_{i',j'} \xrightarrow{\quad g \quad} X_{i'',j''}$$

Now, $g \circ f$ belongs to the j-th component of $T(\beta)\circ T(\alpha)$ and $g \circ f'$ to the j-th component of $T(\beta)\circ T(\alpha')$. Then, since $T(\beta\circ\alpha) = T(\beta\circ\alpha')$, we have that $(g\circ f) \sim (g\circ f')$, which means that there exists a pair of arrows in $T(i'')$ with common target that equalize them. Finally, since $T(i'')$ is filtered, these two arrows can be equalized themselves, so getting a single arrow which equalizes (α, f) and (α', f') as required. ✓

Observe that, since we are assuming $\aleph$ infinite and since in this case a $\aleph$-indexed union of sets of cardinality $\aleph$ has cardinality $\aleph$, if I and J_i have cardinality not greater than $\aleph$, so does K. Therefore, the following proposition applies to $\mathtt{Ind}(\underline{\mathsf{C}})_\aleph$ as well.

PROPOSITION 3.5.18
$U\colon \mathsf{K} \to \underline{\mathsf{C}}$ is the colimit in $\mathtt{Ind}(\underline{\mathsf{C}})$ of $T\colon \mathsf{I} \to \mathtt{Ind}(\underline{\mathsf{C}})$.

Proof. For any $i \in \mathsf{I}$ we can consider the functor

$$\begin{array}{ccc} T(i) & \xrightarrow{\quad u_i \quad} & \mathsf{K} \\ j & \longmapsto & (i,j) \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle (id,f)} \\ j' & \longmapsto & (i,j') \end{array}$$

Of course, we have $U \circ u_i = T(i)\colon \mathsf{J}_i \to \underline{\mathsf{C}}$, and therefore u_i induces a morphism $\lambda_i\colon \varinjlim(\mathsf{Y}\circ T(i)) \to \varinjlim(\mathsf{Y}\circ U)$, i.e., an ind-morphism $\lambda_i\colon T(i) \to U$.

It is easy to see that the λ_i's form a cocone with vertex U. First of all, observe that, by definition, $(\lambda_i)_j$ the j-th component of λ_i is the class of the identity of $X_{i,j}$. Then, $(\lambda_i)_j$ contains any $f\colon X_{i,j} \to X_{h,j'}$ such that $(i,j) \xrightarrow{(\alpha,f)} (h,j')$ is in K. It is now immediate to conclude that for any $\alpha\colon i \to h$ in I we must have $\lambda_h \circ T(\alpha) = \lambda_i$. Consider now another cocone $\{\tau_i\}$, $\tau_i\colon T(i) \to Y$. Explicitly, we have $\tau_i\colon (X_{i,j})_{j\in\mathsf{J}_i} \to (Y_j)_{j\in\mathsf{J}}$. Then, by collecting together these arrows we have $\bar\tau\colon (X_{i,j})_{i\in\mathsf{I}, j\in\mathsf{J}_i} \to (Y_j)_{j\in\mathsf{J}}$, which, thanks to the naturality of the τ's, is easily shown to be an ind-morphism $\bar\tau\colon U \to Y$. Of course we have $\bar\tau \circ \lambda_i = \tau_i$ for any $i \in \mathsf{I}$, and that $\bar\tau$ is the unique ind-morphism $U \to Y$ which enjoys this property. ✓

Thus, we have the following.

PROPOSITION 3.5.19
$\mathtt{Ind}(\underline{\mathsf{C}})$ ($\mathtt{Ind}(\underline{\mathsf{C}})_\aleph$) is ($\aleph$-)filtered cocomplete.

The following proposition matches the ones about presheaves and ind-representable functors. It says that every ind-object is built as a colimit of the constant ind-objects.

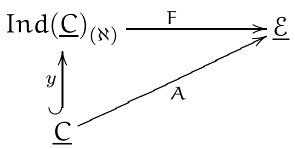
Proposition 3.5.20
For any ind-object $X\colon \mathsf{I} \to \underline{\mathsf{C}}$, *it is* $X = \varinjlim_{\mathsf{I}} y(X_i)$, *the limit being in* $\mathtt{Ind}(\underline{\mathsf{C}})$.

Proof. Apply the construction given above. ✓

Observe, however, that in general it is false that y preserves the filtered colimits which exist in $\underline{\mathsf{C}}$. In other words, $c = \varinjlim_{\mathsf{I}} c_i$ does not imply $\underline{c} \cong \varinjlim_{\mathsf{I}} \underline{c_i}$. We shall get back to this point in a following subsection.

Our claim that y plays the role which Y plays in the case of presheaves can now be fully justified by the following proposition which states the pseudo universal property enjoyed by $\mathtt{Ind}(\underline{\mathsf{C}})$.

Proposition 3.5.21
Let $\underline{\mathsf{C}}$ *be a locally small category. For any* ($\aleph$-)*filtered cocomplete category* $\underline{\mathcal{E}}$ *and any functor* $\mathsf{A}\colon \underline{\mathsf{C}} \to \underline{\mathcal{E}}$, *there is a functor* $\mathsf{F}\colon \mathtt{Ind}(\underline{\mathsf{C}})_{(\aleph)} \to \underline{\mathcal{E}}$ *which preserves the* ($\aleph$-)*filtered colimits and such that the following diagram commutes.*



Moreover, F *is unique up to isomorphism.*

Proof. Consider F which sends the ($\aleph$-)ind-object $(X_i)_{i\in\mathsf{I}}$ to $\varinjlim_{\mathsf{I}}(A \circ X)$ in $\underline{\mathcal{E}}$ and whose behaviour on the morphisms is induced by the universal property of colimits. Since without loss of generality we may assume that $\mathsf{F}(\underline{c}) = \varinjlim y(\mathsf{A}(c)) = \mathsf{A}(c)$, it follows immediately that the diagram commutes. Moreover, by exploiting the explicit definition of ($\aleph$-)filtered colimits in $\mathtt{Ind}(\underline{\mathsf{C}})_{(\aleph)}$ given in Proposition 3.5.18, it is easy to check directly that F preserves them.

Now, suppose that $\mathsf{K}\colon \mathtt{Ind}(C) \to \underline{\mathcal{E}}$ renders the diagram commutative and preserves ($\aleph$-)filtered colimits. Then, for any $(X_i)_{i\in\mathsf{I}}$ in $\mathtt{Ind}(\underline{\mathsf{C}})_{(\aleph)}$, we have:

$$\begin{aligned} \mathsf{F}(X) &\cong \varinjlim \mathsf{A}(X_i) \\ &\cong \varinjlim \mathsf{K}(y(X_i)) \cong \mathsf{K}(\varinjlim(y(X_i))) \cong \mathsf{K}(X), \end{aligned}$$

where the last equality follows from $(X_i)_{i\in\mathsf{I}} = \varinjlim_{\mathsf{I}}(y(X_i))$ in $\mathtt{Ind}(\underline{\mathsf{C}})_{(\aleph)}$, as the reader can check directly. Thus, we have $\mathsf{K} \cong \mathsf{F}$. ✓

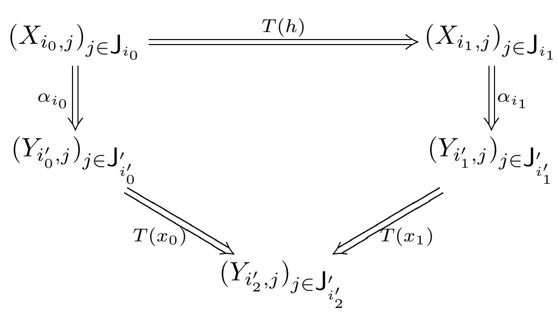
Our next step is to remark that the construction given above is functorial. More precisely, since a filtered diagram in $\mathtt{Ind}(\underline{\mathsf{C}})$ is an object of $\mathtt{Ind}^2(\underline{\mathsf{C}})$, the colimit construction identifies a function $m_{\underline{C}}$ from the objects of $\mathtt{Ind}^2(\underline{C})$ to the objects of $\mathtt{Ind}(\underline{C})$. It follows by the very definition of colimits that $m_{\underline{C}}$ can be extended canonically to a left adjoint functor $\mathtt{Ind}^2(\underline{C}) \to \mathtt{Ind}(\underline{C})$. Moreover, by Proposition 3.5.12, the right adjoint to $m_{\underline{C}}$ is $y_{\mathtt{Ind}(\underline{C})}$. In other words, the proof of the functoriality of $m_{\underline{C}}$ has been implicitly given in Proposition 3.5.18. Nevertheless, in the following we shall make explicit the definition of $m_{\underline{C}}$ on the morphisms of $\mathtt{Ind}^2(\underline{C})$.

Consider $T\colon \mathsf{I} \to \mathtt{Ind}(\underline{C})$ and $T'\colon \mathsf{I}' \to \mathtt{Ind}(\underline{C})$ in $\mathtt{Ind}^2(\underline{C})$. Suppose that $T(i) = (X_{i,j})_{j\in \mathsf{J}_i}$ and $T'(i') = (Y_j^{i'})_{j\in \mathsf{J}'_{i'}}$. Thus, in the indexed notation, we write $((X_{i,j})_{j\in \mathsf{J}_i})_{i\in \mathsf{I}}$ for T and $((Y_{i,j})_{j\in \mathsf{J}'_i})_{i\in \mathsf{I}'}$ for T'. Consider now a morphism $\alpha\colon T \to T'$ in $\mathtt{Ind}^2(\underline{C})$. By definition $\alpha = ([\alpha_i])_{i\in \mathsf{I}}$, is a compatible family of equivalence classes (wrt. I') of ind-morphisms $\alpha_i = ([\alpha_{i,j}])_{j\in \mathsf{J}_i}\colon T(i) \to T'(i')$ (the equivalence being now wrt. $\mathsf{J}'_{i'}$) in $\mathtt{Ind}(\underline{C})$, which in the indexed notation can be written as

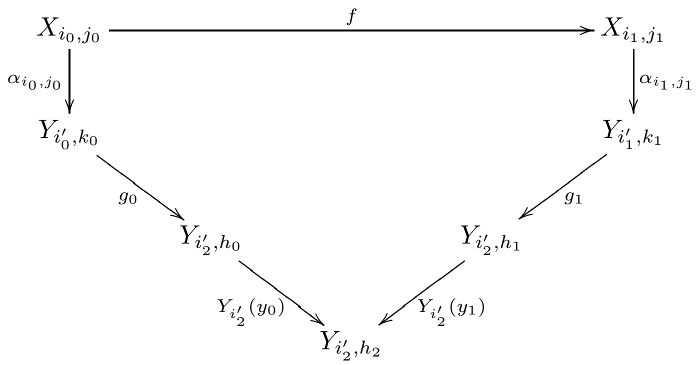
$$\Big(\Big[([\alpha_{i,j}])_{j\in \mathsf{J}_i}\Big]\Big)_{i\in \mathsf{I}}.$$

Then, as it appears neatly in the proof of Proposition 3.5.18, the collection $\bar{\alpha} = ([\alpha_{i,j}])_{(i,j)\in \mathsf{K}}$ of all the equivalence classes (wrt. K') of representatives $\alpha_{i,j}$ of the j-th class of some representative α_i of the i-th component of α, is an ind-morphism from $m_{\underline{C}}(T) = U\colon \mathsf{K} \to \underline{C}$ to $m_{\underline{C}}(T') = U'\colon \mathsf{K}' \to \underline{C}$, where U and U' are the colimits of T and T' determined as earlier in this section. We shall take $m_{\underline{C}}(\alpha)$ to be $\bar{\alpha}$.

In order to show that this is well defined, we only need to verify that the $[\alpha_{i,j}]$'s are compatible, i.e., that for any $(h,f)\colon (i_0,j_0) \to (i_1,j_1)$ in K and for any pairs of representatives of the (i_0,j_0)-th and of the (i_1,j_1)-th component of α, say $\alpha_{i_0,j_0}\colon X_{i_0,j_0} \to Y_{i'_0,k_0}$ and $\alpha_{i_1,j_1}\colon X_{i_1,j_1} \to Y_{i'_1,k_1}$, we have $[\alpha_{i_1,j_1} \circ U(h,f)]_\sim = [\alpha_{i_0,j_0}]_\sim$. Recall that (h,f) is an arrow of K if $h\colon i_0 \to i_1$ is an arrow of I and $f\colon X_{i_0,j_0} \to X_{i_1,j_1}$ is a representative of the j_0-th class of $T(h)\colon T(i_0) \to T(i_1)$. Then, consider the representatives of the i_0-th and the i_1-th components of α, say $\alpha_{i_0}\colon T(i_0) \to T'(i'_0)$ and $\alpha_{i_1}\colon T(i_1) \to T'(i'_1)$, which α_{i_0,j_0} and α_{i_1,j_1}, respectively, come from. Then, since α is an ind-morphism, there exist $x_0\colon i'_0 \to i'_2$ and $x_1\colon i'_1 \to i'_2$ in I' such that $T'(x_0) \circ \alpha_{i_O} = T'(x_1) \circ \alpha_{i_1} \circ T(h)$, as described in the picture below



Now, looking at the representative f of the j_0-th class of $T(h)$, we have that $[g_0 \circ \alpha_{i_0,j_0}]_\sim = [g_1 \circ \alpha_{i_1,j_1} \circ f]_\sim$ where $g_0 \colon Y_{i'_0,k_0} \to Y_{i'_2,h_0}$ and $g_1 \colon Y_{i'_1,k_1} \to Y_{i'_2,h_1}$ are, respectively, representatives of the k_0-th class of $T(x_0)$ and of the k_1-th class of $T(x_1)$, and $\sim$ is wrt. $\mathsf{J}'_{i'_2}$. Now, by definition of $\sim$, we have $y_0 \colon h_0 \to h_2$ $y_1 \colon h_1 \to h_2$ in $\mathsf{J}'_{i'_2}$ such that $Y_{i'_2}(y_0) \circ g_0 \circ \alpha_{i_0,j_0} = Y_{i'_2}(y_1) \circ g_1 \circ \alpha_{i_1,j_1} \circ f$.

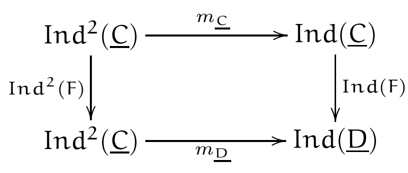


Finally, observe that $(x_0, g_0) \colon (i'_0, k_0) \to (i'_2, h_0)$ and $(x_1, g_1) \colon (i'_1, k_1) \to (i'_2, h_1)$ are morphisms in K', and moreover that $Y_{i'_2}(y_0)$ and $Y_{i'_2}(y_1)$ are representatives, respectively of the h_0-th and the h_1-th class of the identity of $T'(i'_2)$. It follows that $(id, Y_{i'_2}(y_0)) \colon (i'_2, h_0) \to (i'_2, h_2)$ and $(id, Y_{i'_2}(y_1)) \colon (i'_2, h_1) \to (i'_2, h_2)$ are morphisms in K'. Therefore, we have $\beta_0 = (x_0, Y_{i'_2}(y_0) \circ g_0) \colon (i'_0, k_0) \to (i'_2, h_2)$ and $\beta_1 = (x_1, Y_{i'_2}(y_1) \circ g_1) \colon (i'_1, k_1) \to (i'_2, h_2)$ in K', such that $U'(\beta_1) \circ \alpha_{i_1,j_1} \circ U(h, f) = U'(\beta_0) \circ \alpha_{i_0,j_0}$, i.e., $[\alpha_{i_1,j_1} \circ U(h, f)]_\sim = [\alpha_{i_0,j_0}]_\sim$, as required. It is easy to verify that this definition coincides with the behaviour induced by the colimit construction in Proposition 3.5.18, and therefore gives a functor. Alternatively, it is easy to check directly that $m_{\underline{\mathsf{C}}}(id) = id$ and $m_{\underline{\mathsf{C}}}(\beta \circ \alpha) = m_{\underline{\mathsf{C}}}(\beta) \circ m_{\underline{\mathsf{C}}}(\alpha)$. Thus, we conclude as follows.

Proposition 3.5.22
$m_{\underline{\mathsf{C}}} \colon \mathtt{Ind}^2(\underline{\mathsf{C}}) \to \mathtt{Ind}(\underline{\mathsf{C}})$ is a functor which is left adjoint to $y_{\mathtt{Ind}(\underline{\mathsf{C}})}$.

Let us now show that the collection $m_{\underline{\mathsf{C}}} \colon \mathtt{Ind}^2(\underline{\mathsf{C}}) \to \mathtt{Ind}(\underline{\mathsf{C}})$ is a 2-natural transformation $\mathtt{Ind}^2(\_) \stackrel{\cdot}{\to} \mathtt{Ind}(\_)$.

First, we have to show that



commutes. Let $T = ((X_{i,j})_{j \in \mathsf{J}_i})_{i \in \mathsf{I}}$ be in $\mathtt{Ind}^2(\underline{\mathsf{C}})$. Then,

$$\mathtt{Ind}^2(\mathsf{F})(T) = \mathtt{Ind}(\mathsf{F}) \circ T = \Big(\mathtt{Ind}(\mathsf{F})\Big((X_{i,j})_{j \in \mathsf{J}_i}\Big)\Big)_{i \in \mathsf{I}} = \Big(\mathsf{F} \circ (X_{i,j})_{j \in \mathsf{J}_i}\Big)_{i \in \mathsf{I}},$$

i.e., $((\mathsf{F}X_{i,j})_{j \in \mathsf{J}_i})_{i \in \mathsf{I}}$. Then, $m_{\underline{\mathsf{D}}}\Big(\mathtt{Ind}^2(\mathsf{F})(T)\Big) = (\mathsf{F}X_{i,j})_{(i,j) \in \mathsf{K}'}$, where K' is built in $\mathtt{Ind}(\underline{\mathsf{D}})$ for the diagram $\mathtt{Ind}(\mathsf{F}) \circ T$.

On the other hand, it is $m_{\underline{\mathsf{C}}}\Big(((X_{i,j})_{j \in \mathsf{J}_i})_{i \in \mathsf{I}}\Big) = (X_{i,j})_{(i,j) \in \mathsf{K}}$, and therefore $\mathtt{Ind}(\mathsf{F})(m_{\underline{\mathsf{C}}}(T)) = \mathsf{F} \circ m_{\underline{\mathsf{C}}}(T) = (\mathsf{F}X_{i,j})_{(i,j) \in \mathsf{K}}$, where K is built in $\mathtt{Ind}(\underline{\mathsf{C}})$ for the diagram T. Observe that K and K' do not need to be isomorphic. More precisely, the objects in K and K' coincide, being the pairs (i,j) for $i \in \mathsf{I}$ and $j \in \mathsf{J}_i$. However, the morphisms of K' are pairs $(\alpha, f) \colon (i,j) \to (i',j')$ for $\alpha \colon i \to i'$ in I and $f \colon \mathsf{F}X_{i,j} \to \mathsf{F}X_{i',j'}$ in $\underline{\mathsf{D}}$ a representative of the j-th class of $\mathsf{F}T(\alpha)$, while in the morphisms of K the component f is a morphism $f \colon X_{i,j} \to X_{i',j'}$ in $\underline{\mathsf{C}}$, representative of the j-th class of $T(\alpha)$. Of course, these do not need to be the same, since, as observed earlier, $[\mathsf{F}f]_\sim$ is not necessarily the image through F of $[f]_\sim$.

However, observe that, if ϕ is the functor defined by

$$\begin{array}{ccc}
\mathsf{K} & \xrightarrow{\quad\phi\quad} & \mathsf{K}' \\
(i,j) & \longmapsto & (i,j) \\
{\scriptstyle (\alpha,f)}\downarrow & & \downarrow{\scriptstyle (\alpha,\mathsf{F}f)} \\
(i',j') & \longmapsto & (i',j')
\end{array}$$

which is clearly well defined, we have $m_{\underline{\mathsf{D}}}\Big(\mathtt{Ind}^2(\mathsf{F})(T)\Big)\phi = \mathtt{Ind}(\mathsf{F})(m_{\underline{\mathsf{C}}}(T))$. We shall see next that ϕ is cofinal. Then, by Definition 3.5.1, we conclude that $m_{\underline{\mathsf{D}}}\Big(\mathtt{Ind}^2(\mathsf{F})(T)\Big) = \mathtt{Ind}(\mathsf{F})(m_{\underline{\mathsf{C}}}(T))$, since we identify such objects in $\mathtt{Ind}(\underline{\mathsf{D}})$.

REMARK. It is worth noticing that the purpose of the identification of ind-objects in Definition 3.5.1 is mainly to have the (strict) naturality of m. However, we shall see in the next subsection that it is also important to make of $m_{\underline{\mathsf{C}}}$ a strict left inverse for $y_{\mathtt{Ind}(\underline{\mathsf{C}})}$.

With respect to Lemma 3.3.14, point *(ii)*, condition $\mathbf{F_1}$ is immediate; so, in order to conclude ϕ is cofinal, we only need to check condition $\mathbf{F_2}$. Consider $(i,j) \underset{(\alpha,g)}{\overset{(\alpha,f)}{\rightrightarrows}} (i',j')$ in K'. This means that $f, g \colon \mathsf{F}X_{i,j} \to \mathsf{F}X_{i',j'}$ are representatives of the j-th equivalence class of $\mathsf{F}T(\alpha)$. Then, by definition of ind-morphism, there exist $j' \underset{x'}{\overset{x}{\rightrightarrows}} j''$ in $\mathsf{J}_{i'}$ such that $f \circ \mathsf{F}X_{i'}(x) = g \circ \mathsf{F}X_{i'}(x')$. Moreover, since $\mathsf{J}_{i'}$ is filtered, we can find y such that $y \circ x = y \circ x'$. Thus, $f \circ \mathsf{F}X_{i'}(y \circ x) = g \circ \mathsf{F}X_{i'}(y \circ x)$. It follows that $(id, y \circ x)$ in K is such that $f \circ \phi((id, y \circ x)) = g \circ \phi((id, y \circ x))$, which means that ϕ is cofinal.

Consider now a morphism $\alpha = \Big(\Big[([\alpha_{i,j}])_{j\in \mathsf{J}_i}\Big]\Big)_{i\in\mathsf{I}}$ from T to T' in $\mathbb{Ind}^2(\underline{\mathsf{C}})$. Then,

$$\mathbb{Ind}^2(\mathsf{F})(\alpha) = \Big(\Big[\mathbb{Ind}(\mathsf{F})(([\alpha_{i,j}])_{j\in\mathsf{J}_i})\Big]\Big)_{i\in\mathsf{I}} = \Big(\Big[([\mathsf{F}(\alpha_{i,j})])_{j\in\mathsf{J}_i}\Big]\Big)_{i\in\mathsf{I}},$$

and thus $m_{\underline{\mathsf{D}}}\Big(\mathbb{Ind}^2(\mathsf{F})(\alpha)\Big) = ([\mathsf{F}(\alpha_{i,j})])_{(i,j)\in\mathsf{K}'}$, K', as before, being built in $\mathbb{Ind}(\underline{\mathsf{D}})$ for $\mathbb{Ind}(\mathsf{F})\circ T$. Following the other edge of the diagram, we have

$$\mathbb{Ind}(\mathsf{F})(m_{\underline{\mathsf{C}}}(\alpha)) = \mathbb{Ind}(\mathsf{F})\Big(([\alpha_{i,j}])_{(i,j)\in\mathsf{K}}\Big) = ([\mathsf{F}(\alpha_{i,j}]))_{(i,j)\in\mathsf{K}},$$

for K in $\mathbb{Ind}(\underline{\mathsf{D}})$ built for T. We shall show that these morphisms are the same.

Let K_0 and K_0' be the categories built along the two different edges of the diagram for T' and let $\sim$ and $\sim'$ indicate, respectively, the equivalences wrt. K_0 and wrt. K_0'. We know that K_0 and K_0' have the same elements and that there is a cofinal $\phi\colon \mathsf{K}_0 \to \mathsf{K}_0'$. Of course, if $f\sim g$ then $f\sim' g$. Therefore, $[\mathsf{F}(\alpha_{i,j})]_\sim \subseteq [\mathsf{F}(\alpha_{i,j})]_{\sim'}$. Now, suppose that $(\mathsf{F}(\alpha_{i,j})\colon X_{i,j}\to i_0, j_0) \sim' (f\colon X_{i,j}\to X_{i_1,j_1})$. Then, there exist

$$(\beta_0, g_0)\colon (i_0,j_0)\to(i_2,j_2) \quad\text{and}\quad (\beta_1,g_1)\colon(i_1,j_1)\to(i_2,j_2)$$

such that $g_0\circ\mathsf{F}(\alpha_{i,j}) = g_1\circ f$. Since (i_0,j_0) and (i_2,j_2) belong to K_0, we find an upper bound $(\gamma_0,x_0)\colon(i_0,j_0)\to(i_3,j_3)$ and $(\gamma_1,x_1)\colon(i_2,j_2)\to(i_3,j_3)$ in K_0. Then, consider the pair of arrows $\phi((\gamma_0,x_0)) = (\gamma_0,\mathsf{F}(x_0))\colon(i_0,j_0)\to(i_3,j_3)$ and $\phi((\gamma_1,x_1)) = (\gamma_1,\mathsf{F}(x_1))\colon(i_2,j_2)\to(i_3,j_3)$ in K_0'. These arrows provide a parallel pair of arrows in K_0', namely $(\gamma_1,\mathsf{F}(x_1))\circ(\beta_0,g_0)$ and $(\gamma_0,\mathsf{F}(x_0))$, and since ϕ is cofinal we find $(\delta_0,y_0)\colon(i_3,j_3)\to(i_5,j_5)$ such that

$$\phi((\delta_0,y_0))\circ(\gamma_0,\mathsf{F}(x_0)) = \phi((\delta_0,y_0))\circ(\gamma_1,\mathsf{F}(x_1))\circ(\beta_0,g_0),$$

i.e., $(\delta_0\circ\gamma_0,\mathsf{F}(y_0\circ x_0)) = (\delta_0\circ\gamma_1\circ\beta_0,\mathsf{F}(y_0\circ x_1)\circ g_0)$.

Applying the same argument above to (i_2,j_2) and (i_1,j_1), one finds the arrows $(\gamma_2,x_2)\colon(i_1,j_1)\to(i_4,j_4)$ and $(\bar\gamma_1,\bar x_1)\colon(i_2,j_2)\to(i_4,j_4)$ in K_0, and then an upper bound $(\delta_1,y_1)\colon(i_4,j_4)\to(i_6,j_6)$ such that

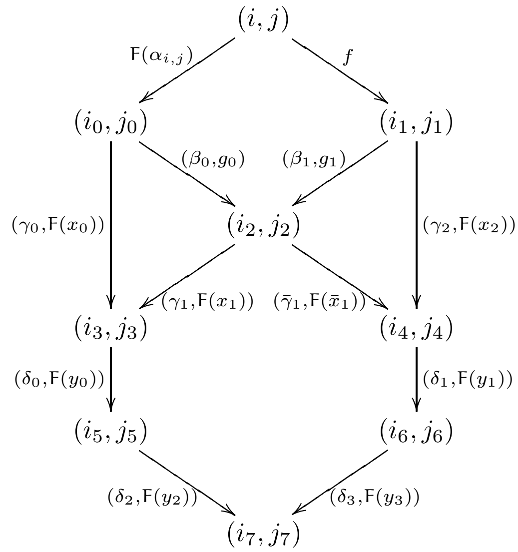
$$\phi((\delta_1,y_1))\circ(\gamma_2,\mathsf{F}(x_2)) = \phi((\delta_1,y_1))\circ(\bar\gamma_1,\mathsf{F}(\bar x_1))\circ(\beta_1,g_1),$$

which is $(\delta_1\circ\gamma_2,\mathsf{F}(y_1\circ x_2)) = (\delta_1\circ\bar\gamma_1\circ\beta_1,\mathsf{F}(y_1\circ\bar x_1)\circ g_1)$.

Finally, since K_0 is filtered, we have

$$(\delta_2,y_2)\colon(i_5,j_5)\to(i_7,j_7) \quad\text{and}\quad (\delta_3,y_3)\colon(i_6,j_6)\to(i_7,j_7)$$

such that $(\delta_2,y_2)\circ(\delta_0,y_0)\circ(\gamma_1,x_1) = (\delta_3,y_3)\circ(\delta_1,y_1)\circ(\bar\gamma_1,\bar x_1)$, which means that $(\delta_2\circ\delta_0\circ\gamma_1, y_2\circ y_0\circ x_1) = (\delta_3\circ\delta_1\circ\bar\gamma_1, y_3\circ y_1\circ\bar x_1)$. Then, we have the following commutative diagram in $\underline{D}$.



Now, consider $(\delta_2\circ\delta_0\circ\gamma_0, y_2\circ y_0\circ x_0)$ from (i_0, j_0) to (i_7, j_7) and $(\delta_3\circ\delta_1\circ\gamma_2, y_3\circ y_1\circ x_2)$ from (i_1, j_1) to (i_7, j_7) in K. We claim that these morphisms show that $\mathsf{F}(\alpha_{i,j}) \sim f$. In fact,

$$\begin{aligned}
\mathsf{F}(y_2 \circ y_0 \circ x_0) \circ \mathsf{F}(\alpha_{i,j}) &= \mathsf{F}(y_2) \circ \mathsf{F}(y_0) \circ \mathsf{F}(x_0) \circ \mathsf{F}(\alpha_{i,j}) \\
&= \mathsf{F}(y_2) \circ \mathsf{F}(y_0) \circ \mathsf{F}(x_1) \circ g_0 \circ \mathsf{F}(\alpha_{i,j}) \\
&= \mathsf{F}(y_2) \circ \mathsf{F}(y_0) \circ \mathsf{F}(x_1) \circ g_1 \circ f \\
&= \mathsf{F}(y_3) \circ \mathsf{F}(y_1) \circ \mathsf{F}(\bar{x}_1) \circ g_1 \circ f \\
&= \mathsf{F}(y_3) \circ \mathsf{F}(y_1) \circ \mathsf{F}(x_2) \circ f \\
&= \mathsf{F}(y_3 \circ y_1 \circ x_2) \circ f.
\end{aligned}$$

Thus, we conclude that $[\mathsf{F}(\alpha_{i,j})]_\sim = [\mathsf{F}(\alpha_{i,j})]_{\sim'}$. In order to complete the identification of the two morphisms above, we should that show that the $\sim$-classes of $\mathsf{F}(\alpha_{i,j})$, which by definition are compatible wrt. K, i.e., for morphisms $(\beta, g)\colon (i, j) \to (i', j')$ in K, are compatible also wrt. K'. However, we omit this proof which can be done exactly as the previous one thanks to the fact that also in this case there exists a cofinal $\phi\colon \mathsf{K} \to \mathsf{K}'$ which is the identity on the objects.

<u>Remark</u>. We would like to stress that the result above proves the naturality of the identification of ind-objects imposed in Definition 3.5.1. In fact, it is important to notice that the proof given above does not rely on the fact that $\phi\colon \mathsf{K} \to \mathsf{K}'$ is the identity on the objects, but just on the fact that it is an isomorphism. Thus, the proof above applies to all the ind-objects identified via the cofinal ϕ.

Next, we have to show that m is 2-natural, i.e., that for any $\alpha\colon \mathsf{F} \overset{\cdot}{\to} \mathsf{G}$ the following equation holds.

$$\mathtt{Ind}^2(\underline{\mathsf{C}}) \xrightarrow{m_{\underline{\mathsf{C}}}} \mathtt{Ind}(\underline{\mathsf{C}}) \underset{\mathtt{Ind}(\mathsf{G})}{\overset{\mathtt{Ind}(\mathsf{F})}{\Downarrow \mathtt{Ind}(\alpha)}} \mathtt{Ind}(\underline{\mathsf{D}}) \quad =$$

$$\mathtt{Ind}^2(\underline{\mathsf{C}}) \underset{\mathtt{Ind}^2(\mathsf{G})}{\overset{\mathtt{Ind}^2(\mathsf{F})}{\Downarrow \mathtt{Ind}^2(\alpha)}} \mathtt{Ind}^2(\underline{\mathsf{D}}) \xrightarrow{m_{\underline{\mathsf{D}}}} \mathtt{Ind}(\underline{\mathsf{D}})$$

Consider an object $T = ((X_{i,j})_{j\in \mathsf{J}_i})_{i\in \mathsf{I}}$ of $\mathtt{Ind}^2(\underline{\mathsf{C}})$, and let $U\colon \mathsf{K} \to \underline{\mathsf{C}}$ be $m_{\underline{\mathsf{C}}}(T)$. Then, the component at T of $\mathtt{Ind}(\alpha) * m_{\underline{\mathsf{C}}}$ is $(\mathtt{Ind}(\alpha) * m_{\underline{\mathsf{C}}})_T = \mathtt{Ind}(\alpha)_U = ([\alpha_{X_{i,j}}])_{(i,j)\in \mathsf{K}}\colon (\mathsf{F}X_{i,j})_{(i,j)\in\mathsf{K}} \to (\mathsf{G}X_{i,j})_{(i,j)\in\mathsf{K}}$.

On the other hand, $\mathtt{Ind}^2(\alpha)_T = ([\mathtt{Ind}(\alpha)_{T(i)}])_{i\in\mathsf{I}} = ([([\alpha_{X_{i,j}}])_{j\in\mathsf{J}_i}])_{i\in\mathsf{I}}$, and thus $m_{\underline{\mathsf{D}}}(\mathtt{Ind}^2(\alpha)_T) = ([\alpha_{X_{i,j}}])_{(i,j)\in\mathsf{K}'}$. But this is again the situation we met before and, thus, we can conclude that the two morphisms coincide. Since the same holds for each T in $\mathtt{Ind}^2(\underline{\mathsf{C}})$, it follows that $\mathtt{Ind}(\alpha) * m_{\underline{\mathsf{C}}} = m_{\underline{\mathsf{D}}} * \mathtt{Ind}^2(\alpha)$.

Then, we have shown the following.

Proposition 3.5.23
$m\colon \mathtt{Ind}^2(\_) \overset{\cdot}{\to} \mathtt{Ind}(\_)$ ($m\colon \mathtt{Ind}^2(\_)_{\aleph} \overset{\cdot}{\to} \mathtt{Ind}(\_)_{\aleph}$) is a 2-natural transformation.

Remark. It is worth noticing the primary role played in establishing the naturality of m by the fact that our index categories are filtered. There is no obvious way to achieve the same result working with chains or directed sets.

We complete this subsection by stating the following relevant fact.

Proposition 3.5.24
Let $\underline{\mathsf{C}}$ and $\underline{\mathsf{D}}$ be locally small categories. Then, for any $\mathsf{F}\colon \underline{\mathsf{C}} \to \underline{\mathsf{D}}$, the functor $\mathtt{Ind}(\mathsf{F})$ preserves filtered colimits.

Proof. We have proved that $m_{\underline{\mathsf{C}}}$ is 2-natural. In particular the following diagram commutes.

$$\begin{array}{ccc} \mathtt{Ind}^2(\underline{\mathsf{C}}) & \xrightarrow{\quad m_{\underline{\mathsf{C}}}\quad} & \mathtt{Ind}(\underline{\mathsf{C}}) \\ {\scriptstyle \mathtt{Ind}^2(\mathsf{F})}\downarrow & & \downarrow{\scriptstyle \mathtt{Ind}(\mathsf{F})} \\ \mathtt{Ind}^2(\underline{\mathsf{C}}) & \xrightarrow[\quad m_{\underline{\mathsf{D}}}\quad]{} & \mathtt{Ind}(\underline{\mathsf{D}}) \end{array}$$

However, since $m_{\underline{C}}$ is an explicit choice of colimits and the colimits of a given diagram are isomorphic, this can be read as $\mathtt{Ind}(\mathsf{F}) \circ \varinjlim_{\underline{C}} \cong \varinjlim_{\underline{D}} \circ \mathtt{Ind}^2(\mathsf{F})$, i.e., $\mathtt{Ind}(\mathsf{F})$ preserves filtered colimits. ✓

Of course, the same holds for $\mathtt{Ind}(\_)_{\aleph}$.

SOME REMARKS ON $\mathtt{Ind}(\underline{C})$

This subsection states some further results which, although not central in our interest, may have future applications and, therefore, be useful. We omit all the proofs.

We have noticed earlier that y does not preserve colimits. The following is a general statement of the properties of y with respect to preservation of limits.

PROPOSITION 3.5.25
For any $\underline{C}$, consider the embedding $y\colon \underline{C} \to \mathtt{Ind}(\underline{C})$ ($y\colon \underline{C} \to \mathtt{Ind}(\underline{C})_{\aleph}$). Then, we have the following.

i) y preserves limits.

ii) y preserves finite colimits.

iii) The following are equivalent:
   (a) y preserves ($\aleph$-)filtered colimits;
   (b) y is an equivalence of categories;
   (c) $\underline{C}$ has filtered colimits and y preserves them;
   (d) $\underline{C}$ has filtered colimits and for any c the functor $\mathsf{Hom}_{\underline{C}}(c, \_)$ preserves them.

One of the most interesting issue in order theory (domain theory) is that, from a complete poset, it is often possible—by considering some special elements—to extract a poset whose completion gives back the complete poset. Examples of this situation are the *principal ideals* in a lattice, the *algebraic* elements of a Scott's domain and the *prime algebraic* elements in Berry's dI-domains. There is a similar issue also on the categorical ground which allows, in some cases, to get back from $\mathtt{Ind}(\underline{C})$ a category which is equivalent to $\underline{C}$. The following notions are studied in great detail in e.g. [28, 64].

Let $\underline{D}$ be a $\aleph$-filtered cocomplete category. An object $c \in \underline{D}$ is ($\aleph$-)*finitely presentable* if the functor $\mathsf{Y}'(c) = \mathsf{Hom}_{\underline{D}}(c, \_)\colon \underline{D}^{op} \to \underline{\mathsf{Set}}$ preserves $\aleph$-filtered colimits. Let $\underline{D}_F$ denote the subcategory of $\underline{D}$ consisting of the finitely presentable (*fp*) objects. A category $\underline{D}$ is ($\aleph$-)*locally finitely presentable* (*lfp*) if $\underline{D}_F$ contains a base which generates $\underline{D}$ by ($\aleph$-)filtered colimits. In other words, $\underline{D}$ is locally finitely presentable if $\mathtt{Ind}(\underline{D}_F) \stackrel{\cdot}{\cong} \underline{D}$ ($\mathtt{Ind}(\underline{D}_F)_{\aleph} \stackrel{\cdot}{\cong} \underline{D}$). The following is stated for $\mathtt{Ind}(\underline{C})$, but, of course, it holds also for $\mathtt{Ind}(\underline{C})_{\aleph}$.

Proposition 3.5.26
For any $c \in \underline{C}$, $y(c)$ is finitely presentable in $\mathtt{Ind}(\underline{C})$.

If the idempotents split in $\underline{C}$, then $\mathtt{Ind}(\underline{C})$ is lfp and $\mathtt{Ind}(\underline{C})_F \stackrel{\cdot}{\cong} \underline{C}$, since it is the subcategory of the essentially constants ind-objects.

The Ind KZ-Doctrine

In this subsection we sum up the results by showing that the data $\mathtt{Ind}(\_)$ ($\mathtt{Ind}(\_)_{\aleph}$), y, and m determine a KZ-doctrine on $\underline{\underline{CAT}}$. We treat mainly the case of $\mathtt{Ind}(\_)$, but everything below can be restated for $\mathtt{Ind}(\_)_{\aleph}$.

We have shown in the previous subsection that $\mathtt{Ind}(\_)$ is a 2-functor and that y and m are 2-natural transformations. We still need to give the family of 2-cells $\lambda_{\underline{C}}$. Recall that, by Proposition 3.5.22, we have a reflection $m_{\underline{C}} \dashv y_{\mathtt{Ind}(\underline{C})}$. Let $\eta_{\underline{C}}: id_{\mathtt{Ind}^2(\underline{C})} \stackrel{\cdot}{\to} y_{\mathtt{Ind}(\underline{C})} \circ m_{\underline{C}}$ be the unit of this adjunction. Then, we take $\lambda_{\underline{C}}$ to be $\eta_{\underline{C}} * \mathtt{Ind}(y_{\underline{C}})$. For general reasons, we know that $\eta_{\underline{C}}$ gives the limit cocones. Let us give it explicitly.

Given $T = ((X_{i,j})_{j\in \mathsf{J}_i})_{i\in \mathsf{I}}$, we have $m_{\underline{C}}(T) = (X_{i,j})_{(i,j)\in \mathsf{K}}$ and therefore we have $y_{\mathtt{Ind}(\underline{C})}(m_{\underline{C}}(T)) = ((X_{i,j})_{(i,j)\in\mathsf{K}})_{\underline{1}}$, where we use the notation $(T)_{\underline{1}}$ for the singleton diagram of value T. Thus, η_T is an I-indexed family of ind-morphisms α_i in $\mathtt{Ind}(\underline{C})$, where α_i is the "inclusion" of $T(i)$ in the colimit of T, which is the J_i-indexed family of the equivalence classes (wrt. K) of the identities of $X_{i,j}$. In other words, we have $\eta_T = ([([id_{X_{i,j}}])_{j\in\mathsf{J}_i}])_{i\in\mathsf{I}}: ((X_{i,j})_{j\in\mathsf{J}_i})_{i\in\mathsf{I}} \to ((X_{i,j})_{(i,j)\in\mathsf{K}})_{\underline{1}}$. Thus, we can conclude the triangular identities, which in the particular case of a reflection take the form

$$\eta_{\underline{C}} * y_{\mathtt{Ind}(\underline{C})} = \mathbf{1} \qquad \text{and} \qquad m_{\underline{C}} * \eta_{\underline{C}} = \mathbf{1}.$$

Let us verify the KZ-doctrine axioms in Definition 3.4.1.

$\mathbf{T_0}$: $m_{\underline{C}} \circ y_{\mathtt{Ind}(\underline{C})} = id$ and $m_{\underline{C}} \circ \mathtt{Ind}(y_{\underline{C}}) = id$.

$m_{\underline{C}}\Big(y_{\mathtt{Ind}(\underline{C})}\Big((X_i)_{i\in\mathsf{I}}\Big)\Big) = m_{\underline{C}}\Big(((X_i)_{i\in\mathsf{I}})_{\underline{1}}\Big) = (X_i)_{(*,i)\in\mathsf{K}}$. Once again, K determined from $\underline{1}$ and I is *not* isomorphic to I. In fact, its objects are pairs $(*, i)$ and its morphisms are pairs $(*, i) \stackrel{(id,f)}{\longrightarrow} (*, i')$, where f is a representative of the i-th class wrt. I of the identity on $(X_i)_{i\in\mathsf{I}}$. Thus, although every $f: i \to i'$ in I corresponds to (id, f) in K, the converse is not true. However, the embedding $\phi: \mathsf{I} \to \mathsf{K}$, which sends i to $(*, i)$ and f to (id, f), is easily shown to be *cofinal*. Thus, the last formula is equal to $(X_i)_{i\in\mathsf{I}}$. The same formal steps prove that $m_{\underline{\mathsf{C}}} \circ y\mathtt{Ind}(\underline{\mathsf{C}})$ is the identity also on the morphisms.

$m_{\underline{\mathsf{C}}}\Big(\mathtt{Ind}(y_{\underline{\mathsf{C}}})((X_i)_{i\in\mathsf{I}})\Big) = m_{\underline{\mathsf{C}}}\Big(((X_i)_{\underline{1}})_{i\in\mathsf{I}}\Big) = (X_i)_{(i,*)\in\mathsf{K}}$. This time the objects of K are pairs $(i,*)$ for $i\in\mathsf{I}$ and the morphisms are pairs (α,k) where $\alpha\colon i\to i'$ is in I and k is a representative of the unique equivalence class wrt. $\underline{1}$ of $X(\alpha)$. However, because of the particular form of $\underline{1}$ there is a unique representative in that class. Therefore, in this case, K is isomorphic to I. Thus, the last formula is equal to $(X_i)_{i\in\mathsf{I}}$. The same argument can be used for the morphisms of $\mathtt{Ind}(\underline{\mathsf{C}})$ to show $m_{\underline{\mathsf{C}}} \circ \mathtt{Ind}(y_{\underline{\mathsf{C}}}) = id$, as required.

Observe now that $\mathtt{Ind}(y_{\underline{\mathsf{C}}})\colon \mathtt{Ind}(\underline{\mathsf{C}}) \to \mathtt{Ind}^2(\underline{\mathsf{C}})$, and thus

$$\lambda_{\underline{\mathsf{C}}} = \eta_{\underline{\mathsf{C}}} * \mathtt{Ind}(y_{\underline{\mathsf{C}}})\colon \mathtt{Ind}(y_{\underline{\mathsf{C}}}) \stackrel{\cdot}{\to} y_{\mathtt{Ind}^2(\underline{\mathsf{C}})} \circ m_{\underline{\mathsf{C}}} \circ \mathtt{Ind}(y_{\mathtt{Ind}(\underline{\mathsf{C}})}) = y_{\mathtt{Ind}^2(\underline{\mathsf{C}})},$$

as required.

Let us proceed to show that the remaining KZ-doctrine axioms hold in our context.

$\mathbf{T_1}$: $\lambda_{\underline{\mathsf{C}}} * y_{\underline{\mathsf{C}}} = \mathbf{1}$.

The left hand side of the equation actually is $\eta_{\underline{\mathsf{C}}} * \mathtt{Ind}(y_{\underline{\mathsf{C}}}) * y_{\underline{\mathsf{C}}}$, which by naturality of y is $\eta_{\underline{\mathsf{C}}} * y_{\mathtt{Ind}(\underline{\mathsf{C}})} * y_{\underline{\mathsf{C}}}$. But the last two elements of this composition are one of the triangular identities for the adjunction, and therefore the formula above is an identity 2-cell.

$\mathbf{T_2}$: $m_{\underline{\mathsf{C}}} * \lambda_{\underline{\mathsf{C}}} = \mathbf{1}$.

The left hand side of the equation is $m_{\underline{\mathsf{C}}} * \eta_{\underline{\mathsf{C}}} * \mathtt{Ind}(y_{\underline{\mathsf{C}}})$, and using the other triangular identity we again can show that it equals $\mathbf{1}$.

$\mathbf{T_3}$: $m_{\underline{\mathsf{C}}} * \mathtt{Ind}(m_{\underline{\mathsf{C}}}) * \lambda_{\mathtt{Ind}^2(\underline{\mathsf{C}})} = \mathbf{1}$.

Consider $T = ((X_{i,j})_{j\in\mathsf{J}_i})_{\in\mathsf{I}}$. We have $\mathtt{Ind}(y_{\mathtt{Ind}(\underline{\mathsf{C}})})(T) = (((X_{i,j})_{j\in\mathsf{J}_i})_{\underline{1}})_{i\in\mathsf{I}}$. Thus, the unit $\eta_{\mathtt{Ind}(\underline{\mathsf{C}})}$ at this object is the I-indexed family α of equivalence classes $[\alpha_i]$ whose representatives are ind-morphisms

$$\alpha_i\colon ((X_{i,j})_{j\in\mathsf{J}_i})_{\underline{1}} \to ((X_{i,j})_{j\in\mathsf{J}_i})_{(i,*)\in\mathsf{K}_0},$$

where K_0 is built by $m_{\mathtt{Ind}(\underline{\mathsf{C}})}$ for I and $\underline{1}$. Each α_i has a unique component which, by definition, is the equivalence class (wrt. K) of the identity ind-morphism of $(X_{i,j})_{j\in\mathsf{J}_i}$. Then $\mathtt{Ind}(m_{\underline{\mathsf{C}}})(\alpha)$ is the I-indexed family β of equivalence classes $[\beta_i]$ whose representatives are ind-morphisms

$$\beta_i\colon (X_{i,j})_{(*,j)\in\mathsf{H}_i} \to ((X_{i,j})_{j\in\mathsf{J}_i})_{(i,*,j)\in\mathsf{L}_i},$$

where H_i is built by $m_{\underline{\mathsf{C}}}$ for $\underline{1}$, and J_i and L_i corresponds to K_0 and J_i. Observe that each component of β_i is the equivalence class (wrt. L_i) of the identity of $X_{i,j}$ in $\underline{\mathsf{C}}$.

Finally, we must compute $m_{\underline{C}}(\beta)$. We have the index categories K built from I and the H_i's and K' built from $\underline{1}$ and the L_i's. Thus, $m_{\underline{C}}(\beta)$ is a $(i,*,j) \in \mathsf{K}$-indexed family γ whose components are equivalence classes (wrt. K') of the identity arrow of $X_{i,j}$. Now let K'' be the index category for $m_{\underline{C}}(T)$. Of course, K, K' and K'' all have isomorphic sets of objects and, as usual, it is not difficult to show that there exist cofinal functors $\phi: \mathsf{K}' \to \mathsf{K}$ and $\phi': \mathsf{K}'' \to \mathsf{K}$. It follows that the component at T of $m_{\underline{C}} * \mathtt{Ind}(m_{\underline{C}}) * \eta_{Ind\underline{C}} * \mathtt{Ind}(y_{\mathtt{Ind}^2(\underline{C})}) = m_{\underline{C}} * \mathtt{Ind}(m_{\underline{C}}) * \lambda_{\mathtt{Ind}^2(\underline{C})}$ is the identity of $m_{\underline{C}}(T)$, i.e., $m_{\underline{C}} * \mathtt{Ind}(m_{\underline{C}}) * \lambda_{\mathtt{Ind}^2(\underline{C})} = \mathbf{1}$, as required.

Thus, we have proved the following.

Proposition 3.5.27
$\left(\mathtt{Ind}(\_), y, m, \{\lambda_{\underline{C}}\}_{\underline{C} \in \underline{\underline{\underline{CAT}}}}\right)$ *is a KZ-doctrine on* $\underline{\underline{\underline{CAT}}}$, *the category of locally small categories.*

Of course, the same holds for $\mathtt{Ind}(\_)_{\aleph}$.

Proposition 3.5.28
$\left(\mathtt{Ind}(\_)_{\aleph}, y, m, \{\lambda_{\underline{C}}\}_{\underline{C} \in \underline{\underline{\underline{CAT}}}}\right)$ *is a KZ-doctrine on* $\underline{\underline{\underline{CAT}}}$, *the category of locally small categories.*

Moreover, by Proposition 3.5.5, we also have the following results concerning $\underline{\underline{\underline{Cat}}}$, the category of the small categories.

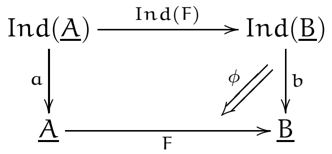
Proposition 3.5.29
$\left(\mathtt{Ind}(\_), y, m, \{\lambda_{\underline{C}}\}_{\underline{C} \in \underline{\underline{\underline{Cat}}}}\right)$ *is a KZ-doctrine on* $\underline{\underline{\underline{Cat}}}$.

Proposition 3.5.30
$\left(\mathtt{Ind}(\_)_{\aleph}, y, m, \{\lambda_{\underline{C}}\}_{\underline{C} \in \underline{\underline{\underline{Cat}}}}\right)$ *is a KZ-doctrine on* $\underline{\underline{\underline{Cat}}}$.

We now turn our attention to the category $\underline{\underline{\underline{\mathtt{Ind}}}}$-$\underline{\underline{\underline{\mathtt{Alg}}}}$ of $\mathtt{Ind}(\_)$-algebras. By Definition 3.4.3, an algebra is a category $\underline{A}$ together with a functor $\mathfrak{a}: \mathtt{Ind}(\underline{A}) \to \underline{A}$ which is a reflection left adjoint for $y_{\underline{A}}: \underline{A} \to \mathtt{Ind}(\underline{A})$. By Proposition 3.5.12 we conclude immediately that the algebras are exactly the locally small filtered cocomplete categories with a *choice* $\mathfrak{a}$ of colimits. Recall that, by the general theory of KZ-doctrines, the same category $\underline{A}$ gives rise to different algebras only via isomorphic $\mathfrak{a}$'s, i.e., via different choices of colimits in $\underline{A}$.

As usual, the same holds for $\mathtt{Ind}(\_)_{\aleph}$, whose algebras are the locally small $\aleph$-filtered cocomplete categories with a choice of colimits.

Let us consider the $\mathtt{Ind}$-homomorphisms. From the theory in Section 3.4 we know that $\mathsf{F}: \underline{A} \to \underline{B}$ is a morphism of the algebras $(\underline{A}, \mathfrak{a})$ and $(\underline{B}, \mathfrak{b})$ if and only if the 2-cell $\phi = b * \mathsf{Ind}(\mathsf{F}) * \mathsf{Ind}(\mathsf{a}) * \lambda_{\underline{\mathsf{A}}}$ is invertible.



Consider $A = (A_i)_{i \in \mathsf{I}}$ in $\mathsf{Ind}(\underline{\mathsf{A}})$. Then, $\mathsf{a}(A)$ is (a choice of) the colimit $\varinjlim_{\underline{\mathsf{A}}} A$, and $\mathsf{F}(\mathsf{a}(A))$ is $\mathsf{F}(\varinjlim_{\underline{\mathsf{A}}} A)$. On the other hand, $\mathsf{Ind}(\mathsf{F})(A) = (\mathsf{F}A_i)_{i \in \mathsf{I}}$ is the translation through F of the diagram A in $\underline{\mathsf{B}}$ and $\mathsf{b}(\mathsf{Ind}(\mathsf{F})(A))$ is (a choice for) its colimit $\varinjlim_{\underline{\mathsf{B}}} \mathsf{F}A$.

In our context we have $\phi = b * \mathsf{Ind}(\mathsf{F}) * \mathsf{Ind}(\mathsf{a}) * \eta_{\underline{\mathsf{A}}} * \mathsf{Ind}(y_{\underline{\mathsf{A}}})$. Moreover, the unit of the reflection $\mathsf{a} \dashv y_{\underline{\mathsf{A}}}$ is given by $\mathsf{Ind}(\mathsf{a}) * \eta_{\underline{\mathsf{A}}} * \mathsf{Ind}(y_{\underline{\mathsf{A}}})$. Observe now that the i-th component of $\eta = (\eta_{\underline{\mathsf{A}}} * \mathsf{Ind}(y_{\underline{\mathsf{A}}}))_A \colon ((A_i)_{\underline{\mathsf{1}}})_{i \in \mathsf{I}} \to ((A_i)_{i \in \mathsf{I}})_{\underline{\mathsf{1}}}$ is the class of $\eta_i \colon (A_i)_{\underline{\mathsf{1}}} \to (A_i)_{i \in \mathsf{I}}$ whose unique component is the class of the identity of A_i. Then,

$$\mathsf{Ind}(\mathsf{a})(\eta) = ([\mathsf{a}(\eta_i)])_{i \in \mathsf{I}} \colon (A_i)_{i \in \mathsf{I}} \to (\mathsf{a}((A_i)_{i \in \mathsf{I}}))_{\underline{\mathsf{1}}}$$

is the limit cocone for A. Therefore, by applying $\mathsf{Ind}(\mathsf{F})$ to $\mathsf{Ind}(\mathsf{a})(\eta)$, we get $([\mathsf{F}\mathsf{a}(\eta_i)])_{i \in \mathsf{I}} \colon (\mathsf{F}A_i)_{i \in \mathsf{I}} \to (\mathsf{F}\mathsf{a}((A_i)_{i \in \mathsf{I}}))_{\underline{\mathsf{1}}}$, which is the translation in $\underline{\mathsf{B}}$ of the cocone, and finally, by applying b, we get $\mathsf{b}(([\mathsf{F}\mathsf{a}(\eta_i)])_{i \in \mathsf{I}}) \colon \mathsf{b}((\mathsf{F}A_i)_{i \in \mathsf{I}}) \to (\mathsf{F}\mathsf{a}((A_i)_{i \in \mathsf{I}}))$ which is the canonical comparison morphism $\varinjlim_{\underline{\mathsf{B}}}(\mathsf{F}A) \to \mathsf{F}(\varinjlim_{\underline{\mathsf{A}}})$. Then, we have that ϕ is invertible if and only if (by definition) F preserves colimits (up to isomorphism).

Therefore, we can conclude with the following proposition.

PROPOSITION 3.5.31
The 2-category of $\mathsf{Ind}(\_)$-algebras ($\mathsf{Ind}(\_)_{\aleph}$-algebras) on $\underline{\underline{\mathsf{CAT}}}$ ($\underline{\underline{\mathsf{Cat}}}$) is the 2-category of the ($\aleph$-)filtered cocomplete locally small (small) categories with a choice of colimits and of the functors which preserve them up to isomorphism.

In equivalent terms, $\underline{\underline{\mathsf{Ind}}}$-$\underline{\underline{\mathsf{Alg}}}$ on $\underline{\underline{\mathsf{CAT}}}$ ($\underline{\underline{\mathsf{Cat}}}$) is the category $\underline{\underline{\aleph}}$-$\underline{\underline{\mathsf{CAT}}}$ ($\underline{\underline{\aleph}}$-$\underline{\underline{\mathsf{Cat}}}$) of the $\aleph$-chain cocomplete locally small (small) categories with a choice of colimits and $\aleph$-cocontinuous functors. It follows from general facts about KZ-doctrines that $\mathsf{Ind}(\_)$ determines a KZ-adjunction from $\underline{\underline{\mathsf{CAT}}}$ ($\underline{\underline{\mathsf{Cat}}}$) to $\underline{\underline{\aleph}}$-$\underline{\underline{\mathsf{CAT}}}$ ($\underline{\underline{\aleph}}$-$\underline{\underline{\mathsf{Cat}}}$).

REMARK. As stressed more than once, the construction we have just given could have been carried out using just chains or, even less, just ω-chains, which after all are our intended application. The difficulty with this is that, although it is easy to guess a"diagonal construction" for it, it is not evident at all how to define the multiplication m to be a (strict) natural transformation. Of course, this is a matter that we would like to study further in the future.

The well developed theory of monads provides a lot of useful results about categories of algebras. Notably, several works by Kock [72, 73, 74, 75] are devoted to the study of conditions which give a (cartesian) closed structure to such categories. Unfortunately, these results are not (yet) available for (KZ-)doctrines. Therefore, we conclude this section by showing directly the following easy result.

Proposition 3.5.32
The category of small (ℵ-)filtered cocomplete categories is cartesian closed.

Proof. Observe that the restriction to the small categories is necessary, since an exponential $\underline{\mathsf{D}}^{\underline{\mathsf{C}}}$ of two locally small categories is not necessarily locally small. However, this is not a problem in our context, since we know that $\mathsf{Ind}(\_)$ restricts to a KZ-doctrine on $\underline{\underline{\mathsf{Cat}}}$, the 2-category of small categories. Thus, in the following, let $\underline{\mathsf{Filt}}$-$\underline{\mathsf{Cat}}$ be the category of small filtered cocomplete categories and $\underline{\mathsf{Filt}_{\aleph}}$-$\underline{\mathsf{Cat}}$ the category of small ℵ-filtered cocomplete categories. We give the proof for $\underline{\mathsf{Filt}}$-$\underline{\mathsf{Cat}}$, but of course, it work also for $\underline{\mathsf{Filt}_{\aleph}}$-$\underline{\mathsf{Cat}}$.

The terminal is the singleton category $\underline{1}$. The cartesian product is just the product in $\underline{\underline{\mathsf{CAT}}}$, since $\underline{\mathsf{C}} \times \underline{\mathsf{D}}$ is cocomplete whenever $\underline{\mathsf{C}}$ and $\underline{\mathsf{D}}$ are. In fact, the colimits are computed pointwise $\varinjlim(\mathsf{F}) = (\varinjlim(\pi_0 \circ \mathsf{F}), \varinjlim(\pi_1 \circ \mathsf{F}))$.

Finally, we have to show that for each filtered cocomplete $\underline{\mathsf{C}}$ the endofunctor $\underline{\mathsf{C}} \times \_$ on $\underline{\mathsf{Filt}}$-$\underline{\mathsf{Cat}}$ has a right adjoint $[\_,\_]$. Clearly, we take $[\underline{\mathsf{C}}, \underline{\mathsf{D}}]$ to be the category of the functors $\mathsf{F}\colon \underline{\mathsf{C}} \to \underline{\mathsf{D}}$ which preserve filtered colimits. Let us see that $[\underline{\mathsf{C}}, \underline{\mathsf{D}}]$ is filtered cocomplete.

Let $D\colon \mathsf{I} \to [\underline{\mathsf{C}}, \underline{\mathsf{D}}]$ be a filtered diagram. Then, $\bar{D} = \varinjlim D$ exists in $\underline{\mathsf{D}}^{\underline{\mathsf{C}}}$ and it is given by $(\varinjlim D)c = \varinjlim(D(i)(c))$. In order to conclude that it belongs to $[\underline{\mathsf{C}}, \underline{\mathsf{D}}]$ we have to see that it is cocontinuous. To this purpose, consider a filtered diagram $X\colon \mathsf{J} \to \underline{\mathsf{C}}$ together with

$$\begin{array}{ccc} \mathsf{I} \times \mathsf{J} & \xrightarrow{\phi} & \underline{\mathsf{D}} \\ (i,j) & \longmapsto & D(i)(X(j)) \\ {\scriptstyle (\alpha,f)}\big\downarrow & & \big\downarrow {\scriptstyle D(i')(X(f))\circ D(\alpha)_{X(j)} = D(\alpha)_{X_{j'}} \circ D(i)(f)} \\ (i',j') & \longmapsto & D(i')(X(j')) \end{array}$$

Observe that, since for any $\mathsf{F} \xrightarrow{\alpha} \mathsf{G} \xrightarrow{\beta} \mathsf{H}$ in $[\underline{\mathsf{C}}, \underline{\mathsf{D}}]$ and any $x \xrightarrow{f} y \xrightarrow{g} z$ in $\underline{\mathsf{C}}$ the following diagram in $\underline{\mathsf{D}}$ is commutative, ϕ is a functor.

$$\begin{array}{ccccc} \mathsf{F}(x) & \xrightarrow{\mathsf{F}(f)} & \mathsf{F}(y) & \xrightarrow{\mathsf{F}(g)} & \mathsf{F}(z) \\ {\scriptstyle \alpha_x}\big\downarrow & & {\scriptstyle \alpha_y}\big\downarrow & & \big\downarrow{\scriptstyle \alpha_z} \\ \mathsf{G}(x) & \xrightarrow{\mathsf{G}(f)} & \mathsf{G}(y) & \xrightarrow{\mathsf{G}(g)} & \mathsf{G}(z) \\ {\scriptstyle \beta_x}\big\downarrow & & {\scriptstyle \beta_y}\big\downarrow & & \big\downarrow{\scriptstyle \beta_z} \\ \mathsf{H}(x) & \xrightarrow[\mathsf{H}(f)]{} & \mathsf{H}(y) & \xrightarrow[\mathsf{H}(g)]{} & \mathsf{H}(z) \end{array}$$

Thus by general results about colimits, the colimits on I and on J commute with each other, i.e., $\varinjlim_{\mathsf{I}} \varinjlim_{\mathsf{J}} \phi = \varinjlim_{\mathsf{J}} \varinjlim_{\mathsf{I}} \phi$. Then we have

$$\begin{array}{rcl} \bar{D}(\varinjlim_{\mathsf{J}} X(j)) & = & (\varinjlim_{\mathsf{I}} D(i))(\varinjlim_{\mathsf{J}} X(j)) \\ & = & \varinjlim_{\mathsf{I}} D(i)(\varinjlim_{\mathsf{J}} X(j)) \\ & = & \varinjlim_{\mathsf{I}} \varinjlim_{\mathsf{J}} D(i)(X(j)) \\ & = & \varinjlim_{\mathsf{J}} \varinjlim_{\mathsf{I}} D(i)(X(j)) \\ & = & \varinjlim_{\mathsf{J}} (\varinjlim_{\mathsf{I}} D(i))(X(j)) = \varinjlim_{\mathsf{J}} \bar{D}(X(j)). \end{array}$$

Finally, we have to see that there is a natural isomorphism

$$\mathsf{Hom}_{\underline{\mathtt{Filt}}\text{-}\underline{\mathtt{Cat}}}(\underline{\mathsf{C}} \times \underline{\mathsf{D}}, \underline{\mathsf{E}}) \cong \mathsf{Hom}_{\underline{\mathtt{Filt}}\text{-}\underline{\mathtt{Cat}}}(\underline{\mathsf{C}}, [\underline{\mathsf{D}}, \underline{\mathsf{E}}]).$$

But this can clearly be obtained via the usual

$$\mathsf{F}\colon \underline{\mathsf{C}} \times \underline{\mathsf{D}} \to \underline{\mathsf{E}} \rightsquigarrow \lambda c\colon \underline{\mathsf{C}}.(\lambda d\colon \underline{\mathsf{D}}.\mathsf{F}(c,d))\colon \underline{\mathsf{C}} \to [\underline{\mathsf{D}}, \underline{\mathsf{E}}].$$

✓

3.6 Ind Completion of Monoidal Categories

In this section we show that the ($\aleph$)-filtered cocompletion of a monoidal category is a monoidal category in a canonical way. Moreover, the KZ-doctrine $(\mathtt{Ind}(\_), y, m, \lambda)$ lifts to KZ-doctrines on any of the 2-categories in Table 3.1 in page 211, giving in this way their "free" cocompletion.

This fact can be proved in some equivalent ways, corresponding to the different characterization of the category $\mathtt{Ind}(\_)$ we have given in the previous sections. In particular, a very elegant approach is to work on categories of fractions. For those categories, in fact, there are recent results [65, 66, 68] extending the seminal work by Lawvere [82, 83], which state that any finitary essentially algebraic structure on a category—that is a structure which can defined by functors which are left Kan adjoints to their restrictions to the category of finite sets viewed as categories, or equivalently functors of the kind $\underline{\mathsf{A}}^n \to \underline{\mathsf{A}}$, for $n \in \omega$, and by natural transformations which enforce equations—is preserved by any "category of fractions" construction.

Since the monoidal structure (but not a closed monoidal one!) on a category consists of a finitary functor $\underline{\mathsf{C}}^2 \to \underline{\mathsf{C}}$ and of three natural transformations subject to a few axioms, the theory above applies to our case. Without using such sophisticated tools, it is easy to prove directly that a monoidal structure on a category $\underline{\mathsf{C}}$ can be lifted *canonically* to the category of fractions $\underline{\underline{\mathsf{C}}}[\Sigma^{-1}]$, for any Σ which is closed for the tensor product. However, such approach would require two steps, since we should firstly extend the monoidality of $\underline{\mathsf{C}}$ to $\underline{\mathtt{Filt}}/\underline{\mathsf{C}}$ and later to $\Sigma^{-1}\underline{\mathtt{Filt}}/\underline{\mathsf{C}}$. Therefore, here we prefer to work directly on ind-morphisms and their indexed representation. We shall illustrate two equivalent approaches, the second being actually a variation of the first. As usual, we state definitions and results preferably for $\mathtt{Ind}(\underline{C})$, although everything which follows can be rephrased for $\mathtt{Ind}(\underline{C})_{\aleph}$.

For the sake of readability, we recall the definitions concerning monoidal categories, functors and transformations.

A *monoidal category* [3, 21, 90] is a structure $(\underline{C}, \otimes, e, \alpha, \lambda, \rho)$, where $\underline{C}$ is a category, $\otimes\colon \underline{C} \times \underline{C} \to \underline{C}$ is a functor, $\alpha\colon {\_}_1 \otimes ({\_}_2 \otimes {\_}_3) \xrightarrow{\sim} ({\_}_1 \otimes {\_}_2) \otimes {\_}_3$ is "the *associativity*" natural isomorphism,[2] $\lambda\colon e \otimes {\_}_1 \xrightarrow{\sim} {\_}_1$ is "the *left unit*" natural isomorphism and $\rho\colon {\_}_1 \otimes e \xrightarrow{\sim} {\_}_1$ is "the *right unit*" natural isomorphism, e is an object in $\underline{C}$, subject to the following Kelly-MacLane *coherence axioms* [87, 62]:

$$\begin{aligned} (\alpha_{x,y,z} \otimes id_k) \circ \alpha_{x,y\otimes z,k} \circ (id_x \otimes \alpha_{y,z,k}) &= \alpha_{x\otimes y,z,k} \circ \alpha_{x,y,z\otimes k}; \\ id_x \otimes \lambda_y \circ \alpha_{x,e,y} &= \rho_x \otimes id_y. \end{aligned} \tag{3.3}$$

A monoidal category is *strict* if α, λ and ρ are the identity natural transformation, i.e., if $\otimes$ is strictly monoidal. It is *symmetric* if it is given a *symmetry* natural isomorphism $\gamma\colon {\_}_1 \otimes {\_}_2 \xrightarrow{\sim} {\_}_2 \otimes {\_}_1$ satisfying the following axioms.

$$\begin{aligned} (\gamma_{x,z} \otimes id_y) \circ \alpha_{x,z,y} \circ (id_x \otimes \gamma_{y,z}) &= \alpha_{z,x,y} \circ \gamma_{x\otimes y,z} \circ \alpha_{x,y,z}; \\ \gamma_{y,x} \circ \gamma_{x,y} &= id_{x\otimes y}; \\ \rho_x \circ \gamma_{e,x} &= \lambda_x. \end{aligned} \tag{3.4}$$

When γ is the identity, $\underline{C}$ is said *strictly symmetric*.

Given $\underline{C} = (\underline{C}, \otimes, e, \alpha, \lambda, \rho, \gamma)$ and $(\underline{D}, \otimes', e', \alpha', \lambda', \rho', \gamma')$, a *monoidal functor* from $\underline{C}$ to $\underline{D}$ is a triple $(\mathsf{F}, \varphi^0, \varphi)$, where $\mathsf{F}\colon \underline{C} \to \underline{D}$ is a functor, $\varphi^0\colon e' \to \mathsf{F}(e)$ is an arrow in $\underline{D}$, and $\varphi\colon \mathsf{F}({\_}_1) \otimes' \mathsf{F}({\_}_2) \stackrel{\cdot}{\to} \mathsf{F}({\_}_1 \otimes {\_}_2)$ is a natural transformation, required to satisfy

$$\begin{aligned} \mathsf{F}\alpha_{x,y,z} \circ \varphi_{x,y\otimes z} \circ (id_{\mathsf{F}x} \otimes' \varphi_{y,z}) &= \varphi_{x\otimes y,z} \circ (\varphi_{x\otimes y} \otimes' id_{Fz}) \circ \alpha'_{\mathsf{F}x,\mathsf{F}y,\mathsf{F}z}; \\ \mathsf{F}\lambda_x \circ \varphi_{e,x} \circ (\varphi^0 \otimes' id_{\mathsf{F}x}) &= \lambda'_{\mathsf{F}x} \\ \mathsf{F}\rho_x \circ \varphi_{x,e} \circ (id_{\mathsf{F}x} \otimes' \varphi^0) &= \rho'_{\mathsf{F}x}. \end{aligned} \tag{3.5}$$

Moreover, $(\mathsf{F}, \varphi^0, \varphi)$ is *symmetric* if

$$\mathsf{F}\gamma_{x,y} \circ \varphi_{x,y} = \varphi_{y,x} \circ \gamma'_{\mathsf{F}x,\mathsf{F}y}. \tag{3.6}$$

If φ^0 and φ are isomorphisms, then $(\mathsf{F}, \varphi^0, \varphi)$ is a *strong* monoidal functor, if they are the identity, then F is a *strict* monoidal functor. The combination of these data give the one-dimensional versions of the categories in Table A.3.

[2] We use the symbols ${\_}_n$ for $n \in \omega$ as placeholders.

A *monoidal transformation* between the functors $(\mathsf{F}, \varphi^0, \varphi)$ and $(\mathsf{F}', \varphi'^0, \varphi')$ is a natural transformation $\sigma: \mathsf{F} \stackrel{\cdot}{\rightarrow} \mathsf{F}'$ such that

$$\begin{array}{rcl} \sigma_{x \otimes y} \circ \varphi_{x,y} & = & \varphi'_{x,y} \circ (\sigma_x \otimes' \sigma_y) \\ \sigma_e \circ \varphi^0 & = & \varphi'^0 \end{array} \tag{3.7}$$

By combining in a sensible way the data above, we get the 2-categories listed in Table 3.1 in page 211.

COCOMPLETION OF MONOIDAL CATEGORIES: FIRST SOLUTION

The first issue is to extend the tensor $\otimes$ to a functor $\hat{\otimes}: \mathtt{Ind}(\underline{\mathsf{C}}) \times \mathtt{Ind}(\underline{\mathsf{C}}) \rightarrow \mathtt{Ind}(\underline{\mathsf{C}})$. Observe that by composing $\otimes$ with $y_{\underline{\mathsf{C}}}$ we get a functor $y_{\underline{\mathsf{C}}} \circ \otimes: \underline{\mathsf{C}} \times \underline{\mathsf{C}} \rightarrow \mathtt{Ind}(\underline{\mathsf{C}})$. Therefore, by the universality of $\mathtt{Ind}(\_)$, we get a functor

$$\otimes': \mathtt{Ind}(\underline{\mathsf{C}} \times \underline{\mathsf{C}}) \rightarrow \mathtt{Ind}(\underline{\mathsf{C}})$$

which is the unique-up-to-isomorphism free extension of $\otimes$ to the ind-objects. It is easy to realize that a possible choice for $\otimes'$ is exactly $\mathtt{Ind}(\otimes)$.

$$\begin{array}{ccc} \mathtt{Ind}(\underline{\mathsf{C}} \times \underline{\mathsf{C}}) & \xrightarrow{\mathtt{Ind}(\otimes)} & \mathtt{Ind}(\underline{\mathsf{C}}) \\ {\scriptstyle y_{\underline{\mathsf{C}} \times \underline{\mathsf{C}}}} \uparrow & & \uparrow {\scriptstyle y_{\underline{\mathsf{C}}}} \\ \underline{\mathsf{C}} \times \underline{\mathsf{C}} & \xrightarrow[\otimes]{} & \underline{\mathsf{C}} \end{array}$$

Thus, we look for a canonical way to relate $\mathtt{Ind}(\underline{\mathsf{C}} \times \underline{\mathsf{C}})$ and $\mathtt{Ind}(\underline{\mathsf{C}}) \times \mathtt{Ind}(\underline{\mathsf{C}})$. We observe that $\mathtt{Ind}(\underline{\mathsf{C}} \times \underline{\mathsf{D}}) \stackrel{\cdot}{\cong} \mathtt{Ind}(\underline{\mathsf{C}}) \times \mathtt{Ind}(\underline{\mathsf{D}})$, although they are not at all isomorphic. Consider the mapping ∇ defined below

$$\begin{array}{ccc} \mathtt{Ind}(\underline{\mathsf{C}} \times \underline{\mathsf{D}}) & \xrightarrow{\nabla} & \mathtt{Ind}(\underline{\mathsf{C}}) \times \mathtt{Ind}(\underline{\mathsf{D}}) \\ \mathsf{I} \xrightarrow{X} \underline{\mathsf{C}} \times \underline{\mathsf{D}} & \longmapsto & (\mathsf{I} \xrightarrow{\pi_0 X} \underline{\mathsf{C}}, \mathsf{I} \xrightarrow{\pi_1 X} \underline{\mathsf{D}}) \\ {\scriptstyle ([f_i])_{i \in \mathsf{I}}} \downarrow & & {\scriptstyle ([fst(f_i)])_{i \in \mathsf{I}}} \downarrow \qquad \downarrow {\scriptstyle ([snd(f_i)])_{i \in \mathsf{I}}} \\ \mathsf{J} \xrightarrow{Y} \underline{\mathsf{C}} \times \underline{\mathsf{D}} & \longmapsto & (\mathsf{J} \xrightarrow{\pi_0 Y} \underline{\mathsf{C}}, \mathsf{J} \xrightarrow{\pi_1 Y} \underline{\mathsf{D}}) \end{array}$$

where $fst\langle f, g \rangle = f$, $snd\langle f, g \rangle = g$ and π_i are the projections associated to the cartesian product.

Given $X: \mathsf{I} \rightarrow \underline{\mathsf{C}} \times \underline{\mathsf{D}}$, suppose $X(i) = (c_i, d_i)$. Then, the identity of X is $([(id_{c_i}, id_{d_i})])_{i \in \mathsf{I}}$, and therefore $\nabla(id_X)$ is the pair $(([id_{c_i}])_{i \in \mathsf{I}}, ([id_{d_i}])_{i \in \mathsf{I}})$ which is $(id_{\pi_0 X}, id_{\pi_1 X})$. Moreover, since $fst(g \circ f) = fst(g) \circ fst(f)$ and $snd(g \circ f) = snd(g) \circ$

$snd(f)$, it is immediate to show that the definition above respects compositions. Thus, ∇ is a functor.

For a quasi-inverse of ∇, we consider the following Δ.

$$\begin{array}{ccc}
\mathtt{Ind}(\underline{\mathsf{C}}) \times \mathtt{Ind}(\underline{\mathsf{D}}) & \xrightarrow{\quad\Delta\quad} & \mathtt{Ind}(\underline{\mathsf{C}} \times \underline{\mathsf{D}}) \\
(\mathsf{I} \xrightarrow{X} \underline{\mathsf{C}}, \mathsf{J} \xrightarrow{Y} \underline{\mathsf{D}}) & \longmapsto & \mathsf{I} \times \mathsf{J} \xrightarrow{X \times Y} \underline{\mathsf{C}} \times \underline{\mathsf{D}} \\
{\scriptstyle ([f_i])_{i\in\mathsf{I}}}\downarrow \qquad \downarrow{\scriptstyle ([g_j])_{j\in\mathsf{J}}} & & \downarrow{\scriptstyle ([f_i \times g_j])_{i\in\mathsf{I}, j\in\mathsf{J}}} \\
(\mathsf{I}' \xrightarrow{X'} \underline{\mathsf{C}}, \mathsf{J}' \xrightarrow{Y'} \underline{\mathsf{D}}) & \longmapsto & \mathsf{I}' \times \mathsf{J}' \xrightarrow{X' \times Y'} \underline{\mathsf{C}} \times \underline{\mathsf{D}}
\end{array}$$

Also in this case it is immediate to see that Δ is a functor. In fact, the image of the identity of the pair X and Y is the $\mathsf{I} \times \mathsf{J}$-indexed family whose component (i, j) is the class of the $id_{X_i} \times id_{Y_j}$ which is $id_{X_i \times Y_j}$, the identity of $(X \times Y)_{i,j}$. Thus, Δ respects the identities. Moreover, since $(f \circ f') \times (g \circ g') = (f \times g) \circ (f' \times g')$, it follows that Δ is a functor.

Now, given $X = ((c_i, d_i))_{i\in\mathsf{I}}$, we have $\Delta\nabla(X) = \pi_0 X \times \pi_1 X{:}\, \mathsf{I} \times \mathsf{I} \to \underline{\mathsf{C}} \times \underline{\mathsf{D}}$. Observe that $\phi_X{:}\, \mathsf{I} \to \mathsf{I} \times \mathsf{I}$ which sends i to (i, i) is clearly cofinal. Moreover, the following diagram commutes.

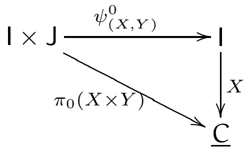
$$\begin{array}{ccc}
\mathsf{I} & \xrightarrow{\quad\phi_X\quad} & \mathsf{I} \times \mathsf{I} \\
 & {\scriptstyle X}\searrow & \downarrow{\scriptstyle \pi_0 X \times \pi_1 X} \\
 & & \underline{\mathsf{C}} \times \underline{\mathsf{D}}
\end{array}$$

It follows that X and $\pi_0 X \times \pi_1 X$ are isomorphic in $\mathtt{Ind}(\underline{\mathsf{C}} \times \underline{\mathsf{D}})$ via the canonical morphism $\bar{\phi}_X$ induced from ϕ_X by colimit, i.e., via the injection $\mathsf{L}(X) = \varinjlim \mathsf{Y}X$ of $\mathtt{Ind}(\underline{\mathsf{C}} \times \underline{\mathsf{D}})$ in the category of presheaves over $\underline{\mathsf{C}} \times \underline{\mathsf{D}}$. Since ϕ_X enjoys a universal property, it is clear that the family $\{\bar{\phi}_X\}_{X\in\mathtt{Ind}(\underline{\mathsf{C}}\times\underline{\mathsf{D}})}$ gives a natural transformation $Id \stackrel{\cdot}{\to} \Delta\nabla$.

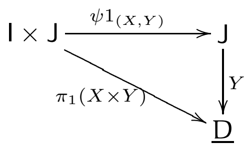
On the other hand, given the pair $((X_i)_{i\in\mathsf{I}}, (Y_j)_{j\in\mathsf{J}})$ in $\mathtt{Ind}(\underline{\mathsf{C}}) \times \mathtt{Ind}(\underline{\mathsf{D}})$, we have $\nabla\Delta((X, Y)) = (\pi_0(X \times Y), \pi_1(X \times Y))$, where $\pi_0(X \times Y){:}\, \mathsf{I} \times \mathsf{J} \to \underline{\mathsf{C}}$ and $\pi_1(X \times Y){:}\, \mathsf{I} \times \mathsf{J} \to \underline{\mathsf{D}}$. Of course, $\mathsf{I} \times \mathsf{J}$ is cofinal both in I and in J, via the functors

$$\begin{array}{ccc}
\mathsf{I} \times \mathsf{J} & \xrightarrow{\quad\psi^0_{(X,Y)}\quad} & \mathsf{I} \\
(i, j) & \longmapsto & i \\
{\scriptstyle (f,g)}\downarrow & & \downarrow{\scriptstyle f} \\
(i', j') & \longmapsto & i'
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
\mathsf{I} \times \mathsf{J} & \xrightarrow{\quad\psi^1_{(X,Y)}\quad} & \mathsf{J} \\
(i, j) & \longmapsto & j \\
{\scriptstyle (f,g)}\downarrow & & \downarrow{\scriptstyle g} \\
(i', j') & \longmapsto & j'
\end{array}$$

Moreover, the following diagrams commute.



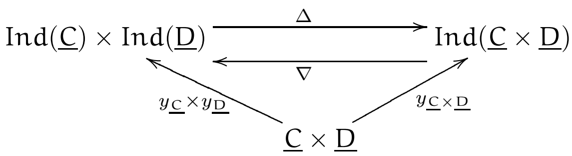
and



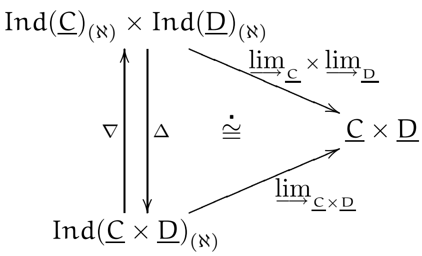
and so we have that there exist invertible ind-morphisms $\bar{\psi}^0_{(X,Y)}\colon \pi_0(X\times Y)\to X$ and $\bar{\psi}^1_{(X,Y)}\colon \pi_1(X\times Y)\to Y$ induced by the universal property of colimits from $\psi^0_{(X,Y)}$ and $\psi^1_{(X,Y)}$. For general reasons, it follows that we have a natural transformation $\bar{\psi}\colon \nabla\Delta \stackrel{\cdot}{\to} Id$, where $\bar{\psi}_{(X,Y)} = (\bar{\psi}^0_{(X,Y)}, \bar{\psi}^1_{(X,Y)})$. In other words we have the following.

PROPOSITION 3.6.1
$Id \cong \Delta\nabla$ via $\bar{\phi}$ and $\nabla\Delta \cong Id$ via $\bar{\psi}$. Therefore, $\mathsf{Ind}(\underline{C}\times\underline{D}) \stackrel{\cdot}{\cong} \mathsf{Ind}(\underline{C})\times\mathsf{Ind}(\underline{D})$.

It is clear from the definition that Δ and ∇ are such that

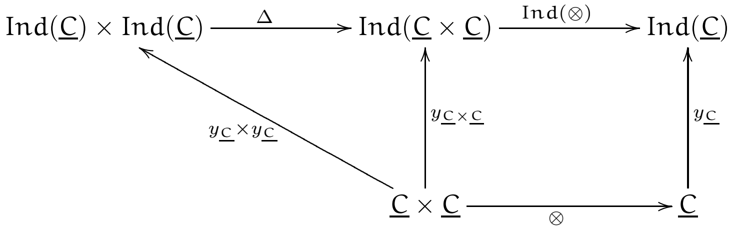


and, if $\underline{C}$ and $\underline{D}$ are ($\aleph$-)filtered cocomplete, then



i.e., $\varinjlim_{\underline{C}\times\underline{D}}(\mathsf{F}) = (\varinjlim_{\underline{C}}\pi_0\mathsf{F}, \varinjlim_{\underline{D}}\pi_1\mathsf{F})$ and $(\varinjlim_{\underline{C}}\mathsf{F}, \varinjlim_{\underline{D}}\mathsf{G}) = \varinjlim_{\underline{C}\times\underline{D}}\mathsf{F}\times\mathsf{G}$.

So we are allowed to define



Observe that this diagram commutes, which means that the tensors of $\underline{C}$ and $\mathbb{Ind}(\underline{C})$ coincide on the (essentially) constant ind-objects. We shall see that actually the entire monoidal structure of $\underline{C}$, and not merely the tensor, is preserved in $\mathbb{Ind}(C)$. In the following we shall denote $\mathbb{Ind}(\otimes) \circ \Delta$ by $\hat{\otimes}$. In terms of indexed representation of ind-object we can then write

$$\begin{array}{c} (X_i)_{i\in \mathsf{I}} \\ {\scriptstyle ([f_i])_{i\in\mathsf{I}}}\big\downarrow \\ (X'_i)_{i\in\mathsf{I}'} \end{array} \quad \hat{\otimes} \quad \begin{array}{c} (Y_j)_{j\in \mathsf{J}} \\ \big\downarrow{\scriptstyle ([g_j])_{j\in\mathsf{J}}} \\ (Y'_j)_{j\in\mathsf{J}'} \end{array} \quad = \quad \begin{array}{c} (X_i\otimes Y_j)_{(i,j)\in \mathsf{I}\times\mathsf{J}} \\ \big\downarrow{\scriptstyle ([f_i\otimes g_j])_{(i,j)\in\mathsf{I}\times\mathsf{J}}} \\ (X'_i\otimes Y'_j)_{(i,j)\in\mathsf{I}'\times\mathsf{J}'} \end{array}$$

In order to get acquainted with the indexed representation, we check again the functoriality axioms directly on such representation of ind-morphisms. Consider the ind-objects X and Y and their identities $id_X = ([id_{X_i}])_{i\in\mathsf{I}}$ and $id_Y = ([id_{Y_j}])_{j\in\mathsf{J}}$. It is

$$id_X \,\hat{\otimes}\, id_Y = ([id_{X_i} \otimes id_{Y_j}])_{(i,j)\in\mathsf{I}\times\mathsf{J}} = ([id_{X_i\otimes Y_j}])_{(i,j)\in\mathsf{I}\times\mathsf{J}},$$

which is $id_{X\hat{\otimes}Y}$. Consider now the ind-morphisms

$$([f_i])_{i\in\mathsf{I}} \colon (X_i)_{i\in\mathsf{I}} \to (X'_i)_{i\in\mathsf{I}'}, \qquad ([f'_i])_{i\in\mathsf{I}'} \colon (X'_i)_{i\in\mathsf{I}'} \to (X''_i)_{i\in\mathsf{I}''}, \tag{3.8}$$

$$([g_j])_{j\in\mathsf{J}} \colon (Y_j)_{j\in\mathsf{J}} \to (Y'_j)_{j\in\mathsf{J}'}, \qquad ([g'_j])_{j\in\mathsf{J}'} \colon (Y'_j)_{j\in\mathsf{J}'} \to (Y''_j)_{j\in\mathsf{J}''} \tag{3.9}$$

We have $([f_i])_{i\in\mathsf{I}} \,\hat{\otimes}\, ([g_j])_{j\in\mathsf{J}} = ([f_i \otimes g_j])_{(i,j)\in\mathsf{I}\times\mathsf{J}}$. Then,

$$\begin{aligned} \Big(([f'_i])_{i\in\mathsf{I}'} \,\hat{\otimes}\, ([g'_i])_{i\in\mathsf{J}'}\Big) \quad &\circ \quad \Big(([f_i])_{i\in\mathsf{I}} \,\hat{\otimes}\, ([g_j])_{j\in\mathsf{J}}\Big) \\ &= \quad ([(f'_{i'_i} \otimes g'_{j'_j}) \circ (f_i \otimes g_j)])_{(i,j)\in\mathsf{I}\times\mathsf{J}} \\ &= \quad ([(f'_{i'_i} \circ f_i) \otimes (g'_{j'_j} \circ g_j)])_{(i,j)\in\mathsf{I}\times\mathsf{J}} \\ &= \quad ([(f'_{i'_i} \circ f_i) \otimes (g'_{j'_j} \circ g_j)])_{(i,j)\in\mathsf{I}\times\mathsf{J}} \\ &= \quad \Big(([f'_i])_{i\in\mathsf{I}'} \circ ([f_i])_{i\in\mathsf{I}}\Big) \,\hat{\otimes}\, \Big(([g'_j])_{j\in\mathsf{J}'} \circ ([g_j])_{j\in\mathsf{J}}\Big), \end{aligned}$$

To make explicit the remaining monoidal structure we have to identify the unit for $\hat{\otimes}$, to lift the coherence natural isomorphisms α, λ, ρ and γ to $\mathbb{Ind}(\underline{C})$, and to prove that the axioms are satisfied. This task is fairly easy now. Concerning the unit, of course we take $\hat{e} = y_{\underline{C}}(e) = \underline{e}$.

$\hat{\alpha}\colon \_{1} \,\hat{\otimes}\, (\_{2} \,\hat{\otimes}\, \_{3}) \xrightarrow{\sim} (\_{1} \,\hat{\otimes}\, \_{2}) \,\hat{\otimes}\, \_{3}$.

For $X = (X_i)_{i\in\mathsf{I}}$, $Y = (Y_j)_{j\in\mathsf{J}}$, $Z = (Z_k)_{k\in\mathsf{K}}$ in $\mathbb{Ind}(\underline{C})$, let H be $\mathsf{I}\times\mathsf{J}\times\mathsf{K}$ and define $\hat{\alpha}_{X,Y,Z}$ as follows

$$([\alpha_{X_i,Y_j,Z_k}])_{(i,j,k)\in\mathsf{H}} \colon (X_i \otimes (Y_j \otimes Z_k))_{(i,j,k)\in\mathsf{H}} \to ((X_i \otimes Y_j) \otimes Z_k)_{(i,j,k)\in\mathsf{H}}.$$

Observe that $\hat{\alpha}$ can be described as $\mathtt{Ind}(\alpha) * \Delta * (Id_{\mathtt{Ind}(\underline{C})} \times \Delta)$. It follows that $\hat{\alpha}$, since it is the image of a natural isomorphism through a 2-functor, is a natural isomorphism.

$\hat{\lambda}\colon \hat{e} \mathbin{\hat{\otimes}} \_{}_1 \xrightarrow{\sim} \_{}_1$.

For $X = (X_i)_{i\in \mathsf{I}}$ in $\mathtt{Ind}(\underline{C})$, the component at X of $\hat{\lambda}$ is

$$\hat{\lambda}_X = ([\lambda_{X_i}])_{i\in \mathsf{I}}\colon (e \otimes X_i)_{i\in \mathsf{I}} \to (X_i)_{i\in \mathsf{I}}.$$

This time $\hat{\lambda}$ can be written as $\mathtt{Ind}(\lambda) * \Delta(\hat{e}, \_)$, which implies that it is a natural isomorphism.

$\hat{\rho}\colon \_{}_1 \mathbin{\hat{\otimes}} \hat{e} \xrightarrow{\sim} \_{}_1$.

Given $X = (X_i)_{i\in \mathsf{I}}$ in $\mathtt{Ind}(\underline{C})$, we define

$$\hat{\rho}_X = ([\rho_{X_i}])_{i\in \mathsf{I}}\colon (X_i \otimes e)_{i\in \mathsf{I}} \to (X_i)_{i\in \mathsf{I}}.$$

Observe that $\hat{\rho}$ is $\mathtt{Ind}(\rho) * \Delta(\_, \hat{e})$, and thus a natural isomorphism.

$\hat{\gamma}\colon \_{}_1 \mathbin{\hat{\otimes}} \_{}_2 \xrightarrow{\sim} \_{}_2 \mathbin{\hat{\otimes}} \_{}_1$.

For $X = (X_i)_{i\in \mathsf{I}}$ and $Y = (X_j)_{j\in \mathsf{J}}$, we define

$$\hat{\gamma}_{X,Y} = ([\gamma_{X_i,Y_j}])_{(i,j)\in \mathsf{I}\times\mathsf{J}}\colon (X_i \otimes Y_j)_{(i,j)\in \mathsf{I}\times\mathsf{J}} \to (Y_j \otimes X_i)_{(j,i)\in \mathsf{J}\times\mathsf{I}},$$

which again is $\mathtt{Ind}(\gamma) * \Delta$, and thus a natural isomorphism.

Now it is really simple to check that these definitions enjoy the Kelly-MacLane coherence axioms [87, 62, 67] (see also Appendix A.2). Thus, we have the following.

Proposition 3.6.2

For any symmetric monoidal category $(\underline{C}, \otimes, e, \alpha, \lambda, \rho, \gamma)$ *the filtered cocomplete category* $(\mathtt{Ind}(\underline{C}), \hat{\otimes}, \hat{e}, \hat{\alpha}, \hat{\lambda}, \hat{\rho}, \hat{\gamma})$ *is a symmetric monoidal category.*

Moreover, if $\underline{C}$ *is monoidal strict, then so is* $\mathtt{Ind}(\underline{C})$*; if* $\underline{C}$ *is strictly symmetric so is* $\mathtt{Ind}(\underline{C})$*.*

Of course, the same holds for $(\mathtt{Ind}(\underline{C})_{\aleph}, \hat{\otimes}, \hat{e}, \hat{\alpha}, \hat{\lambda}, \hat{\rho}, \hat{\gamma})$*.*

Proof. Concerning the cases where $\underline{C}$ is strict monoidal or strictly symmetric, observe that the structure transformations $\hat{\alpha}$, $\hat{\lambda}$, $\hat{\rho}$ and $\hat{\gamma}$ are identities when the corresponding transformations of $\underline{C}$ are so. ✓

As anticipated above, the embedding $y_{\underline{C}}$ preserves the monoidal structure of $\underline{C}$. Therefore, $\mathtt{Ind}(\underline{C})$ can be considered the "free" cocomplete monoidal category on $\underline{C}$.

Proposition 3.6.3

The subcategory $y_{\underline{C}}(\underline{C})$ *of* $\mathtt{Ind}(\underline{C})$ *is isomorphic to* $\underline{C}$ *in the monoidal sense, i.e.,* $y_{\underline{C}}$ *is a strict monoidal functor.*

Proof. Of course, $y_{\underline{C}}(c \otimes d) = y_{\underline{C}}(c) \hat{\otimes} y_{\underline{C}}(d)$, since we identify $\underline{1}$ and $\underline{1} \times \underline{1}$. For the rest, observe that

$$\begin{aligned} y_{\underline{C}}(e) &= \hat{e}; \\ y_{\underline{C}}(\alpha_{x,y,z}) &= \hat{\alpha}_{y_{\underline{C}}(x), y_{\underline{C}}(y), y_{\underline{C}}(z)}; \\ y_{\underline{C}}(\lambda_x) &= \hat{\lambda}_{y_{\underline{C}}(x)}; \\ y_{\underline{C}}(\rho_x) &= \hat{\rho}_{y_{\underline{C}}(x)}; \\ y_{\underline{C}}(\gamma_{x,y}) &= \hat{\gamma}_{y_{\underline{C}}(x), y_{\underline{C}}(y)}; \end{aligned}$$

which is enough to conclude the desired result. ✓

We conclude this subsection by studying the behaviour of $\mathtt{Ind}(\_)$ on monoidal functors and monoidal transformations. Let $(\mathsf{F}, \varphi^0, \varphi)$ be a monoidal functor between the monoidal categories $\underline{C} = (\underline{C}, \otimes, e, \alpha, \lambda, \rho, \gamma)$ and $(\underline{D}, \otimes', e', \alpha', \lambda', \rho', \gamma')$. Consider the triple $(\mathtt{Ind}(\mathsf{F}), y_{\underline{C}}(\varphi), \mathtt{Ind}(\delta) * \Delta)$, i.e., a functor $\mathtt{Ind}(\mathsf{F})\colon \mathtt{Ind}(\underline{C}) \to \mathtt{Ind}(\underline{D})$, a morphism $y_{\underline{C}}(\varphi)\colon \hat{e}' \to \mathtt{Ind}(\mathsf{F})(\hat{e})$ and a natural transformation from $\mathtt{Ind}(\mathsf{F})(\_{}_1) \hat{\otimes}' \mathtt{Ind}(\mathsf{F})(\_{}_2) \to \mathtt{Ind}(\mathsf{F})(\_{}_1 \hat{\otimes} \_{}_2)$, whose component at the ind-objects $X\colon \mathsf{I} \to \underline{C}$ and $Y\colon \mathsf{J} \to \underline{C}$ is

$$([\delta_{X_i, Y_j}])_{(i,j)\in \mathsf{I}\times\mathsf{J}}\colon (\mathsf{F}(X_i) \otimes' \mathsf{F}(Y_j))_{(i,j)\in\mathsf{I}\times\mathsf{J}} \to (\mathsf{F}(X_i \otimes Y_j))_{(i,j)\in\mathsf{I}\times\mathsf{J}}.$$

It is just a matter of a few calculations to verify that the axioms (3.5) hold for $(\mathtt{Ind}(\mathsf{F}), y_{\underline{C}}(\varphi), \mathtt{Ind}(\delta) * \Delta)$. Moreover, if $(\mathsf{F}, \varphi, \delta)$ is symmetric, then (3.6) also holds, i.e., $(\mathtt{Ind}(\mathsf{F}), y_{\underline{C}}(\varphi), \mathtt{Ind}(\delta) * \Delta)$ is symmetric. Clearly, strongness and strictness are also preserved. Let $\sigma\colon (\mathsf{F}, \varphi^0, \varphi) \xrightarrow{\cdot} (\mathsf{F}', \varphi'^0, \varphi')$ be a monoidal transformation. Recall that the component of $\mathtt{Ind}(\sigma)$ at the $\aleph$-ind-objects $X = (X_i)_{i\in\mathsf{I}}$ is $([\sigma_{X_i}])_{i\in\mathsf{I}}\colon (\mathsf{F}X_i)_{i\in\mathsf{I}} \to (\mathsf{F}'X_i)_{i\in\mathsf{I}}$. Therefore, it follows easily that, when σ satisfies (3.7), $\mathtt{Ind}(\sigma)$ is a monoidal transformation from $(\mathtt{Ind}(\mathsf{F}), y_{\underline{C}}(\varphi^0), \mathtt{Ind}(\varphi) * \Delta)$ to $(\mathtt{Ind}(\mathsf{F}'), y_{\underline{C}}(\varphi'^0), \mathtt{Ind}(\varphi') * \Delta)$. Therefore, we can state the following proposition.

Proposition 3.6.4
The KZ-doctrine $\mathtt{Ind}(\_)$ $(\mathtt{Ind}(\_)_{\aleph})$ *on* $\underline{\underline{\mathsf{CAT}}}$ $(\underline{\underline{\mathsf{Cat}}})$ *lifts to KZ-doctrines on* $\underline{\underline{B}}$, *for any* $\underline{\underline{B}}$ *appearing in Table 3.1.*

Proof. Concerning the categories of locally small categories, the result follows immediately from the previous considerations about monoidal functors and transformations and from Proposition 3.6.2. In the cases where $\underline{\underline{B}}$ is a category of small categories, it follows from the above and Proposition 3.5.29. ✓

For each $\underline{\underline{B}}$ appearing in Table 3.1, let $\underline{\underline{\omega\text{-}B}}$ be the category consisting of the ($\aleph$-)chain cocomplete categories in $\underline{\underline{B}}$ with a choice of colimits and of the functors in $\underline{\underline{B}}$ which preserve ($\aleph$-)chain colimits up to isomorphism. Then, by general facts in the theory of KZ-doctrines, we have that any $\underline{\underline{B}}$ in Table 3.1, $\mathtt{Ind}(\_)$ $(\mathtt{Ind}(\_)_{\aleph})$ determines a KZ-adjunction from $\underline{\underline{B}}$ to $\underline{\omega}$-$\underline{\underline{B}}$.

Cocompletion of Monoidal Categories: Second Solution

The extension of the monoidal structure of $\underline{C}$ to $\mathtt{Ind}(\underline{C})$ given in the previous subsection may look rather far from our intended interpretation motivated in Section 3.1. For instance, the tensor of two ω-chains is (represented by) a two-dimensional structure $\omega \times \omega$. However, this is just a comfortable representation for the tensor. One could consider another representation taking for example the "diagonal" cofinal chain in $\omega \times \omega$, which is isomorphic as ind-object to the "whole square", and which corresponds to the motivating diagram shown in Section 3.1. In this subsection, we study an alternative description of the monoidal structure of $\mathtt{Ind}(\underline{C})$ which is more intuitive and better suited for our intend applications. Since to be $\aleph$-filtered cocomplete is equivalent to be $\aleph$-chain cocomplete, we could consider the subcategory of $\mathtt{Ind}(\underline{C})$ consisting of $\aleph$-chains. This choice essentially does not change the category. However, it is not clear whether we could or not define a KZ-doctrine out of it, because of the problem with the multiplication we have already mentioned. In the following, we focus only on ω-chains.

Let $\underline{C}^{\omega}$ be the full subcategory of $\mathtt{Ind}(\underline{C})$ consisting of the ind-objects indexed by $\underline{\omega}$. Then we have the following.

Proposition 3.6.5
$\underline{C}^{\omega} \overset{\cdot}{\cong} \mathtt{Ind}(\underline{C})_{\omega}$.

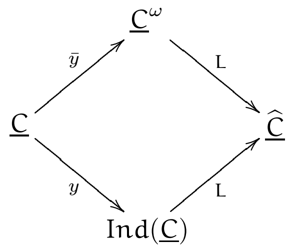
Proof. The inclusion functor $\underline{C}^{\omega} \hookrightarrow \mathtt{Ind}(\underline{C})_{\omega}$ is by definition full and faithfull. We show that is replete image is $\mathtt{Ind}(\underline{C})$. Then, by exploting the results of [26] as in Proposition 3.5.3, we have the desired result.

Let X be an ω-ind-object, i.e., a *countable* filtered diagram in $\underline{C}$. We have to show that it is isomorphic in $\mathtt{Ind}(\underline{C})_{\omega}$ to an ω-chain. By applying the lemma (from [43]) stated in Proposition 3.3.17, and thanks to Proposition 3.5.6, we may assume that X is indexed over a countable directed set D. Then, using the lemma (from [93]) given in Proposition 3.3.6, we find a countable sequence of *finite* directed subsets $\{D_i\}_{i\in\omega}$ such that, $D_i \subset D_{i+1}$, for any $i \in \omega$, and $D = \bigcup_{i\in\omega} D_i$. Then, we can extract from $\{D_i\}_{i\in\omega}$ a sequence of $\{c_i\}_{i\in\omega}$, where c_i is the greatest element of D_i, which exists since D_i is directed and finite. Now, define the functors $\phi{:}\,\underline{\omega} \to D$ and $Y{:}\,\underline{\omega} \to \underline{C}$ as follows:

$$\begin{array}{rcl} \phi(i) = c_i & \text{and} & Y(i < i+1) = c_i < c_{i+1}; \\ Y(i) = X(c_i) & \text{and} & Y(i < i+1) = X(c_i < c_{i+1}). \end{array}$$

Clearly, by Lemma 3.3.14 *(iii)*, we have that ϕ is cofinal, and since $Y\phi = X$, by Proposition 3.5.6, we conclude that X and Y are isomorphic in $\mathtt{Ind}(\underline{C})_{\omega}$. Since Y is an ω-chain, this concludes the proof. ✓

Observe that, as immediate consequence of the proposition above, we have that $\underline{\mathsf{C}}^\omega$ is ω-chain cocomplete. Of course, working with$\underline{\mathsf{C}}^\omega$, we have to redefine y. We shall consider the obvious choice $\bar{y}(c) = \underline{\underline{c}} = \underline{\omega} \stackrel{c}{\longrightarrow} \underline{\mathsf{C}}$, the constant chain. Of course, we still have that $\bar{y}\colon \underline{\mathsf{C}} \to \underline{\mathsf{C}}^\omega$ is full and faithful. Moreover, the following diagram is commutative



Restricting our attention to $\underline{\omega}$ makes possible expressing the commutativity of $(\_)^\omega$ and $\_ \times \_$ by an isomorphism.

Proposition 3.6.6
There is an isomorphism $(\underline{\mathsf{C}} \times \underline{\mathsf{D}})^\omega \cong \underline{\mathsf{C}}^\omega \times \underline{\mathsf{D}}^\omega$, *defined by* Δ *and* ∇ *below.*

$$
\begin{array}{ccc}
(\underline{\mathsf{C}} \times \underline{\mathsf{D}})^\omega & \xrightarrow{\quad\nabla\quad} & \underline{\mathsf{C}}^\omega \times \underline{\mathsf{D}}^\omega \\
\underline{\omega} \xrightarrow{X} \underline{\mathsf{C}} \times \underline{\mathsf{D}} & \longmapsto & (\underline{\omega} \xrightarrow{\pi_0 X} \underline{\mathsf{C}}, \underline{\omega} \xrightarrow{\pi_1 X} \underline{\mathsf{D}}) \\
\downarrow {\scriptstyle ([f_i])_{i\in\omega}} & & {\scriptstyle ([fst(f_i)])_{i\in\omega}} \downarrow \quad \downarrow {\scriptstyle ([snd(f_i)])_{i\in\omega}} \\
\underline{\omega} \xrightarrow{Y} \underline{\mathsf{C}} \times \underline{\mathsf{D}} & \longmapsto & (\underline{\omega} \xrightarrow{\pi_0 Y} \underline{\mathsf{C}}, \underline{\omega} \xrightarrow{\pi_1 Y} \underline{\mathsf{D}})
\end{array}
$$

fst and snd being as in the previous subsection.

$$
\begin{array}{ccc}
\underline{\mathsf{C}}^\omega \times \underline{\mathsf{D}}^\omega & \xrightarrow{\quad\Delta\quad} & (\underline{\mathsf{C}} \times \underline{\mathsf{D}})^\omega \\
(\underline{\omega} \xrightarrow{X} \underline{\mathsf{C}}, \underline{\omega} \xrightarrow{Y} \underline{\mathsf{D}}) & \longmapsto & \underline{\omega} \xrightarrow{\langle X,Y\rangle} \underline{\mathsf{C}} \times \underline{\mathsf{D}} \\
{\scriptstyle ([f_i])_{i\in\omega}} \downarrow \quad \downarrow {\scriptstyle ([g_i])_{i\in\omega}} & & \downarrow {\scriptstyle ([f_i \times g_i])_{i\in\omega}} \\
(\underline{\omega} \xrightarrow{X'} \underline{\mathsf{C}}, \underline{\omega} \xrightarrow{Y'} \underline{\mathsf{D}}) & \longmapsto & \underline{\omega} \xrightarrow{\langle X',Y'\rangle} \underline{\mathsf{C}} \times \underline{\mathsf{D}}
\end{array}
$$

Proof. We have

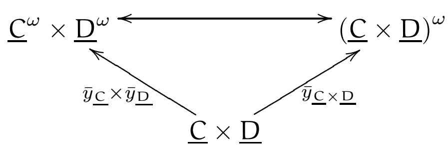
$$
\begin{array}{ccc}
\Delta\nabla(\underline{\omega} \xrightarrow{X} \underline{\mathsf{C}} \times \underline{\mathsf{D}}) & = & (\underline{\omega} \xrightarrow{\langle \pi_0 X, \pi_1 X\rangle} \underline{\mathsf{C}} \times \underline{\mathsf{D}}) \\
{\scriptstyle \Delta\nabla(([f_i])_{i\in\omega})} \downarrow & & \downarrow {\scriptstyle ([fst(f_i) \times snd(f_i)])_{i\in\omega}} \\
\Delta\nabla(\underline{\omega} \xrightarrow{Y} \underline{\mathsf{C}} \times \underline{\mathsf{D}}) & = & (\underline{\omega} \xrightarrow{\langle \pi_0 Y, \pi_1 Y\rangle} \underline{\mathsf{C}} \times \underline{\mathsf{D}})
\end{array}
$$

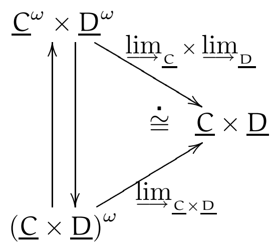
which is a strict equality, since $\langle \pi_0 X, \pi_1 X\rangle = X$ and $fst(f_i) \times snd(f_i) = f_i$. On the other hand, we have

$$\begin{array}{ccc}
\nabla\Delta(\underline{\omega} \xrightarrow{X} \underline{\mathsf{C}}, \underline{\omega} \xrightarrow{Y} \underline{\mathsf{D}}) & = & (\underline{\omega} \xrightarrow{\pi_0\langle X,Y\rangle} \underline{\mathsf{C}}, \underline{\omega} \xrightarrow{\pi_1\langle X,Y\rangle} \underline{\mathsf{D}}) \\
\Big\downarrow {\scriptstyle \nabla\Delta(([f_i])_{i\in\omega},([g_i])_{i\in\omega})} & & \Big\downarrow {\scriptstyle (([f_i])_{i\in\omega},([g_i])_{i\in\omega})} \\
\nabla\Delta(\underline{\omega} \xrightarrow{X'} \underline{\mathsf{C}}, \underline{\omega} \xrightarrow{Y'} \underline{\mathsf{D}}) & = & (\underline{\omega} \xrightarrow{\pi_0\langle X',Y'\rangle} \underline{\mathsf{C}}, \underline{\omega} \xrightarrow{\pi_1\langle X',Y'\rangle} \underline{\mathsf{D}})
\end{array}$$

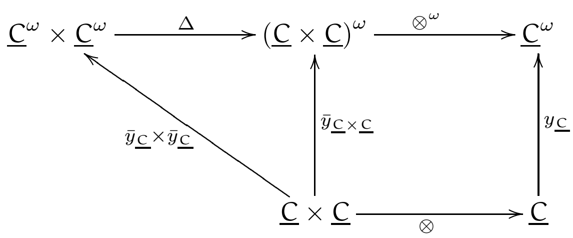
which is of course the identity functor. ✓

As in the previous subsection, we have the following commutative diagrams.





the second diagram existing when $\underline{\mathsf{C}}$ and $\underline{\mathsf{D}}$ are ω-chain cocomplete, and thus we can define



In the following, $\otimes^\omega \circ \Delta$ will be denoted by $\tilde{\otimes}$. Writing the tensor in terms of the indexed representation of ind-object and morphisms makes clear the correspondence of this approach with the discussion in Section 3.1.

$$\begin{array}{ccccc}
(X_i)_{i\in\omega} & & (Y_i)_{i\in\omega} & & (X_i \otimes Y_i)_{i\in\omega} \\
{\scriptstyle ([f_i])_{i\in\omega}} \Big\downarrow & \tilde{\otimes} & \Big\downarrow {\scriptstyle ([g_i])_{i\in\omega}} & = & \Big\downarrow {\scriptstyle ([f_i \otimes g_i])_{i\in\omega}} \\
(X'_i)_{i\in\omega} & & (Y'_i)_{i\in\omega} & & (X'_i \otimes Y'_i)_{i\in\omega}
\end{array}$$

So, given the symmetric monoidal category $(\underline{\mathsf{C}}, \otimes, e, \alpha, \lambda, \rho, \gamma)$, the monoidal structure on $\underline{\mathsf{C}}^\omega$ is $(\underline{\mathsf{C}}^\omega, \tilde{\otimes}, \tilde{e}, \tilde{\alpha}, \tilde{\lambda}, \tilde{\rho}, \tilde{\gamma})$, where

- $\tilde{e} = \bar{y}_{\underline{\mathsf{C}}}(e) = \underline{e}$;

- $\tilde{\alpha}_{X,Y,Z} = ([\alpha_{X_i,Y_i,Z_i}])_{i\in\omega}$;
- $\tilde{\lambda}_X = ([\lambda_{X_i}])_{i\in\omega}$;
- $\tilde{\rho}_X = ([\rho_{X_i}])_{i\in\omega}$;
- $\tilde{\gamma}_{X,Y} = ([\gamma_{X_i,Y_i}])_{i\in\omega}$.

Showing that these data form a symmetric monoidal category is a routine task. Therefore, we can summarize the results in the following propositions.

Proposition 3.6.7
For any symmetric monoidal category $(\underline{\mathsf{C}}, \otimes, e, \alpha, \lambda, \rho, \gamma)$ *the* ω*-filtered cocomplete category* $(\underline{\mathsf{C}}^\omega, \tilde{\otimes}, \tilde{e}, \tilde{\alpha}, \tilde{\lambda}, \tilde{\rho}, \tilde{\gamma})$ *is a symmetric monoidal category.*
Moreover, if $\underline{\mathsf{C}}$ *is monoidal strict, then so is* $\underline{\mathsf{C}}^\omega$*; if* $\underline{\mathsf{C}}$ *is strictly simmetric so is* $\underline{\mathsf{C}}^\omega$.

Proposition 3.6.8
The subcategory $\bar{y}_{\underline{\mathsf{C}}}(\underline{\mathsf{C}})$ *of* $\underline{\mathsf{C}}^\omega$ *is isomorphic to* $\underline{\mathsf{C}}$ *in the monoidal sense, i.e.,* $\bar{y}_{\underline{\mathsf{C}}}$ *is a strict monoidal functor.*

Proposition 3.6.9
$\underline{\mathsf{C}}^\omega$ *is, up to equivalence, the free* ω*-chain cocomplete monoidal category on* $\underline{\mathsf{C}}$.

Proof. It follows easily from Proposition 3.6.5 and the pseudo universal property of $\mathtt{Ind}(\underline{\mathsf{C}})_\omega$ (see, e.g., Proposition 3.5.21). ✓

Of course, the results above can be restated for any chain α and the corresponding subcategory $\underline{\mathsf{C}}^\alpha$ of $\mathtt{Ind}(\underline{\mathsf{C}})$.

3.7 Applications to Petri Nets

The previous sections have shown how we can build the (pseudo) free ($\aleph$)-filtered cocomplete category $\mathtt{Ind}(\underline{\mathsf{C}})$ over a given $\underline{\mathsf{C}}$. In particular, in Section 3.6 we have proved that the construction lifts to a KZ-doctrine on $\underline{\mathtt{SsMonCat}}$, giving in this way the completion of symmmetric strict monoidal categories (SSMC's). The theories we have formalized and the arguments we have discussed in Chapter 1 of this thesis, and in particular in Sections 1.1, 1.2, 1.3 and 1.9 support the claim that this result brings us close again to Petri nets. This section, which matches in style Section 3.1, explains further these facts.

It is important to observe that Petri nets are not precisely symmetric strict monoidal categories, since they enjoy other important axioms, the principal one being the *free monoidal structure* on the objects. In order to fix the ideas, we recall that Chapter 1 presented four relevant monoidal constructions for net computations, namely *i)* $\mathcal{T}[N]$, which gives a <u>CatPetri</u> object, i.e., a *strictly* symmetric strict monoidal category whose objects form a free commutative monoid, corresponding to the notion of *commutative processes* of N (see Definition 1.1.7 and Proposition 1.1.12).

ii) $\mathcal{P}[N]$, which gives the symmetric strict monoidal category obtained by quotienting the free symmetric strict monoidal category on N via the axioms

$$\begin{array}{rcll} \gamma_{a,b} & = & id_{a\otimes b} & \text{if } a,b \in S \text{ and } a \neq b, \\ (id \otimes \gamma_{a,a} \otimes id); t & = & t & \text{if } t \in T \text{ and } a \in S, \\ t; (id \otimes \gamma_{a,a} \otimes id) & = & t & \text{if } t \in T \text{ and } a \in S, \end{array}$$

and corresponds to the *concatenable processes* of N. (See Definition 1.1.16, Propositions 1.1.19, and 1.2.5 and Corollary 1.2.7.) We shall refer to the category of small symmetric monoidal categories with the properties above as <u>CatProc</u>.

iii) $\mathcal{DP}[N]$, which is as above, but satisfying only the first two axioms, and corresponds to the *decorated concatenable processes* of N. (See Definition 1.9.13 and Proposition 1.9.14.) Let <u>CatDecProc</u> denote the category of the small categories with the properties above.

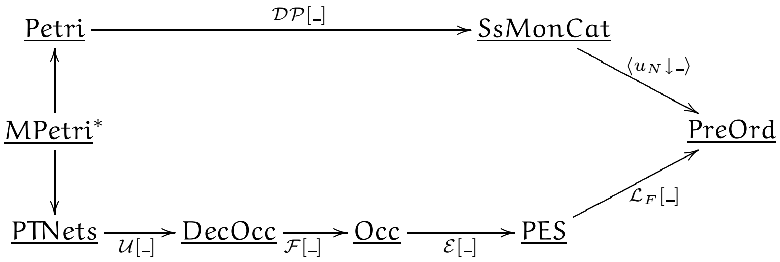
iv) $\mathcal{Q}[N]$, which gives the quotient of the free symmetric strict monoidal category on the *graph* of the "linearizations" of the transitions of N modulo the axiom

$$s; t_{u',v'}; s = t_{u,v} \quad \text{for } s\colon u \to u', \quad s'\colon v' \to v \text{ symmetries.}$$

and corresponds to the *strong concatenable processes* of N. (See Definition 1.3.8 and Proposition 1.4.5.) Proposition 1.1.19). We shall refer to the category of small symmetric monoidal categories with the properties above as <u>CatStrProc</u>.

Since we already know that the monoidal structure of such categories is preserved by the $\aleph$-ind-completion process, the remaining question is whether the additional structure is preserved by the ($\aleph$-)ind-completion or, in other words, if $\mathsf{Ind}(\_)$ lifts to KZ-monads on <u>CatPetri</u>, <u>CatProc</u>, <u>CatDecProc</u> and <u>CatStrProc</u>. Clearly, this would be the best possible result from our point of view, since it would allow a full application of the theory of the cocompletion of monoidal categories to the case of Petri nets, thus giving a full account of infinite behaviours of nets. Unfortunately, this is not the case. More precisely, only the objects of <u>CatPetri</u> are rather close to keep their structure under the cocompletion construction. In fact, we know from Proposition 3.6.2 that $\mathsf{Ind}(\mathcal{T}[N])$ is a strictly symmetric strict monoidal category. However, $\mathsf{Ind}(\mathcal{T}[N])$ does not belong to <u>CatPetri</u> since its monoid of objects is *not* free. The situation is worse for the other categories. Observe, in fact, that when $\underline{C}$ is not strictly symmetric, then the primary requirement about the monoid of objects being commutative fails immediately. In fact, in this case, the tensor of diagrams in general will not be commutative because of the arrows.

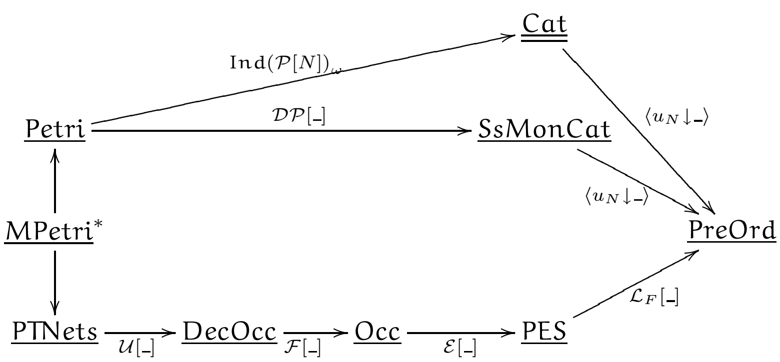
Thus, $\mathsf{Ind}(\_)$ does *not* restrict to an endofunctor on the categories we are mainly concerned with. A possible way out of this problem, which is currently under investigation, consists of looking for an alternative presentation of the cocompletion doctrine, i.e., for a doctrine whose functor is isomorphic to $\mathsf{Ind}(\_)$ but better suited for the case of Petri nets. For the time being, however, we present some considerations about the relationships between Petri nets and the cocompletion of their categories of processes which aim at showing that, at the level of a single net, $\mathsf{Ind}(\_)$ behaves as expected, giving a faithful description of infinite processes. In particular, we shall present the following considerations about the relationships between Petri nets and the cocompletion of their categories of processes. We shall focus on decorated concatenable processes, although all the following, apart from Proposition 3.7.4, applies also to $\mathcal{T}[\_]$, $\mathcal{P}[\_]$ and $\mathcal{Q}[\_]$. Consider again the diagram



(1.26)

of Section 1.9, where we have shown that for any net (N, u_N) in $\underline{\mathsf{MPetri}}^*$ we have that the decorated concatenable processes of N leaving from u coincide with the finite marked processes of $\mathcal{FU}[N]$, i.e., with the finite configurations of $\mathcal{EFU}[N]$.

As a simple application of the theory illustrated in this chapter, we shall extend the result to infinite processes and infinite configurations, as shown by the diagram below.



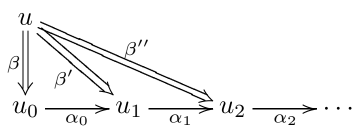
(3.10)

First of all, we need to show that $\mathtt{Ind}(\mathcal{DP}[N])$ can be considered the category of infinite decorated concatenable processes of N. Consider a net $N \in \underline{\mathtt{Petri}}$ and an ω-chain

$$u_0 \xrightarrow{\alpha_0} u_1 \xrightarrow{\alpha_1} u_2 \cdots u_n \xrightarrow{\alpha_n} u_{n+1} \cdots$$

i.e., an ind-object U in $\mathcal{DP}[N]^\omega$. We look at this chain as a limit point for an infinite computation (and not as the infinite computation itself!), i.e., a sort of generalized *infinite marking* represented by the computation which produces it from the finite markings. Observe that the adjective "generalized" is appropriate, since, in general, the infinite marking above depends on the transitions which appear in the chain, not just on their sources and targets. For instance, if we consider a net with two transitions $t, t' \colon a \to a$, then the chains consisting respectively of a sequence of t and a sequence of t' represent different infinite markings.

In order to substantiate the intuition about morphisms, let us start with the following case. Let $\underline{u}$ be the standard representative of u in $\mathtt{Ind}(\mathcal{DP}[N])$, i.e. $y(u)$ the diagram with value u^I on the singleton filtered category $\underline{1}$. Given the particular shapes of $\underline{1}$ and $\underline{\omega}$, an arrow from $\underline{u}$ to U in $\mathtt{Ind}(\mathcal{DP}[N])$, is an equivalence class $[\beta] \colon \underline{u} \to U$ of arrows $\beta \colon u \to u_n$ in $\mathcal{DP}[N]$, where $(\beta \colon u \to u_n) \sim (\beta' \colon u \to u_k)$ with $n \leq k$ if and only if $\beta; \alpha_{n+1}; \cdots; \alpha_k = \beta'$.



Now, recalling the characterization of arrows in $\mathcal{DP}[N]$ as decorated concatenable processes (see also Proposition 1.9.16), we conclude that an arrow from $\underline{u}$ to U in $\mathtt{Ind}(\mathcal{DP}[N])$ is an ω-chain of decorated concatenable processes embedded into each other, i.e., it represents a unique infinite decorated process. Before getting to the generality of arrows between ind-objects, it is worthwhile to point out the following particular case. Recall that each ind-object is the limit in $\mathtt{Ind}(\underline{\mathsf{C}})$ of its component constant ind-objects. Then, it follows immediately from the discussion above that, for any $n \in \omega$, the component at n of the limit cocone for U, say $\lambda_n \colon u_n \to U$, contains the set

$$\{\alpha_n,\, \alpha_n; \alpha_{n+1},\, \alpha_n; \alpha_{n+1}; \alpha_{n+2}, \ldots\}$$

as a cofinal subset. Then, $\lambda_0 \colon u_0 \to U$ represents the limit of the sequence of processes α_i, as expected.

Consider now the ind-objects

$$U = u_0 \xrightarrow{\alpha_0} u_1 \xrightarrow{\alpha_1} \cdots \quad \text{and} \quad V = v_0 \xrightarrow{\beta_0} v_1 \xrightarrow{\beta_1} \cdots$$

and an arrow $([\sigma_i])_{i\in\omega}: U \to V$. As explained above, each component $[\sigma_i]$ represents an infinite process leaving from u_i, i.e., leaving from the i-th approximation of the generalized marking U. Now, the "compatibility" condition on the components of $([\sigma_i])_{i\in\omega}$ means that for any $n \leq k$ and for any $\sigma_i: u_n \to v_{n'}$ and $\sigma_k: u_k \to v_{k'}$, representatives of, respectively, the i-th and the k-th component of the ind-morphism, assuming without loss of generality $n' \leq k'$, we must have $\alpha_n; \cdots; \alpha_{k-1}; \sigma_k = \sigma_n; \beta_{n'}; \cdots; \beta_{k'-1}$. It follows that the infinite processes (corresponding to) $[\sigma_i]$ form a sequence of embedded processes which leave from better and better approximations of U. Then, this chain admits a colimit process which is the decorated infinite process corresponding to $([\sigma_i])_{i\in\omega}$. In other words, morphisms from generalized infinite markings are defined via continuity from "finite" approximation morphisms. The same, of course, happens for the composition of decorated infinite processes.

It is now easy to translate the previous informal discussion into a formal proof of the following.

Of course, the previous informal discussion could be easily translated into a formal proof of the fact that $\mathcal{P}[N]^\omega$ captures the usual intuitive notion of infinite processes, thus yielding a smooth extension of the algebraic theory of Petri nets of [97, 16] to an axiomatization in terms of monoidal categories of the infinite causal behaviour of N. For the purpose of this section, however, we simply claim that the following definitions are completely adequate.

Definition 3.7.1
$\mathsf{Ind}(\mathcal{T}[N])_\omega$ *is, up to equivalence, the (strictly symmetric strict monoidal) category of infinite commutative processes of* N*.*

$\mathsf{Ind}(\mathcal{P}[N])_\omega$ *is, up to equivalence, the (symmetric strict monoidal) category of infinite concatenable processes of* N*.*

$\mathsf{Ind}(\mathcal{DP}[N])_\omega$ *is, up to equivalence, the (symmetric strict monoidal) category of infinite decorated concatenable processes of* N*.*

$\mathsf{Ind}(\mathcal{Q}[N])_\omega$ *is, up to equivalence, the (symmetric strict monoidal) category of infinite strong concatenable processes of* N*.*

Moreover, since $\underline{\mathsf{C}}^\omega$ and $\mathsf{Ind}(\underline{\mathsf{C}})$ are equivalent categories, we have also the following proposition.

Proposition 3.7.2
$\mathcal{T}[N]^\omega$ *is the category of infinite commutative processes of* N*.*

$\mathcal{P}[N]^\omega$ *is the category of infinite concatenable processes of* N*.*

$\mathcal{DP}[N]^\omega$ *is the category of infinite decorated concatenable processes of* N*.*

$\mathcal{Q}[N]^\omega$ *is the category of infinite strong concatenable processes of* N*.*

In addition, by comparing Definition 1.9.6 of decorated processes and Definition 1.9.10 of decorated concatenable processes, the following proposition is easily proved.

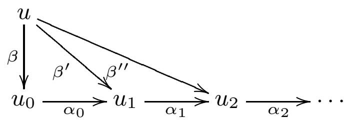
Proposition 3.7.3
The comma category $\langle \underline{u}_N \!\downarrow\! \mathtt{Ind}(\mathcal{DP}[N]) \rangle$ *is the category of infinite marked processes of* $(N,u)_N$ *in* $\underline{\mathsf{MPetri}}$.
The comma category $\langle \underline{\underline{u}}_N \!\downarrow\! \mathcal{DP}[N]^\omega \rangle$ *is the category of infinite marked processes of* (N, u_N) *in* $\underline{\mathsf{MPetri}}$.

Finally, we get back to diagram 3.10, and we conclude with the following extension of Proposition 1.9.16.

Proposition 3.7.4
For any (N,u) *in* $\underline{\mathsf{MPetri}}^*$,

$$\langle \underline{u} \downarrow \mathtt{Ind}(\mathcal{DP}[N]) \rangle \cong \langle \underline{\underline{u}} \downarrow \mathcal{DP}[N]^\omega \rangle \cong DP[(N,u)]^\omega \cong \mathcal{LEU}'[(N,u)].$$

Proof. By Proposition 1.9.16, we have that $\langle u \!\downarrow\! \mathcal{DP}[N] \rangle \cong DP[(N,u)] \cong \mathcal{L}_F\mathcal{EU}'[(N,u)]$, whence it follows that $\langle u \downarrow \mathcal{DP}[N] \rangle^\omega \cong DP[(N,u)]^\omega \cong \mathcal{L}_F\mathcal{EU}'[(N,u)]^\omega$. Now, from Proposition 3.5.10, $\mathcal{L}_F\mathcal{EU}'[(N,u)]^\omega = \mathcal{LEU}'[(N,u)]$, the domain of configurations of $\mathcal{EU}'[N]$. Finally, we are left to see that $\langle u \downarrow \mathcal{DP}[N] \rangle^\omega \cong \langle \underline{\underline{u}} \downarrow \mathcal{DP}[N]^\omega \rangle$. The objects of $\langle u \downarrow \mathcal{DP}[N] \rangle^\omega$ are commutative diagrams of the kind



which give rise to the ind-morphism $([\beta])_{i \in \omega} \colon \underline{\underline{u}} \to U$, U being the chain above. Thus, the thesis follows. ✓

Conclusions and Further Work

Of course, besides the problems with the semantics of nets we have noticed in the previous section, there are many other applications to be investigated, and we plan to explore many of these in the near future. For example, a mainstream in the research on infinite computations focuses on *topology*—more precisely on metric spaces [13, 118, 78]. Roughly, the approach consists of defining a suitable distance between finite computations and applying to the resulting metric space a standard *Cauchy completion*, thus yielding a complete metric space where the infinite computations are the cluster points. One of the most valuable aspects of this approach is that, by choosing appropriate metrics, it is possible to factor out those infinite computations which do not enjoy certain properties, in particular *fairness* properties [18]. It is indeed a very interesting question whether these results can be recovered in the categorical framework building on the seminal paper [85].

Moreover, since by now there are several categorical approaches to the semantics of computing systems in which objects represent states and arrows computations, this also yields a general method to construct and manipulate infinite computations of those systems. A notable example is given by Meseguer's concurrent rewriting systems [96]. This issue deserves to be fully investigated in future.

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Let us come to references to authors.
You have nothing else to do but
to look for a book that quotes them all, from A to Z.
Then you put this same alphabet in yours.
Perhaps there will be even someone silly enough
to believe that you have made use of them all
in your simple and straightforward story.

Miguel de Cervantes, Don Quixote

Appendices

These appendices are mainly intended to fix the basic notations, even though we tried to stick always to the standard ones. They also can serve as a quick reference to simple categorical notions not quite diffused among the computer scientists, such as monoidal categories, 2-categories and categories of fractions.

Se comprendere è impossibile,
conoscere è necessario.

Primo Levi, I sommersi ed i salvati

Neglect of mathematics
works injury to all knowledge,
since he who is ignorant of it
cannot know the other sciences
or the things of this world.

Roger Bacon

There is no branch of mathematics,
however abstract,
which may not some day be applied
to phenomena of the real world.

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A.1 Categories

In this section, we recall the classical concepts from Category Theory that we use in the paper. The appendix is not intended to be an introduction to the theory, but rather to fix the notation. For a complete introduction to category theory the reader is referred to [90, 1, 103, 127, 26, 92].

We assume, as in [90, 89], the existence of a *universe* $\mathfrak{U}$ of small sets upon which categories are built. In particular, $\mathfrak{U}$ is a *set*, whose elements are called *small sets*, which is closed under the usual set theoretic constructions. The idea is that $\mathfrak{U}$ is "big enough" in order for the *small* categories to model all the "ordinary" mathematical objects. Therefore, a set theory which contains $\mathfrak{U}$ is "big enough" to construct all the "needed" categories. In order to be more precise about $\mathfrak{U}$ and its role in the *foundations* of category theory, the following discussion—in which some knowledge of set theory is assumed—can be useful. The uninterested reader may safely skip the remark below.

Remark. The issue of foundations for category theory raised some problems for the set theoretic approach to the foundations of mathematics, in the precise sense of providing an example in which *set theory*, in the classical formulations of Zermelo-Fraenkel [147, 25] and Gödel-Bernays [7, 34] (see [55, 136] for a thorough exposition), does *not* fit directly to the common practise of (naive) category theory. The need of foundations arises from the facts that the definition of category is based on the notion of set—actually it includes that of sets—and it is *self-applicative*, i.e., that the collection of all categories has itself the structure of a category. Clearly, considering such totalities as the category of all sets, of all groups, of all categories, and so on, leads to the usual paradoxes. But of course, this is not new at all! Similar situations arise often and they are well-known to logicians, as for instance in the cases of the set structure of the collection of all sets, the well-ordering structure of well-orderings, the semigroup structure of semigroups, and so on. This consideration, together with the fact that "being a category" can be simply viewed as a first order property of sets, has prevented logicians to take seriously the matter for a long while.

However, the novelty presented by categories is that category theorist intend (well, a categorist would say "need" and a logician "want") to consider the aforesaid illegitimate categories very seriously. In other words, the key resides in the fact that category theory aim at working in the "small", i.e., at describing mathematical objects via categories, and in the "large", i.e., at studying the global structure of the collection of such objects. Therefore, any foundation for category theory, in order to be useful and to reflect the very aspiration of the theory, has to take into account this issue.

The first approaches to the foundations of category theory consisted, as directly suggested by the discussion above, of working with the Gödel-Bernays axiomatization of set theory, were both the notions of *set* and *class* are present. Details can be found in [86]. This solution, however, is not satisfactory at all, since it does not allow to construct *functor categories*. In fact, the collection of all class functions between two classes is not a class. This easy observation makes it clear that the axioms of set theory have to be strengthen if one wants to fullfil the requirements. The Grothendieck school has suggested then to postulate the existence of a *hierarchy of universes* [27, 43] by means of which all the paradoxes are avoided: considering the collection of all sets of a universe, or similar constructions, brings to a larger universe. Although this provides sound foundations based on a simple intuition, it still does not allow to consider the category of all sets, while presenting the annoying need of relating different universes. In other words, the approach is probably more complicated than needed, especially with respect to its achievements. In fact, as pointed out by MacLane [89], one can dispense with the hierarchy above and enrich system ZFC with axioms which guarantee the existence of a *single universe* $\mathfrak{U}$ playing the role already mentioned. This approach has the clear advantages of abandoning classes and being as simple as possible. Yet, the admissible collections are only those consisting of "small objects". Working on the same guidelines, Feferman [24] axiomatizes very smoothly the notion of small sets, under fairly weak conditions, by enriching the language of ZFC with a new symbol $\mathfrak{S}$ representing the set of small sets and five axioms. Remarkably, Feferman's system allows to show the following relevant facts about the set of small sets which fully justify the claim that they are "enough" to model the "ordinary" mathematical objects.

- $\mathfrak{U}$ is an initial segment of the *cumulative hierarchy* $\mathfrak{R}_\alpha$, where, for any ordinal α, $\mathfrak{R}_\alpha$ is obtained by iterating α times the cumulative power set operator $\mathfrak{R}\colon x \mapsto x \cup \wp(x)$.
- There exists the smaller ordinal σ which does not belong to $\mathfrak{S}$; moreover σ is a limit ordinal greater that ω and $\mathfrak{S} = \mathfrak{R}_\sigma$.
- If $\alpha < \sigma$ then $\aleph_\alpha < \sigma$; therefore

  $$\aleph_0, \aleph_1, \ldots, \aleph_{\aleph_0}, \aleph_{\aleph_1}, \ldots \aleph_{\aleph_{\aleph_0}}, \ldots$$

  can all be proved to belong to $\mathfrak{S}$.

The current trends in the mathematical development clearly show the growth of the conviction that the relevant features of mathematical objects are those given by their abstract stucture rather than those residing in the individuality of the elements they are made of. This brings immediately to the consideration that such a viewpoint should be reflected, if possible, by providing a foundation for mathematics in which membership and sets do not play any role. It is self-evident that, if such a foundation is to exist, it must be dealt with category theory. Therefore, instead of trying to found category theory on set theory, one could try to establish an *axiomatic* theory of categories upon which the whole mathematics could be based. The considerable amount of work recently devoted to *topos theory* [57, 92] seems to indicate that the latter may be possible.

A very interesting attempt along the lines discussed above is Lawvere's axiomatization of the category of all categories [84]. Although it is somehow incomplete in the sense that it still needs some metatheorems supporting the claim of adequacy for the proposed axiomatization, the cited work has, at the very least, the merit of giving good insights on the structure of the category of categories. In [6], Bénabou moves from very well motivated and strong criticisms to the set theoretic approaches to category theory illustrating, in particular, the misjudged relevance of enriching the language of set theory in order to make *definable* the usual categorical constructions. Moreover, he proposes *fibrations* [42, 41, 4] as a natural vehicle to provide foundations of category theory independently of sets. As for Lawvere's paper, and for the same reason, the aforecited work is somehow incomplete. However, it is a very instructive, almost pedagogical, paper.

Getting back to our immediate interests, in this thesis we follow the MacLane by fixing a universe $\mathfrak{U}$ of small sets. This choice is motivated by the simplicity of the approach, besides the fact that it suffices to our aims. In the following, we shall briefly list the relevant definitions and their most immediate consequences.

The Zermelo-Fraenkel theory of sets is a first order theory whose language $\mathcal{L}$ has a symbol of a binary relation $\in$ representing membership. The axioms of the theory ZFC can be *informally* stated as follows.

Extensionality Axiom:	$\forall x. (x \in a \leftrightarrow x \in b) \to a = b$;
Sum Axiom:	for every set a there exists a set $\bigcup a = \{x \mid x \in b \in a\}$;
Power Set Axiom:	for every set a there exists a set $\wp(a) = \{x \mid x \subseteq a\}$;
Axiom of Regularity:	for every non empty set a $\exists x. (x \in a \wedge x \cap a \neq \varnothing)$;
Axiom of Infinity:	there exists a set ω such that $\varnothing \in \omega \wedge \forall b. (b \in \omega \to b \cup \{b\} \in \omega)$;
Replacement Schema:	for every formula $\varphi(x, y)$ of $\mathcal{L}$ $\dfrac{\forall x. \forall y. \forall z. (x \in a \wedge \varphi(x, y) \wedge \varphi(x, z) \to y = z)}{\exists b. \forall y. (y \in b \leftrightarrow \exists x. (x \in a \wedge \varphi(x, y)))}$;
Axiom of Choice:	for every set a there exists a function f such that $\forall x. (x \subseteq a \wedge x \neq \varnothing \to f(x) \in x)$;

A *universe* $\mathfrak{U}$ is a set with the following properties:

i) $x \in y \wedge y \in \mathfrak{U} \to x \in \mathfrak{U}$;

ii) $\omega \in \mathfrak{U}$;

iii) $x \in \mathfrak{U} \to \wp(x) \in \mathfrak{U}$;

iv) $x \in \mathfrak{U} \to \bigcup x \in \mathfrak{U}$;

v) If $f : x \to a$ is a surjective function and $x \in \mathfrak{U}$ and $a \subseteq \mathfrak{U}$, then $a \in \mathfrak{U}$.

Among the immediate consequences of the definition of universe in the setting of ZFC, we have that all the natural numbers belong to $\mathfrak{U}$; if x and y are in $\mathfrak{U}$, then the cartesian product $x \times y$ and the set of all functions from x to y are in $\mathfrak{U}$. Moreover, if y is a subset of x and x is in $\mathfrak{U}$ then y belongs to $\mathfrak{U}$. Finally, we have that if $I \in \mathfrak{U}$ and x_i is a I-indexed family of sets in $\mathfrak{U}$, then $\bigcup_{i \in I} x_i$ also belongs to $\mathfrak{U}$.

Then, the theory proposed as foundations for category theory is Zermelo-Fraenkel ZFC plus one axiom which postulate the existence of a universe $\mathfrak{U}$. To conclude, it is worthwhile to remark that the sets we consider in the definition of categories and throughout this thesis are not limited just to the elements of $\mathfrak{U}$—which includes only the small sets, intended to model all the "ordinary" mathematical objects—and not even just to the subsets of $\mathfrak{U}$—which in this view may be considered as playing the role classes played in the Grothedieck's approach—but it also includes many other things, such as $\wp(\mathfrak{U})$, $\wp(\wp(\mathfrak{U}))$ and so on. For instance, we also have the set $\{\mathfrak{U}\}$. Observe that, as far cardinality is concerned, this is a very small set, namely a singleton, but it is not a small set in our formal sense. If fact, if it were $\{\mathfrak{U}\} \in \mathfrak{U}$, it would be $\mathfrak{U} \in \mathfrak{U}$ which is impossible by the Regularity Axiom. This is to remark that the term small does not refer to a set with a small cardinality, but to members of $\mathfrak{U}$.

Definition A.1.1 *(Graphs)*
A graph *is a structure* $(dom, cod \colon A \to O)$, *where* A *is a set of arrows,* O *is a set of objects and* dom *and* cod *are functions which associate to each arrow, respectively, a domain and a codomain.*

A graph is small *if its arrows form a* small set, *i.e., it belongs to* $\mathfrak{U}$, *it is* locally small *if, for any* o, o' *in* O, *the arrows of domain* o *and codomain* o' *form a small set, and* large *otherwise.*

Given a graph G, the set of its *composable arrows* is

$$A \times_O A = \{\langle g, f\rangle \mid g, f \in A \text{ and } dom(g) = cod(f)\}.$$

A *category* is a graph in which the set of arrows is closed under a given associative operation of composition $\circ \colon A \times_O A \to A$, and each object has an assigned arrow, the identity, which is a unit for such operation.

Definition A.1.2 *(Categories)*
A category $\underline{\mathsf{C}}$ *is a graph together with two additional functions*

$$id \colon O \to A \quad \text{and} \quad \circ \colon A \times_O A \to A,$$

called, respectively, identity and composition, such that

$$\text{for all } a \in O,\ cod(id(a)) = a = dom(id(a)),$$

$$\text{for all } \langle g, f\rangle \in A \times_O A,\ cod(g \circ f) = cod(g) \quad \text{and} \quad dom(g \circ f) = dom(g).$$

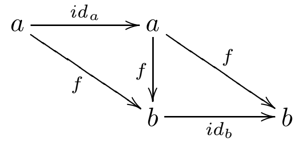
Moreover, $\circ$ *is associative and for all* $f \in A$, *given* $a = dom(f)$ *and* $b = cod(f)$, *we have* $f \circ id(a) = f = id(b) \circ f$.

A category is *small*, *locally small* or *large*, when its underlying graph is respectively small, locally small or large. Usually, the arrows of a category, also called *morphisms*, are denoted by $f \colon a \to b$, where $a = dom(f)$ and $b = cod(f)$. Identities are denoted by id_a, or simply id when the object is clear from the context. The set of arrows $f \colon a \to b$ in $\underline{\mathsf{C}}$ is denoted by $\mathsf{Hom}_{\underline{\mathsf{C}}}(a, b)$ or equivalently $\underline{\mathsf{C}}[a, b]$. Moreover, in dealing with a category $\underline{C}$ the actual sets A and O are never mentioned: we write $c \in \underline{C}$ for objects and f in $\underline{C}$ for arrows. A *subcategory* $\underline{B}$ of $\underline{C}$ is a category whose sets of objects and arrows are contained in the respective sets of $\underline{C}$. A subcategory $\underline{B}$ is *full* if for each $a, b \in \underline{B}$, $\underline{B}[a, b] = \underline{C}[a, b]$.

The first examples of categories are $\underline{0}$, the empty category, and $\underline{1}$, the category consisting of a single object and its identity arrow. Categories whose only arrows are identity arrows are called *discrete*. Clearly, any set S can be regarded as a discrete category. Further natural examples of categories are provided by preordered sets. These are particular kinds of categories $\underline{C}$ in which the homset $\mathsf{Hom}_{\underline{C}}(a, b)$ is the singleton set exactly when a is below b and it is empty otherwise. Partial orders are preordered categories whose only isomorphisms are identities. A *monoid* may be viewed as a category possessing a *single* object and viceversa. Clearly, the elements of the monoid are the morphisms of the category, the operation of the monoid is the composition of arrows and the neutral element is the identity arrow. In [80, 81], categories with additional structure are used to model deductive systems. Interesting examples of *large* categories are $\underline{\mathsf{Set}}$, the category of *small* sets and functions between them, $\underline{\mathsf{Mon}}$, the category of *small* monoids and homomorphisms, $\underline{\mathsf{Grp}}$, the category of *small* groups, and so on. These categories are large in the sense that the collection of their arrows form a subset of the universe $\mathfrak{U}$. Observe, however, that they are locally small. The category $\underline{\mathsf{Set}}_f$ of finite sets and functions between them is a *full* subcategory of $\underline{\mathsf{Set}}$.

An arrow $i: a \to b$ in $\underline{C}$ is *invertible* if there exists (a necessarily unique) $i': b \to a$ in $\underline{C}$ which is left and right inverse to i, i.e., $i' \circ i = id_a$ and $i \circ i' = id_b$. An invertible arrow is also said *isomorphism*, or simply *iso*. The set theoretic notions of injective and surjective functions are captured by the categorical concepts of *monic* and *epic* morphisms. An arrow $m: b \to c$ is a *monomorphism*, or simply *monic*, if for any parallel pair of morphisms $f, g: a \to b$ in $\underline{C}$ whenever $m \circ f = m \circ g$ it is $f = g$. Conversely, $e: a \to b$ is an *epimorphism*, or *epic*, if for any pair $f, g: b \to c$ in $\underline{C}$, $f \circ e = g \circ e$ implies $f = g$. Given $f: a \to b$ and $g: b \to a$, if $g \circ f = id_a$, i.e., if g is a (not necessarily unique) left inverse for f, then f is called *section* and g *retraction*.

Equation in category theory are expressed by imposing the commutativity of *diagrams*. A diagram consist of nodes, labelled by objects, and arcs, labelled by arrows. It commutes if for each pair of paths leaving from and leading to the same nodes, the compositions of the actual functions which label the paths yield the same arrow. For instance, given $\underline{C}$, the unit law of the identities in Definition A.1.2 can be expressed by saying that for all $f: a \to b$ in $\underline{C}$ the diagram



commutes.

Statement ψ	Dual statement ψ^*
$f: a \to b$	$f: b \to a$
$a = dom(f)$	$a = cod(f)$
$i = id(a)$	$i = id(a)$
f is monic	f is epic
f is right inverse to h	f is left inverse to h
f is a section	f is a retraction
f is invertible	f is invertible
t is a terminal object	t is an initial object
$\eta: \mathsf{F} \stackrel{\cdot}{\to} \mathsf{G}$ is a natural transformation	$\eta: \mathsf{G} \stackrel{\cdot}{\to} \mathsf{F}$ is a natural transformation
$\langle \mathsf{F}, \mathsf{G}, \varphi \rangle$ is an adjunction	$\langle \mathsf{G}, \mathsf{F}, \varphi^{-1} \rangle$ is an adjunction
η is the unit of $\langle \mathsf{F}, \mathsf{G}, \varphi \rangle$	η is the counit of $\langle \mathsf{G}, \mathsf{F}, \varphi^{-1} \rangle$

Table A.2: The principle of duality

A relevant notion in category theory is that of *duality principle*. Given $\underline{\mathsf{C}}$, its *opposite* category $\underline{\mathsf{C}}^{op}$ is the category obtained from $\underline{\mathsf{C}}$ by "formally" reversing its arrows. More precisely, the objects of $\underline{\mathsf{C}}^{op}$ are the objects of $\underline{\mathsf{C}}$, its homsets are given by the correspondence $\mathsf{Hom}_{\underline{\mathsf{C}}^{op}}(a, b) = \mathsf{Hom}_{\underline{\mathsf{C}}}(b, a)$, and the composition $f \circ g$ in $\underline{\mathsf{C}}^{op}$ is the arrow $g \circ f$, the latter composition being in $\underline{\mathsf{C}}$. Of course $(\underline{\mathsf{C}}^{op})^{op}$ is again $\underline{\mathsf{C}}$.

Statements about categories are easily formalized in a simple first order language that, as already mentioned, serves as the formal framework for the first steps towards an axiomatic approach to category theory [84]. In its elementary form, such a language is nothing but the formalization of the one we used to introduce the notion of category. Clearly, it has variables for objects and arrows and symbols for the functions domain, codomain, identity and composition. Thus, its atomic statements are of the form "$a = dom(f)$", "$b = cod(f)$", "$f = id(a)$", "$h = g \circ f$" and so on. Any statement ψ of the language admits a dual statement ψ^* which is obtained simply by replacing in ψ "$dom(\_)$" with "$cod(\_)$", "$cod(\_)$" with "$dom(\_)$" and "$g \circ f$" with "$f \circ g$". The duals of some of the most used statements are given in Table A.2. (The still undefined notions which appear in the table will be introduced shortly.) Observe that the dual of the dual is again the original statement.

Now, the *duality principle* of category theory is expressed by the following.

> METATHEOREM. *Let ψ the a sentence of the language of categories, i.e., a formula without free variables. Then, ψ is a theorem, i.e., it holds for every category, if and only if its dual ψ^* is so.*
>
> *Let ψ be any formula of the language of categories. Then, ψ is true in $\underline{\mathsf{C}}$ for a given instantiation of its free variables if and only if ϕ^* is true in $\underline{\mathsf{C}}^{op}$ for the same instantiation of the free variables.*

This principle implies that it is never necessary to prove a statement and its dual. For instance, the proof that f is monic in a category $\underline{\mathsf{C}}$ is also a proof that f is epic in $\underline{\mathsf{C}}^{op}$. Of course, since definitions may be dualizated as well, every notion "X" introduced in the theory comes with a "dual" notion "coX". Instances of this pattern are products and coproduct, equalizers and coequalizers, limits and colimits, and many more.

Functors are essentially morphisms of categories, in the informal sense that they are mappings which respect the relevant structure of categories.

DEFINITION A.1.3 *(Functors)*
A functor $\mathsf{F}\colon \underline{\mathsf{A}} \to \underline{\mathsf{B}}$ *is a function which maps objects of* $\underline{\mathsf{A}}$ *to objects of* $\underline{\mathsf{B}}$ *and arrows in* $\underline{\mathsf{A}}$ *to arrows in* $\underline{\mathsf{B}}$*, in such a way that*

i) for all $f\colon a \to b$ *in* $\underline{\mathsf{A}}$, $\mathsf{F}(f)\colon \mathsf{F}(a) \to \mathsf{F}(b)$ *in* $\underline{\mathsf{B}}$*;*

ii) for all $a \in \underline{\mathsf{A}}$, $\mathsf{F}(id_a) = id_{\mathsf{F}(a)}$*;*

iii) for all composable pairs $\langle g, f\rangle$, $\mathsf{F}(g \circ f) = \mathsf{F}(g) \circ \mathsf{F}(f)$ *.*

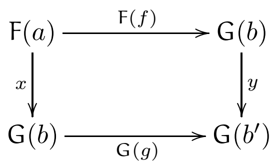
A functor $\mathsf{F}\colon \underline{\mathsf{A}} \to \underline{\mathsf{B}}$ is *full* if, for all $a, b \in \underline{\mathsf{A}}$, its restriction to $\mathsf{Hom}_{\underline{\mathsf{A}}}(a,b)$ is a surjective function., i.e., if for any $g\colon \mathsf{F}(a) \to \mathsf{F}(b)$ in $\underline{\mathsf{B}}$ there exists $f\colon a \to b$ in $\underline{\mathsf{A}}$ such that $\mathsf{F}(f) = g$. Analogously, F is *faithful*, or an *embedding*, if for any parallel pair $f, g\colon a \to b$ in $\underline{\mathsf{A}}$, if $\mathsf{F}(f) = \mathsf{F}(g)$ then $f = g$, i.e, if F restricted to any homset $\mathsf{Hom}_{\underline{\mathsf{A}}}(a,b)$ is injective. A functor F which is full and faithful is also called a *full embedding*.

Since some situations of great relevance occur frequently in the categorical practise, there are some kinds of functors which are named after their characteristics, as in the case of the aforesaid *embeddings* and in the case of *forgetful* and *inclusion* functors. Given a category $\underline{\mathsf{B}}$ whose objects are objects of $\underline{\mathsf{A}}$ with some additional structure and whose morphisms are the morphisms of $\underline{\mathsf{A}}$ which in addition preserve such structure, a functor from $\underline{\mathsf{B}}$ to $\underline{\mathsf{A}}$ which sends each object to its "underlying" $\underline{\mathsf{A}}$-object a category and each morphism to its $\underline{\mathsf{A}}$-version is called a forgetful functor. Strictly speaking, an inclusion functor is a functor from a subcategory $\underline{\mathsf{A}}$ of $\underline{\mathsf{B}}$ to $\underline{\mathsf{B}}$ whose object and arrow components are all *set inclusions*. However, it is easy to realize that the key feature of such functors is that they are full and faithful, i.e., full embeddings, and, in addition, they are injective on the objects. Thus, with a little abuse, one often refers to any functor with this properties as to an inclusion

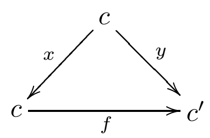
A functor $\mathsf{F}\colon \underline{\mathsf{A}}^{op} \to \underline{\mathsf{B}}$ is usually called a *contravariant* functor from $\underline{\mathsf{A}}$ to $\underline{\mathsf{B}}$. In fact, looking at them as "mappings" from $\underline{\mathsf{A}}$ to $\underline{\mathsf{B}}$, for $f\colon a \to b$ in $\underline{\mathsf{A}}$, we have that $\mathsf{F}(f)\colon \mathsf{F}(b) \to \mathsf{F}(a)$ and $\mathsf{F}(g \circ f) = \mathsf{F}(f) \circ \mathsf{F}(g)$, whence their name.

A composition operation for functors is easily obtained by composing their object and arrows components as functions. It is immediate to see that this operation is associative and admits identities, namely the (endo)functors $Id_{\underline{A}}: \underline{A} \to \underline{A}$ whose components are the identities on the sets of objects and arrows of $\underline{A}$. Therefore, we can define the category $\underline{\mathsf{Cat}}$ whose objects are small categories and whose arrows are functors. Analogously, we can define $\underline{\mathsf{CAT}}$, the category of locally small categories and functors.

Given two functors $\mathsf{F}: \underline{A} \to \underline{C}$ and $\mathsf{G}: B \to \underline{C}$ with a common target category $\underline{C}$ we can consider the *comma category* $\langle \mathsf{F}\downarrow\mathsf{G} \rangle$.[3] This is a standard categorical construction which makes (a selected subset of) the arrows of $\underline{C}$ be objects of a new category. More precisely, $\langle \mathsf{F}\downarrow\mathsf{G} \rangle$ is the category whose objects are the arrows $\mathsf{F}(a) \to \mathsf{G}(b)$ in $\underline{C}$, for $a \in \underline{A}$ and $b \in \underline{B}$, and whose arrows $h: (x: \mathsf{F}(a) \to \mathsf{G}(b)) \to (y: \mathsf{F}(a') \to \mathsf{G}(b'))$ are pairs $(f: a \to a', g: b \to b')$ of arrows respectively of $\underline{A}$ and $\underline{B}$, such that the following diagram commutes.



Clearly, by varying the choice of F and G, we obtain several different "comma constructions". For example, if $\underline{A}$ is $\underline{1}$, then F reduces to select an object c of $\underline{C}$. If moreover $\underline{B} = \underline{C}$ and $\mathsf{G} = Id_{\underline{C}}$, we have that $\langle \mathsf{F}\downarrow\mathsf{G} \rangle$, denoted in this case by $\langle c\downarrow\underline{C} \rangle$, is the category whose objects are the arrows $c \to c'$ of $\underline{C}$ and whose arrows from $(x: c \to c')$ to $(y: c \to c'')$ are the arrows $f: c' \to c''$ in $\underline{C}$ which make the following diagram commute.



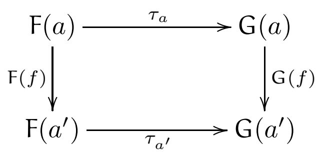
The category $\langle c\downarrow\underline{C} \rangle$ is commonly called the category of object under c. Dually, in case $\mathsf{F} = Id_{\underline{C}}$, $\underline{B} = \underline{1}$ and $\mathsf{G}(*) = c$, we obtain the comma category $\langle \underline{C}\downarrow c \rangle$, called the category of objects over c.

We can think of *natural transformations* as morphisms between functors: they respect the structure of functors in the sense that they provide a *uniform* way to *traslate* images of objects and arrows through F to images through G. They are *natural* in this sense.

DEFINITION A.1.4 *(Natural Transformations)*
Given two functors $\mathsf{F}, \mathsf{G}: \underline{A} \to \underline{B}$, *a natural transformation* $\tau: \mathsf{F} \xrightarrow{\cdot} \mathsf{G}$ *is a family of*

[3] The notation originally used by Lawvere in [82, 83] is (F, G), whence the name "comma" category.

morphisms $\tau = \{\tau_a \colon \mathsf{F}(a) \to \mathsf{G}(a)\}_{a \in \underline{\mathsf{A}}}$ *such that the following diagram commutes for any* $f \colon a \to a'$ *in* $\underline{\mathsf{B}}$.



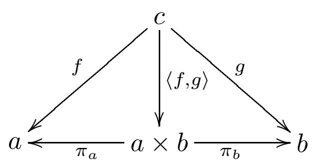
Given two natural transformations $\tau \colon \mathsf{F} \stackrel{\cdot}{\to} \mathsf{G}$ *and* $\mu \colon \mathsf{G} \stackrel{\cdot}{\to} \mathsf{H}$ *we can define their (vertical) composition as the natural transformation* $\mu \circ \tau = \{\mu_a \circ \tau_a\} \colon \mathsf{F} \stackrel{\cdot}{\to} \mathsf{H}$.

The operation of vertical composition of natural transformations is associative and has identities $Id_F \colon \mathsf{F} \stackrel{\cdot}{\to} \mathsf{F}$. So, if we take the functors between two categories $\underline{\mathsf{A}}$ and $\underline{\mathsf{B}}$ as objects and the natural transformations as morphisms, we obtain a category denoted by $\underline{\mathsf{B}}^{\underline{\mathsf{A}}}$. Categories as such, called *functor categories*, are very relevant. Later on, we shall get back to them.

A fundamental concept in mathematics is that of *universal construction.* It arises in many forms. Here are some classical universal constructions.

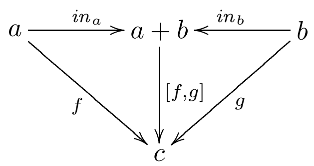
DEFINITION A.1.5 *(Initial and Terminal Objects)*
An object $i \in \underline{\mathsf{C}}$ *is* initial *if for any object* $c \in \underline{\mathsf{C}}$ *there exists a unique morphism* $f \colon i \to c$ *in* $\underline{\mathsf{C}}$. *Dually,* $t \in \underline{\mathsf{C}}$ *is* terminal *if for any* $c \in \underline{\mathsf{C}}$ *there exists a unique arrow* $f \colon c \to t$ *in* $\underline{\mathsf{C}}$.

DEFINITION A.1.6 *(Products and Coproducts)*
The product *in* $\underline{\mathsf{C}}$ *of objects* $a, b \in \underline{\mathsf{C}}$ *is an object* $a \times b \in \underline{\mathsf{C}}$ *together with two arrows* $\pi_a \colon a \times b \to a$ *and* $\pi_b \colon a \times b \to b$, *called* projections, *such that for any pair of arrows* $f \colon c \to a$, $g \colon c \to b$ *there exists a unique arrow* $\langle f, g\rangle \colon c \to a \times b$ *in* $\underline{\mathsf{C}}$ *such that the diagram*



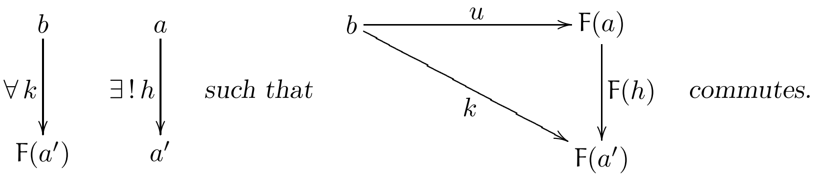
commutes.

Dually, the coproduct *of* $a, b \in \underline{\mathsf{C}}$ *is* $a + b \in \underline{\mathsf{C}}$ *and two arrows* $in_a \colon a \to a + b$ *and* $in_b \colon b \to a + b$, *called* injections, *such that for any pair of arrows* $f \colon a \to c$, $g \colon b \to c$ *there exists a unique arrow* $[f, g] \colon a + b \to c$ *in* $\underline{\mathsf{C}}$ *such that the diagram*

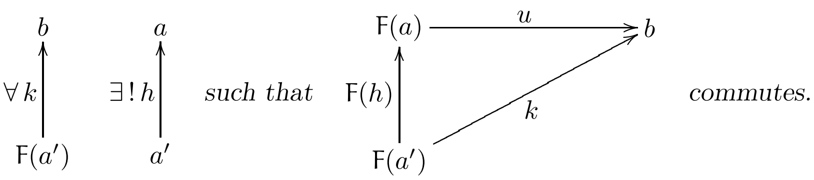


commutes.

DEFINITION A.1.7 *(Universal Arrows)*
Given a functor $\mathsf{F}: \underline{\mathsf{A}} \to \underline{\mathsf{B}}$ *and an object* $b \in \underline{\mathsf{B}}$, *a universal arrow from* b *to* F *is a pair* $\langle a, u \rangle$ *consisting of an object* $a \in \underline{\mathsf{A}}$ *and an arrow* $u: b \to \mathsf{F}(a)$ *such that for all* $a' \in \underline{\mathsf{A}}$



i.e., every arrow k *from* b *to (an object in the image of)* F *factors uniquely through* u.
Dually, a universal arrow from F *to* b *is a pair* $\langle a, u \rangle$, *where* $a \in \underline{\mathsf{A}}$ *and* $u: \mathsf{F}(a) \to b$, *such that for all* $a' \in \underline{\mathsf{A}}$

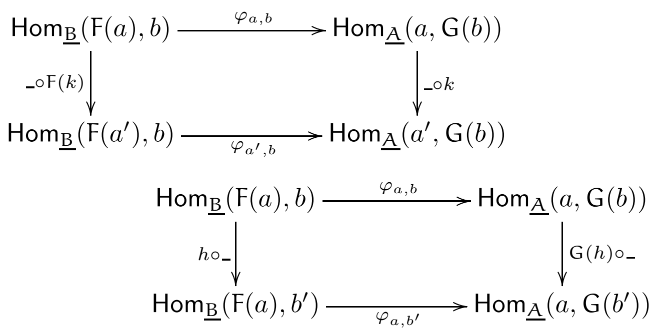


A basic concept in Category Theory is that of adjunction, due to Kan [59]. It provides a different formulation for the concept of universal constructions.

DEFINITION A.1.8 *(Adjunction)*
An adjunction from $\underline{\mathsf{A}}$ *to* $\underline{\mathsf{B}}$ *is a triple* $\langle \mathsf{F}, \mathsf{G}, \varphi \rangle: \underline{\mathsf{A}} \rightharpoonup \underline{\mathsf{B}}$, *where* $\mathsf{F}: \underline{\mathsf{A}} \to \underline{\mathsf{B}}$ *and* $\mathsf{G}: \underline{\mathsf{B}} \to \underline{\mathsf{A}}$ *are functors and* φ *is a function which assigns to each pair of objects* $a \in \underline{\mathsf{A}}$ *and* $b \in \underline{\mathsf{B}}$ *a bijection*

$$\varphi_{a,b}: \mathsf{Hom}_{\underline{\mathsf{B}}}(\mathsf{F}(a), b) \cong \mathsf{Hom}_{\underline{\mathsf{A}}}(a, G(b)),$$

which is natural both in a *and* b, *i.e., such that for all* $k: a' \to a$ *and* $h: b \to b'$ *the following diagrams commute.*



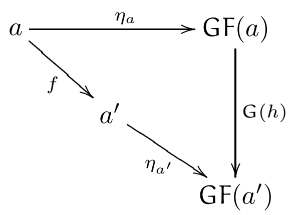
If $\langle \mathsf{F}, \mathsf{G}, \varphi \rangle : \underline{\mathsf{A}} \rightharpoonup \underline{\mathsf{B}}$ is an adjunction, F is called *left adjoint* to G and, viceversa, G is called *right adjoint* to F. This is denoted by $\mathsf{F} \dashv \mathsf{G}$.

The following two theorems state the relation between universal arrows and adjunctions.

THEOREM A.1.9 *(Adjoints and Universal Arrows, part I)*
Let $\mathsf{G} : \underline{\mathsf{B}} \to \underline{\mathsf{A}}$ *a functor such that*

$$\forall a \in \underline{\mathsf{A}}\ \exists \mathsf{F}(a) \in \underline{\mathsf{B}} \text{ and } \eta_a : a \to \mathsf{GF}(a) \text{ in } \underline{\mathsf{A}} \text{ which is universal from } a \text{ to } \mathsf{G}.$$

Then F *extends to a functor* $\mathsf{F} : \underline{\mathsf{A}} \to \underline{\mathsf{B}}$, *where for each* $f : a \to a' \in \underline{\mathsf{A}}$, $\mathsf{F}(f)$ *is defined to be the unique morphism* $h : \mathsf{F}(a) \to \mathsf{F}(a')$ *in* $\underline{\mathsf{B}}$ *which makes*



commute.

Moreover, the pair $\langle \mathsf{F}, \mathsf{G}, \varphi \rangle : \underline{\mathsf{A}} \rightharpoonup \underline{\mathsf{B}}$ *is an adjunction, where*

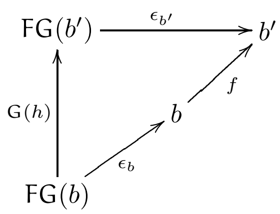
$$\varphi_{a,b} = \mathsf{G}(\_) \circ \eta_a : \mathsf{Hom}_{\underline{\mathsf{B}}}(\mathsf{F}(a), b) \to \mathsf{Hom}_{\underline{\mathsf{A}}}(a, \mathsf{G}(b)),$$

$\mathsf{F} \dashv \mathsf{G}$, *i.e.,* F *is left adjoint to* G *and* G *is right adjoint to* F *and* $\eta = \{\eta_a\}_{a \in \underline{\mathsf{A}}}$ *is a natural transformation* $Id_{\underline{\mathsf{A}}} \stackrel{\cdot}{\to} \mathsf{GF}$, *called the unit of the adjunction.*

THEOREM A.1.10 *(Adjoints and Universal arrows, part II)*
Let $\mathsf{F} : \underline{\mathsf{A}} \to \underline{\mathsf{B}}$ *a functor such that*

$$\forall b \in \underline{\mathsf{B}}\ \exists \mathsf{G}(b) \in \underline{\mathsf{A}} \text{ and } \epsilon_b : \mathsf{FG}(b) \to b \text{ in } \underline{\mathsf{A}} \text{ which is universal from } \mathsf{F} \text{ to } b.$$

Then G *extends to a functor* $G : \underline{\mathsf{B}} \to \underline{\mathsf{A}}$, *where for each* $f : b \to b' \in \underline{\mathsf{B}}$, $\mathsf{G}(f)$ *is defined to be the unique morphism* $h : \mathsf{G}(b) \to \mathsf{G}(b')$ *in* $\underline{\mathsf{A}}$ *which makes*



Moreover, the pair $\langle \mathsf{F}, \mathsf{G}, \varphi\rangle\colon \underline{\mathsf{A}} \rightharpoonup \underline{\mathsf{B}}$ *is an adjunction, where*

$$\varphi_{a,b} = \epsilon_b \circ \mathsf{F}(\_)\colon \mathsf{Hom}_{\underline{\mathsf{A}}}(a, \mathsf{G}(b)) \to \mathsf{Hom}_{\underline{\mathsf{B}}}(\mathsf{F}(a), b),$$

$\mathsf{F} \dashv \mathsf{G}$, *i.e.* F *is left adjoint to* G *and* G *is right adjoint to* F *and* $\epsilon = \{\epsilon_b\}_{b\in\underline{\mathsf{B}}}$ *is a natural transformation* $\mathsf{FG} \stackrel{\cdot}{\to} Id_{\underline{\mathsf{B}}}$, *called the counit of the adjunction.*

An adjunction $\langle F, \mathsf{G}, \eta, \epsilon\rangle\colon \underline{\mathsf{A}} \rightharpoonup \underline{\mathsf{B}}$ is called (generalized) *reflection* of $\underline{\mathsf{A}}$ in $\underline{\mathsf{B}}$, or $\underline{\mathsf{B}}$ is said *reflective* in $\underline{\mathsf{A}}$, if the counit is a natural isomorphism. Dually, it is a (generalized) *coreflection* of $\underline{\mathsf{B}}$ in $\underline{\mathsf{A}}$, or $\underline{\mathsf{A}}$ is *coreflective* in $\underline{\mathsf{B}}$, if the unit is a natural isomorphism.

We end this section by recalling the concept of *colimit*, which is a generalization of that of coproduct. Of course, there is the dual notion of *limit* that we shall not give here. However, later on we shall consider again both limits and colimits.

DEFINITION A.1.11 *(Diagrams)*
Given a small category J, *the index, and a category* $\underline{\mathsf{C}}$ *a diagram (of shape* J*) in* $\underline{\mathsf{C}}$ *is a functor* $D\colon \mathsf{J} \to \underline{\mathsf{C}}$.

Informally speaking, a diagram selects an object $c_j \in \underline{\mathsf{C}}$ for any object j in J and an arrow $u_i\colon c_j \to c_k$ in $\underline{\mathsf{C}}$ for any arrow $i\colon j \to k$ in J.

The diagonal functor $\Delta^{\mathsf{J}}\colon \underline{\mathsf{C}} \to \underline{\mathsf{C}}^{\mathsf{J}}$ associates to each object c the functor $\Delta^{\mathsf{J}}_c\colon \mathsf{J} \to \underline{\mathsf{C}}$ such that

i) $\Delta^{\mathsf{J}}_c(j) = c$ for any $j \in \mathsf{J}$;

ii) $\Delta^{\mathsf{J}}_c(h) = id_c$ for any $h\colon i \to j$ in J.

and associates to each arrow $f\colon c \to c'$ the natural transformation $\{f_j\colon \Delta^{\mathsf{J}}_c \to \Delta^{\mathsf{J}}_{c'}\}$ where $f_j = f$ for any $j \in \mathsf{J}$. Intuitively, for each c in $\underline{\mathsf{C}}$, Δ builds the diagram consisting of the unique point c, i.e. $\{c_j\}_{j\in\mathsf{J}} = \{c\}$ and $\{u_h\}_{h \text{ in } \mathsf{J}} = \{id_c\}$.

DEFINITION A.1.12 *(Cocones and Colimits)*
A cocone of a diagram D *in* $\underline{\mathsf{C}}^{\mathsf{J}}$ *is a natural transformation* $\tau\colon D \stackrel{\cdot}{\to} \Delta^{\mathsf{J}}_c$ *for some* c *in* $\underline{\mathsf{C}}$, *i.e., an object* c *in* $\underline{\mathsf{C}}$ *and a family of arrows* $\{\tau_j\}_{j\in\mathsf{J}}$ *such that* $\tau_j = \tau_k \circ u_i$ *for any* $u_i\colon c_j \to c_k$.

The colimit (or inductive or direct limit) of a diagram D *is an universal cocone, i.e., a cocone* $\mu\colon D \stackrel{\cdot}{\to} \Delta^{\mathsf{J}}_c$ *such that for each cocone* $\tau\colon D \stackrel{\cdot}{\to} \Delta^{\mathsf{J}}_{c'}$ *there exists a unique* $f\colon c \to c'$ *in* $\underline{\mathsf{C}}$ *such that* $\Delta^{\mathsf{J}}(f) \circ \mu = \tau$, *that is* $f \circ \mu_j = \tau_j$ *for each* $j \in \mathsf{J}$.

By abuse of notation, we will call c itself colimit of D and will denote it by $\varinjlim D$. Moreover, when J is clear from the context, we write simply Δ and often we omit Δ itself writing $\tau\colon D \stackrel{\cdot}{\to} c$ for $\tau\colon D \stackrel{\cdot}{\to} \Delta_c$.

Theorem A.1.13 *(Uniqueness of the Colimit)*
The colimit of a diagram D, if it exists, is unique up to isomorphisms.

Functor Categories

Given categories $\underline{\mathsf{C}}$ and $\underline{\mathsf{D}}$, the category $\underline{\mathsf{D}}^{\underline{\mathsf{C}}}$ has objects the functors $\mathsf{F}\colon \underline{\mathsf{C}} \to \underline{\mathsf{D}}$ and has morphisms the natural transformations between them. (See also, e.g. [92, 14].) Observe that, since the category of categories is actually a 3-category (i.e., we have *modifications* $\rho\colon \alpha \Rrightarrow \beta\colon \mathsf{F} \Rightarrow \mathsf{G}\colon \underline{\mathsf{C}} \to \underline{\mathsf{D}}$ between natural transformations) $\underline{\mathsf{D}}^{\underline{\mathsf{C}}}$ is a 2-category (see Appendix A.3).

There is an interesting connection between functor categories and limits and colimits that we briefly recall. Let J be a small category. There is an obvious embedding $\Delta_{\mathsf{J}}\colon \underline{\mathsf{C}} \to \underline{\mathsf{C}}^{\mathsf{J}}$ defined as follows to $\underline{\mathsf{Set}}^{\underline{\mathsf{C}}^{\mathrm{op}}}$.

$$\begin{array}{ccc} \underline{\mathsf{C}} & \xrightarrow{\quad\Delta_{\mathsf{J}}\quad} & \underline{\mathsf{C}}^{\mathsf{J}} \\ c & \longmapsto & \mathsf{J} \xrightarrow{\underline{c}} \underline{\mathsf{C}} \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle \underline{f}} \\ c' & \longmapsto & \mathsf{J} \xrightarrow{\underline{c'}} \underline{\mathsf{C}} \end{array}$$

where $\underline{c}$ is the constant functor with value c and $\underline{f}$ is the natural transformation with each component equal to f.

By definition, Δ_{J} has a right adjoint $\Delta_{\mathsf{J}} \dashv \varprojlim_{\mathsf{J}}$ if and only if $\underline{\mathsf{C}}$ admits limits of type J. In this case, the component at F of the counit $\varepsilon_{\mathsf{F}}\colon \Delta_{\mathsf{J}} \varprojlim_{\mathsf{J}} \to \mathsf{F}$ is the limit cone.

Dually, Δ_{J} has a left adjoint $\varinjlim_{\mathsf{J}} \dashv \Delta_{\mathsf{J}}$ if and only $\underline{\mathsf{C}}$ admits colimits of type J, and in this case $\eta_{\mathsf{F}}\colon \mathsf{F} \to \varprojlim_{\mathsf{J}} \Delta_{\mathsf{J}}$, the component at F of the unit η, is the limit cocone.

Given categories $\underline{\mathsf{C}}$ and $\underline{\mathsf{D}}$ which admits limits of type J, and given a functor $\mathsf{F}\colon \underline{\mathsf{C}} \to \underline{\mathsf{D}}$, consider the following diagram

$$\begin{array}{ccc} \underline{\mathsf{C}}^{\mathsf{J}} & \xrightarrow{\quad\varprojlim_{\underline{\mathsf{C}}}\quad} & \underline{\mathsf{C}} \\ {\scriptstyle \mathsf{G}^{\mathsf{J}}}\big\downarrow & & \big\downarrow{\scriptstyle \mathsf{G}} \\ \underline{\mathsf{D}}^{\mathsf{J}} & \xrightarrow[\quad\varprojlim_{\underline{\mathsf{D}}}\quad]{} & \underline{\mathsf{D}} \end{array}$$

where G^{J} is the functor which sends $\mathsf{F}\colon \mathsf{J} \to \underline{\mathsf{C}}$ to $\mathsf{GF}\colon \mathsf{J} \to \underline{\mathsf{D}}$. From the universal property of limits, we have a "canonical" natural transformation

$$\alpha_{\mathsf{J}}\colon \mathsf{G} \circ \varprojlim_{\underline{\mathsf{C}}} \stackrel{\cdot}{\to} \varprojlim_{\underline{\mathsf{D}}} \circ \mathsf{G}^{\mathsf{J}}$$

which gives the comparison $\mathsf{G}(\varprojlim \mathsf{F}) \to \varprojlim(\mathsf{GF})$. If α_{J} is an isomorphism, we say that G preserves limits of type J.

Dually, we can consider

$$\begin{array}{ccc} \underline{\mathsf{C}}^{\mathsf{J}} & \xrightarrow{\varinjlim_{\underline{\mathsf{C}}}} & \underline{\mathsf{C}} \\ {\scriptstyle \mathsf{G}^{\mathsf{J}}}\downarrow & & \downarrow{\scriptstyle \mathsf{G}} \\ \underline{\mathsf{D}}^{\mathsf{J}} & \xrightarrow[\varinjlim_{\underline{\mathsf{D}}}]{} & \underline{\mathsf{D}} \end{array}$$

and obtain the canonical comparison

$$\beta_{\mathsf{J}} \colon \varinjlim_{\underline{\mathsf{D}}} \circ \mathsf{G}^{\mathsf{J}} \xrightarrow{\cdot} \mathsf{G} \circ \varinjlim_{\underline{\mathsf{C}}},$$

i.e., the comparison $\varinjlim(\mathsf{GF}) \to \mathsf{G}(\varprojlim \mathsf{F})$. If β_{J} is an isomorphism, we say that G preserves colimits of type J.

It is interesting to recall that limits and colimits in functor categories are computed "pointwise". It follows that, when $\underline{\mathsf{D}}$ admits limits (colimits) of type J, then $\underline{\mathsf{D}}^{\underline{\mathsf{C}}}$ admits limits (colimits) of type J. Moreover, for any $c \in \underline{\mathsf{C}}$, the *evaluation functor* $(\_)_c \colon \underline{\mathsf{D}}^{\underline{\mathsf{C}}} \to \underline{\mathsf{D}}$ which applies $\mathsf{F} \colon \underline{\mathsf{C}} \to \underline{\mathsf{D}}$ to c, preserves such limits (colimits). More precisely, let $\mathsf{F} \colon \mathsf{J} \to \underline{\mathsf{D}}^{\underline{\mathsf{C}}}$ be a functor. For any $c \in \underline{\mathsf{C}}$, consider the functor $\mathsf{F}_c = (\_)_c \circ \mathsf{F}$, i.e.,

$$\begin{array}{ccc} \mathsf{J} & \xrightarrow{\mathsf{F}_c} & \underline{\mathsf{D}} \\ j & \longmapsto & \mathsf{F}(j)(c) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathsf{F}(f)_c} \\ j' & \longmapsto & \mathsf{F}(j')(c) \end{array}$$

If $\varprojlim \mathsf{F}_c$ ($\varinjlim \mathsf{F}_c$) exists for all $c \in \underline{\mathsf{C}}$, then $\varprojlim \mathsf{F}$ ($\varinjlim \mathsf{F}$) exists and

$$(\varprojlim \mathsf{F})c \cong \varprojlim \mathsf{F}_c \qquad ((\varinjlim \mathsf{F})c \cong \varinjlim \mathsf{F}_c).$$

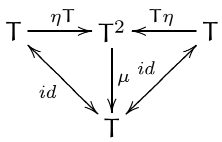
Monads

Monads [22, 10, 72, 73, 74, 75] are a formalization of the notion of algebraic structures on sets. They are a very important and pervasive concept in category theory and they are intimately connected to adjunctions. Formally monads are a generalization of the notion of monoid. Actually, a monad *is* a monoid in the category of endofunctors on a given category $\underline{\mathsf{C}}$, as their definition shows.

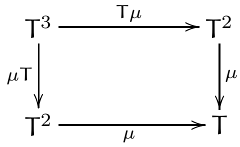
Definition A.1.14
A monad on a category $\underline{\mathsf{C}}$ *is a triple* (T, η, μ)*, where*

- T *is a endofunctor* $\mathsf{T}\colon \underline{\mathsf{C}} \to \underline{\mathsf{C}}$*;*
- $\eta\colon Id \stackrel{\cdot}{\to} T$ *is a natural transformation, called the unit of the monad;*
- $\mu\colon \mathsf{T}^2 \stackrel{\cdot}{\to} \mathsf{T}$ *is a natural transformation, called the multiplication of the monad.*

such that the following two diagrams commute



Unit Laws

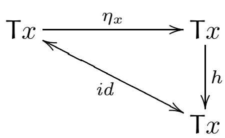


Associative law

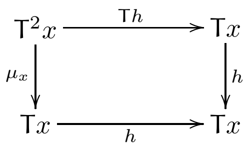
In order to realize how a monad is connected with algebra, one should think of T as being the "signature", in the precise sense of associating to each object $c \in \underline{\mathsf{C}}$ a "free algebra" $\mathsf{T}c$, should think of η as the injection of c in the "free algebra" on it, and should think of μ as providing the interpretation for the operations in $\mathsf{T}c$. In this view, the following definition is self-explaining.

A first connection between monads and adjunctions is obtained by noticing that any adjunction $\eta, \epsilon\colon \mathsf{F} \dashv \mathsf{G}\colon \underline{\mathsf{C}} \rightharpoonup \underline{\mathsf{D}}$ determines a "canonical" monad $(\mathsf{GF}, \eta, \mathsf{G}\epsilon\mathsf{F})$ on $\underline{\mathsf{C}}$.

Definition A.1.15
A T*-algebra is a pair* (x, h)*, where* $x \in \underline{\mathsf{C}}$ *and* $h\colon \mathsf{T}x \to x$ *is a morphism, called the structure map, such that*



Unit



Associativity

commute.

In terms of the algebraic interpretation above, one could rephrase this definition by saying that an algebra for T is an object together with a map h which interprets the "free operations" of $\mathsf{T}x$ in actual operations in x. Of course, while doing this, one must interpret the generators of the "free algebra" in the corresponding "elements" of x, and this is what the unit axiom says, and must "respect" the signature, which is the associativity axiom.

Observe that the "free algebras" are algebras, i.e., for any x we have that $(\mathsf{T}x, \mu_x)$ is a algebra.

DEFINITION A.1.16
A morphism of T*-algebras, or a* T*-homomorphism,* $f\colon (x,h) \to (x',h')$ *is a morphism* $f\colon x \to x'$ *in* $\underline{\mathsf{C}}$ *such that the following diagram commute.*

$$\begin{array}{ccc} \mathsf{T}x & \xrightarrow{\mathsf{T}f} & \mathsf{T}x' \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h'} \\ x & \xrightarrow[f]{} & x' \end{array}$$

Observe that the condition above just says that a morphism in $\underline{\mathsf{C}}$ is a T-homomorphism if it respects the algebraic structure given by T and μ. As in the case of algebra, morphisms of the kind $\mathsf{T}f$ are always T-homomorphisms.

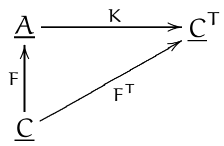
T-algebras and their morphisms define the category $\underline{\mathsf{C}}^{\mathsf{T}}$, which is related to $\underline{\mathsf{C}}$ by an adjunction $\eta^{\mathsf{T}}, \epsilon^{\mathsf{T}} \colon \mathsf{F}^{\mathsf{T}} \dashv \mathsf{G}^{\mathsf{T}} \colon \underline{\mathsf{C}} \rightharpoonup \underline{\mathsf{C}}^{\mathsf{T}}$ defined as follows.

- $$\begin{array}{ccc} \underline{\mathsf{C}} & \xrightarrow{\mathsf{F}^{\mathsf{T}}} & \underline{\mathsf{C}}^{\mathsf{T}} \\ & & \\ x & \longmapsto & (\mathsf{T}x, \mu_x) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathsf{T}f} \\ x' & \longmapsto & (\mathsf{T}x', \mu_{x'}) \end{array}$$
- $$\begin{array}{ccc} \underline{\mathsf{C}}^{\mathsf{T}} & \xrightarrow{\mathsf{G}^{\mathsf{T}}} & \underline{\mathsf{C}} \\ & & \\ (x,h) & \longmapsto & x \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ (x',h') & \longmapsto & x' \end{array}$$
- $\eta^{\mathsf{T}} = \eta$;
- ϵ^{T} is the natural transformation whose element at (x,h) is h.

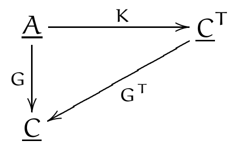
Observe that the monad $(\mathsf{G}^{\mathsf{T}}\mathsf{F}^{\mathsf{T}}, \eta^{\mathsf{T}}, \mathsf{G}^{\mathsf{T}}\epsilon^{\mathsf{T}}\mathsf{F}^{\mathsf{T}})$ determined by the adjunction above is again (T, η, μ).

Thus, an adjunction determines a monad and a monad determines an adjunction. It is easy to see that this correspondence is not one-to-one. In fact, given an adjunction $\mathsf{F} \dashv \mathsf{G} \colon \underline{\mathsf{C}} \rightharpoonup \underline{\mathsf{D}}$, consider the category $\underline{\mathsf{D}}_{\mathsf{F}}$ image of F in $\underline{\mathsf{D}}$. Of course, F and the restriction of G to $\underline{\mathsf{D}}_F$ still form an adjunction which determines exactly the same monad on $\underline{\mathsf{C}}$. However, among the adjunctions which identify a given monad, it is possible to distinguish two best choices, i.e., those which enjoys the two kinds of universal properties. The first of these is the adjunction $\mathsf{F}^{\mathsf{T}} \dashv \mathsf{G}^{\mathsf{T}}$ just given.

Theorem A.1.17
Let $\mathsf{F} \dashv \mathsf{G} \colon \underline{\mathsf{C}} \rightharpoonup \underline{\mathsf{A}}$ *be an adjunction which determines the monad* (T, η, μ)*. Then, there exists a unique* $\mathsf{K} \colon \underline{\mathsf{A}} \to \underline{\mathsf{C}}^{\mathsf{T}}$ *such that the following diagrams commute.*



and



In other words, one could say that $\mathsf{F}^{\mathsf{T}} \dashv \mathsf{G}^{\mathsf{T}}$ *is terminal in the category of the adjunctions* $\mathsf{F} \dashv \mathsf{G}$ *which determines* T*.*

The functor K of theorem above is called *comparison* functor. When K is an isomorphism, one says that F is *monadic*, which means that F is isomorphic to a "category of algebras" construction.

The other canonical adjunction which identifies T is given by the Kleisli category $\underline{\mathsf{C}}_{\mathsf{T}}$ of free algebras for T [70], which is again motivated by algebraic considerations.

A free algebra is characterized by the well-know property that each homomorphism from it to any other algebra is uniquely determined by its behaviour on the generators. Thus, given the monad (T, η, μ) on $\underline{\mathsf{C}}$, the category of "free algebras" for T can be reasonably defined as follows:

Objects: for any $c \in \underline{\mathsf{C}}$ there is a (formal) object c_{T} in $\underline{\mathsf{C}}_{\mathsf{T}}$, representing the free algebra on c;

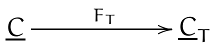
Arrows: for any arrow $f \colon c \to \mathsf{T}c'$ there is a morphism $f^* \colon c_{\mathsf{T}} \to c'_{\mathsf{T}}$; observe that this makes c_{T} a free algebra.

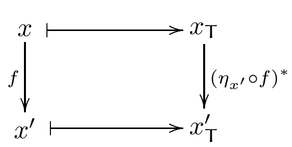
The composition of arrow in $\underline{\mathsf{C}}_{\mathsf{T}}$ is given by the following rule

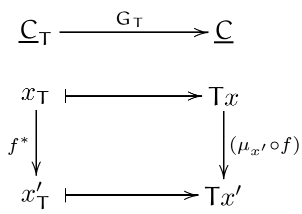
$$x_{\mathsf{T}} \xrightarrow{f^*} y_{\mathsf{T}} \xrightarrow{g^*} z_{\mathsf{T}} \rightsquigarrow \left(x \xrightarrow{f} \mathsf{T}y \xrightarrow{\mathsf{T}g} \mathsf{T}^2 z \xrightarrow{\mu_z} \mathsf{T}z \right)^*.$$

Observe that the identity on x_{T} is η_x^*

As announced, there is an adjuction $\eta_{\mathsf{T}}, \epsilon_{\mathsf{T}} \colon \mathsf{F}_{\mathsf{T}} \dashv \mathsf{G}_{\mathsf{T}} \colon \underline{\mathsf{C}} \rightharpoonup \underline{\mathsf{C}}_{\mathsf{T}}$. Let us give the relevant definitions.

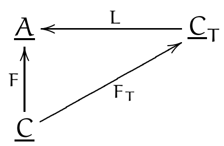


- 

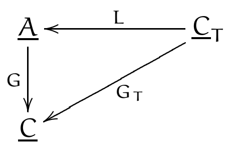
- 
- $\eta_{\mathsf{T}} = \eta$;
- ϵ^{T} is the natural transformation whose element at x_{T} is $id^*_{\mathsf{T}x}$.

Of course,the monad $(\mathsf{G_T F_T}, \eta_{\mathsf{T}}, \mathsf{G_T}\epsilon_{\mathsf{T}}\mathsf{F_T})$ which this adjunction determines is (T, η, μ), and we have the following theorem.

THEOREM A.1.18
Let $\mathsf{F} \dashv \mathsf{G}\colon \underline{\mathsf{C}} \rightharpoonup \underline{\mathsf{A}}$ *be an adjunction which determines the monad* (T, η, μ)*. Then, there exists a unique* $\mathsf{L}\colon \underline{\mathsf{C}}_{\mathsf{T}} \to \underline{\mathsf{A}}$ *such that the following diagrams commute.*

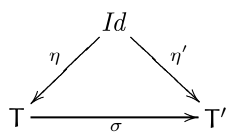


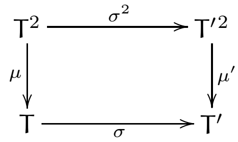
and



which means that $\mathsf{F_T} \dashv \mathsf{G_T}$ *is initial in the category of the adjunctions* $\mathsf{F} \dashv \mathsf{G}$ *which determines* T*.*

We conclude this section by recalling that a natural transformation $\sigma\colon \mathsf{T} \to \mathsf{T}'$ is a morphisms of monads between (T, η, μ) and $(\mathsf{T}', \eta', \mu')$ if the following diagrams commute.





Monad T is a *submonad* of T' is σ is mono.

A.2 Monoidal Categories

A *monoidal category* [3, 21, 90] is a structure $\underline{V} = (\underline{V}_0, \otimes, e, \alpha, \lambda, \rho)$, where

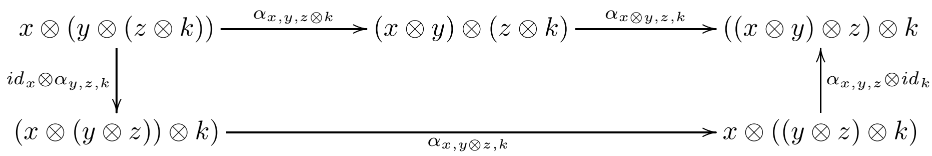
- $\underline{V}_0$ is the underlying category and $e \in \underline{V}$;
- $\otimes\colon \underline{V}_0 \times \underline{V}_0 \to \underline{V}_0$ is a functor;

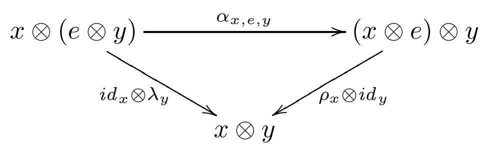
- $\alpha\colon {\_}_1 \otimes ({\_}_2 \otimes {\_}_3) \xrightarrow{\sim} ({\_}_1 \otimes {\_}_2) \otimes {\_}_3$ is "the *associativity*" natural isomorphism;

- $\lambda\colon e \otimes {\_}_1 \xrightarrow{\sim} {\_}_1$ is "the *left unit*" natural isomorphism;

- $\rho\colon {\_}_1 \otimes e \xrightarrow{\sim} {\_}_1$ is "the *right unit*" natural isomorphism;

subject to the Kelly-MacLane *coherence axioms* [87, 62, 67] expressed by the commutativity of the diagrams below.





Such axioms are needed in order to guarantee the coherence of the structural isomorphisms α, λ and ρ. In fact, these are meant to express the usual laws of monoids "up to isomorphism" by giving explicitly the isomorphism, e.g., between $e \otimes x$ and x. It is reasonable then to require that, between two given objects, there is at most one such isomorphism, that is to say that the isomorphisms are "well-given".

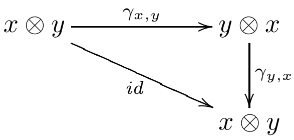
> Theorem. *Every diagram of natural transformations each arrow of which is obtained by repeatedly applying $\otimes$ to "instances" of α, λ, ρ, their inverses and identities, where in turn "instances" means components of the natural transformations at objects of $\underline{V}_0$ obtained by repeated applications of $\otimes$ to e and to "variables", commutes.*

A monoidal category is *strict* if α, λ and ρ are the identity natural transformation, i.e., if $\otimes$ is strictly monoidal. An interesting example of a strict monoidal category is the category of endofunctors on a category $\underline{C}$, the tensor product being given by the composition of functors and horizontal composition of natural transformations.

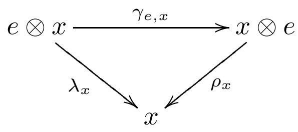
A monoidal category is *symmetric* if it is given a *symmetry* natural isomorphism

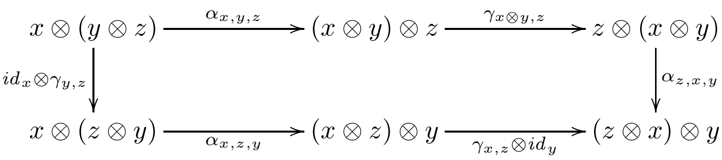
$$\gamma\colon {\_}_1 \otimes {\_}_2 \xrightarrow{\sim} {\_}_2 \otimes {\_}_1$$

satisfying the axioms expressed by the commutativity of the following diagrams.



and





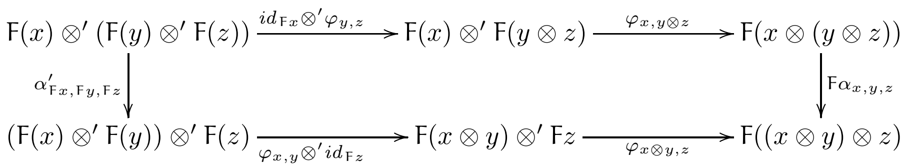
Observe that the first axiom says that the inverse of γ is $\gamma\Delta$, where Δ is the bifunctor which swaps its arguments, which is the coherence of γ wih itself. The other two axioms express the coherence of γ respectively with λ, ρ and α.

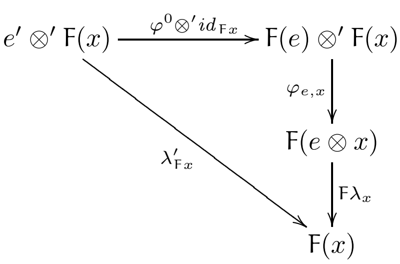
When γ is the identity, $\underline{V}$ is said *strictly symmetric.*

A *monoidal functor* $(\mathsf{F}, \varphi^0, \varphi)\colon \underline{V} \to \underline{V'}$ is given by

- a functor $\mathsf{F}\colon \underline{V}_0 \to \underline{V'}_0$;
- an arrow $\varphi^0\colon e' \to \mathsf{F}(e)$ in $\underline{V'}_0$;
- a natural transformation $\varphi\colon \mathsf{F}(\__1) \otimes' \mathsf{F}(\__2) \stackrel{\cdot}{\to} \mathsf{F}(\__1 \otimes \__2)$;

which make the following diagrams commutative





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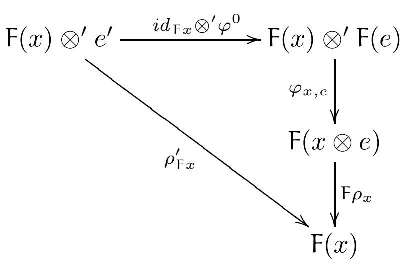


A monoidal functor $(\mathsf{F}, \varphi, \delta)$ is *symmetric* if, in addition, the following commutes.

$$\begin{array}{ccc} \mathsf{F}(x)\otimes'\mathsf{F}(y) & \xrightarrow{\varphi_{x,y}} & \mathsf{F}(x\otimes y)\\ {\scriptstyle \gamma'_{\mathsf{F}x,\mathsf{F}y}}\big\downarrow & & \big\downarrow{\scriptstyle \mathsf{F}\gamma_{x,y}} \\ \mathsf{F}(y)\otimes'\mathsf{F}(x) & \xrightarrow[\varphi_{y,x}]{} & \mathsf{F}(y\otimes x)\end{array}$$

When φ^0 and φ are isomorphisms, respectively identities, then $(\mathsf{F}, \varphi^0, \varphi)$ is called *strong*, respectively *strict*, (symmetric) monoidal functor. Combining strictness and non-strictness conditions, one obtains the category of small and locally small monoidal categories listed in Table A.3

A *monoidal transformation* is a natural transformation $\sigma\colon \mathsf{F} \stackrel{\bullet}{\to} \mathsf{F}'$ such that

$$\begin{array}{ccc} \mathsf{F}(x)\otimes'\mathsf{F}(y) & \xrightarrow{\varphi_{x,y}} & \mathsf{F}(x\otimes y)\\ {\scriptstyle \sigma_x\otimes'\sigma_y}\big\downarrow & & \big\downarrow{\scriptstyle \sigma_{x\otimes y}} \\ \mathsf{F}'(x)\otimes'\mathsf{F}'(y) & \xrightarrow[\varphi'_{x,y}]{} & \mathsf{F}'(x\otimes y)\end{array} \qquad \text{and} \qquad \begin{array}{ccc} \mathsf{F}(e) & \xrightarrow{\sigma_e} & \mathsf{F}'(e)\\ {\scriptstyle \varphi^0}\nwarrow & & \nearrow{\scriptstyle \varphi'^0} \\ & e' & \end{array}$$

Clearly, all the categories $\underline{\mathcal{B}}$ appearing in Table A.3 have a corresponding 2-category $\underline{\underline{\mathcal{B}}}$ obtained by providing $\underline{\mathcal{B}}$ with monoidal transformations as 2-cells. (See, e.g., Table B.)

A.3 2-Categories

A relevant part of the work in category theory has been devoted to the study of an interesting and conceptually relevant generalization of the notion of category, namely the *enriched categories* [63]. The idea is to replace the homsets in a category by objects of a chosen monoidal closed category $\underline{\mathsf{V}}$ [21], giving rise to $\underline{\mathsf{V}}$-categories, or categories enriched over $\underline{\mathsf{V}}$. This idea is supported by the observation that what makes of sets a suitable place where the hom-*objects* can live is their monoidal closed structure, whose tensor product allows the definition of arrow composition.

2-categories [69, 5] are a special case of enriched categories, namely the categories enriched over $\underline{\mathsf{Cat}}$. Of course this choice is a bit special, since $\underline{\mathsf{Cat}}$ is a rather special category which enjoys many properties and thus allows much more than a generic monoidal category. Consequently, the theory of 2-categories has developed on its own way.

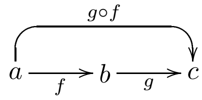
Formally, a 2-category $\underline{\underline{\mathsf{C}}}$ consists of a collection $\{a, b, c, \ldots\}$ of *objects*, or 0-cells, a collection $\{f, g, h, \ldots\}$ of *morphisms*, or 1-cells, and a collection $\{\alpha, \beta, \gamma, \ldots\}$ of *transformations*, or 2-cells. 1-cells are assigned a source and a target 0-cell, written

		small		locally small	
		monoidal	strict monoidal	monoidal	strict monoidal
STRICT	non symmetric	MonCat	sMonCat	MonCAT	sMonCAT
	symmetric	SMonCat	SsMonCat	SMonCAT	SsMonCAT
	strictly symmetric	sSMonCat	sSsMonCat	sSMonCAT	sSsMonCAT
STRONG	non symmetric	MonCat*	sMonCat*	MonCAT*	sMonCAT*
	symmetric	SMonCat*	SsMonCat*	SMonCAT*	SsMonCAT*
	strictly symmetric	sSMonCat*	sSsMonCat*	sSMonCAT*	sSsMonCAT*
MONOIDAL	non symmetric	MonCat**	sMonCat**	MonCAT**	sMonCAT**
	symmetric	SMonCat**	SsMonCat**	SMonCAT**	SsMonCAT**
	strictly symmetric	sSMonCat**	sSsMonCat**	sSMonCAT**	sSsMonCAT**

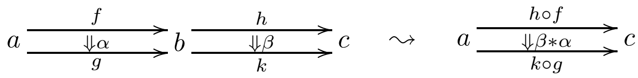
Legenda: The data in the definition of monoidal categories and functors give rise to many combinations according to whether the monoidality and the symmetry are strict or not and so on. To fix notation, we propose the nomenclature above. The idea is that, since we consider the categories with *strict* monoidal functors as the "normal" categories, we explicitly indicate with simple and double superscripted $\star$'s the categories with, respectively, *strong* monoidal functors and simply *monoidal* functors. This is indicated by the leftmost column in the table. Clearly, the categories of symmetric monoidal categories consists always of *symmetric* monoidal functors. Moreover, sS means *strictly symmetric* while sMon means *monoidal strict.* We distinguish between categories of locally small and of small categories by using uppercase letters in the first case. Enriching the categories above with monoidal transformations between monoidal functors, one gets an analogous table for the categories above considered as 2-categories.

Table A.3: A nomenclature for categories of monoidal categories as $f: a \to b$, and 2-cells are assigned a source and a target 1-cell, say f and g, in such a way that $f, g: a \to b$, and this is indicated as $\alpha: f \Rightarrow g: a \to b$, or simply $\alpha: f \Rightarrow g$. Moreover, the following operations are given:

- a partial operation $\_ \circ \_$ of *horizontal composition* of 1-cells, which assignes to each pair $(g: b \to c, f: a \to b)$ a 1-cell $g \circ f: a \to c$;



- a partial operation $\_ * \_$ of *horizontal composition* of 2-cells, which assigns to each pair $(\beta: h \Rightarrow k: b \to c, \alpha: f \Rightarrow g: a \to b)$ a 2-cell $\beta * \alpha: h \circ f \Rightarrow k \circ g: a \to c$.



- a partial operation $\_ \centerdot \_$ of *vertical composition* of 2-cells, which assignes to each pair $(\beta: g \Rightarrow h: a \to b, \alpha: f \Rightarrow g: a \to b)$ a 2-cell $(\beta \centerdot \alpha: f \Rightarrow h: a \to b$.

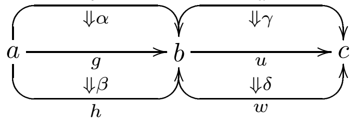
Moreover, to each object a there is an associated identity 1-cell id_a and to each morphism f there is an associated 2-cell identity $\mathbf{1}_f$. These data satisfy the following axioms.

- The objects and the morphisms with the horizontal composition of 1-cells and the identities id_a form a category $\underline{\mathsf{C}}$, called the underlying category of $\underline{\underline{\mathsf{C}}}$;

- For any pair of objects a and b, the morphisms of the kind $f: a \to b$ and their 2-cells form a category under the given operations of vertical composition of 2-cells with the identities $\mathbf{1}_f$;

- the objects and the 2-cells form a category under the operation of horizontal composition of 2-cells with identities $\mathbf{1}_{id_a}$;

- for all $f: a \to b$ and $g: b \to c$, it $\mathbf{1}_g * \mathbf{1}_f = \mathbf{1}_{(g \circ f)}$. Finally, for all the situations of the kind

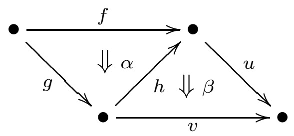


it is $(\delta * \beta) \centerdot (\gamma * \alpha) = (\delta \centerdot \gamma) * (\beta \centerdot \alpha)$.

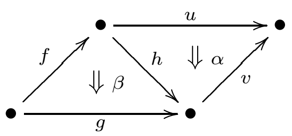
All the above is the long way to say that a 2-category $\underline{\underline{\mathsf{C}}}$ is a category "enriched with morphisms" between the morphisms of fixed source and target, in such a way that each homset becomes a category $\mathsf{Hom}_{\underline{\underline{\mathsf{C}}}}(a,b)$ and, moreover, the composition $\mathsf{Hom}_{\underline{\underline{\mathsf{C}}}}(b,c) \times \mathsf{Hom}_{\underline{\underline{\mathsf{C}}}}(a,b) \to \mathsf{Hom}_{\underline{\underline{\mathsf{C}}}}(a,c)$ is a functor.

<u>NOTATIONS</u>. Very often horizontal composition is denoted just by juxtaposition. Moreover, $\_ * \_$ takes precedence over $\_ \centerdot \_$. Identities 2-cells are almost always denoted by the correspondent 1-cell. Therefore $\mathbf{1}_g * \alpha * \mathbf{1}_f$ is written as $g * \alpha * f$, or even $g\alpha f$.

One of the most used techniques in the theory of 2-categories is the *pasting* of diagrams of 2-cells [5, 69]. The two basic situations are



$$= \quad \beta g \centerdot u\alpha \colon uf \Rightarrow uhg \Rightarrow vg$$



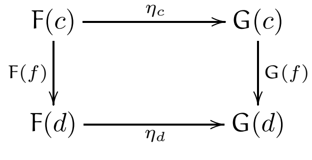
$$= \quad v\alpha \centerdot \beta f \colon uf \Rightarrow vhf \Rightarrow vg.$$

The pasting of a general diagram can be obtained from these two cases by reducing the diagram step by step. It can be shown that the result of the pasting does not depend on the order in which it is reduced, and that is what makes of this a powerful technique.

The notion of 2-functors and 2-natural transformations are the natural extensions of the corresponding one dimensional version when 2-cells are present. Here follow the definitions.

DEFINITION A.3.1
Given the 2-categories $\underline{\underline{\mathsf{C}}}$ *and* $\underline{\underline{\mathsf{D}}}$, *a 2-functor* $\mathsf{F}\colon \underline{\underline{\mathsf{C}}} \to \underline{\underline{\mathsf{D}}}$ *is a function which maps objects to objects, morphisms to morphisms and 2-cells to 2-cells, preserving identities and composition of all kinds.*

DEFINITION A.3.2
Given the 2-functors $\mathsf{F}, \mathsf{G}\colon \underline{\underline{\mathsf{C}}} \to \underline{\underline{\mathsf{D}}}$, *a 2-natural transformation* $\eta\colon \mathsf{F} \stackrel{\cdot}{\to} \mathsf{G}$ *is a* $\underline{\underline{\mathsf{C}}}$ *indexed families of arrows* $\eta_c\colon \mathsf{F}(c) \to \mathsf{G}(c)$ *in* $\underline{\underline{\mathsf{D}}}$ *such that for any* $f\colon c \to d$ *in* $\underline{\underline{\mathsf{C}}}$, *the following diagram commutes*



and, for any $\alpha\colon f \to g$ *in* $\underline{\underline{\mathsf{C}}}$,

$$\mathsf{F}(c) \begin{array}{c}\xrightarrow{\mathsf{F}(f)}\\ \Downarrow\mathsf{F}(\alpha)\\ \xrightarrow[\mathsf{F}(g)]{}\end{array} \mathsf{F}(d) \xrightarrow{\eta_d} \mathsf{G}(d) \quad = \quad \mathsf{F}(c) \xrightarrow{\eta_c} \mathsf{G}(c) \begin{array}{c}\xrightarrow{\mathsf{G}(f)}\\ \Downarrow\mathsf{G}(\alpha)\\ \xrightarrow[\mathsf{G}(g)]{}\end{array} \mathsf{G}(d)$$

It is a general fact from the theory of enriched categories that, for any pair of $\underline{\mathsf{V}}$-functors, the $\underline{\mathsf{V}}$-natural transformations between them form an object of $\underline{\mathsf{V}}$. Since in the case of 2-categories $\underline{\mathsf{V}}$ is $\underline{\mathfrak{Cat}}$, we have that the 2-natural transformations $\mathsf{F} \stackrel{\bullet}{\to} \mathsf{G}$ are a category. In other words, the category of the small 2-categories is a 3-category. It also follows that the category of functors between two 2-categories is still a 2-category We state the notion of "morphism" between natural transformations.

DEFINITION A.3.3

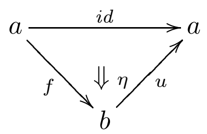
Given the 2-natural transformations $\eta, \sigma\colon \mathsf{F} \to \mathsf{G}$, *a modification* $\rho\colon \eta \Rrightarrow \sigma$ *is a* $\underline{\underline{\mathsf{C}}}$ *indexed family of 2-cells* $\rho_c\colon \eta_c \Rightarrow \sigma_c\colon \mathsf{F}(c) \to \mathsf{G}(c)$ *in* $\underline{\underline{\mathsf{D}}}$, *such that, for any* $f\colon c \to d$ *in* $\underline{\underline{\mathsf{C}}}$, *it is*

$$\mathsf{F}(c) \xrightarrow{\mathsf{F}(f)} \mathsf{F}(d) \begin{array}{c}\xrightarrow{\eta_d}\\ \Downarrow\rho_d\\ \xrightarrow[\sigma_d]{}\end{array} \mathsf{G}(d) \quad = \quad \mathsf{F}(c) \begin{array}{c}\xrightarrow{\eta_c}\\ \Downarrow\rho_c\\ \xrightarrow[\sigma_c]{}\end{array} \mathsf{G}(c) \xrightarrow{\mathsf{G}(f)} \mathsf{G}(d)$$

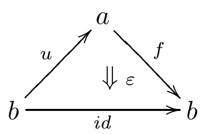
We conclude this section by recalling the definition of adjunction in a 2-category, which the usual definition of adjunction in $\underline{\mathfrak{Cat}}$ is a particular case of.

DEFINITION A.3.4

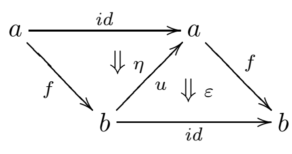
An adjunction $\eta, \varepsilon\colon f \dashv u\colon a \to b$ *in* $\underline{\underline{\mathsf{C}}}$ *is a pair of 1-cells* $f\colon a \to b$ *and* $u\colon b \to a$ *together with two cells*



and

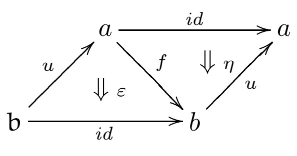


called respectively unit and counit of the adjunction, which are inverse to each other wrt. the two possible pastings of them, i.e.,



$$= \quad \mathbf{1}_f$$

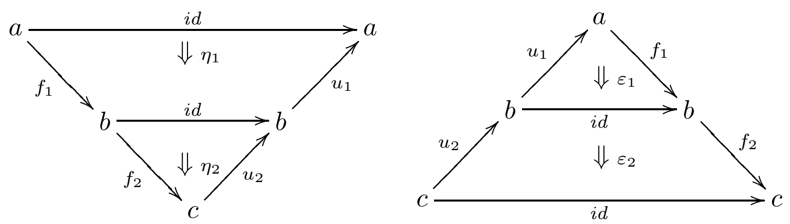
and



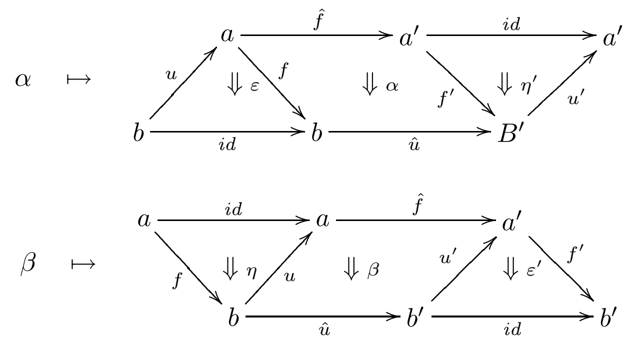
$$= \quad \mathbf{1}_u$$

which are called the triangular equalities of the adjunction.

Clearly, adjunctions in $\underline{\underline{C}}$ can be composed and, actually, they form a category whose identities are the adjunctions $\mathbf{1}, \mathbf{1}\colon id_a \dashv id_a\colon a \to a$. Let $\eta_1, \varepsilon_1\colon f_1 \dashv u_1\colon a \to b$ and $\eta_2, \varepsilon_2\colon f_2 \dashv u_2\colon b \to c$ be adjunctions. Their composite is $\eta, \varepsilon\colon f_2 f_1 \dashv u_1 u_2\colon a \to c$ where η and ε are given by the pastings of diagrams below.



A relevant notion linked to adjunction is that of 2-cells *mates under adjunctions*. Given the adjunctions $\eta, \varepsilon\colon f \dashv u\colon a \to b$ and $\eta', \varepsilon'\colon f' \dashv u'\colon a' \to b'$, and given the 1-cells $\hat{f}\colon a \to a'$ and $\hat{u}\colon b \to b'$, there is a bijection between the 2-cells $\alpha\colon f' \circ \hat{f} \Rightarrow \hat{u} \circ f$ and $\beta\colon \hat{f} \circ u \Rightarrow u' \circ \hat{u}$ given by the following correspondence:



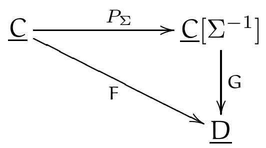
In other words, the mate of a 2-cell is obtained by pasting the appropriate unit and counit at its ends. The interesting fact is that the bijection above respects composition, both horizontal and vertical. Correspondent α and β are said to be *mates* under the adjunctions $f \dashv u$ and $f' \dashv u'$ (wrt. $\hat{f}$ and $\hat{u}$).

A.4 Categories of Fractions

An important technique in many fields of mathematics is "taking the quotient", which, generally speaking, permits to reduce an "object" by identifying some of "its elements". The kind of device one needs in order to be able to quotient an object of course depends on the structure on the object itself. For instance, in set theory one uses *equivalence relations*, while an algebra is quotiented via *congruences*.

There is also an obvious way to quotient *categories*, namely, instead of considering equivalence classes of objects and/or morphisms, leave the categorical structure do the work for you, and just rend *isomorphic* some objects. Observe that this is completely in tune with category theory, where the individuality of the objects is given by the morphisms.

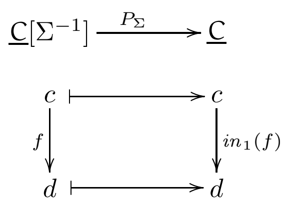
Given a category $\underline{\mathsf{C}}$ we say that a functor $\mathsf{F}\colon \underline{\mathsf{C}} \to \underline{\mathsf{D}}$ makes a morphism f in $\underline{\mathsf{C}}$ invertible, or simply inverts f, if $\mathsf{F}(f)$ is invertible. If Σ is a class of arrows $\underline{\mathsf{C}}$, $\underline{\mathsf{C}}[\Sigma^{-1}]$, the category of fractions [29, 127] of $\underline{\mathsf{C}}$ for Σ, is a category with a functor $P_\Sigma\colon \underline{\mathsf{C}} \to \underline{\mathsf{C}}[\Sigma^{-1}]$ which is universal among the functors which make all the morphisms in Σ invertible. In explicit terms, this means that for any $\mathsf{F}\colon \underline{\mathsf{C}} \to \underline{\mathsf{D}}$ which inverts Σ there is a *unique* functor $\mathsf{G}\colon \underline{\mathsf{C}}[\Sigma^{-1}] \to \underline{\mathsf{D}}$ which makes the following diagram commutative.



If $\underline{\mathsf{C}}$ is locally small then $\underline{\mathsf{C}}[\Sigma^{-1}]$ exists for any Σ and it is given by the following construction.

> Let A be the disjoint union of the arrows of $\underline{\mathsf{C}}$ and Σ, and let in_1 an in_2 denote the respective injections. Let G be the (large) graph whose nodes are the objects of $\underline{\mathsf{C}}$, whose class of arcs is A and where the source and target relations are given as follows
>
> $$\begin{array}{lcl} f\colon c \to d \text{ in } \underline{\mathsf{C}} & \Rightarrow & c \stackrel{in_1(f)}{\longrightarrow} d \text{ in } G \\ f\colon c \to d \text{ in } \Sigma & \Rightarrow & d \stackrel{in_2(f)}{\longrightarrow} c \text{ in } G \end{array}$$
>
> Then, $\underline{\mathsf{C}}[\Sigma^{-1}]$ is the category freely generated from G modulo the following equations:
>
> $$\begin{array}{l} in_1(g) \circ in_1(f) = in_1(g \circ f) \quad \text{and} \quad id_c = in_1(id_c); \\ in_1(f) \circ in_2(f) = id = in_2(f) \circ in_1(f) \end{array}$$

Now, P_Σ is given by



This makes evident that P_Σ inverts Σ. Consider now any $\mathsf{F}: \underline{\mathsf{C}} \to \underline{\mathsf{D}}$ which inverts Σ. Define $\mathsf{G}: \underline{\mathsf{C}}[\Sigma^{-1}] \to \underline{\mathsf{D}}$ as follows

$$\begin{array}{ccc} c & \longmapsto & \mathsf{F}(c) \\ {\scriptstyle in_1(f)}\downarrow & & \downarrow{\scriptstyle \mathsf{F}(f)} \\ d & \longmapsto & \mathsf{F}(d) \end{array} \quad \text{and} \quad \begin{array}{ccc} c & \longmapsto & \mathsf{F}(c) \\ {\scriptstyle in_2(f)}\downarrow & & \downarrow{\scriptstyle \mathsf{F}(f)^{-1}} \\ d & \longmapsto & \mathsf{F}(d) \end{array}$$

which immediately gives $\mathsf{G}P_\Sigma = \mathsf{F}$. If G' has the same property, then it coincides with G (and with F) on the objects. Moreover, $\mathsf{G}'(in_1(f)) = \mathsf{F}(f) = \mathsf{G}(in_1(f))$, while, by definition of functor $\mathsf{G}'(in_2(f)) = \mathsf{G}'(in_2(f)^{-1})^{-1} = \mathsf{G}'(in_1(f))^{-1} = \mathsf{G}(in_1(f))^{-1} = \mathsf{G}(in_2(f)^{-1})$, i.e., $\mathsf{G}' = \mathsf{G}$.

Of course, the morphisms in Σ will not necessarily be the unique morphisms inverted by P_Σ. The class of morphisms f such that $P_\Sigma(f)$ is invertible, is called the *saturation* of Σ.

Example A.4.1

Consider the monoid $\mathbb{N}^*$ of the non-zero natural numbers with the usual product, viewed as a category, and take Σ to be the set of primes. Then, $\underline{\mathbb{N}^*}[\Sigma^{-1}]$ is $\mathbb{Q}$, the group of the non-zero rational numbers, and the saturation of Σ is $\mathbb{N}^*$ itself.

The following is an interesting fact which relates categories of fractions, reflections and coreflections. The reader is referred to [29] for the proof.

Proposition A.4.2

Let $\mathsf{F} \dashv \mathsf{G}: \underline{\mathsf{C}} \rightharpoonup \underline{\mathsf{D}}$ be an adjunction. Consider $\Sigma_\mathsf{F} = \{f \text{ in } \underline{\mathsf{C}} \mid \mathsf{F}(f) \text{ invertible}\}$ and $\Sigma_\mathsf{G} = \{f \text{ in } \underline{\mathsf{D}} \mid \mathsf{G}(f) \text{ invertible}\}$. Then

i) $\mathsf{F} \dashv \mathsf{G}$ *is a reflection* $\Leftrightarrow$ $\mathsf{H}: \underline{\mathsf{C}}[\Sigma_\mathsf{F}^{-1}] \to \underline{\mathsf{D}}$ *s.t.* $\mathsf{H} \circ P_{\Sigma_\mathsf{F}} = \mathsf{F}$ *is an equivalence;*

ii) $\mathsf{F} \dashv \mathsf{G}$ *is a coreflection* $\Leftrightarrow$ $\mathsf{H}: \underline{\mathsf{D}}[\Sigma_\mathsf{G}^{-1}] \to \underline{\mathsf{C}}$ *s.t.* $\mathsf{H} \circ P_{\Sigma_\mathsf{G}} = \mathsf{G}$ *is an equivalence;*

Although the description of $\underline{\mathsf{C}}[\Sigma^{-1}]$ is very simple, managing its morphisms can sometime be awkward. However, there is a particularly happy case in which such morphisms can be characterized very smoothly.

We say that Σ admits a *calculus of left fractions* if

i) All the identities of $\underline{\mathsf{C}}$ belong to Σ;

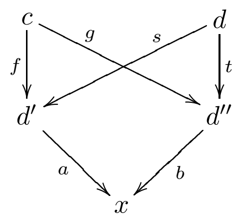
ii) If $s: x \to y$ and $t: y \to z$ are in Σ, then $t \circ s$ belongs to Σ.

iii) For any $\begin{array}{cc} x & \xrightarrow{f} y \\ {\scriptstyle s}\downarrow & \\ x' & \end{array}$ with $s \in \Sigma$, there exists a commutative $\begin{array}{ccc} x & \xrightarrow{f} & y \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle t} \\ x' & \xrightarrow[g]{} & y' \end{array}$ with $t \in \Sigma$;

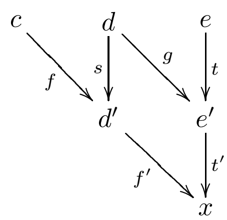
iv) For any $x \xrightarrow{s} y \underset{g}{\overset{f}{\rightrightarrows}} z$, $s \in \Sigma$ and $f \circ s = g \circ s$, there exists $y \underset{g}{\overset{f}{\rightrightarrows}} z \xrightarrow{t} y'$ with $t \in \Sigma$ and $t \circ f = t \circ g$.

Suppose that Σ admits a calculus of left fractions and for any $c, d \in \underline{C}$ consider the set $\mathbf{H}(c, d)$ of pairs of morphisms (s, f) with $f : c \to d'$ in $\underline{C}$ and $s : d \to d'$ in Σ. Then, define the binary relation $\sim$ on $\mathbf{H}(c, d)$ as follows:

$(s, f) \sim (t, g)$ if there exist $d' \xrightarrow{a} x$ $d'' \xrightarrow{b} x$ such that $a \circ f,\ b \circ t \in \Sigma$

and  commutes.

Thanks to the fact that Σ admits a calculus of left fractions, it is very easy to see that $\sim$ is an *equivalence* relation on $\mathbf{H}(c, d)$. Consider $(s, f) \in \mathbf{H}(c, d)$ and $(t, g) \in \mathbf{H}(d, e)$. By points *(ii)* and *(iii)* above, we can push it out to an element $(t' \circ t, f' \circ f) \in \mathbf{H}(c, e)$, as shown by the diagram below.



(A.11)

Exploiting again the four properties of Σ, one can shown quite easily that a different choice of $f' : d' \to x$ and $t' : e' \to x$ in the diagram (A.11) would give a pair $\sim$-equivalent to $(t' \circ t, f' \circ f)$ in $\mathbf{H}(c, e)$. Moreover, it is again elementary to verify that replacing (s, f) and (g, t) by $\sim$-equivalent pairs, does not affect the equivalence class of the resulting pair, which remains $[(t' \circ t, f' \circ f)]_\sim$. Thus, for any triple $c, d, e \in \underline{C}$, we have a well-defined function $* : \mathbf{H}(d, e)/\sim \times \mathbf{H}(c, d)/\sim \to \mathbf{H}(c, e)/\sim$, which, denoting with $s|f$ the $\sim$-equivalence class of (s, f), can be expressed in elementary terms as $t|g * s|f = (t' \circ t)|(f' \circ f)$ for some f' in $\underline{C}$ and $t' \in \Sigma$ for which the diagram (A.11) commutes.

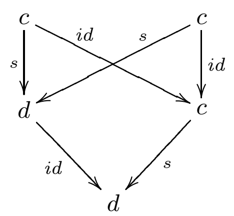
It is now just a matter of a few calculations to see that $\_ * \_$ is associative and that the elements $id_c|id_c$ behaves as identities. Therefore, we can consider the category $\Sigma^{-1}\underline{C}$ whose objects are those of $\underline{C}$ and whose homsets are the quotient sets $\mathbf{H}(c, d)/\sim$.

PROPOSITION A.4.3

Suppose that Σ admits a calculus of left fractions. Then $\Sigma^{-1}\underline{C} \cong \underline{C}[\Sigma^{-1}]$.

Proof. It is enough to see that there exists $\Sigma P: \underline{C} \to \Sigma^{-1}\underline{C}$ which inverts Σ and enjoys the same universal property as $\underline{C}[\Sigma^{-1}]$. To this aim, consider ΣP which acts as the identity on the objects and sends $f: c \to d$ in $\underline{C}$ to $id_d|f$ in $\Sigma^{-1}\underline{C}$.

Then, concerning the first point, consider the following diagram, which shows that, for any $s \in \Sigma$, it is $s|s = id|id$.



Then, it is easy to see that $s|id$ is the inverse of $id|s$. In fact, considering the appropriate diagrams of the kind of diagram (A.11), we obviously have $s|id * id|s = s|s$ and $id|s * s|id = id|id$.

Let now $\mathsf{F}: \underline{C} \to \underline{D}$ be a functor which inverts Σ. Then, $\mathsf{G}: \Sigma^{-1}\underline{C} \to \underline{D}$ such that $\mathsf{G}(c) = \mathsf{F}(c)$ and $\mathsf{G}(s|f) = \mathsf{F}(s)^{-1} \circ \mathsf{F}(f)$ is the unique functor such that $\mathsf{G} \circ \Sigma P = \mathsf{F}$. ✓

EXAMPLE A.4.4

Let us consider again the situation of Example A.4.1, but with $\Sigma = \mathbb{N}^*$. Then, we can look at the pairs (n, m) as fractions. Then $n|m$ is the rational number represented by (n, m). In fact, it is easy to see that $(n, m) \sim (p, q)$ if and only if $nq = pm$, i.e., if and only if $n/m = p/q$.

Glossary

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