Axiomatizing Petri Net Concatenable Processes

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Abstract. The concatenable processes of a Petri net N can be characterized abstractly as the arrows of a symmetric monoidal category $\mathcal{P}[N]$. Yet, this is only a partial axiomatization, since $\mathcal{P}[N]$ is built on a concrete, ad hoc chosen, category of symmetries. In this paper we give a fully equational description of the category of concatenable processes of N, thus yielding an axiomatic theory of the noninterleaving behaviour of Petri nets.

Introduction

Concatenable processes of Petri nets have been introduced in [3] to account, as their name indicates, for the issue of process concatenation. Let us briefly reconsider the ideas which led to their definition.

The development of theory Petri nets, focusing on the noninterleaving aspects of concurrency, brought to the foreground various notions of process, e.g. [14, 5, 2, 12, 3]. Generally speaking, Petri net processes—whose standard version is given by the Goltz-Reisig *non-sequential processes* [5]—are structures needed to account for the *causal relationships* which rule the occurrence of events in computations. Thus, ideally, processes are simply computations in which explicit information about such causal connections is added. More precisely, since it is a well-established idea that, as far as the theory of computation is concerned, causality can be faithfully described by means of partial orderings—though interesting 'heretic' ideas appear sometimes—abstractly, the processes of a net N are ordered sets whose elements are labelled by transitions of N. Concretely, in order to describe exactly which multisets of transitions are processes, one defines a process of N to be a map $\pi\colon \Theta \to N$ which maps transitions to transitions and places to places respecting the 'bipartite graph structure' of nets. Here Θ is a *finite deterministic occurrence net*, i.e., roughly speaking, a finite conflict-free 1-safe acyclic net such that the minimal and maximal elements of the partial ordering $\preccurlyeq$ naturally induced by the 'flow relation' on the elements of Θ are places. The role of π is to 'label' the places and the (partially ordered) transitions of Θ with places and transitions of N compatibly with the structure of N.

Given this definition, one can assign the natural *source* and *target* states to a process $\pi\colon \Theta \to N$ by considering the multisets of places of N which are the image via π of, respectively, the minimal and maximal (wrt. $\preccurlyeq$) places of Θ. Now, the simple minded attempt to concatenate a process $\pi_1\colon \Theta_1 \to N$ with source u to a process $\pi_0\colon \Theta_0 \to N$ with target u by merging the maximal places of Θ_0 with the minimal places of Θ_1 in a way which preserves the labellings fails immediately. In fact, if more than one place of u is labelled by a single place of N, there are many ways to put in one-to-one correspondence the maximal places of Θ_0

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and the minimal places of Θ_1 preserving the labels, i.e., there are many possible concatenations of π_0 and π_1, each of which gives a possibly different process of N. In other words, as the above argument shows, process concatenation has to do with *merging tokens*, i.e., instances of places, rather than *merging places*.

Therefore, any attempt to deal with process concatenation must disambiguate the *identity* of each token in a process. This is exactly the idea of *concatenable processes*, which are simply Goltz-Reisig processes in which the minimal and maximal places carrying the same label are linearly ordered. This yields immediately an operation of concatenation, since the ambiguity about the identity of tokens is resolved using the additional information given by the orderings. Moreover, the existence of concatenation leads easily to the definition of the category of concatenable processes of N. It turns out that such a category is a *symmetric monoidal category* whose tensor product is provided by the parallel composition of processes [3]. The relevance of this result is that it describes Petri net behaviours as *algebras* in a remarkably smooth way.

Naturally linked to the fact that they are algebraic structures, concatenable processes are amenable to abstract descriptions. In [3] the authors deal with this by associating to each net N a symmetric monoidal category $\mathcal{P}[N]$ isomorphic to the category of concatenable processes of N; such a characterization, however, is not completely abstract and it provides only a partial axiomatization of the algebra of concatenable processes of N, since in the cited work $\mathcal{P}[N]$ is built on a concrete, ad hoc constructed, category Sym_N. In this paper we show that Sym_N can be characterized abstractly, thus yielding a *purely algebraic* and *completely abstract* axiomatization of the category of concatenable processes of N. Namely, we shall prove that $\mathcal{P}[N]$ is the *free symmetric strict monoidal* category on the net N modulo two simple additional axioms.[1] This result complements the investigation of [3] on the structure of net computations by showing that they can be described by an *essentially algebraic theory* (whose models are symmetric monoidal categories), which, in our opinion, is a remarkable fact. In addition, our axiomatization of $\mathcal{P}[N]$ naturally provides a *term algebra* and an *equational theory* of concatenable processes of N, by means of which one can 'compute' with and 'reason' about them. The relevance of this is evident when one thinks of N as modelling a complex system whose behaviour is to be analysed.

Concerning the organization of the paper, Section 1 recalls the needed definitions; the reader acquainted with [12, 3] and with monoidal categories can safely skip it. In Section 2 we sketch the proof of our result. The present paper intends to be an extended abstract; therefore, most of the proofs are omitted and those remaining are just sketched. Full expositions can be found in [15, 16].

1 Background

The notion of *monoidal category* dates back to [1] (see [11] for an easy thorough introduction and [4] for advanced topics). In this paper we shall be concerned only with a particular kind of symmetric monoidal categories, name-

[1] We remark that the existence of a similar axiomatization was conjectured also in [6].

ly those which are *strict monoidal* and whose objects form a *free commutative monoid*. Remarkably, a very similar kind of categories have appeared as distinguished algebraic structures also in [10], where they are called PROP's (for Product and Permutation categories), and in [8].

A *symmetric strict monoidal category* (SSMC in the following) is a structure $(\underline{\mathsf{C}}, \otimes, e, \gamma)$, where $\underline{\mathsf{C}}$ is a category, e is an object of $\underline{\mathsf{C}}$, called the *unit* object, $\otimes : \underline{\mathsf{C}} \times \underline{\mathsf{C}} \to \underline{\mathsf{C}}$ is a functor, called the *tensor product*, subject to the equations

$$\otimes \circ \langle \otimes \times 1_{\underline{\mathsf{C}}} \rangle = \otimes \circ \langle 1_{\underline{\mathsf{C}}} \times \otimes \rangle, \quad (1)$$

$$\otimes \circ \langle \underline{e}, 1_{\underline{\mathsf{C}}} \rangle = 1_{\underline{\mathsf{C}}}, \quad (2)$$

$$\otimes \circ \langle 1_{\underline{\mathsf{C}}}, \underline{e} \rangle = 1_{\underline{\mathsf{C}}}, \quad (3)$$

where $\underline{e} : \underline{\mathsf{C}} \to \underline{\mathsf{C}}$ is the constant functor which associate e and id_e respectively to each object and each morphism of $\underline{\mathsf{C}}$, $\langle \_, \_ \rangle$ is the pairing of functors induced by the cartesian product, and $\gamma : \_1 \otimes \_2 \xrightarrow{\sim} \_2 \otimes \_1$ is a natural isomorphism, called the *symmetry* of $\underline{\mathsf{C}}$, subject to the Kelly-MacLane *coherence axioms* [9, 7]:

$$(\gamma_{x,z} \otimes id_y) \circ (id_x \otimes \gamma_{y,z}) = \gamma_{x \otimes y, z}, \quad (4)$$

$$\gamma_{y,x} \circ \gamma_{x,y} = id_{x \otimes y}. \quad (5)$$

Equation (1) states that the tensor is associative on both objects and arrows, while (2) and (3) state that e and id_e are, respectively, the unit object and the unit arrow for $\otimes$. Concerning the coherence axioms, axiom (5) says that $\gamma_{y,x}$ is the inverse of $\gamma_{x,y}$, while (4), the *real key* of symmetric monoidal categories, links the symmetry at composed objects to the symmetry at the components. A *symmetry* s in a symmetric monoidal category $\underline{\mathsf{C}}$ is any arrow obtained as composition and tensor of *identities* and *components* of γ. We use $Sym_{\underline{\mathsf{C}}}$ to denote the subcategory of the symmetries of $\underline{\mathsf{C}}$.

A *symmetric strict monoidal functor* from $(\underline{\mathsf{C}}, \otimes, e, \gamma)$ to $(\underline{\mathsf{D}}, \otimes', e', \gamma')$, is a functor $\mathsf{F} : \underline{\mathsf{C}} \to \underline{\mathsf{D}}$ such that

$$\mathsf{F}(e) = e', \quad (6)$$

$$\mathsf{F}(x \otimes y) = \mathsf{F}(x) \otimes' \mathsf{F}(y), \quad (7)$$

$$\mathsf{F}(\gamma_{x,y}) = \gamma'_{\mathsf{F}x, \mathsf{F}y}. \quad (8)$$

Let $\underline{\mathsf{SSMC}}$ be the category of SSMC's and symmetric strict monoidal functors and let $\underline{\mathsf{SSMC}}^{\oplus}$ be the full subcategory consisting of the monoidal categories whose objects form *free* commutative monoids.

Next, we recall the definition of Petri nets formulated in [12].

Notation. We denote by $S^{\oplus}$ the *free commutative monoid* on S, i.e., the monoid of *finite multisets* of S. Recall that a finite multiset is a functions from S to ω which yields nonzero values at most on finitely many elements of S. As usual, we represent $u \in S^{\oplus}$ as a formal sum $\bigoplus_{i \in I} u(a_i) \cdot a_i$ where only the $a_i \in S$ such that $u(a_i) > 0$ appear.

A *Petri net* is a structure $N = (\partial_N^0, \partial_N^1 : T_N \to S_N^{\oplus})$, where T_N is a set of *transitions*, S_N is a set of *places*, and ∂_N^0 and ∂_N^1 are functions assigning to each transition, respectively, a source and a target multiset. A *morphism* of PT nets

$f\colon N_0 \to N_1$ is a pair $\langle f_t, f_p\rangle$, where $f_t\colon T_{N_0} \to T_{N_1}$, the transition component, is a *function* and $f_p\colon S^{\oplus}_{N_0} \to S^{\oplus}_{N_1}$, the place component, is a *monoid homomorphism* which respect source and target, i.e., the two diagrams below commute.

$$
\begin{array}{ccc}
T_{N_0} & \xrightarrow{\partial^0_{N_0}} & S^{\oplus}_{N_0} \\
{\scriptstyle f_t}\downarrow & & \downarrow{\scriptstyle f_p} \\
T_{N_1} & \xrightarrow[\partial^0_{N_1}]{} & S^{\oplus}_{N_1}
\end{array}
\qquad\qquad
\begin{array}{ccc}
T_{N_0} & \xrightarrow{\partial^1_{N_0}} & S^{\oplus}_{N_0} \\
{\scriptstyle f_t}\downarrow & & \downarrow{\scriptstyle f_p} \\
T_{N_1} & \xrightarrow[\partial^1_{N_1}]{} & S^{\oplus}_{N_1}
\end{array}
$$

The data above define the category <u>Petri</u> of PT nets.

Let N be a net. We recall now the construction of the symmetric strict monoidal category $\mathcal{P}[N]$. We start by introducing the *vectors of permutations* (*vperms*) of N,[2] which from the categorical viewpoint play the role of the symmetry isomorphism for $\mathcal{P}[N]$.

Notation. We denote by $\Pi(n)$ the group of permutations of n elements and we write $|\sigma| = n$ when $\sigma \in \Pi(n)$. To simplify notation, we shall assume that the empty function $\varnothing\colon \varnothing \to \varnothing$ is the (unique) permutation of zero elements.

For $u \in S^{\oplus}$, a *vperm* $s\colon u \to u$ is a function which assigns to each $a \in S$ a permutation $s(a) \in \Pi(u(a))$. Given $u = n_1 \cdot a_1 \oplus \ldots \oplus n_k \cdot a_k$ in $S^{\oplus}_N$, we shall represent a vperm s on u as a vector of permutations, $\langle \sigma_{a_1}, \ldots, \sigma_{a_k}\rangle$, where $s(a_j) = \sigma_{a_j}$, whence their name. One can define the operations of sequential and parallel composition of vperms, so that they can be organized as the arrows of a SSMC. The details follow (see also Figure 1).

Given the vperms $s = \langle \sigma_{a_1}, \ldots, \sigma_{a_k}\rangle\colon u \to u$ and $s' = \langle \sigma'_{a_1}, \ldots, \sigma'_{a_k}\rangle\colon u \to u$ their *sequential composition* $s;s'\colon u \to u$ is the vperm $\langle \sigma_{a_1};\sigma'_{a_1}, \ldots, \sigma_{a_k};\sigma'_{a_k}\rangle$, where $\sigma;\sigma'$ is the composition of permutation which we write in the diagrammatic order from left to right. Given the vperms $s = \langle \sigma_{a_1}, \ldots, \sigma_{a_k}\rangle\colon u \to u$ and $s' = \langle \sigma'_{a_1}, \ldots, \sigma'_{a_k}\rangle\colon v \to v$ (where possibly $\sigma_{a_j} = \varnothing$ for some j), their *parallel composition* $s \otimes s'\colon u \oplus v \to u \oplus v$ is the vperm $\langle \sigma_{a_1} \otimes \sigma'_{a_1}, \ldots, \sigma_{a_k} \otimes \sigma'_{a_k}\rangle$, where

$$
(\sigma \otimes \sigma')(x) = \begin{cases} \sigma(x), & \text{if } 0 < x \le |\sigma|, \\ \sigma'(x - |\sigma|) + |\sigma|, & \text{if } |\sigma| < x \le |\sigma| + |\sigma'|. \end{cases}
$$

Let γ be $\{1 \leftrightarrows 2\} \in \Pi(2)$ and consider $u_i = n^i_1 \cdot a_1 \oplus \ldots \oplus n^i_k \cdot a_k$, $i = 1, 2$, in $S^{\oplus}$, the *interchange vperm* $\gamma(u_1, u_2)$ is the vperm $\langle \sigma_{a_1}, \ldots, \sigma_{a_k}\rangle\colon u_1 \oplus u_2 \to u_1 \oplus u_2$ where

$$
\sigma_{a_j}(x) = \begin{cases} x + n^2_j, & \text{if } 0 < x \le n^1_j, \\ x - n^1_j, & \text{if } n^1_j < x \le n^1_j + n^2_j. \end{cases}
$$

It is now immediate to see that $\_;\_$ is associative. Moreover, for each $u \in S^{\oplus}$ the vperm $u = \langle id_{a_1}, \ldots, id_{a_n}\rangle\colon u \to u$, where id_{a_j} is the identity permutation, is an identity for sequential composition. Let 0 be the empty multiset on S. Then, the vperm $s\colon 0 \to 0$ is a unit for parallel composition. Now, given a net N, let

[2] Vperms are called *symmetries* in [3]. Here, in order to avoid confusion with the general notion of symmetry in a symmetric monoidal category, we prefer to use another term.

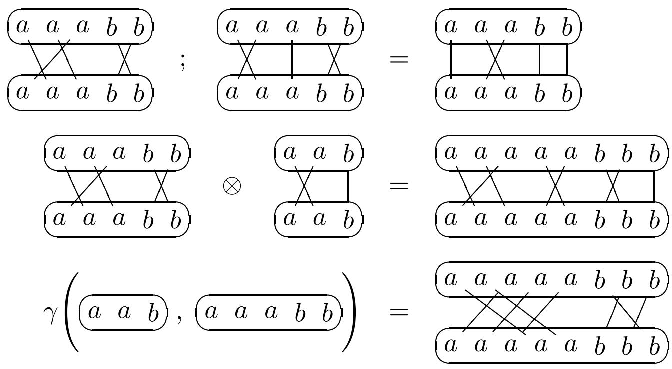

Figure 1: The monoidal structure of vperms

Sym_N be the category whose objects are the elements of $S_N^{\oplus}$ and whose arrows are the vperms $s: u \to u$ for $u \in S_N^{\oplus}$. Then, it is easy to show that Sym_N is a SSMC with respect to the given composition and tensor product, with identities and unit element as explained above and the symmetry natural isomorphism given by the collection $\gamma = \{\gamma(u, v)\}_{u,v \in Sym_N}$ of the interchange vperms.

We can now define $\mathcal{P}[N]$ as the category which includes Sym_N as subcategory and has as additional arrows those defined by the following inference rules:

$$\frac{t: u \to v \text{ in } T_N}{t: u \to v \text{ in } \mathcal{P}[N]}$$

$$\frac{\alpha: u \to v \text{ and } \beta: u' \to v' \text{ in } \mathcal{P}[N]}{\alpha \otimes \beta: u \oplus u' \to v \oplus v' \text{ in } \mathcal{P}[N]} \qquad \frac{\alpha: u \to v \text{ and } \beta: v \to w \text{ in } \mathcal{P}[N]}{\alpha; \beta: u \to w \text{ in } \mathcal{P}[N]}$$

plus axioms expressing the fact that $\mathcal{P}[N]$ is a SSMC with composition $\_;\_$, tensor $\_\otimes\_$ (extending those already defined on vperms) and symmetry isomorphism γ, and the following axioms

$$\begin{array}{ll} t; s = t, & \text{where } t: u \to v \text{ in } T_N \text{ and } s: v \to v \text{ in } Sym_N, \\ s; t = t, & \text{where } t: u \to v \text{ in } T_N \text{ and } s: u \to u \text{ in } Sym_N. \end{array} \qquad (\Psi)$$

In other words, $\mathcal{P}[N]$ is built on the category Sym_N by adding the transitions of N and freely closing with respect to sequential and parallel composition of arrows, so that $\mathcal{P}[N]$ is made symmetric strict monoidal and the axioms (Ψ) hold. The relevant fact about $\mathcal{P}[N]$ is that its arrows can be interpreted precisely as concatenable processes of N, i.e., $\mathcal{P}[N]$ represents exactly the noninterleaving behaviour of N, including its algebraic structure. (See [3] for the details.)

Theorem 1.1 ($\mathcal{P}[N]$ vs. *Concatenable Processes [3]*)
Let N be a net. Then there exists a one-to-one correspondence between the arrows of $\mathcal{P}[N]$ and the concatenable processes of N such that, for each $u, v \in S_N^{\oplus}$, the arrows of the kind $u \to v$ correspond to the processes enabled by u and producing v, and such that sequential and parallel composition (tensor product) of processes (arrows) are respected.

2 Axiomatizing Concatenable Processes

In this section we show that the category of vperms Sym_N can be described abstractly, thus yielding a fully axiomatic characterization of concatenable processes. We start by associating a free SSMC to each net N. Although this may not be very surprising, our proof will identify a 'minimal' description of the free category on N which will be useful later on.

PROPOSITION 2.1 ($\mathcal{F} \dashv \mathcal{U}$)
The forgetful functor $\mathcal{U}: \underline{\mathrm{SSMC}}^{\oplus} \to \underline{\mathsf{Petri}}$ *has a left adjoint* $\mathcal{F}: \underline{\mathsf{Petri}} \to \underline{\mathrm{SSMC}}^{\oplus}$.

Proof. (Sketch.) Consider the category $\mathcal{F}(N)$ whose objects are the elements of $S_N^{\oplus}$ and whose arrows are generated by the inference rules

$$\frac{u \in S_N^{\oplus}}{id_u : u \to u \text{ in } \mathcal{F}(N)} \qquad \frac{a \text{ and } b \text{ in } S_N}{c_{a,b} : a \oplus b \to b \oplus a \text{ in } \mathcal{F}(N)} \qquad \frac{t : u \to v \text{ in } T_N}{t : u \to v \text{ in } \mathcal{F}(N)}$$

$$\frac{\alpha : u \to v \text{ and } \beta : u' \to v' \text{ in } \mathcal{F}(N)}{\alpha \otimes \beta : u \oplus u' \to v \oplus v' \text{ in } \mathcal{F}(N)} \qquad \frac{\alpha : u \to v \text{ and } \beta : v \to w \text{ in } \mathcal{F}(N)}{\alpha ; \beta : u \to w \text{ in } \mathcal{F}(N)}$$

modulo the axioms expressing that $\mathcal{F}(N)$ is a strict monoidal category, namely,

$$\begin{array}{rcl} \alpha ; id_v = \alpha = id_u ; \alpha & \text{and} & (\alpha ; \beta) ; \gamma = \alpha ; (\beta ; \gamma), \\ (\alpha \otimes \beta) \otimes \gamma = \alpha \otimes (\beta \otimes \gamma) & \text{and} & id_0 \otimes \alpha = \alpha = \alpha \otimes id_0, \\ id_u \otimes id_v = id_{u \oplus v} & \text{and} & (\alpha \otimes \alpha') ; (\beta \otimes \beta') = (\alpha ; \beta) \otimes (\alpha' ; \beta'), \end{array}$$

the latter whenever the righthand term is defined, and the following axioms

$$\begin{array}{rcll} c_{a,b} ; c_{b,a} & = & id_{a \oplus b}, & \\ c_{u,u'} ; (\beta \otimes \alpha) & = & (\alpha \otimes \beta) ; c_{v,v'}, & \text{for } \alpha : u \to v, \ \beta : u' \to v', \end{array} \qquad (9)$$

where $c_{u,v}$ for $u, v \in S_N^{\oplus}$ denote *any* term obtained from $c_{a,b}$ for $a, b \in S_N$ by applying recursively the following rules (compare with axiom (4)):

$$\begin{array}{rcl} c_{0,u} & = & id_u \quad = \quad c_{u,0}, \\ c_{a \oplus u, v} & = & (id_a \otimes c_{u,v}) ; (c_{a,v} \otimes id_u), \\ c_{u, v \oplus a} & = & (c_{u,v} \otimes id_a) ; (id_v \otimes c_{u,a}). \end{array} \qquad (10)$$

Observe that equation (9), in particular, equalizes all the terms obtained from (10) for fixed u and v. In fact, let $c_{u,v}$ and $c'_{u,v}$ be two such terms and take α and β to be, respectively, the identities of u and v. Now, since $id_u \otimes id_v = id_{u \oplus v} = id_v \otimes id_u$, from (9) we have that $c_{u,v} = c'_{u,v}$ in $\mathcal{F}(N)$. Then, it can be shown that the collection $\{c_{u,v}\}_{u,v \in S_N^{\oplus}}$ is a symmetry natural isomorphism which makes $\mathcal{F}(N)$ into a SSMC which is free on N. This means that $\mathcal{F}$ extends to a functor left adjoint to $\mathcal{U}$. ✓

Thus, establishing the adjunction $\underline{\mathsf{Petri}} \rightharpoonup \underline{\mathrm{SSMC}}^{\oplus}$, we have identified the free SSMC on N as a category generated, modulo appropriate equations, from the net N viewed as a graph enriched with formal arrows id_u, which play the role of the identities, and $c_{a,b}$ for $a, b \in S_N$, which generate all the needed symmetries. In the following, we speak of the *free* SSMC on N to mean $\mathcal{F}(N)$ as constructed above.

The following is the adaptation to SSMC's of the usual notion of quotient algebras characterized, as usual, by a universal property.

Proposition 2.2 (*Monoidal Quotient Categories*)
For a given SSMC $\underline{C}$*, let* $\mathcal{R}$ *be a function which assigns to each pair of objects* a *and* b *of* $\underline{C}$ *a binary relation* $\mathcal{R}_{a,b}$ *on the homset* $\underline{C}(a,b)$*. Then, there exist a* SSMC $\underline{C}/\mathcal{R}$ *and a symmetric strict monoidal functor* $\mathsf{Q}_{\mathcal{R}}\colon \underline{C} \to \underline{C}/\mathcal{R}$ *such that*

i) If $f\mathcal{R}_{a,b}f'$ *then* $\mathsf{Q}_{\mathcal{R}}(f) = \mathsf{Q}_{\mathcal{R}}(f')$*;*

ii) For each symmetric strict monoidal $\mathsf{H}\colon \underline{C} \to \underline{D}$ *such that* $\mathsf{H}(f) = \mathsf{H}(f')$ *whenever* $f\mathcal{R}_{a,b}f'$*, there exists a unique functor* $\mathsf{K}\colon \underline{C}/\mathcal{R} \to \underline{D}$*, which is necessarily symmetric strict monoidal, such that* $\mathsf{K} \circ \mathsf{Q}_{\mathcal{R}} = \mathsf{H}$*.*

Proof. (Sketch.) Say that $\mathcal{R}$ is a $\otimes$-*congruence* if $\mathcal{R}_{a,b}$ is an equivalence for each a and b and if $\mathcal{R}$ respects composition and tensor, i.e., whenever $f\mathcal{R}_{a,b}f'$ then, for all $h\colon a' \to a$ and $k\colon b \to b'$, we have $(k \circ f \circ h)\mathcal{R}_{a',b'}(k \circ f' \circ h)$ and for all $h\colon a' \to b'$ and $k\colon a'' \to b''$, we have $(h \otimes f \otimes k)\mathcal{R}_{a'\otimes a\otimes a'',b'\otimes b\otimes b''}(h \otimes f' \otimes k)$. Clearly, if $\mathcal{R}$ is a $\otimes$-congruence, the following definition is well-given: $\underline{C}/\mathcal{R}$ is the category whose objects are those of $\underline{C}$, whose homset $\underline{C}/\mathcal{R}(a,b)$ is $\underline{C}(a,b)/\mathcal{R}_{a,b}$, i.e., the quotient of the corresponding homset of $\underline{C}$ modulo the appropriate component of $\mathcal{R}$, and whose arrow composition and tensor product are given by $[g]_{\mathcal{R}} \circ [f]_{\mathcal{R}} = [g \circ f]_{\mathcal{R}}$ and $[f]_{\mathcal{R}} \otimes [g]_{\mathcal{R}} = [f \otimes g]_{\mathcal{R}}$, respectively. Moreover, it is easy to check that $\underline{C}/\mathcal{R}$ is a SSMC with symmetry isomorphism given by the natural transformation whose component at (u,v) is $[\gamma_{u,v}]_{\mathcal{R}}$ and unit object e.

Observe now that, given $\mathcal{R}$ as in the hypothesis, it always possible to find the least $\otimes$-congruence $\mathcal{R}'$ which includes (componentwise) $\mathcal{R}$. Then, take $\underline{C}/\mathcal{R}$ to be $\underline{C}/\mathcal{R}'$ and $\mathsf{Q}_{\mathcal{R}}$ to be the obvious projection of $\underline{C}$ into $\underline{C}/\mathcal{R}$. Clearly, $\mathsf{Q}_{\mathcal{R}}$ is a symmetric strict monoidal functor. Moreover, it is not difficult to show that it enjoys the properties *(i)* and *(ii)* above. ✓

Our next step is to show that $\mathcal{P}[N]$ is the quotient of $\mathcal{F}(N)$ modulo two simple additional axioms. In order to show this, we need the following lemma.

Lemma 2.3 (*Axiomatizing* Sym_N)
The arrows of Sym_N *are generated via sequential composition by the vperms of the kind* $id_u \otimes \gamma(a,a) \otimes id_v\colon u \oplus 2 \cdot a \oplus v \to u \oplus 2 \cdot a \oplus v$*. Moreover, two such compositions yield the same vperm if and only if this can be shown by using the axioms*

$$\begin{aligned}((id_{u\oplus a} \otimes \gamma(a,a) \otimes id_v)\,;\,(id_u \otimes \gamma(a,a) \otimes id_{a\oplus v}))^3 &= id_{u\oplus 3\cdot a\oplus v},\\ ((id_u \otimes \gamma(a,a) \otimes id_{2\cdot b\oplus v})\,;\,(id_{u\oplus 2\cdot a} \otimes \gamma(b,b) \otimes id_v))^2 &= id_{u\oplus 2\cdot a\oplus 2\cdot b\oplus v},\\ (id_u \otimes \gamma(a,a) \otimes id_v)^2 &= id_{u\oplus 2\cdot a\oplus v}.\end{aligned} \tag{11}$$

where f^n *indicates the composition of* f *with itself* n *times.*

Proof. (Sketch.) Concerning the first claim, a vperm $p = \langle \sigma_{a_1}, \ldots, \sigma_{a_n} \rangle$ coincides with the tensor $\sigma_{a_1} \otimes \cdots \otimes \sigma_{a_n}$ which, exploiting the functoriality of $\otimes$, can be written as $(\sigma_{a_1} \otimes \cdots \otimes id_{u_n}); \cdots ; (id_{u_1} \otimes \cdots \otimes \sigma_{a_n})$. Now, since σ_{a_i} is a permutation, it is a composition of transpositions of adjacent elements, and since the transposition $\tau_i\colon n\cdot a \to n\cdot a$, $1 \le i < n$, can be written as $id_{(i-1)\cdot a} \otimes \gamma(a,a) \otimes id_{(n-i-1)\cdot a}$ in Sym_N, we have that $\sigma_{a_i} = (id_{u'_1} \otimes \gamma(a_i,a_i) \otimes id_{u''_1}); \cdots ; (id_{u'_k} \otimes \gamma(a_i,a_i) \otimes id_{u''_k})$. Therefore, the vperms $id_u \otimes \gamma(a,a) \otimes id_v$ generate via composition all the vperms of Sym_N. Concerning the axiomatization, it is easy to verify that the equations (11) hold in Sym_N. On the other hand, the completeness of axioms (11) follows non trivially from a non-trivial axiomatization of the groups of permutations [13]. ✓

Proposition 2.4 (*Axiomating $\mathcal{P}[N]$*)
$\mathcal{P}[N]$ is the monoidal quotient of the free SSMC *on N modulo the axioms*

$$c_{a,b} = id_{a\oplus b}, \quad \text{if } a,b \in S_N \text{ and } a \neq b, \tag{12}$$
$$s;t;s' = t, \quad \text{if } t \in T_N \text{ and } s,s' \text{ are symmetries.} \tag{13}$$

Proof. (Sketch.) We show that $\mathcal{P}[N]$ enjoys the universal property of $\mathcal{F}(N)/\mathcal{R}$ stated in Proposition 2.2, where $\mathcal{R}$ is the congruence generated from equations (12) and (13). It follows then from general facts about universal constructions that $\mathcal{P}[N]$ is isomorphic to $\mathcal{F}(N)/\mathcal{R}$.

First of all observe that $\mathcal{P}[N]$ belongs to $\underline{\mathsf{SSMC}}^{\oplus}$. Therefore, corresponding to the Petri net *inclusion* morphism $N \to \mathcal{U}\mathcal{P}[N]$, there is a symmetric strict monoidal functor $\mathsf{Q}\colon \mathcal{F}(N) \to \mathcal{P}[N]$ which is the identity on the places and on the transitions of N. It follows easily from the definition of $\mathcal{P}[N]$ that Q equalizes the pairs in $\mathcal{R}$. Then, we have to show that Q is universal among such functors.

We start by observing that Sym_N can be embedded in $Sym_{\mathcal{F}(N)}$ via a monoidal functor. Consider the mapping G of objects and arrows of Sym_N to, respectively, objects and arrows of $Sym_{\mathcal{F}(N)}$ which is the identity on the objects and such that

$$\begin{aligned} \mathsf{G}(id_u \otimes \gamma(a,a) \otimes id_v) &= id_u \otimes c_{a,a} \otimes id_v, \\ \mathsf{G}(p;q) &= \mathsf{G}(p);\mathsf{G}(q), \\ \mathsf{G}(id_u) &= id_u. \end{aligned}$$

It follows from Lemma 2.3 that the equations above define G on all vperms. Thus, to conclude that G is a functor we only need to show that it is well-defined; exploiting Lemma 2.3, this can be seen by showing that it respects axioms (11). Clearly, G is not symmetric strict monoidal, since $\mathsf{G}(\gamma(a,b)) = id_{a\oplus b} \neq c_{a,b}$, i.e., axiom (8) does not hold. However, G is strict monoidal in the sense that (6) and (7) hold.

Let $\underline{\mathsf{C}} = (\underline{\mathsf{C}}, \otimes, e, \gamma)$ be a SSMC and suppose that there exists a symmetric strict monoidal functor $\mathsf{H}\colon \mathcal{F}(N) \to \underline{\mathsf{C}}$ such that, for any pair $a \neq b \in S_N$ and for any symmetries s and s', $\mathsf{H}(c_{a,b}) = \mathsf{H}(id_{a\oplus b})$ and $\mathsf{H}(s;t;s') = \mathsf{H}(t)$. We have to show that there exists a unique $\mathsf{K}\colon \mathcal{P}[N] \to \underline{\mathsf{C}}$ such that $\mathsf{H} = \mathsf{KQ}$. We consider the following definition of K on objects and generators

$$\begin{aligned} \mathsf{K}(u) &= \mathsf{H}(u), && \text{if } u \in S_N^{\oplus}, \\ \mathsf{K}(s) &= \mathsf{H}(\mathsf{G}(s)), && \text{if } s \text{ is a symmetry} \\ \mathsf{K}(t) &= \mathsf{H}(t), && \text{if } t \in T_N, \end{aligned}$$

extendend to $\mathcal{P}[N]$ by $\mathsf{K}(\alpha;\beta) = \mathsf{K}(\alpha);\mathsf{K}(\beta)$ and $\mathsf{K}(\alpha \otimes \beta) = \mathsf{K}(\alpha) \otimes \mathsf{K}(\beta)$.

First of all, we have to show that K is well-defined, i.e., that the equations which hold in $\mathcal{P}[N]$ are preserved by K. Since H and G are strict monoidal, it follows that the functoriality of $\otimes$, axioms (1)–(3) and (Ψ) are preserved. The key to show that the same holds for the naturality of the symmetry, for (4) and for (5) is to show that $\mathsf{K}(\gamma(u,v)) = \gamma_{\mathsf{K}(u),\mathsf{K}(v)}$, which can be done by induction on the least of the sizes of u and v. Once this fact is established, the aforesaid points follow from fact that $\underline{\mathsf{C}}$ is a SSMC.

The next task is to show that $\mathsf{H} = \mathsf{KQ}$. It follows from the fact that Q is symmetric strict monoidal and from the definition of the symmetries of $\mathcal{F}(N)$ that H and KQ coincide on $Sym_{\mathcal{F}(N)}$. Then, one proves that $\mathsf{H} = \mathsf{KQ}$ by proving, by easy induction on the structure of the terms, that each arrow of $\mathcal{F}(N)$ can be written as the composition of symmetries and arrows of the kind $id_u \otimes t \otimes id_v$, for $t \in T_N$. Finally, concerning the uniqueness condition on K, observe that it must necessarily be $\mathsf{K}(id_u \otimes \gamma(a,a) \otimes id_v) = id_{\mathsf{H}(u)} \otimes \gamma_{\mathsf{H}(a),\mathsf{H}(a)} \otimes id_{\mathsf{H}(v)}$, which, by Lemma 2.3, defines K uniquely on Sym_N. Moreover, the behaviour of K on the arrows formed as composition and tensor of transitions is uniquely determined by H. ✓

The next corollary gives an alternative form for axiom (13).

Corollary 2.5 (*Axiom (13) revisited*)
Axiom (13) in Proposition 2.4 can be replaced by the axioms

$$\begin{aligned} t; (id_u \otimes c_{a,a} \otimes id_v) &= t \qquad \text{if } t \in T_N \text{ and } a \in S_N, \\ (id_u \otimes c_{a,a} \otimes id_v); t &= t \qquad \text{if } t \in T_N \text{ and } a \in S_N. \end{aligned}$$

Proof. Since $(id_u \otimes \gamma_{a,a} \otimes id_v)$ and all the identities are symmetries, axiom (13) implies the present ones. It is easy to see that, on the contrary, the axioms above, together with axiom (12), imply (13). ✓

Finally, in the next corollary, we sum up the purely algebraic characterization of the category of concatenable processes that we have proved in the paper.

Corollary 2.6 (*Axiomatizing Concatenable Processes*)
The category $\mathcal{P}[N]$ of concatenable processes of N is the category whose objects are the elements of $S_N^{\oplus}$ and whose arrows are generated by the inference rules

$$\frac{u \in S_N^{\oplus}}{id_u : u \to u \text{ in } \mathcal{P}[N]} \qquad \frac{a \text{ in } S_N}{c_{a,a} : a \oplus a \to a \oplus a \text{ in } \mathcal{P}[N]} \qquad \frac{t : u \to v \text{ in } T_N}{t : u \to v \text{ in } \mathcal{P}[N]}$$

$$\frac{\alpha : u \to v \text{ and } \beta : u' \to v' \text{ in } \mathcal{P}[N]}{\alpha \otimes \beta : u \oplus u' \to v \oplus v' \text{ in } \mathcal{P}[N]} \qquad \frac{\alpha : u \to v \text{ and } \beta : v \to w \text{ in } \mathcal{P}[N]}{\alpha ; \beta : u \to w \text{ in } \mathcal{P}[N]}$$

modulo the axioms expressing that $\mathcal{P}[N]$ is a strict monoidal category, namely,

$$\begin{aligned} \alpha; id_v = \alpha = id_u; \alpha \quad &\text{and} \quad (\alpha;\beta);\gamma = \alpha;(\beta;\gamma), \\ (\alpha \otimes \beta) \otimes \gamma = \alpha \otimes (\beta \otimes \gamma) \quad &\text{and} \quad id_0 \otimes \alpha = \alpha = \alpha \otimes id_0, \\ id_u \otimes id_v = id_{u \oplus v} \quad &\text{and} \quad (\alpha \otimes \alpha');(\beta \otimes \beta') = (\alpha;\beta) \otimes (\alpha';\beta'), \end{aligned}$$

the latter whenever the righthand term is defined, and the following axioms

$$\begin{aligned} c_{a,a}; c_{a,a} &= id_{a \oplus a}, \\ t; (id_u \otimes c_{a,a} \otimes id_v) &= t, \quad \text{if } t \in T_N, \\ (id_u \otimes c_{a,a} \otimes id_v); t &= t, \quad \text{if } t \in T_N, \\ c_{u,u'}; (\beta \otimes \alpha) &= (\alpha \otimes \beta); c_{v,v'}, \quad \text{for } \alpha : u \to v, \ \beta : u' \to v', \end{aligned}$$

where $c_{u,v}$ *for* $u, v \in S_N^{\oplus}$ *is obtained by repeatedly applying the following rules:*

$$\begin{aligned} c_{a,b} &= id_{a\oplus b}, \quad \textit{if } a = 0 \textit{ or } b = 0 \textit{ or } (a, b \in S_N \textit{ and } a \neq b), \\ c_{a\oplus u,v} &= (id_a \otimes c_{u,v}); (c_{a,v} \otimes id_u), \\ c_{u,v\oplus a} &= (c_{u,v} \otimes id_a); (id_v \otimes c_{u,a}). \end{aligned}$$

Proof. Easy from Proposition 2.1, Proposition 2.4 and Corollary 2.5. ✓

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