Comparing Transition Systems with Independence and Asynchronous Transition Systems

Thomas T. Hildebrandt[*] *Vladimiro Sassone*[⋆,*]

[*] **BRICS** – Computer Science Dept., University of Aarhus
[⋆] Dipartimento di Informatica, Università di Pisa

Abstract. Transition systems with independence and asynchronous transition systems are *noninterleaving* models for concurrency arising from the same simple idea of decorating transitions with events. They differ for the choice of a *derived* versus a *primitive* notion of event which induces considerable differences and makes the two models suitable for different purposes. This opens the problem of investigating their mutual relationships, to which this paper gives a fully comprehensive answer.

In details, we characterise the category of *extensional* asynchronous transitions systems as the largest full subcategory of the category of (labelled) asynchronous transition systems which admits TSI, the category of transition systems with independence, as a *coreflective* subcategory. In addition, we introduce *event-maximal* asynchronous transitions systems and we show that their category is *equivalent* to TSI, so providing an exhaustive characterisation of transition systems with independence in terms of asynchronous transition systems.

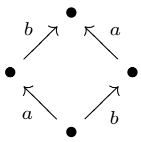
Introduction

Following the leading idea of CCS [11] and related process calculi [10, 2, 12, 9], the behaviour of concurrent systems is often specified *extensionally* by describing their 'state-transitions' and the observable behaviours that such transitions produce. The simplest formal model of computation able to express naturally this idea is that of *labelled transition systems*, where the labels on the transitions are thought of as the actions of the system at its 'external ports', or, more generally, the observable part of its behaviour.

Transition systems are an *interleaving* model of concurrency, which means that they do not allow to draw a natural distinction between interleaved and concurrent execution of actions. More precisely, transition systems do not model the fact that concurrent actions can overlap in time and reduce concurrency to a nondeterministic choice of action interleavings, so loosing track of the casual dependencies between actions and, consequently, of the fact that computations that differ only for the order of independent actions represent, actually, the same behaviour. In other words, interleaving models abstract away from the difference between the factual *temporal* occurrence order and the more conceptual *causal* ordering of actions. The simplest exemplification of this situation is provided by the CCS terms $a \mid b$ and $a.b + b.a$, both described by the following transition

[*] **B**asic **R**esearch **in** **C**omputer **S**cience, Centre of the Danish National Research Foundation.
[⋆] This author was supported by EU **H**uman **C**apital and **M**obility grant ERBCHBGCT920005.

system.

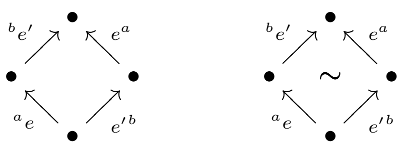


(1)

Although for many applications this level of abstraction is appropriate, for several other kinds of analysis a model may be desirable that takes full account of concurrency. For instance, apart from any philosophical consideration about the semantic relevance of cause/effect relationships, knowing that different interleavings represent the same behaviour can reduce considerably the state-space explosion problem when checking system properties such as safety properties and fairness [8, 20, 16].

Several efforts have been devoted to the search of transition-based *noninterleaving* models, e.g., transition systems enriched with additional features that make expressing concurrency explicitly possible (cf., e.g., [17, 4, 6, 7, 5, 3]). The present paper focuses on two such models, namely *asynchronous transition systems*, introduced independently by Bednarczyk [1] and Shields [19], and *transitions systems with independence*, proposed by Winskel and Nielsen [21]. These two competing approaches are, among the others, those building on the simplest idea: endow transition systems with some formal notion of 'similarity' of transitions that enables to distinguish whether or not the opposite edges in diagrams such as (1) represent the same action. Intuitively, this is achieved in both approaches by thinking of transitions as *occurrences* of *events*, two transitions representing the same event if they correspond to the same action. However, the differences induced on the models by the different choices of how to assign events to transitions are definitely not trivial. And so are the relationships that these models bear to each other.

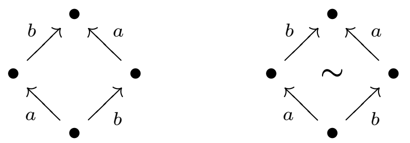
Getting to the details, asynchronous transition systems assign events to transitions explicitly and enrich the structure further by adding an *independence relation* on the events which describes their causal relationships. This clearly makes distinguishing nondeterminism and concurrency possible; $a.b+b.a$ and $a|b$ can be represented respectively by, e.g., the following *labelled* asynchronous transition systems, where $\sim$ indicates whether or not the events e and e' (labelled by a and b) are independent.



Observe that here and in the rest of the paper we consider *labelled* asynchronous transition systems [1, 21], i.e., asynchronous transition systems with a further labelling of events, as the proper extension of labelled transition systems.

The expressive power of asynchronous transition systems is clearly not limited to the example above; for instance, Bednarczyk [1] and Mukund and Nielsen [14] have shown that noninterleaving related issues for CCS processes—such as *localities*—can be modelled faithfully using this model. However, it can be argued that assigning both the independence relation and the decoration of transitions with events explicitly means assigning too much. In fact, this obviously introduces some *redundancies* in the model: there are, for instance, many non-isomorphic variations of the asynchronous transitions systems above which can still be reasonably thought as models of $a|b$ and $a.b + b.a$. Moreover, although it is usually easy to tell about independence of transitions, in many important cases it is at least *not* immediate to assign events to transitions: it might very well be the goal of the entire semantic analysis to understand what the events of the system and their mutual relationships are. This consideration seems to indicate that asynchronous transitions systems cannot have a significant impact in Plotkin's SOS style semantics, unless the independence relation is promoted to a greater role.

Transition systems with independence are an attempt to answer to the previous observation. Here events are *not* introduced explicitly. They are rather *derived* from the structure of the 'simply-labelled' transitions, upon which the independence relation is directly layered. In such a model, each of the CCS terms discussed above admits only one transition system which can faithfully represent it, viz., respectively,

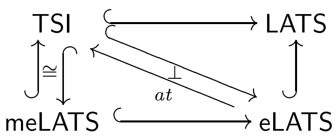


The implicit information about events can be easily deduced from the presence (or the absence) of $\sim$, making the achieved expressive power comparable to that of asynchronous transition systems. Moreover, avoiding a primitive notion of event makes providing a *'noninterleaving'* operational semantics in the SOS style a relatively simple task (cf. [21]).

However, in order to be consistent with the computational intuition, the axiomatics of transition systems with independence involves (apparently necessarily [18]) *one* condition expressed 'globally' in terms of all the transitions representing occurrences of the same event. This contrasts with the 'local' conditions defining asynchronous transition systems and can make hard checking that a given structure is a transitions system with independence. Thus, the differences induced on the two models by the choice of a *primitive* versus a *derived* notion of event are far-reaching and seem to make them suitable for different applications. This indicates that it is not wise to choose *once and for all* between asynchronous transition systems and transition systems with independence, which, in turn, opens the issue of investigating *formally* their analogies and differences. The contribution of this paper is to answer exhaustively such a question, which, actually, escaped the thorough analysis of models for concurrency carried out in [21, 15, 18]. Precisely, we prove that transition systems with independence besides being nicely related to a class of asynchronous transition systems that we call *extensional*, are *equivalent* to the so-called *event-maximal* asynchronous transition systems. These latter can be seen at the same time as those transition systems that make as few identifications of transitions as possible, i.e., contain no confusion about event identities, and as those in which such identities are derivable from the independence relation, i.e., reduce the redundancy. It is worth mentioning that the converse does not hold: the asynchronous transitions systems for which the independence relation is in turn derivable from the structure of events, and therefore redundant, are slightly less general. They correspond to the transitions systems with independence for which 'independence is concurrency' considered in [15, 18].

Concerning the organization of the paper and its technical contributions, after recalling in Section 1 the definitions of LATS and TSI, respectively the categories of labelled asynchronous transitions systems and of transitions systems with independence, in Section 2 we look for a functor adjoint to the obvious embedding $\mathsf{TSI} \hookrightarrow \mathsf{LATS}$. In particular, we identify the category of extensional asynchronous transitions systems, eLATS, as the largest subcategory of LATS which admits TSI as a *coreflective* subcategory. It is worth noticing here that $at\colon \mathsf{eLATS} \to \mathsf{TSI}$, the right adjoint of the coreflection, complements and corrects a non-well-defined construction sketched in [21]: as a matter of fact, due to the greater generality of asynchronous transition systems, eLATS happens to be the largest subcategory of LATS on which such a construction makes sense. Finally, Section 3 introduces event-maximal asynchronous transitions systems and their category meLATS, providing the proof of the *equivalence* $\mathsf{TSI} \cong \mathsf{meLATS}$. This yields a complete description of TSI in terms of LATS which can be useful in practise to translate back and forth between the two models when the application one has in mind requires it.

Summing up our results, this paper presents the following commutative diagram, which makes completely formal and precise the relationships between transition systems with independence and asynchronous transition systems.



1 Preliminaries

In this section we recall briefly the definitions of asynchronous transition systems, transition systems with independence, and their respective categories [1, 21].

As discussed in the introduction, asynchronous transition systems are simply transition systems whose transitions are decorated by events equipped with an independence relation. Four axioms (A1–A4) are needed to guarantee the intended meaning for the events and the independence relation.

Definition 1.1 (*Labelled Asynchronous Transition Systems*)
*A labelled asynchronous transition system (*lats *for short) is a structure*

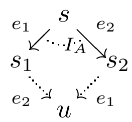
$$A = (S_A, i_A, E_A, Tran_A, I_A, L_A, \ell_A),$$

where $(S_A, i_A, E_A, Tran_A)$ *is a transition system with set of states* S_A*, initial state* $i_A \in S_A$*, and transitions* $Tran_A \subseteq S_A \times E_A \times S_A$*, and where* E_A *is a set of events,* L_A *a set of labels,* $\ell_A : E_A \to L_A$ *a labelling function, and* $I_A \subseteq E_A \times E_A$*, the independence relation, is an irreflexive, symmetric relation such that*

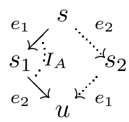
A1. $e \in E_A \quad \Rightarrow \quad \exists s_1, s_2 \in S_A.\ (s_1, e, s_2) \in Tran_A;$

A2. $(s, e, s_1), (s, e, s_2) \in Tran_A \quad \Rightarrow \quad s_1 = s_2;$

A3. $e_1\ I_A\ e_2\ \&\ (s, e_1, s_1), (s, e_2, s_2) \in Tran_A \quad \Rightarrow$
$\exists u.\ (s_1, e_2, u), (s_2, e_1, u) \in Tran_A;$



A4. $e_1\ I_A\ e_2\ \&\ (s, e_1, s_1), (s_1, e_2, u) \in Tran_A \quad \Rightarrow$
$\exists s_2.\ (s, e_2, s_2), (s_2, e_1, u) \in Tran_A.$



In the rest of the paper we shall let $I(e)$ denote the set $\{e' \mid e\ I_A\ e'\}$ and, for convenience, use (s, e^a, s') as a shorthand for a transition (s, e, s') with $\ell_A(e) = a$.

The following is the standard definition of morphisms for lats, which essentially captures the idea of *simulation* (cf. [1, 21]).

Definition 1.2 (*Asynchronous Transition System Morphisms*)
For A *and* A' lats*, a morphism from* A *to* A' *is a triple of (partial) functions*[1] $(\sigma : S_A \to S_{A'}, \eta : E_A \rightharpoonup E_{A'}, \lambda : L_A \rightharpoonup L_{A'})$*, where* (σ, η) *is a morphism of labelled transition systems, i.e.,*

- ▷ $\sigma(i_A) = i_{A'};$
- ▷ $(s_1, e, s_2) \in Tran_A\ \&\ \eta(e)\downarrow \quad \Rightarrow \quad \big(\sigma(s_1), \eta(e), \sigma(s_2)\big) \in Tran_{A'};$
  $(s_1, e, s_2) \in Tran_A\ \&\ \eta(e)\uparrow \quad \Rightarrow \quad \sigma(s_1) = \sigma(s_2);$

which preserves the labelling, i.e., makes the following diagram commutative

$$\begin{array}{ccc} E_A & \xrightharpoonup{\ \eta\ } & E_{A'} \\ {\scriptstyle \ell_A}\downarrow & & \downarrow{\scriptstyle \ell_{A'}} \\ L_A & \xrightharpoonup[\ \lambda\ ]{} & L_{A'}; \end{array}$$

and the independence, i.e.,

$$e_1\ I_A\ e_2\ \&\ \eta(e_1)\downarrow,\ \eta(e_2)\downarrow \quad \Rightarrow \quad \eta(e_1)\ I_{A'}\ \eta(e_2).$$

[1] We use, respectively, $f : A \to B$ and $f : A \rightharpoonup B$ to indicate total and partial functions. For f a partial function, $f(x)\downarrow$ $(f(x)\uparrow)$ means that f is (un)defined at x.

It is immediate to see that lats and their morphisms form a category, which we shall refer as LATS.

Starting from Definition 1.1, transition systems with independence attempt to simplify the structure retaining explicitly only the independence, now layered directly on the transitions. As already mentioned, the notion of event becomes implicit, determined by the independence relation through the equivalence $\sim$.

Definition 1.3 (*Transition Systems with Independence*)
A transition system with independence (tsi for short) is a structure

$$T = (S_T, i_T, L_T, \mathit{Tran}_T, I_T),$$

where $(S_T, i_T, L_T, \mathit{Tran}_T)$ is a transition system and $I_T \subseteq \mathit{Tran}_T \times \mathit{Tran}_T$, the independence relation, is an irreflexive, symmetric relation, such that, denoting by $\prec$ the binary relation on transitions given as

$$\begin{aligned}(s,a,s_1) \prec (s_2,a,u) \quad \Leftrightarrow \\ \exists b \in L_T.\ (s,a,s_1)\ I_T\ (s,b,s_2)\ \& \\ (s,a,s_1)\ I_T\ (s_1,b,u)\ \&\ (s,b,s_2)\ I_T\ (s_2,a,u)\end{aligned}$$

and by $\sim$ the least equivalence on transitions which includes it, we have

- T1. $(s,a,s_1) \sim (s,a,s_2) \ \Rightarrow\ s_1 = s_2$;
- T2. $(s,a,s_1)\ I_T\ (s,b,s_2) \ \Rightarrow\ \exists u.\ (s,a,s_1)\ I_T\ (s_1,b,u)\ \&\ (s,b,s_2)\ I_T\ (s_2,a,u)$;
- T3. $(s,a,s_1)\ I_T\ (s_1,b,u) \ \Rightarrow\ \exists s_2.\ (s,a,s_1)\ I_T\ (s,b,s_2)\ \&\ (s,b,s_2)\ I_T\ (s_2,a,u)$;
- T4. $(s,a,s_1) \prec \cup \succ (s_2,a,u)\ I_T\ (w,b,w') \ \Rightarrow\ (s,a,s_1)\ I_T\ (w,b,w')$.

The $\sim$-equivalence classes, in the following denoted by $[(s,a,s')]$, for (s,a,s') a representative of the class, are to be thought of as events, i.e., $t_1 \prec t_2$ means that t_1 and t_2 are part of a 'concurrency diamond', whilst $t_1 \sim t_2$ means that they are occurrences of the same event. Concerning the axioms, notice then that T1 (the global condition mentioned earlier) corresponds to A2 and axioms T2 and T3 correspond, respectively, to A3 and A4. The role of T4 is to ensure that the independence relation is actually well defined as a relation on events. In the rest of the paper we shall see that this view of $[(s,a,s')]$ agrees with the notion of events for lats and that, in fact, it identifies an interesting subclass of them.

Using $I(t)$ to denote the set $\{t' \mid t\ I_T\ t'\}$, we can state the following lemma which will be useful later on. As a matter of notations, we shall use π_i to denote projections, i.e., if t is (s,a,s'), then $\pi_1(t) = s$, $\pi_2(t) = a$ and $\pi_3(t) = s'$.

Lemma 1.4
Axiom T4 is equivalent to

$$t_1 \sim t_2 \quad \Rightarrow \quad I(t_1) = I(t_2). \tag{T4$'$}$$

Proof. Easy, by induction. ✓

The following definition of morphisms for transition systems with independence resembles closely that given earlier for lats.

Definition 1.5 (*Transition System with Independence Morphisms*)
For T and T' tsi, a morphism from T to T' consists of a pair of (partial) functions $(\sigma\colon S_T \to S_{T'}, \lambda\colon L_T \rightharpoonup L_{T'})$ which is a morphism of transition systems and, in addition, preserves independence, i.e.,

$$(s_1, a, s_2)\ I_T\ (s_1', b, s_2')\ \&\ \lambda(a)\downarrow, \lambda(b)\downarrow \quad \Rightarrow \\ \big(\sigma(s_1), \lambda(a), \sigma(s_2)\big)\ I_{T'}\ \big(\sigma(s_1'), \lambda(b), \sigma(s_2')\big).$$

We shall use TSI to denote the category of tsi and their morphisms.

The following lemma states that tsi morphisms are well defined as maps of events, an easy consequence of the fact that they preserve independence that we shall use in order to embed TSI into LATS.

Lemma 1.6 (*Morphisms map Events to Events*)
For $(\sigma, \lambda)\colon T \to T'$ a morphism of tsi and $(s_1, a, s_2) \sim (s_1', a, s_2')$ equivalent transitions of T, if $\lambda(a)\downarrow$, then $\big(\sigma(s_1), \lambda(a), \sigma(s_2)\big) \sim \big(\sigma(s_1'), \lambda(a), \sigma(s_2')\big)$, i.e., lats morphisms preserve $\sim$.

2 From LATS to TSI: a coreflection

The scene is now set to expose the adjunction between TSI and a full subcategory of LATS. First, we define an inclusion $ta\colon$ TSI $\hookrightarrow$ LATS in the obvious way.

On the objects, ta acts by decorating each transition with the event identified by the $\sim$-class the transition belongs to. The label of such an event is, of course, the label originally carried in the tsi by the transition. Observe that, in force of Definition 1.3 of $\sim$, this labelling is well defined. Finally, the independence relation of $ta(T)$ is inherited directly from the one of T. The formal definition is as follows.

Definition 2.1 (TSI $\hookrightarrow$ LATS)
For T a tsi, let $ta(T)$ be the structure $(S_T, i_T, E, Tran, I, L_T, \ell)$, where, denoting by $\sim$ the equivalence relation induced by I_T as in Definition 1.3,

- ▷ *$E = Tran_T/\sim$, the set of $\sim$-classes of $Tran_T$;*
- ▷ *$Tran = \big\{\big(s_1, [(s_1, a, s_2)], s_2\big) \bigm| (s_1, a, s_2) \in Tran_T\big\}$;*
- ▷ *$[(s_1, a, s_2)]\ I\ [(s_1', a, s_2')]$ if and only if $(s_1, a, s_2)\ I_T\ (s_1', a, s_2')$;*
- ▷ *$\ell\big([(s_1, a, s_2)]\big) = a$.*

It follows from Lemma 1.4 that the definition of the independence on the events of $ta(T)$ is well given. It is now easy to verify the following.

Proposition 2.2
The transition system $ta(T)$ is a lats.

Proof. Axiom A1 is trivially satisfied. Axiom A2 is satisfied because of T1, for, by definition of ta, two transitions carry the same event if and only if they belong to the same $\sim$-class in T. Concerning A3 and A4, they correspond directly to T2 and T3. ✓

In order to define ta as a functor, we need to assign its action on the morphisms in TSI.

Definition 2.3 (TSI $\hookrightarrow$ LATS)
For $(\sigma, \lambda)\colon T \to T'$ *a morphism of* tsi, *let* $ta\big((\sigma,\lambda)\big)$ *be* (σ, η, λ), *where*

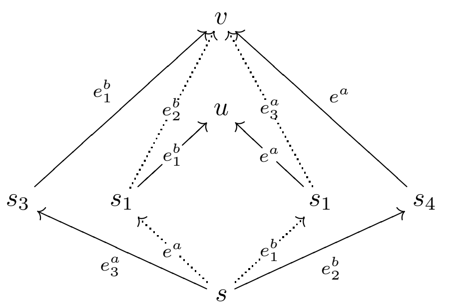
$$\eta\big([(s,a,s')]\big) = \begin{cases} \big[(\sigma(s), \lambda(a), \sigma(s'))\big] & \text{if } \lambda(a)\downarrow, \\ \textit{undefined} & \text{if } \lambda(a)\uparrow. \end{cases}$$

That Definition 2.3 is well given follows from Lemma 1.6; it is then easy to check that ta is a *full* and *faithful* functor, i.e., an embedding of TSI in LATS.

The obvious idea for a map at left inverse to ta, as hinted also in [21], is to forget the events and bring the independence from the events down to the transitions, i.e., for A a lats, to take $at(A)$ to be $(S_A, i_A, L_A, \mathit{Tran}, I)$, where

- ▷ $(s,a,s') \in \mathit{Tran}$ if and only if $(s, e^a, s') \in \mathit{Tran}_A$,
- ▷ $(s,a,s_1)\ I\ (s_2,b,s_3)$ if and only if $(s, e_1^a, s_1), (s_2, e_2^b, s_3) \in \mathit{Tran}_A$ & $e_1\ I_A\ e_2$.

This construction, however, contrarily to the claims of [21], is not well defined on the whole LATS, since the interplay between the explicitly given independence and events in lats allows rather complicated situations—of dubious computational significance—which cannot be expressed with tsi. A counterexample is illustrated by the following lats.



The independent events are $e\ I_A\ e_1$, $e_3\ I_A\ e_1$, $e\ I_A\ e_2$, and $e_2\ I_A\ e_3$, i.e., the system consists of three independency diamonds 'on top of each other'. It is easy to check that this is an object of LATS. However, by applying at we create a 'ghost' independency diamond (the one highlighted by the dotted lines), so violating condition T1. In fact, $(s,a,s_3) \sim (s,a,s_1)$ with $s_1 \neq s_3$. This demonstrates that the combination of independence and events makes it hard to define 'uniformly' a map from LATS to TSI to act as left inverse to $ta\colon$ TSI $\hookrightarrow$ LATS.

However, it is not hard to check that things go smoothly for those lats belonging to the *image* of ta. In such a case, at lands in TSI and, of course, we have the following result.

Lemma 2.4
For any T *in* TSI, *we have* $at \circ ta(T) = T$.

At this point, the issue arises of identifying suitable conditions which, imposed on lats, constrain them down to a category which bears good relationships with TSI. Possibly, one should also like to find a nice characterisation of the image of ta in LATS. We shall do so next, by focusing on *extensional* asynchronous transition systems.

We start by considering lats A satisfying

$$(s_1, e_1^a, s_2) \neq (s_1, e_2^b, s_2) \in Tran_A \quad \Rightarrow \quad a \neq b. \tag{Ex}$$

In words, these are lats where no two transitions between the same states can carry the same label. This is a kind of extensionality condition that, in view of Definition 1.3, is clearly necessary for our purposes. In fact, without (Ex), the one-to-one correspondence between morphisms of the kinds $ta(T) \to A$ and $T \to at(A)$—required by the adjointness conditions—would not exist. Next, we let the counterexample discussed above guide us to identify two simple additional conditions—strengthening A3 and A4 with uniqueness criteria—that we shall prove to be *necessary* and *sufficient* in order for at to be well defined on lats satisfying (Ex). As a notation, for $(s, e^a, s') \in Tran_A$, we shall use $at(s, e^a, s')$ to refer to the (unique) transition $(s, a, s') \in Tran_{at(A)}$ it corresponds to.

Proposition 2.5
For A a lats satisfying (Ex), $at(A)$ belongs to TSI if and only if

i) for $e_1\ I_A\ e_2$ and $(s, e_1^a, s_1), (s, e_2^b, s_2) \in Tran_A$, there exists a unique pair $(s_1, x_2^b, u), (s_2, x_1^a, u) \in Tran_A$ such that $e_1\ I_A\ x_2$, $e_2\ I_A\ x_1$, and $x_1\ I_A\ x_2$.

ii) for $e_1\ I_A\ e_2$ and $(s, e_1^a, s_1), (s_1, e_2^b, u) \in Tran_A$, there exists a unique pair $(s, x_2^b, s_2), (s_2, x_1^a, u) \in Tran_A$ such that $e_1\ I_A\ x_2$, $e_2\ I_A\ x_1$, and $x_1\ I_A\ x_2$.

Proof. If $at(A) \in$ TSI, the pairs of transitions in *i)* and *ii)* exist because of axioms A3 and A4. Their uniqueness is needed in order for $at(A)$ to satisfy axiom T1. Suppose that, on the contrary, in case *ii)* there are two pairs $(s_1, x_2^b, u), (s_2, x_1^a, u)$ and $(s_1, y_2^b, w), (s_2, y_1^a, w)$ satisfying the condition. Since A satisfies (Ex), we have $w \neq u$, which implies that $at(s, e_2^b, s_2) \prec at(s_1, y_2^b, w)$ and $at(s, e_2^b, s_2) \prec at(s_1, x_2^b, u)$, i.e., that $at(s_1, x_2^b, u) \sim at(s_1, y_2^b, w)$, which contradicts T1. The case for *ii)* can be proved along the same lines, thus showing the necessity of the conditions.

Concerning their sufficiency, the extensionality guarantees that $I_{at(A)}$ is irreflexive, whilst the property of symmetry for $I_{at(A)}$ is inherited from I_A. It remains check that the axioms T1–T4 defining tsi hold for $at(A)$. Axioms A3, A4 and conditions *i)* and *ii)* above ensure that if $at(t) \prec at(t')$, then $\pi_2(t) = \pi_2(t')$, i.e., t and t' represent the same event. It follows then by induction that $at(t) \sim at(t')$ implies $\pi_2(t) = \pi_2(t')$, for all $at(t), at(t') \in Tran_{at(A)}$. If in addition $\pi_1\big(at(t)\big) = \pi_1\big(at(t')\big)$, then also $\pi_1(t) = \pi_1(t')$ and axiom A2 implies that $\pi_3(at(t)) = \pi_3(at(t'))$. So T1 is satisfied. Actually, this also implies that T4 holds. For, since the independence in $at(A)$ is inherited from that on the events in A, and t and t' carry the same event, we have that $at(t) \sim at(t')$ implies $I(at(t)) = I(at(t'))$. This, as proved by Lemma 1.4, is equivalent to T4. Finally, T2 and T3 hold because of the corresponding A3 and A4. ✓

We call *extensional* the lats satisfying (Ex) and the conditions of Proposition 2.5, and we denote by eLATS the full subcategory of LATS they determine.

Clearly, at can be extended to a functor from eLATS to TSI which simply 'forgets' the event component of LATS morphisms, i.e., for $(\sigma, \eta, \lambda) \colon A \to A'$, take $at\big((\sigma, \eta, \lambda)\big)$ to be (σ, λ). We shall see next that such a functor is right adjoint to $ta \colon \mathsf{TSI} \hookrightarrow \mathsf{eLATS}$.

Proposition 2.6 ($ta \dashv at \colon \mathsf{TSI} \rightharpoonup \mathsf{eLATS}$)
For any $A \in \mathsf{eLATS}$ and any morphism $m \colon T \to at(A)$ in TSI, there exists a unique morphism $m^T \colon ta(T) \to A$ such that $at(m^T) = m$.

Proof. Let m be (σ, λ). Clearly, by definition of at, m^T must be of the form $(\sigma, \gamma, \lambda)$ for some $\gamma \colon E_{ta(T)} \to E_A$. It is easy to realize that the only possible choice for γ is the following: for $(s, a, s') \in Tran_T$ and $\lambda(a)\downarrow$, let $\gamma([(s, a, s')])$ be the event $e \in E_A$ of the unique transition $\big(\sigma(s), e^{\lambda(a)}, \sigma(s')\big) \in Tran_A$. This is a well given definition, for Lemma 1.6 ensures that m maps all transitions in $[(s, a, s')]$ to the same $\sim$-class of $Tran_{at(A)}$, and the proof of Proposition 2.5 shows that if two transitions belong to the same $\sim$-class of $Tran_{at(A)}$, they originate from transitions in $Tran_A$ carrying the same event. This proves both existence and uniqueness of m^T. ✓

Proposition 2.6 proves that the identity natural transformation

$$\eta = \big\{ id_T \colon T \to at \circ ta(T) \big\}_{T \in \mathsf{TSI}}$$

is the *unit* of an adjunction involving ta and at. Moreover, since η is an isomorphism, by standard results in category theory, we have that the adjunction $ta \dashv at \colon \mathsf{TSI} \rightharpoonup \mathsf{eLATS}$ is a coreflection, i.e., TSI is *coreflective* in eLATS. This, together with Proposition 2.5 and the discussion at the beginning of the present section, shows that eLATS is the largest subcategory of LATS on which at can be defined as a functor to TSI, yielding a right adjoint to ta.

3 meLATS: A category of LATS equivalent to TSI

In this section we identify the *replete* image of ta in LATS, i.e., the full subcategory meLATS of eLATS consisting of the objects isomorphic to $ta(T)$, for some $T \in \mathsf{TSI}$. In addition, we characterise those lats for which the independence can be recovered from the structure of events, and relate them to a relevant subcategory of TSI considered in [15, 18].

Recall from basic category theory that meLATS is determined by the coreflection: it consists of those $A \in \mathsf{eLATS}$ for which the corresponding component ϵ_A of the *counit* of $ta \dashv at$ is iso. Applying standard categorical results to derive ϵ from $(-)^T$ and η, we find that it is the natural transformation

$$\epsilon = \big\{ (id_{S_A}, \gamma, id_{L_A}) \colon ta \circ at(A) \to A \big\}_{A \in \mathsf{eLATS}},$$

where for $(s, a, s') \in Tran_{at(A)}$, $\gamma([(s, a, s')]) =_{def} e$, for $e \in E_A$ the event of the unique $(s, e^a, s') \in Tran_A$. Clearly, ϵ_A is iso if and only if γ is such, i.e.,

$$\forall t, t' \in Tran_A,\ \pi_2(t) = \pi_2(t') \quad \Rightarrow \quad at(t) \sim at(t'),$$

which means that two transitions carry the same event if and only if they belong to the same $\sim$-class of A (viewed as a tsi). Although this characterises meLATS $\subset$ LATS equivalent to TSI, it would of course be better to find a more direct description of it, one not referring to $at(A)$. This is the purpose of the notion of *event-maximal* asynchronous transitions systems introduced next.

Intuitively, a lats is *event-maximal* if its events and independence are 'tightly coupled', so that one cannot 'split' events without destroying the global lats structure. More precisely, A is event-maximal if for any $\bar{e} \in E_A$ and any subset T of transitions carrying $\bar{e}$, the structure resulting from replacing $\bar{e}$ on the transitions in T by a *fresh* event $\tilde{e}$ is *no* longer a lats.

Definition 3.1 (*Event-Maximal Asynchronous Transition Systems*)
For A a LATS, $\bar{e} \in E_A$, and $T \subset T_{\bar{e}} = \{t \in Tran_A \mid \pi_2(t) = \bar{e}\}$, let $A[T]$ denote the replacement of $\bar{e}$ on the transitions in T for a fresh event $\tilde{e} \notin E_A$, i.e., $A[T] = (S_A, i_A, E_A \cup \{\tilde{e}\}, Tran, I, L_A, \ell)$, where

- $\triangleright$ $Tran = \big(Tran_A \smallsetminus T\big) \cup \big\{(s_1, \tilde{e}, s_2) \bigm| (s_1, \bar{e}, s_2) \in T\big\}$;
- $\triangleright$ $I = I_A \cup \big\{(\tilde{e}, e) \bigm| \bar{e}\ I_A\ e\big\}$;
- $\triangleright$ $\ell(e) = \begin{cases} \ell_A(e) & \text{if } e \in E_A, \\ \ell_A(\bar{e}) & \text{if } e = \tilde{e}. \end{cases}$

A lats A is event-maximal if for each $\bar{e} \in E_A$ and each nonempty $T \subset T_{\bar{e}}$, the transition systems $A[T]$ is not a lats.
The category meLATS is the full subcategory of LATS consisting of the extensional, event-maximal lats.

Observe that the interesting, nontrivial choices for T are those such that $\varnothing \subset T \subset T_{\bar{e}}$, i.e., those in which at least one $\tilde{e}$-transition is added and at least one $\bar{e}$-transition is kept in $A[T]$. The definition above, stating that any such structure must fail to be a lats, is our way to express that—as remarked in the introduction—the identity of the events in event-maximal lats is forced by the independence relation. This provides us with the direct characterisation of TSI in terms of LATS that we sought.

Proposition 3.2 (meLATS $\cong$ TSI)
meLATS *is equivalent to* TSI.

Proof. Let A be an extensional lats. We prove that the counit ϵ_A is iso if and only if A belongs to meLATS. To this purpose, let γ be the event component of ϵ_A.

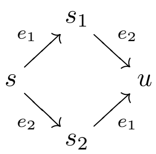
If γ is iso, i.e., for all $t, t' \in Tran_A$ we have that $\pi_2(t) = \pi_2(t')$ implies $at(t) \sim at(t')$, for any choice of $\bar{e} \in E_A$ and any $\varnothing \subset T \subset T_{\bar{e}}$, then the condition in Definition 3.1 is satisfied, since, by the extensionality of A, either A3 or A4 must fail for $A[T]$. In fact, in order for $A[T]$ to be a LATS, extensionality implies that $t' \in T$ whenever a $at(t') \sim at(t)$ for some $t \in T$, i.e., by the hypothesis on γ, T should be $T_{\bar{e}}$. So A is event-maximal.

If γ is not iso, i.e., if there exist t and t' such that which $at(t) \not\sim at(t')$ but $\pi_2(t) = \pi_2(t')$, then $T = \{t'' \mid at(t'') \sim at(t)\} \subset T_{\pi_2(t)}$ is a nonempty set for which the 'splitting' of $\pi_2(t)$ yields a lats, i.e., A is not event-maximal. ✓

To conclude this exposition, we observe that the independence relation in event-maximal lats is *not* uniquely determined by rest of the structure. This is due to the fact that the independence on events is still rather *intensional* notion: events may be independent and still never occur in the same path, i.e., intuitively, be mutually exclusive. Observing that such situations have little computational relevance, one may consider on lats the property

$$e_1 \; I_A \; e_2 \quad \Rightarrow \quad \exists (s, e_1, s_1), (s, e_2, s_2) \in \mathit{Tran}_A, \tag{E}$$

which can be seen as an extensionality condition on I_A. It is easy to prove that, if $A \in$ meLATS satisfies (E), then $e_1 \; I_A \; e_2$ if and only if there exists a square in A involving e_1 and e_2, i.e.,



Thus, for such lats the independence is completely redundant and can be omitted: all the information is already contained in $(S_A, i_A, E_A, \mathit{Tran}_A, L_A, \ell_A)$.

It is worth remarking here that a condition corresponding to (E) for TSI—viz., whenever $t \; I_T \; t'$, there exist $(s, a, s') \sim t$ and $(s, b, s'') \sim t'$ in Tran_T—was identified in [15, 18] while investigating the tight relationships between tsi and event structures. Such a condition yields $\mathsf{TSI_E}$, a very good-behaved full subcategory of TSI for which we can state the following corollary of Proposition 3.2, which concludes the paper. Here we use $\mathsf{meLATS_E}$ to denote the full subcategory of meLATS consisting of the structures satisfying (E).

Proposition 3.3 ($\mathsf{meLATS_E} \cong \mathsf{TSI_E}$)
$\mathsf{meLATS_E}$ *is equivalent to* $\mathsf{TSI_E}$.

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