

# Authors' response to the referees' reports

Paper title: Algorithms for deterministic balanced subspace identification  
Authors: Ivan Markovsky, Jan C. Willems, Paolo Rapisarda, and Bart L.M. De Moor

13 May 2004

We thank the referees for their efforts to improve the manuscript.

In this document, we quote in **bold face** statements from the reports. Our replies follow in ordinary print.

## Review #1 (ID 8557)

**But a construction of balanced realization stated in Section 5 is well-known.**

Yes, the procedure is outlined in the introduction as the “basic algorithm” and is not (claimed to be) new. It can be found in [VD96, Ch. 5]. Section 5 gives the details for clarity of the exposition. We added the following note in the text of Section 5: “Steps 3, 4, and 5 follow from standard derivations, which we now repeat for completeness.”

**As stated in the paper, once the impulse responses are computed, then we can easily compute (balanced) realizations. Thus I cannot understand why the authors returns to the original observed data to construct realizations.**

Kung's algorithm is indeed one possible way to go. Another way, however, is by returning to the data and a state sequence, which is the approach of the N4SID-type algorithms. Yet another way is the approach of the MOESP-type algorithms: the  $A$  and  $C$  parameters are extracted from the extended observability matrix without going back to the data but the  $B$  and  $D$  parameters are obtained by solving a system of equation using the original data. Clearly for exact data all methods are equivalent. They have different performance, however, in the presence of noise and they differ in the amount of computations needed.

We decided to follow the approach of the N4SID-type algorithms. The reason is partially that we want to be in the same framework as the previous algorithms [MR93] and [VD96, Ch. 5], deriving balanced realization directly from input/output data, and partially because we believe (confirmed only by numerical experience) that the N4SID-type algorithms give better accuracy than the alternative using Kung's algorithm.

**Definition 1: Is “the PE of order  $L$ ” correct for a vector signal.**

Yes, because  $\mathcal{H}_L(\tilde{u})$  has  $L$  block rows and not  $L$  rows.

**the term system lag: It seems to me that this term may not be standard.**

We define it at the first place where it is used: “The system lag 1 is defined as the observability index of  $\mathcal{S}$ .”

**Definition of block rows: “row dim” is used inconsistently in p. 6 and in Note 3 of p. 7.**

Corrected.

**line 6 from below: “which” should read “of which”.**

Corrected to “whose”.

**“The proposed algorithm computes a representation arbitrary close to an infinite time balanced one.” But this is difficult to follow. Please restate it.**

Corrected to: “By choosing the parameter  $\Delta$  large enough, the proposed algorithm computes a representation arbitrary close to an infinite time balanced one.”

**A.2: For many people, the second proof will be enough as far as the subspace identification method is concerned.**

We elaborated the fundamental lemma and its implications in [WMRM04]. For this reason, we find it appropriate to skip altogether the proof from the present manuscript and refer to [WMRM04].

**Moore, B. C.: For discrete-time systems, Pernebo and Silverman (1982) may be more appropriate.**

The reference suggested is added.

## Review #3 (ID 8559)

**In every subspace algorithm there is a kind of model reduction step involved, in particular for approximate modelling and for the case of noisy data! The authors do not mention how they deal with model reduction in the proposed algorithm.**

Quoting from the introduction (page 2):

... in a balanced basis one can apply truncation as a very effective heuristic for model reduction, which yields a method for approximate identification. Thus exact identification of a balanced model is a prelude to approximate identification.

We consider identification of the model in a balanced basis precisely because it is suitable for model reduction. The model reduction step need not a proper part of the balancing algorithm.

**I suspect that this is done in the SVD decomposition step of the algorithm by keeping the  $n$  largest singular values and by skipping all others. As an alternative one could use the fact that (for exact data) the matrix  $R_{21}$  (defined in Note 3) is rank deficient.**

If the model reduction step has to be implemented in the algorithm (*e.g.*, because the identification of the high order model is not feasible), then it is done on step 5 of the basic algorithm: the solution of the linear

system of equations. Aiming at a reduced  $r$ th order model, we truncate the balanced state sequence  $\{\tilde{x}_{\text{bal}}\}$  to the first  $r$  rows, call the result  $\{\tilde{x}_{\text{red}}\}$ , and solve the *least squares problem*

$$\begin{bmatrix} \tilde{x}_{\text{red}}(\mathbf{n}_{\text{max}} + 2) & \cdots & \tilde{x}_{\text{red}}(\mathbf{n}_{\text{max}} + T - \Delta + 1) \\ \tilde{y}(\mathbf{n}_{\text{max}} + 1) & \cdots & \tilde{y}(\mathbf{n}_{\text{max}} + T - \Delta) \end{bmatrix} = \begin{bmatrix} A_{\text{red}} & B_{\text{red}} \\ C_{\text{red}} & D_{\text{red}} \end{bmatrix} \begin{bmatrix} \tilde{x}_{\text{red}}(\mathbf{n}_{\text{max}} + 1) & \cdots & \tilde{x}_{\text{red}}(\mathbf{n}_{\text{max}} + T - \Delta) \\ \tilde{u}(\mathbf{n}_{\text{max}} + 1) & \cdots & \tilde{u}(\mathbf{n}_{\text{max}} + T - \Delta) \end{bmatrix}$$

instead of the *exact system of equations*

$$\begin{bmatrix} \tilde{x}_{\text{bal}}(\mathbf{n}_{\text{max}} + 2) & \cdots & \tilde{x}_{\text{bal}}(\mathbf{n}_{\text{max}} + T - \Delta + 1) \\ \tilde{y}(\mathbf{n}_{\text{max}} + 1) & \cdots & \tilde{y}(\mathbf{n}_{\text{max}} + T - \Delta) \end{bmatrix} = \begin{bmatrix} A_{\text{bal}} & B_{\text{bal}} \\ C_{\text{bal}} & D_{\text{bal}} \end{bmatrix} \begin{bmatrix} \tilde{x}_{\text{bal}}(\mathbf{n}_{\text{max}} + 1) & \cdots & \tilde{x}_{\text{bal}}(\mathbf{n}_{\text{max}} + T - \Delta) \\ \tilde{u}(\mathbf{n}_{\text{max}} + 1) & \cdots & \tilde{u}(\mathbf{n}_{\text{max}} + T - \Delta) \end{bmatrix}.$$

We added this clarification in the manuscript, see Note 2.

**In the abstract the authors state that this paper gives an answer to the long standing question on how to split the data into "past" and "future". In my opinion this is overstated for the following reasons: One of the main arguments is the simulation presented in section 7.**

We do not agree that "one of the main arguments is the simulation presented". The understanding how to partition the data into "past" and "future" comes from theoretical considerations (weaving lemma: Lemma 2), see the explanation on page 14. We do not refer to simulation results in the argumentation.

**In particular all of the presented measures of performance are somewhat questionable. It would be more natural to compare different algorithms on the basis of some system invariants, like poles, zeros or the parameters of a canonical form.**

We choose performance measures that are indicative for what we want to show:  $\|Y_0 - \hat{Y}_0\|_F$  for the accuracy of the approximation of the impulse response,

$$e_{\text{bal}}^2 := \frac{\|C - \hat{C}\|_F^2 + \|\mathcal{O} - \hat{\mathcal{O}}\|_F^2}{\|C\|_F^2 + \|\mathcal{O}\|_F^2}$$

for the closeness to balancing, and the flops counts for the efficiency. We do find these measures objective.

Concerning the use of other system invariants: they will be indicative for the consistency properties of the algorithms, which we do not discuss. Note that with exact data the estimated system is up to the numerical errors equivalent to the true system, so that the comparison in terms of system invariants makes sense only for noisy data or for checking the accuracy of the numerical implementations.

**The new algorithm is based on three design parameters:  $p$ ,  $f$  ( $= L$ ) and  $\Delta$ . So there is even more freedom and the authors only give some very basic guidelines on page 14 on how to choose these parameters.**

The reviewer misunderstood the meaning of the subscripts  $p$  and  $f$ ; they are not variables but refer to "past" and "future". The design parameters are only  $L$  and  $\Delta$ . The parameter  $\Delta$  determines the closeness to balancing. An alternative to  $\Delta$  is the convergence tolerance  $\varepsilon$  for the impulse response sequence. We find that either  $\Delta$  or  $\varepsilon$  is a natural user defined parameter because the basic algorithm computes only an approximation of a true balanced system and a tolerated approximation error has to be specified.

We note in the manuscript that this feature of our algorithm is an advantage over the existing algorithms for balanced subspace identification [MR93, VD96] that do not allow to control the closeness to balancing and have inherent limitation in achieving infinite time balanced model from finite amount of input/output data.

The parameter  $L$  is indeed not a natural user defined parameter. We do not know what is the "best" choice for it and instead give some general comments like the following one.  $L$  can be chosen from an efficiency

point of view to minimize the amount of work for solving the system of equations on step 3.1 of the iterative algorithms Algorithms 2 and 3. However, other criteria as numerical stability have to be taken into account as well. In a concrete numerical implementation  $L$  can be chosen from numerical results on test examples. It seems that the “optimal” performance depends on the particular computer architecture used, so the choice of  $L$  could be left to the implementation level.

**One of the interpretations of subspace algorithms is that the state is estimated from the intersection of the ”past” and the ”future” spaces. From that interpretation it seems reasonable to choose  $\mathbf{p} = \mathbf{f}$ .**

This interpretation originates from [Wil86, Wil87], but even there the equal past and future horizons is not needed and not always natural. In fact in [Wil86, Wil87], the past and future horizons are not taken to be equal.

**It is not quite clear how  $\mathbf{p}$ ,  $\mathbf{f}$  were chosen for the Van Overschee-De Moor algorithm.**

In the algorithms of Van Overschee-De Moor and Moonen-Ramos, the past and future horizons are indeed equal to the design parameter  $i$ . This is specified in the corresponding algorithms: Algorithm 5 and Algorithm 6.

**It seems that  $\mathbf{p} = \mathbf{3}$  and  $\mathbf{f} = \Delta$  was chosen. However the choice  $\mathbf{p} = \mathbf{f} = \Delta$  would be more natural from what was said above and this would lead to fairer comparison.**

The algorithms of Van Overschee-De Moor and Moonen-Ramos are implemented as outlined in the paper with past and future horizons equal to  $i$ . In the experiment  $i = \Delta$ , so we did the simulation as the reviewer suggests.

**In order to answer the above question a much more detailed analysis and/or simulation study would be necessary.**

Our answer to the question of the choice of the past and future horizons is clear and simple from the perspective of the deterministic identification problem but we do not claim that it is universal for all subspace algorithms. For example, we do not know whether the same interpretation will be appropriate in the combined deterministic-stochastic case.

**The importance of the weaving algorithm is not clear to me. For the case of exact data the system may be identified from the first  $L(>= n)$  zero inputs, impulse responses only. Choosing  $\Delta > L$  in this case just gives a different realization of the same system.**

This is correct, but our purpose is to derive (finite time) balanced realization, so we do need more samples of the impulse response. The weaving lemma, being at the heart of the recursive algorithms Algorithms 2 and 3, allows to use the same data in order to continue a response once a leading piece of it has been computed.

**However in the case of noisy data it is not clear what happens.**

Our problem is stated and solved in the purely deterministic setting. We only do some simulations with noisy data in order to give a rough idea how the estimates degrade in the presence of noise. Analysis of the algorithm in the presence of non-exact data is a question of further study.

**In this referees opinion the importance of balanced realizations is a little bit overemphasised. Of course it may be nice to get a balanced realisation directly from data. But on the other**

**hand, given some estimate of the system, it is not hard to get a (finite or infinite time) balanced realisation.**

The idea is to find balanced realization *directly* from data because if one aims to do model reduction of a very complicated model, with an indirect approach one has to identify first the high order model which may not be feasible. In contrast, the direct algorithm can do internally the model reduction step before the high order model is derived, so the approach could still be feasible.

**Why does  $\text{col}(f(0), \dots, f(T))$  start with  $f(0)$ ?**

This is a typo. We corrected it.

**Moreover the col operator is used in the paper also for stacking of matrices.**

Yes, it is used for both vectors and matrices. We do not find this notation confusing because in the matrix case, a block partitioning is always specified that makes the operation well defined.

**... This gives a system theoretic meaning to the oblique projection.**

Yes, we agree but none of the classical references on subspace identification emphasizes this useful and insightful interpretation. Moreover from our personal communication we found that for many people working on system identification, this interpretation of the oblique projection is indeed new.

**It should be mentioned that  $R_{12} = 0$ , since this is essential for the solution of the system of equations (3) via the QR decomposition.**

As suggested,  $R_{12}$  is replaced with 0 for concreteness.

**note that row  $\dim(R_{11}) = m(l_{\max} + 2\Delta) + pl_{\max}$**

Corrected.

**The stopping rule of the algorithm should be  $kL \geq 2\Delta$ .**

Corrected.

**Are the two noise components acting on  $u(t)$  and  $y(t)$  uncorrelated?**

Yes. An explanation is added.

The remaining corrections are taken into account.

## **Review #4 (ID 9084)**

**The deterministic realization problem is a very simple problem to the reviewer**

An enormous amount of papers are published on the classical realization problem (impulse response data), which is a special case of the deterministic identification problem. In our opinion and presumably in the opinion of the authors of these papers, the classical realization problem is a nontrivial problem. We deal with the following questions:

What is the shortest time sequence that allows to recover the system?  
What persistency of excitation of the input is necessary and sufficient?  
What is the most efficient in the sense of the length of the time series algorithm?

The current literature does not answer or gives only partial answers to these important issues. As elaborated in [WMRM04], the fundamental lemma of our manuscript gives quite complete answers.

In addition to the above, comment we want to point that the considered problem is the deterministic *balanced* realization problem. To the best of our knowledge there are only two solutions to this problem proposed in the literature, namely [MR93] and [VD96, Ch. 5]. In the manuscript, we give an extensive discussion in what respects the proposed algorithm is better than the ones in [MR93, VD96].

Of course, stochastic and/or approximate system identification algorithms are more involved and perhaps more useful. But one should not underestimate the problem of understanding exact system identification.

**the presentation of the paper appears to be difficult to follow due partly to the use of complicated notation and partly to the “hide-and-seek” presentation style**

We did not really understand this. We feel that the presentation in the paper is quite logical.

**For example, the significance of Theorem 1 is that it leads to the useful part of Algorithm 1, which is not new.**

To the best of our knowledge Algorithm 1 is original. We would like to know what are the references that substantiate the claim of the reviewer.

**In section 4 after (6) is presented, it is then show that it is the same as a well known oblique projection. Again nothing new here, but it is considered notable in the abstract.**

We are not sure to what the second “it” in the first sentence above refers. If the reference is to the oblique projection, then indeed as Note 6 points out, the algorithm derived from our point of view (the fundamental lemma) leads to the well known oblique projection. In this sense it is indeed not new. If the reference is to Algorithm 3, however, we disagree with the reviewer. Algorithm 3 is original and as commented in the manuscript, it gives important advantages over the block computation (*i.e.*, the classical way of computing the oblique projection). Summarizing: we show what the oblique projection does, as far as computing a system response is concerned, and use this interpretation in order to set up a more efficient algorithm to compute the oblique projection.

**The Algorithm 1, which is not considered new in the opinion of the reviewer, ...**

We disagree. See our earlier replay.

**... is hardly representative of the typical subspace identification practice.**

Yes, indeed it is not and partly for this reason our algorithm is novel.

**In fact in this algorithm only the first block column of the block Toeplitz matrix, which contains the Markov parameters, is extracted.**

This is in fact the strong point of the algorithm because, contrary to the subspace algorithms [MR93, VD96], it need not recompute the same elements of the block Toeplitz matrix repeatedly.

The paper claims that it answered the “long standing question in subspace identification on how to partition the data into “past” and “future””. This problem, however, is of most interest in the context of deterministic-stochastic realization, . . .

In the purely deterministic case the data is partitioned into “past” and “future”, so that the question how to partition is a valid question in this context too. We do not believe that there is such a thing as “*the* right question” and think that ours is *a* right question of interest.

## References

- [MR93] M. Moonen and J. Ramos. A subspace algorithm for balanced state space system identification. *IEEE Trans. on Aut. Control*, 38:1727–1729, 1993.
- [VD96] P. Van Overschee and B. De Moor. *Subspace Identification for Linear Systems: Theory, Implementation, Applications*. Kluwer Academic Publishers, 1996.
- [Wil86] J. C. Willems. From time series to linear system—Part II. Exact modelling. *Automatica*, 22(6):675–694, 1986.
- [Wil87] J. C. Willems. From time series to linear system—Part III. Approximate modelling. *Automatica*, 23(1):87–115, 1987.
- [WMRM04] J. C. Willems, I. Markovskiy, P. Rapisarda, and B. De Moor. A note on persistency of excitation. Technical Report 04–54, Dept. EE, K.U. Leuven, 2004.