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Recursive exact H_∞-identification from impulse-response measurements

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Abstract

We study the H_∞-partial realization problem from a behavioral point of view; we give necessary and sufficient conditions for solvability, and a characterization of all solutions. Instrumental in such analysis is the notion of time- and space-symmetrization of the data, which allows to transform the realization problem with metric- and stability constraints into an unconstrained behavioral modeling one.

Keywords: Behavioral approach; Most powerful unfalsified model; Quadratic difference forms; H_∞-partial realization; Data symmetrization

1. Introduction

In this paper we deal with the following problem: Let $w_0, w_1, \ldots, w_N$ be real numbers; find polynomials a and b such that

1. $a(\xi)/b(\xi) = \sum_{j=0}^{N} w_j \xi^{-j} + \varphi(\xi)\xi^{-(N+1)}$ for some proper rational function φ;
2. b is Hurwitz;
3. $\|a/b\|_\infty < 1$.

This is the Schur interpolation problem of analytic interpolation theory; we refer to [4,10–12], for a treatment along the classical lines.

In this paper we call (1)–(3) the H_∞*-partial realization problem*. In order to solve it, we adopt the point of view of partial realization as exact modeling of time series in the behavioral framework (see [1,3,7,8,13]). We model a finite number of time series derived from the impulse-response measurement, with the most powerful unfalsified model (MPUM in the following), which can be constructed iteratively; from a suitable representation of the MPUM, a solution to the H_∞-partial realization problem is easily computed. The novel

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aspect of our approach lies in how the metric and stability constraints are accommodated in such framework: we transform the H_∞-partial realization problem into an *unconstrained* behavioral modeling problem as follows. Besides modeling the "primal" data derived from $\{w_i\}_{i=0,1,\dots,N}$, we implicitly model also the *dualized data*, on which a special structure, symmetric in time and space with respect to the primal ones, has been imposed. The result of such modeling procedure is a kernel representation of the MPUM $\mathfrak{B}^*$ for the primal data, from which a kernel representation of the MPUM $\mathfrak{B}'^*$ for the dualized data is easily obtained. In this paper we also re-derive the well-known necessary and sufficient condition for the existence of a solution to the H_∞-realization problem, based on the positivity of a certain Stein matrix derived from the data (see [5]). In order to derive and express effectively such condition, the framework for quadratic difference forms developed by the first author and Fujii (see [6]) is instrumental. The last result presented in this paper is a characterization of all solutions to the H_∞-partial realization problem by means of a special representation of the MPUM $\mathfrak{B}^*$.

The approach closer to the one proposed in this paper is that of [2], in which the stable partial realization problem with metric constraints is solved. The main difference between the two approaches lies in the fact that in our case, identification is performed in the time-domain. Moreover, the proof of the correctness of our procedure for performing H_∞-identification (see Remark 2 in Section 4) is entirely self-contained, and need not refer to the algorithm of Schur.

The paper is organized as follows: in Section 2 we illustrate the basics of exact behavioral modeling. In Section 3 we introduce the concept of dualization of the data and the notion of Stein matrix associated with the data. Necessary and sufficient conditions for the existence of a solution are stated in Section 4, while Section 5 provides a characterization of all solutions, based on a special representation of the MPUM. We conclude the paper with Section 6, discussing the possible generalizations of the algorithm.

Notation. In this paper we denote the set of nonnegative integers with $\mathbb{Z}_+$, similarly the set of nonpositive integers is denoted with $\mathbb{Z}_-$. We denote with $\mathbb{D}_e = \{z \in \mathbb{C} \mid |z| \geqslant 1\}$ the exterior of the open unit disk. The space of n-dimensional real vectors is denoted by $\mathbb{R}^n$, and the space of $m \times n$ real matrices, by $\mathbb{R}^{m\times n}$. If $A \in \mathbb{R}^{m\times n}$, then $A^{\mathrm{T}} \in \mathbb{R}^{n\times m}$ denotes its transpose. The set consisting of all sequences from $\mathbb{Z}_+$ to $\mathbb{R}^q$ (from $\mathbb{Z}_-$ to $\mathbb{R}^q$) is denoted with $(\mathbb{R}^q)^{\mathbb{Z}_+}$ ($(\mathbb{R}^q)^{\mathbb{Z}_-}$, respectively). On such space we define the *left*, i.e. *backward*, *shift* $(\sigma w)(t) := w(t+1)$ for all $t \in \mathbb{N}$. On $(\mathbb{R}^q)^{\mathbb{Z}_-}$ we define the *right*, i.e. *forward*, *shift* σ^*, defined as $(\sigma^* w)(t) = w(t-1)$. The ring of polynomials with real coefficients in the indeterminate ξ is denoted by $\mathbb{R}[\xi]$; the ring of two-variable polynomials with real coefficients in the indeterminates ζ and η is denoted by $\mathbb{R}[\zeta,\eta]$. The space of all $n \times m$ polynomial matrices in the indeterminate ξ is denoted by $\mathbb{R}^{n\times m}[\xi]$. Given a polynomial matrix $R(\xi) = R_0 + \cdots + R_L\xi^L \in \mathbb{R}^{n\times m}[\xi]$ with $R_L \neq 0$, we define its *reciprocal matrix* $R^{\mathrm{r}}(\xi)$ as $R^{\mathrm{r}}(\xi) = R_0\xi^L + \cdots + R_L \in \mathbb{R}^{n\times m}[\xi]$.

2. Modeling with behaviors

In this paper we consider linear, shift-invariant behaviors with time-axis $\mathbb{Z}_+$ or $\mathbb{Z}_-$, in other words, subspaces of $(\mathbb{R}^q)^{\mathbb{Z}_+}$ (respectively, $(\mathbb{R}^q)^{\mathbb{Z}_-}$) consisting of trajectories $w : \mathbb{Z}_+ \to \mathbb{R}^q$ (respectively, $w : \mathbb{Z}_- \to \mathbb{R}^q$) such that if w belongs to the behavior, then also $\sigma^k w$ ($\sigma^{*k} w$) belong to the behavior for all $k \in \mathbb{Z}_+$. We denote the set of such behaviors with $\mathscr{L}^q(\mathbb{Z}_+)$ ($\mathscr{L}^q(\mathbb{Z}_-)$, respectively).

In [13] a framework has been developed for modeling in a behavioral framework; in such approach, the notion of most powerful unfalsified model is of fundamental importance, and we briefly review it now. In the following we limit ourselves to the treatment of sequences in $(\mathbb{R}^q)^{\mathbb{Z}_+}$, the case of $(\mathbb{R}^q)^{\mathbb{Z}_-}$ being completely analogous.

In this paper we consider sets $\mathscr{D} = \{d_i\}_{i=0,\dots,N}$ of *data*, where $d_i \in (\mathbb{R}^q)^{\mathbb{Z}_+}$ for $0 \leqslant i \leqslant N$, and we pick our models from the *model class* $\mathscr{M} = \mathscr{L}^q(\mathbb{Z}_+)$. A model $\mathfrak{B}$ is called an *unfalsified model* for $\mathscr{D}$ if $\mathscr{D} \subseteq \mathfrak{B}$. We call a model $\mathfrak{B}^*$ the MPUM for $\mathscr{D}$ if $\mathfrak{B}^* \supseteq \mathscr{D}$ and any other unfalsified model $\mathfrak{B}$ for $\mathscr{D}$ satisfies

$\mathfrak{B}^* \subseteq \mathfrak{B}$. Observe that the MPUM restricts the possible outcomes of the phenomenon under study to the smallest possible set not refuted by the data.

The MPUM does not need to exist in general; it can be shown, however, that for the model classes $\mathscr{L}^q(\mathbb{Z}_+)$ and $\mathscr{L}^q(\mathbb{Z}_-)$ considered in this paper, the MPUM always exists and is uniquely determined; we denote such behavior with $\mathfrak{B}^*$. It can also be proved that $\mathfrak{B}^*$ can be represented as the kernel of a matrix polynomial operator $R(\sigma)$ in the shift σ, with the property that R is square and nonsingular as a polynomial matrix.

In order to compute such a representation, the following iterative algorithm can be used (see [1]). Define $R_{-1} := I_q$ and proceed iteratively as follows for $k = 0, 1, \ldots, N$. At step k, define the kth *error trajectory* $\varepsilon_k := R_{k-1}(\sigma)d_k$. Compute (for example using the algorithm on p. 679 of [13]) the polynomial matrix corresponding to a kernel representation E_k of the MPUM for ε_k, i.e. $E_k(\sigma)\varepsilon_k = 0$. Then define $R_k := E_k R_{k-1}$. After $N+1$ steps, if N is the number of time series in $\mathscr{D}$, such algorithm produces a $q \times q$ polynomial matrix R_N such that $R_N(\sigma)d_i = 0$ for $1 \leqslant i \leqslant N$; then $\mathfrak{B}^*_{\mathscr{D}} = \ker R_N(\sigma)$.

Such procedure can be refined in order to provide a solution to the minimal partial realization problem *without* stability and metric constraints; we refer the reader to [8], where the Kuijper–Berlekamp–Massey (KBM) algorithm for behavioral modeling of partial realization data is presented. In order to accommodate the stability- and metric constraints of the H_∞-partial realization problem, the concept of dualization of the data, introduced in the next section, is key.

3. Data dualization, and the Stein matrix

As in the KBM algorithm, from the impulse response samples w_i, $i = 0, \ldots, N$ we define the following time series with time axis $\mathbb{Z}_+$:

$$d = \left\{ \begin{pmatrix} w_N \\ 0 \end{pmatrix}, \begin{pmatrix} w_{N-1} \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} w_1 \\ 0 \end{pmatrix}, \begin{pmatrix} w_0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \ldots \right\}. \tag{1}$$

In the following we call (1) the *primal data*, and we refer to the problem of finding a representation of the MPUM for such time series and its left (backwards) shifts $\sigma^k d$, $k = 0, 1, \ldots,$ as the *primal modeling problem*. Observe that the primal problem is a behavioral modeling problem with model class $\mathscr{L}^2(\mathbb{Z}_+)$.

We associate to the primal time series (1) its *dual*, defined as the time series with time axis $\mathbb{Z}_-$:

$$d' = \left\{ \ldots, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ w_0 \end{pmatrix}, \begin{pmatrix} 0 \\ w_1 \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ w_{N-1} \end{pmatrix}, \begin{pmatrix} 0 \\ w_N \end{pmatrix} \right\}. \tag{2}$$

Observe that d' is obtained from d by reversing the direction of time and multiplying the result by

$$\Pi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{3}$$

In the following we refer to the problem of finding (a representation of) the MPUM for (2) and its right (forward) shifts $\sigma^{*k} d'$, $k = 0, 1, \ldots,$ as the *dual modeling problem*. Observe that the dual modeling problem is a behavioral modeling problem with model class $\mathscr{L}^2(\mathbb{Z}_-)$.

Finally, we introduce the notion of Stein matrix associated with the impulse-response data w_i, $i = 0, \ldots, N$. Such matrix will be instrumental in stating one of the necessary and sufficient conditions for the existence of a solution to the H_∞-partial realization problem which is presented in the next section. Denote the successive left (backwards) shifts of the primal time series d as

$$d_k := \sigma^{N-k} d = \left\{ \begin{pmatrix} w_k \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} w_1 \\ 0 \end{pmatrix}, \begin{pmatrix} w_0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \ldots \right\}, \tag{4}$$

$k = 0, 1, \ldots,$ and consider the MPUM $\mathfrak{B}^*$ for $\{d_k\}_{k=0,1,\ldots,N}$; observe that $\mathfrak{B}^*$ consists of all linear combination of the d_k's. Let $v, w \in \mathfrak{B}^*$, and let J denote the 2×2 matrix

$$J = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{5}$$

Such matrix induces the indefinite inner product on $\mathfrak{B}^*$ defined by $\langle v, w \rangle_J := \sum_{k=0}^{\infty} v(k)^{\mathrm{T}} J w(k)$; such inner product is well-defined, since all trajectories in $\mathfrak{B}^*$ have compact support. The *Stein matrix* associated with $\{w_i\}_{i=0,\ldots,N}$ is the symmetric $(N+1) \times (N+1)$ matrix defined as

$$S_{\{d_i\}_{i=0,\ldots,N}} = (\langle d_i, d_j \rangle_J). \tag{6}$$

Whenever the dimensions of the Stein matrix will be clear from the context, we will simply denote it with $S_{\{d_i\}}$.

4. Necessary and sufficient conditions for solvability

In this section we state two necessary and sufficient conditions for the existence of a solution to the H_∞-partial realization problem. The first, classic, condition consists of the positive definiteness of the Stein matrix of the data introduced in Section 3. The second one is new, and relates the solvability of the H_∞-partial realization problem with the existence of a special representation $\mathfrak{B}^* = \ker R(\sigma)$ of the MPUM for the primal data $\{\sigma^k d\}_{k=0,1,\ldots}$, which after reciprocation ($R \to R^{\mathrm{r}}$) yields a representation $\mathfrak{B}'^* = \ker R^{\mathrm{r}}(\sigma^*)$ of the MPUM for the dual data $\{\sigma^{*k} d'\}_{k=0,1,\ldots}$.

Theorem 1. *The following three statements are equivalent*:

1. *The Stein matrix* $S_{\{d_i\}_{i=0,\ldots,N}}$ *is positive definite*;
2. *The MPUM* $\mathfrak{B}^* \in \mathscr{L}^2(\mathbb{Z}_+)$ *for the primal data* $\{\sigma^k d\}_{k=0,1,\ldots} \subseteq (\mathbb{R}^2)^{\mathbb{Z}_+}$ *and the MPUM* $\mathfrak{B}'^* \in \mathscr{L}^2(\mathbb{Z}_-)$ *for the dual data* $\{\sigma^{*k} d'\}_{k=0,1,\ldots} \subseteq (\mathbb{R}^2)^{\mathbb{Z}_-}$ *have kernel representations* $\mathfrak{B}^* = \ker R(\sigma)$ *and* $\mathfrak{B}'^* = \ker R^{\mathrm{r}}(\sigma^*)$ *respectively*, *induced by a matrix* $R := (r_{ij})_{i,j=1,2}$ *satisfying the following properties*:
   (a) $(r_{11} \;\; r_{12}) = (r_{21} \;\; r_{22})^{\mathrm{r}} \Pi$;
   (b) r_{22} *is a Hurwitz polynomial*;
   (c) $R(\xi) J R(\xi^{-1})^{\mathrm{T}} = R(\xi^{-1})^{\mathrm{T}} J R(\xi) = J = R(\xi^{-1}) J R(\xi)^{\mathrm{T}} = R(\xi)^{\mathrm{T}} J R(\xi^{-1})$;
   (d) $\|r_{21}/r_{22}\|_\infty < 1$;
   (e) $\|r_{12}/r_{22}\|_\infty < 1$;
3. *There exists a solution to the* H_∞*-partial realization problem.*

Proof. We first prove the implication (1) $\Rightarrow$ (2), using induction on the number K of time series to be modeled.

For $K=0$, consider the model $\mathfrak{B}_0 \in \mathscr{L}^q(\mathbb{Z}_+)$ represented in kernel form by

$$R_0(\xi) = -\frac{1}{1-w_0^2}\begin{pmatrix} -1 & w_0 \\ -w_0 & w_0^2 \end{pmatrix} + \frac{1}{1-w_0^2}\begin{pmatrix} -w_0^2 & w_0 \\ -w_0 & 1 \end{pmatrix}\xi. \tag{7}$$

Observe that $R_0(\sigma)d_0 = 0$ and consequently $\ker R_0(\sigma)$ is an unfalsified model for $\{\sigma^k d_0\}_{k=0,1,\ldots}$. Since $\det(R_0(\xi)) = \xi$, the dimension of $\ker R_0(\sigma)$ is one, and consequently $\{\sigma^k d_0\}_{k=0,1,\ldots} \supseteq \ker R_0(\sigma)$. This shows that R_0 is a representation of the MPUM $\mathfrak{B}_0$ for d_0. It is easy to verify in an analogous way, that the model $\mathfrak{B}'_0 = \ker R_0^{\mathrm{r}}(\sigma^*) \in \mathscr{L}^q(\mathbb{Z}_-)$ represented in kernel form by $R_0^{\mathrm{r}}(\xi)$ is the MPUM for $\{\sigma^{*k} d'_0\}_{k=0,1,\ldots}$.

It is a matter of straightforward verification to prove that the matrix R_0 defined in (7) satisfies (2a) and (2c). Denote the entries of R_0 as r_{ij}^0, $i,j=1,2$. In order to prove (2b), note that $r_{22}^0(\xi)$ has its root in w_0^2; since the Stein matrix $1-w_0^2>0$, we conclude that r_{22}^0 is Hurwitz.

In order to prove (2d), note that $\|r_{21}^0/r_{22}^0\|_\infty < 1$ if and only if $(r_{21}^0(\mathrm{e}^{\mathrm{i}\omega})/r_{22}^0(\mathrm{e}^{\mathrm{i}\omega}))(r_{21}^0(\mathrm{e}^{-\mathrm{i}\omega})/r_{22}^0(\mathrm{e}^{-\mathrm{i}\omega})) < 1$ for all $\omega \in \mathbb{R}$, or equivalently $r_{22}^0(\mathrm{e}^{\mathrm{i}\omega})r_{22}^0(\mathrm{e}^{-\mathrm{i}\omega}) - r_{21}^0(\mathrm{e}^{\mathrm{i}\omega})r_{21}^0(\mathrm{e}^{-\mathrm{i}\omega}) > 0$ for all $\omega \in \mathbb{R}$. Now observe that $r_{22}^0(\mathrm{e}^{\mathrm{i}\omega})r_{22}^0(\mathrm{e}^{-\mathrm{i}\omega}) - r_{21}^0(\mathrm{e}^{\mathrm{i}\omega})r_{21}^0(\mathrm{e}^{-\mathrm{i}\omega})$ is the (2,2)-entry of $R_0(\mathrm{e}^{\mathrm{i}\omega})JR_0(\mathrm{e}^{-\mathrm{i}\omega})^{\mathrm{T}}$ and from property (2c), such entry equals 1, which proves the claim.

In order to prove (2e), note that $r_{22}^0(\mathrm{e}^{\mathrm{i}\omega})r_{22}^0(\mathrm{e}^{-\mathrm{i}\omega}) - r_{12}^0(\mathrm{e}^{\mathrm{i}\omega})r_{12}^0(\mathrm{e}^{-\mathrm{i}\omega})$ is the (2,2)-entry of $R_0(\mathrm{e}^{\mathrm{i}\omega})^{\mathrm{T}}JR_0(\mathrm{e}^{-\mathrm{i}\omega})$ and from property (2c), such entry equals 1. Then proceed as in the proof of (2d).

This concludes the proof of properties (2a)–(2e) for the representation (7) of the MPUM for $K=0$.

Before proceeding with the inductive step, we prove a property of the error time-series $\varepsilon_1 = R_0(\sigma)d_1$ which is essential in order to apply the one-step model (7) iteratively. We claim that $S_{\{d_i\}} > 0$ implies that the second component of $\varepsilon_1(0)$ is nonzero; in such case ε_1 can be multiplied by a suitable constant so that the one-step model (7) can be applied iteratively. We call such property of the error time-series the *nondegeneracy property*. In order to prove such claim, observe that the second component of $\varepsilon_1(0) = (R_0(\sigma)d_1)(0)$ equals $(1/(1-w_0^2))w_0w_1 + 1$. Assume by contradiction that it is zero, equivalently, $-w_0w_1 = 1-w_0^2$. Observe that $-w_0w_1$ equals the (1,2)- and (2,1)-entry of $S_{\{d_i\}}$. Now substitute $1-w_0^2$ for such entries, and transform $S_{\{d_i\}}$ as $T^{\mathrm{T}}S_{\{d_i\}}T$ with the nonsingular matrix T defined as

$$T = \begin{pmatrix} 1 & -(1-w_0^2) & 0_{1\times(N-1)} \\ 0 & 1-w_0^2 & 0_{1\times(N-1)} \\ 0_{(N-1)\times 1} & 0_{(N-1)\times 1} & I_{N-1} \end{pmatrix}.$$

It can be verified that the upper-left 2×2 submatrix of $T^{\mathrm{T}}S_{\{d_i\}}T$ is $\mathrm{diag}(1-w_0^2, -w_1^2(w_0^2-1)^2)$; since the (2,2)-entry of such submatrix is $\leqslant 0$, this contradicts the positivity of $S_{\{d_i\}}$. We conclude that the nondegeneracy property holds.

We now go to the inductive step. Define the new set of data $\hat{d}_{k-1} = \sigma^{N-k}R_0(\sigma)d$, $k=1,2,\ldots,K-1$ and assume that if necessary, the normalization of the second component of $\hat{d}_0(0) = (\sigma^{N-1}R_0(\sigma)d)(0) = (R_0(\sigma)d_1)(0)$ has been carried out. In order to apply the inductive assumption, we now show that the Stein matrix $S_{\{\hat{d}_k\}_{k=0,1,\ldots,K-2}}$ of the $\hat{d}_k$'s is positive definite.

In order to do this, denote the $K\times K$ Stein matrix $S_{\{d_k\}_{k=0,1,\ldots,K-1}}$ of the original data with

$$S_{\{d_k\}_{k=0,1,\ldots K-1}} = \begin{pmatrix} \langle d_0, d_0\rangle_J & b^{\mathrm{T}} \\ b & S' \end{pmatrix}$$

where $b^{\mathrm{T}} = (\langle d_0, d_1\rangle_J \cdots \langle d_0, d_{K-1}\rangle_J)$ and $S'_{ij} = \langle d_i, d_j\rangle_J,\ i,j = 1,\dots,K-1$. Let

$$T = \begin{pmatrix} 1 & -\langle d_0, d_0\rangle_J^{-1} b^{\mathrm{T}} \\ 0_{(K-1)\times 1} & I_{(K-1)\times(K-1)} \end{pmatrix}$$

and observe that T is well-defined and nonsingular because $\langle d_0, d_0\rangle_J > 0$. Under the congruence transformation induced by T, the matrix $S_{\{d_k\}_{k=0,1,\dots,K-1}}$ becomes

$$T^{\mathrm{T}} S_{\{d_k\}_{k=0,1,\dots,K-1}} T = \begin{pmatrix} \langle d_0, d_0\rangle_J & 0_{1\times(K-1)} \\ 0_{(K-1)\times 1} & -\langle d_0, d_0\rangle_J^{-1} b b^{\mathrm{T}} + S' \end{pmatrix}.$$

We now show that such (2,2)-block of $T^{\mathrm{T}} S_{\{d_i\}_{i=0,\dots,K-1}} T$ is the Stein matrix of the new data $\{\hat{d}_k\}_{k=0,\dots,K-2}$; from this, its positive definiteness follows directly. Consider that $\hat{d}_i = \sigma^{N-i-1} R_0(\sigma) d = R_0(\sigma)\sigma^{N-i-1} d = R_0(\sigma) d_{i+1}$, $i = 0,\dots,K-2$ and consequently $\langle \hat{d}_i, \hat{d}_j\rangle_J = \langle R_0(\sigma) d_{i+1},\ R_0(\sigma) d_{j+1}\rangle_J = \sum_{k=0}^{\infty} L_\Phi(d_{i+1}, d_{j+1})(k)$ where L_Φ is the bilinear difference form induced by

$$\Phi(\zeta,\eta) = R_0(\zeta)^{\mathrm{T}} J R_0(\eta) \in \mathbb{R}^{2\times 2}[\zeta,\eta].$$

From the second equality of property (2c) it follows that $\delta\Phi(\xi) := \Phi(\xi^{-1},\xi) = J$; applying the argument used in the necessity part of Lemma 3.1 of [6] we conclude that $\Phi(\zeta,\eta) - J$ is divisible by $\zeta\eta - 1$, and consequently there exists $\Psi(\zeta,\eta) \in \mathbb{R}^{2\times 2}[\zeta,\eta]$ such that $\Phi(\zeta,\eta) - J = (\zeta\eta - 1)\Psi(\zeta,\eta)$. Indeed, such equality holds with $\Psi(\zeta,\eta)$ defined as

$$\Psi(\zeta,\eta) = \frac{1}{1-w_0^2}\begin{pmatrix} w_0^2 & -w_0 \\ -w_0 & 1 \end{pmatrix}.$$

We conclude that

$$\begin{aligned}\langle \hat{d}_i, \hat{d}_j\rangle_J &= -L_\Psi(d_{i+1}, d_{j+1})(0) + \sum_{k=0}^{\infty} L_J(d_{i+1}, d_{j+1})(k) \\ &= -L_\Psi(d_{i+1}, d_{j+1})(0) + \langle d_{i+1}, d_{j+1}\rangle_J = -\langle d_0, d_0\rangle^{-1}\langle d_0, d_{i+1}\rangle_J \langle d_0, d_{j+1}\rangle_J + \langle d_{i+1}, d_{j+1}\rangle_J,\end{aligned}$$

$i,j = 0,\dots,K-2$. It is a matter of straightforward verification to check that the last expression is indeed the $(i+1,\ j+1)$th entry of $-\langle d_0, d_0\rangle_J^{-1} b b^{\mathrm{T}} + S'$ as was to be proved. This yields immediately the positivity of the matrix $S_{\{\hat{d}_k\}_{k=0,1,\dots,K-2}}$.

Having shown this, we apply the inductive assumption, and conclude that there exists a representation R' of the MPUM for the $\hat{d}_k$'s, $k = 0,\dots,K-2$, satisfying properties (2a)–(2e) above, and such that the nondegeneracy property holds, i.e. that the second component of $(R'(\sigma)\hat{d}_{K-1})(0)$ is nonzero; observe that by inductive assumption, such representation induces also a representation of the MPUM for the dual problem. A representation of the MPUM for d_i, $i = 0,\dots,K-1$, is obtained as

$$\underbrace{\begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix}}_{R} := \underbrace{\begin{pmatrix} r'_{11} & r'_{12} \\ r'_{21} & r'_{22} \end{pmatrix}}_{R'} \underbrace{\begin{pmatrix} r^0_{11} & r^0_{12} \\ r^0_{21} & r^0_{22} \end{pmatrix}}_{R_0}$$

(cf. the iterative behavioral modeling algorithm presented in Section 2). We now prove that such representation satisfies properties (2a)–(2d) of the theorem and the nondegeneracy property (i.e. that the second component of $(R(\sigma)d_K)(0) = (R(\sigma)\sigma^{N-K}d)(0)$ is nonzero).

In order to prove that (2a) is satisfied, verify first that $\Pi R_0 = R_0^{\mathrm{r}}\Pi$. Now use the inductive assumption (2a) on R' and then the fact that the reciprocal of the product of two matrix polynomials is the product of the reciprocals, in order to prove that

$$(r_{11} \quad r_{12}) = (r'_{11} \quad r'_{12})R_0 = (r'_{21} \quad r'_{22})^{\mathrm{r}}\Pi R_0$$
$$= (r'_{21} \quad r'_{22})^{\mathrm{r}}R_0^{\mathrm{r}}\Pi = ((r'_{21} \quad r'_{22})R_0)^{\mathrm{r}}\Pi = (r_{21} \quad r_{22})^{\mathrm{r}}\Pi.$$

In order to prove (2b), that is, $r_{22} = r'_{21}r^0_{12} + r'_{22}r^0_{22}$ is Hurwitz, assume by contradiction that there exists $\mu \in \mathbb{D}_e$ such that $0 = r_{22}(\mu) = r'_{21}(\mu)r^0_{12}(\mu) + r'_{22}(\mu)r^0_{22}(\mu)$. Since by inductive assumption r'_{22} and r^0_{22} are Hurwitz, $r'_{22}(\mu)r^0_{22}(\mu) \neq 0$ and consequently we can write $|r'_{21}(\mu)r^0_{12}(\mu)/r'_{22}(\mu)r^0_{22}(\mu)| = |r'_{21}(\mu)/r'_{22}(\mu)|\,|r^0_{12}(\mu)/r^0_{22}(\mu)| = 1$, which yields a contradiction with $\|r^0_{12}/r^0_{22}\|_\infty < 1$, $\|r'_{21}/r'_{22}\|_\infty < 1$. Consequently, r_{22} is Hurwitz.

In order to prove (2c), observe that

$$R(\xi)JR(\xi^{-1})^{\mathrm{T}} = R'(\xi)\underbrace{R_0(\xi)JR_0(\xi^{-1})^{\mathrm{T}}}_{=J}R'(\xi^{-1})^{\mathrm{T}} = R'(\xi)JR'(\xi^{-1})^{\mathrm{T}} = J.$$

Analogous computations yield the other equalities of (2c).

In order to prove (2d), and (2e), proceed with the same argument used in the proof of the $K = 0$ step.

Finally, we prove the nondegeneracy property, namely that the second component of $(R(\sigma)d_K)(0) = (R(\sigma)\sigma^{N-K}d)(0)$ is nonzero. This follows from

$$R(\sigma)d_K = R(\sigma)\sigma^{N-K}d = R'(\sigma)R_0(\sigma)\sigma^{N-K}d = R'(\sigma)\underbrace{\sigma^{N-K}R_0(\sigma)d}_{\hat{d}_{K-1}} = R'(\sigma)\hat{d}_{K-1}$$

and from the inductive assumption. This concludes the proof of (1) ⇒ (2).

We now prove (2) ⇒ (3). Let a representation of the MPUM for $\{\sigma^k d\}_{k=0,1,\ldots}$ satisfying (a)–(e) be given. We prove now that $-r_{21}/r_{22}$ is a solution to the H_∞-partial realization problem. Observe that since $\|r_{21}/r_{22}\|_\infty < 1$, the metric constraint is satisfied, and moreover, r_{22} is Hurwitz from (2b). We now show that $-r_{21}/r_{22}$ has the required expansion at infinity. In order to do this, first write

$$-r_{22}(\xi) =: q_0 + q_1\xi + \cdots + q_n\xi^n = q(\xi), \quad r_{21}(\xi) =: p_0 + p_1\xi + \cdots + p_m\xi^m = p(\xi).$$

Observe that since $\|r_{21}/r_{22}\|_\infty < 1$, $\deg(r_{22}) = n \geqslant \deg(r_{21}) = m$. Consider that $-r_{21}(\xi)/r_{22}(\xi) = w_0 + w_1\xi^{-1} + \cdots + w_N\xi^{-N} + \cdots$ if and only if

$$p_0 + \cdots + p_m\xi^m = (q_0 + \cdots + q_n\xi^n)(w_0 + w_1\xi^{-1} + \cdots + w_N\xi^{-N} + \cdots).$$

Equating the corresponding powers of ξ on both sides of such expression, we obtain three series of equalities:

$$p_j = q_jw_0 + \cdots + q_nw_{n-j}, \quad (\text{coefficients of } \xi^j,\ j = 0,\ldots,m), \tag{8}$$

$$0 = q_jw_0 + \cdots + q_nw_{n-j}, \quad (\text{coefficients of } \xi^j,\ j = m+1,\ldots,n), \tag{9}$$

$$0 = q_0w_j + \cdots + q_nw_{n+j}, \quad (\text{coefficients of } \xi^{-j},\ j > 0). \tag{10}$$

Now observe that from (2a) and from the fact that R is a representation of the MPUM satisfying (a)–(e) it follows that $(-q \;\; p)^{\mathrm{r}} = (r_{11} \;\; r_{12})$ represents an unfalsified model for $\{\sigma^k d\}_{k=0,1,\ldots}$, that is $((r_{11}(\sigma) \;\; r_{12}(\sigma))d)(k) = 0$ for all $k \geqslant 0$. Writing down such equalities explicitly, we find that (10) corresponds to the terms with $k = 0,\ldots,N-n$; (8) correspond to those with $k = N-n+1,\ldots,N-n+m$, while those for $k > N-n+m$ correspond to (9). Observe incidentally that the above argument shows that p/q has the expansion at infinity characterized by the Markov parameters w_j, $j = 0,\ldots,N$, if and only if $(q \;\; -p)^{\mathrm{r}}$ represents an unfalsified model for d. The proof of the implication (2) ⇒ (3) is thus concluded.

Finally, we prove (3) ⇒ (1). Let f/g be a solution to the H_∞-partial realization problem; without loss of generality, assume that $\mathrm{GCD}(f,g) = 1$. Define from f and g the two-variable polynomial

$\Phi(\zeta,\eta) := g(\zeta)g(\eta) - f(\zeta)f(\eta)$. It follows from $\|f/g\|_\infty < 1$ that $\Phi(\mathrm{e}^{-\mathrm{i}\omega},\mathrm{e}^{\mathrm{i}\omega}) > 0$ for all $\omega \in \mathbb{R}$. We conclude from point (2) of Theorem 3.1 of [6] that the quadratic difference form Q_Φ induced by Φ is strictly average positive, i.e. $\sum_{k=-\infty}^{+\infty} Q_\Phi(\ell(k)) > 0$ for all square-summable ℓ. This implies $\sum_{k=-\infty}^{0} Q_\Phi(\ell(k)) > 0$ for all negative half-line square-summable ℓ; we now prove that such inequality implies the positivity of the Stein matrix S_{d_k}. Now consider the system $\mathfrak{B} \subseteq (\mathbb{R}^2)^{\mathbb{Z}}$ represented in observable image form as

$$\begin{pmatrix} y \\ u \end{pmatrix} = \begin{pmatrix} f(\sigma) \\ g(\sigma) \end{pmatrix} \ell \tag{11}$$

and observe that $\sum_{k=-\infty}^{0} Q_\Phi(\ell(k)) > 0$ for all half-line square-summable ℓ implies that

$$\left\langle \begin{pmatrix} y \\ u \end{pmatrix}, \begin{pmatrix} y \\ u \end{pmatrix} \right\rangle_{J,-} := \sum_{k=-\infty}^{0} (y(k) \quad u(k)) J \begin{pmatrix} y(k) \\ u(k) \end{pmatrix} > 0,$$

for all y, u related as in (11) to ℓ.

Since the impulse response of the system described by the transfer function f/g has the values w_i, $i = 0, \ldots, N$, it follows that the impulse response trajectory

$$w = \ldots, \underbrace{\begin{pmatrix} 0 \\ 0 \end{pmatrix}}_{k=-1}, \underbrace{\begin{pmatrix} w_0 \\ 1 \end{pmatrix}}_{k=0}, \underbrace{\begin{pmatrix} w_1 \\ 0 \end{pmatrix}}_{k=1}, \ldots, \underbrace{\begin{pmatrix} w_N \\ 0 \end{pmatrix}}_{k=N}, \underbrace{\begin{pmatrix} v_{N+1} \\ 0 \end{pmatrix}}_{k=N+1}, \underbrace{\begin{pmatrix} v_{N+2} \\ 0 \end{pmatrix}}_{k=N+1}, \ldots$$

belongs to $\mathfrak{B}$ for suitable values $v_j \in \mathbb{R}$, $j \in \mathbb{Z}$, $j \geqslant N+1$. It is easy to see that such trajectory corresponds to a square-summable ℓ. Now consider any linear combination $\sum_{i=0}^{N} \alpha_i \sigma^i w$ of the left-shifts of w up to the Nth one, with $\alpha_i \in \mathbb{R}$. Observe that the truncation of such trajectories to the time axis $(-\infty, 0]$ belongs to $\mathfrak{B}_{|(-\infty,0]}$, and that $\langle \sum_{i=0}^{N} \alpha_i \sigma^i w, \sum_{i=0}^{N} \alpha_i \sigma^i w \rangle_{J,-} > 0$. Now denote the $(N+1) \times 1$ vector of the coefficients α_i, $i = 0, \ldots, N$ with $\bar{\alpha}$, and consider that $\langle \sum_{i=0}^{N} \alpha_i \sigma^i w, \sum_{i=0}^{N} \alpha_i \sigma^i w \rangle_{J,-} = \bar{\alpha}^{\mathrm{T}} \; (\langle \sigma^i w, \sigma_j w \rangle_{J,-})_{i,j=0,\ldots,N} \bar{\alpha}$. Observe that $\langle \sigma^i w, \sigma^j w \rangle_{J,-} = \langle d_i, d_j \rangle_J$ with d_i defined by (4) and consequently

$$\left\langle \sum_{i=0}^{N} \alpha_i \sigma^i w, \sum_{i=0}^{N} \alpha_i \sigma^i w \right\rangle_{J,-} = \bar{\alpha}^{\mathrm{T}} S_{\{d_k\}} \bar{\alpha} > 0.$$

Since the coefficients α_i are arbitrary, the positive definiteness of $S_{\{d_k\}}$ follows. This concludes the proof of the theorem. □

The above result connects the Stein matrix, the solvability of the H_∞-partial realization problem, and the notion of MPUM. The first connection is well-known, see for example [5], while the second is reminiscent of the results of [9], where a "symmetrization" procedure on the data is used in place of the dualization considered above. Observe also that the structure of the one-step model (7) is the same as that of Section 1.1.4 of [5] in a non-iterative approach to the Schur interpolation problem; for an iterative approach to the solution of such problem, see also [12].

Remark 2. From the proof of the implication (1) ⇒ (2) in Theorem 1, an iterative algorithm arises, that takes as inputs the impulse response samples and provides as output a representation R of the MPUM satisfying (a)–(e), from which a solution to the H_∞-partial realization problem is easily computed. We state such an algorithm explicitly:

Input: $N+1$ impulse response samples

$w_0, w_1, \ldots, w_N$.

Output: an MPUM representation satisfying (a)–(e), if such a representation exists.

Compute the Stein matrix S of the data, and check whether $S > 0$; if no, then exit: no representation satisfying (a)–(e) exists.

Let $R_{-1}(\xi) = I_{2\times 2}$;
For $i = 0, \ldots, N$ do
  Compute ith error series $\varepsilon_i := R_{i-1}(\sigma)d_i$, with d_i defined as in (4);
  Normalize second component of $\varepsilon_i(0)$ to one;
  Compute one-step model $V_i(\xi)$ for ε_i as in (7);
  Define $R_i(\xi) := V_i(\xi)\ R_{i-1}(\xi)$;
end for;
Return $R_N(\xi)$;

5. Characterizing all solutions

It is easy to see that if a representation R of the MPUM for d is available, then all unfalsified models M for the data can be represented as $M = R'R$ for some polynomial matrix R'. We use this fact in order to derive a characterization of all solutions to the H_∞-partial realization problem, given in terms of an MPUM representation satisfying (2a)–(2e) of Theorem 1.

Proposition 3. *Let a kernel representation R of the MPUM for the data be given as in Theorem* 1, *and let $p, q \in \mathbb{R}[\xi]$. Then p/q is a solution to the H_∞-partial realization problem if and only if there exists π, φ, with φ Hurwitz and $\|\pi/\varphi\|_\infty < 1$, such that*

$$(p \quad -q) = (\pi \quad -\varphi)R. \tag{12}$$

Proof. We first prove the following auxiliary result.

Lemma 4. *If R satisfies property* (2a) *of Theorem* 1, *then $R^{\mathrm{r}}\Pi = \Pi R$.*

Proof. Denote the rows of R as r_1 and r_2. Observe first that from the definition of reciprocal it follows that $r_1 = r_2^{\mathrm{r}}\Pi$ implies that $r_1(0) \neq 0$. It follows that either both rows of R are nonzero at $\xi = 0$, or $r_2(0) = 0$.

In the first case, observe that $\deg(r_i) = \deg(r_i^{\mathrm{r}})$, $i = 1, 2$. Together with $r_1 = r_2^{\mathrm{r}}\Pi$ this implies that $\deg r_1 = \deg r_2 = \deg R$, so that

$$\begin{pmatrix} r_2^{\mathrm{r}} \\ r_1^{\mathrm{r}} \end{pmatrix}^{\mathrm{r}} = \begin{pmatrix} r_2 \\ r_1 \end{pmatrix}.$$

Observe that since $r_2(0) \neq 0$, it holds $(r_2^{\mathrm{r}})^{\mathrm{r}} = r_2$; consequently from (2a) of Theorem 1 it follows that $r_1^{\mathrm{r}} = r_2\Pi$ and therefore

$$R^{\mathrm{r}}\Pi = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}^{\mathrm{r}} \Pi = \begin{pmatrix} r_1\Pi \\ r_2\Pi \end{pmatrix}^{\mathrm{r}} = \begin{pmatrix} r_2^{\mathrm{r}} \\ r_1^{\mathrm{r}} \end{pmatrix}^{\mathrm{r}} = \begin{pmatrix} r_2 \\ r_1 \end{pmatrix} = \Pi R$$

as was to be proved. Let us consider the second case, and write $r_2 = \xi^k r_2'$, where $r_2'(0) \neq 0$. Observe that $r_2^{\mathrm{r}} = r_2'^{\mathrm{r}}$. Using this relation, it follows that $r_1 = r_2^{\mathrm{r}}\Pi = r_2'^{\mathrm{r}}\Pi$ and since $r_2'(0) \neq 0$, it follows also $r_1^{\mathrm{r}} = r_2'\Pi$. Observe that $\deg r_1 = \deg r_2'$, since $r_2'(0) \neq 0$. From these considerations and property (2a) of Theorem 3, we obtain

$$
\begin{aligned}
R^{\mathrm{r}} &= \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}^{\mathrm{r}} = \begin{pmatrix} r_1 \\ \xi^k r_2' \end{pmatrix}^{\mathrm{r}} = \begin{pmatrix} 1 & 0 \\ 0 & \xi^k \end{pmatrix}^{\mathrm{r}} \begin{pmatrix} r_1 \\ r_2' \end{pmatrix}^{\mathrm{r}} = \begin{pmatrix} \xi^k & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} r_2^{\mathrm{r}} \Pi \\ r_2' \end{pmatrix}^{\mathrm{r}} \\
&= \begin{pmatrix} \xi^k & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} r_2^{\mathrm{r}} \\ r_2' \Pi \end{pmatrix}^{\mathrm{r}} \Pi = \begin{pmatrix} \xi^k & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} r_{2'}^{\mathrm{r}} \\ r_1^{\mathrm{r}} \end{pmatrix}^{\mathrm{r}} \Pi \\
&= \begin{pmatrix} \xi^k & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} r_2' \\ r_1 \end{pmatrix} \Pi = \begin{pmatrix} r_2 \\ r_1 \end{pmatrix} \Pi = \Pi R \Pi
\end{aligned}
$$

from which the claim easily follows. □

(*Necessity*.) Without loss of generality, assume that p and q are coprime; observe that this implies that $(-q(0) \;\; p(0)) \neq 0$. It follows from an argument analogous to the one used to prove (2) ⇒ (3) in Theorem 1 that if p/q has the expansion $w_0 + w_1\xi^{-1} + w_2\xi^{-2} + \cdots + w_N\xi^{-N} + \cdots$ then $(-q \;\; p)^{\mathrm{r}}$ represents an unfalsified model for the data d of (1). Since R is an MPUM for d, $(-q \;\; p)^{\mathrm{r}}$ represents an unfalsified model for the data d of (1) if and only if there exist polynomials π' and φ' such that $(-q \;\; p)^{\mathrm{r}} = (-\varphi' \;\; \pi')R$. Now take the reciprocal of both sides of such equality, and observe that since $(-q(0) \;\; p(0)) \neq 0$, it holds $((-q \;\; p)^{\mathrm{r}})^{\mathrm{r}} = (-q \;\; p)$. Conclude that

$$
(-q \quad p) = (-\varphi' \quad \pi')^{\mathrm{r}} R^{\mathrm{r}}. \tag{13}
$$

Now multiply both sides of Eq. (13) on the right by Π and use Lemma 4 in order to conclude that Eq. (12) holds with polynomials π and φ defined by $(\pi \;\; -\varphi) := (-\varphi' \;\; \pi')^{\mathrm{r}}\Pi$.

We proceed to prove that (12) and $\|p/q\|_\infty < 1$ imply that $\|\pi/\varphi\|_\infty < 1$. Use (12) and property (2c) of Theorem 1 to write $q(\xi)q(\xi^{-1}) - p(\xi)p(\xi^{-1})$ as

$$
(\pi(\xi) \quad -\varphi(\xi))\underbrace{R(\xi)JR(\xi^{-1})^{\mathrm{T}}}_{J} \begin{pmatrix} \pi(\xi^{-1}) \\ -\varphi(\xi^{-1}) \end{pmatrix} = \varphi(\xi)\varphi(\xi^{-1}) - \pi(\xi)\pi(\xi^{-1}).
$$

Now let $\xi = \mathrm{e}^{\mathrm{i}\omega}$ and conclude that $q(\mathrm{e}^{\mathrm{i}\omega})q(\mathrm{e}^{-\mathrm{i}\omega}) - p(\mathrm{e}^{\mathrm{i}\omega})p(\mathrm{e}^{-\mathrm{i}\omega}) = \varphi(\mathrm{e}^{\mathrm{i}\omega})\varphi(\mathrm{e}^{-\mathrm{i}\omega}) - \pi(\mathrm{e}^{\mathrm{i}\omega})\pi(\mathrm{e}^{-\mathrm{i}\omega})$. Consequently, $\varphi(\mathrm{e}^{\mathrm{i}\omega})\varphi(\mathrm{e}^{-\mathrm{i}\omega}) - \pi(\mathrm{e}^{\mathrm{i}\omega})\pi(\mathrm{e}^{-\mathrm{i}\omega}) > 0 \;\forall\; \omega \in \mathbb{R}$ iff $q(\mathrm{e}^{\mathrm{i}\omega})q(\mathrm{e}^{-\mathrm{i}\omega}) - p(\mathrm{e}^{\mathrm{i}\omega})p(\mathrm{e}^{-\mathrm{i}\omega}) > 0 \;\forall\; \omega \in \mathbb{R}$; in other words, $\|\pi/\varphi\|_\infty < 1$ iff $\|p/q\|_\infty < 1$.

We now prove that φ is Hurwitz. Observe first that since $\|\pi/\varphi\|_\infty < 1$ and $\|r_{12}/r_{22}\|_\infty < 1$, the function $1/[(\pi/\varphi)(r_{12}/r_{22}) - 1]$ is well-defined. Now conclude from $-q = \pi r_{12} - \varphi r_{22}$ that $-1/q = 1/(\pi r_{12} - \varphi r_{22}) = (1/\varphi)1/[(\pi/\varphi)(r_{12}/r_{22}) - 1](1/r_{22})$. In order to prove that φ is Hurwitz, we prove that $1/\varphi$ has no poles in $\mathbb{D}_e$ by computing its winding number wno from the last expression. Using the logarithmic property of the winding number, we conclude

$$
\mathrm{wno}\left(-\frac{1}{q}\right) = \mathrm{wno}\left(\frac{1}{\varphi}\right) + \mathrm{wno}\left(\frac{1}{(\pi/\varphi)(r_{12}/r_{22}) - 1}\right) + \mathrm{wno}\left(\frac{1}{r_{22}}\right).
$$

Since q and r_{22} are Hurwitz, $\mathrm{wno}(-1/q) = \mathrm{wno}(1/r_{22}) = 0$ and consequently,

$$
0 = \mathrm{wno}\left(\frac{1}{\varphi}\right) + \mathrm{wno}\left(\frac{1}{(\pi/\varphi)(r_{12}/r_{22}) - 1}\right). \tag{14}
$$

In order to complete the argument, let $0 \leqslant \alpha \leqslant 1$ be a real number, and consider

$$\mathrm{wno}\left(\frac{1}{(\pi/\varphi)(r_{12}/r_{22})\alpha - 1}\right).$$

Such expression is well-defined for all $0 \leqslant \alpha \leqslant 1$, as can be readily verified. Moreover, it is a continuous function of α that takes integer values. It follows that its value is independent of α, and since for $\alpha = 0$ its value is $\mathrm{wno}(-1) = 0$, we conclude that $\mathrm{wno}(1/[(\pi/\varphi)(r_{12}/r_{22})(\alpha - 1)]) = 0$. From (14) and the last equality we conclude that $0 = \mathrm{wno}(1/\varphi)$. This yields that φ is Hurwitz, which completes the proof of necessity.

(*Sufficiency*). In order to prove sufficiency, take the reciprocals of both sides of the equality (12) and multiply on the right by Π in order to show that

$$(p \quad -q)^{\mathrm{r}}\Pi = (-q \quad p)^{\mathrm{r}} = (\pi \quad -\varphi)^{\mathrm{r}}R^{\mathrm{r}}\Pi.$$

Use Lemma 4 in order to conclude that $(-q \quad p)^{\mathrm{r}} = (\pi' \quad -\varphi')R$ for suitable $\pi', \varphi' \in \mathbb{R}[\xi]$. It follows that $\ker(-q \quad p)^{\mathrm{r}}(\sigma)$ is an unfalsified model for $\{\sigma^k d\}_{k=0,\ldots}$. Now apply the same argument used in the proof of (2) $\Rightarrow$ (3) of Theorem 1 to conclude that p/q has the right expansion at infinity. This shows that p/q is a solution to the partial realization problem.

In order to prove that $\|\pi/\varphi\|_\infty < 1$ implies $\|p/q\|_\infty < 1$, use the same argument applied to the converse implication in the "necessity" part above.

We conclude the proof showing that φ Hurwitz implies q is Hurwitz. First, use (12) to conclude that $q = \varphi r_{22} - \pi r_{12}$. Assume by contradiction that there exists $\mu \in \mathbb{D}_e$ such that $q(\mu) = 0$; then $\varphi(\mu)r_{22}(\mu) - \pi(\mu)r_{12}(\mu) = 0$. Since φ and r_{22} are Hurwitz, $\varphi(\mu) \neq 0$ and $r_{22}(\mu) \neq 0$; then the last equality implies $(\pi(\mu)/\varphi(\mu))(r_{12}(\mu)/r_{22}(\mu)) = -1$ from which

$$\left|\frac{\pi(\mu)}{\varphi(\mu)}\frac{r_{12}(\mu)}{r_{22}(\mu)}\right| = \left|\frac{\pi(\mu)}{\varphi(\mu)}\right|\left|\frac{r_{12}(\mu)}{r_{22}(\mu)}\right| = 1$$

follows. The last equality, however, is a contradiction with $\|\pi/\varphi\|_\infty < 1$, with statement (2e) of Theorem 1, and with the maximum modulus theorem. This concludes the proof of the Proposition. □

6. Conclusions

We have considered the H_∞-partial realization problem, namely that of computing a stable rational function whose power series expansion at infinity matches a finite number of given Markov parameters, and whose ∞-norm is less than one. The main result of this paper is Theorem 1, which connects the solutions to the H_∞-partial realization problem with the MPUM and the Stein matrix associated with the data. As a ramification of such result, we obtained the iterative algorithm presented in Section 4, which computes a special representation for the MPUM from which a solution to the H_∞-partial realization problem is easily obtained. By means of such representation of the MPUM, one can characterize all solutions to the H_∞-partial realization problem (Proposition 3).

A direction in which the results presented in this paper are being extended and generalized, is solving the problem of how to obtain a model for a general set of (input, output) data (i.e. not necessarily impulse response samples, and in principle, not separated a priori into inputs and outputs) corresponding to a stable transfer function of ∞-norm less than one: *the H_∞-data modeling problem*. Another important issue to investigate is that of *minimality* of the solution of the H_∞-partial realization problem, obtained from the representation of the MPUM obtained from the algorithm presented in Section 4.

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