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Editorial

The Behavioral Approach to Systems and Modeling

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ABSTRACT

An introduction to behavioral system theory, and a brief review of the content of the Special Issue are given.

Keywords: Behaviors, modeling, linear differential systems, polynomial matrix algebra.

1. INTRODUCTION

For nearly three decades, mathematicians and engineers alike have identified linear system theory with the transfer function and the state-space framework. The importance of such paradigms in our field of study is usually justified claiming that the input-output framework fits the practitioner's point of view of a system as a black-box transforming input signals into output signals; while the notion of state provides insight in the internal structure of a system, and it makes for efficient computational techniques.

While the importance of state-space and transfer function techniques cannot be denied, their hegemony in the current discourse of and about linear system theory is puzzling, if not downright inexplicable for the unprejudiced scholar. Indeed, the shortcomings of the transfer-function approach are evident when considering those situations in which the variables of a system cannot be classified in inputs and outputs, or those in which the point of view of the system as a ‘‘signal processor’’ transforming inputs in outputs is untenable on rational grounds (see, for example, the analysis of a simple door-closing mechanism illustrated in [1]). The shortcomings of the state-space approach are no less evident: for example, modeling a physical system from first principles hardly ever results in a state-space description, which indeed usually

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needs to be constructed from the set of higher-order differential equations (possibly with static constraints among the variables) describing the model. Thus, state-space descriptions cannot be considered to be the most characteristic nor the most natural starting point for the study of linear systems. Another unsatisfactory aspect of the state-space approach is that properties such as controllability and observability, which, as common sense dictates, are related to the essential nature of a system, are instead highly non-intrinsic, being defined as properties of the particular state-space representation at hand.

Issues such as these brought Jan C. Willems to seek a theoretical framework for the modeling and analysis of systems that would be general, so as to encompass also the transfer-function and the state-space approaches, conceptually sound, and simple. In [2] and in the subsequent works [3, 4, 5] (see also the textbook [6]) he identified the *behavior*, that is, the set of trajectories that satisfy the laws of the system, its properties, and the way in which such properties are reflected in those of its representations (whether transfer function, or state-space, or set of differential equations), as the central object of study in system theory. The adoption of such paradigm fostered a great deal of work in the classical tradition of the mathematical sciences, making possible a precise and logically consistent formulation of some general principles of dynamics and of some fundamental properties of dynamical systems through the use of novel, simple, and efficient mathematical techniques: *behavioral system theory* was born. In the course of time, a growing group of students and of researchers came to appreciate the potential of such theoretical framework and made significant contributions to it, also extending its influence in areas other than that of systems described by linear, constant-coefficient differential equations, witness for example the use of behavioral concepts in coding theory, in data modeling (see also the seminal papers [7, 8]), in the theory of systems described by partial differential equations (see [9]), in the theory and practice of simulation, to name but a few of the directions involved in such investigations.

The purpose of this Editorial is to familiarize the reader with the main concepts and the basic ideas of behavioral system theory which will be used throughout this issue. No attempt at being exhaustive has been made: indeed, we have been forced to leave out several interesting notions and applications for reasons of space. The interested reader should consult the literature quoted at the end of this article and the references therein.

We begin our exposition by introducing the notions at the very center of the behavioral approach, those of *dynamical system* and of *behavior*.

2. DYNAMICAL SYSTEMS AND BEHAVIORS

In modeling a dynamical system, one aims at describing how a set of variables of interest, in the sequel called *manifest variables* and denoted with $w = (w_1, w_2, \ldots, w_q)$, evolves as a function of another set of independent variables, say

$x = (x_1, x_2, \ldots, x_n)$. In general, any modeling procedure will lead to a set of (algebraic, differential, difference, partial differential) equations usually called *the model* of the system under study. However, experience dictates that different sets of equations can describe the same dynamical system, so that in fact there are *many* models for the same system. The starting point of the behavioral approach is to avoid this fallacy, and to identify a model with the set of the manifest variable trajectories that can occur (called the *behavior* of the system). Any set of equations in terms of which the behavior is described is then called a *representation of the behavior.* More precisely, in the behavioral approach a dynamical system consists of three objects:

1. A subset $\mathbb{I}$ of $\mathbb{R}^n$, called the *index set*, in which the independent variable $x = (x_1, x_2, \ldots, x_n)$ takes its values. In many instances there is only one independent variable, that is, $n = 1$, and this variable often has the interpretation of *time*. In this case the independent variable is denoted by t. The index set is then a subset of $\mathbb{R}$, typically $\mathbb{I} = \mathbb{R}$ or $\mathbb{I} = [0, \infty)$, in which case we speak about a *continuous time system*. If, for example, $\mathbb{I} = \mathbb{Z}$ of $\mathbb{N}$, we speak about a *discrete time system*. If $n > 1$ the manifest variables depend on more than one independent variable. For example, if $n = 4$, then we might have $x = (t, x, y, z)$, where t is time, and (x, y, z) is position in a three-dimensional space.
2. A set $\mathbb{W}$, called the *signal space*. This is the set in which the manifest variable $w = (w_1, w_2, \ldots, w_q)$ takes its values. For example, if w only takes real values, then $\mathbb{W}$ can be a subset of $\mathbb{R}^q$. In the case of distributed dynamical systems it often occurs that the components of $w(x_1, x_2, \ldots, x_n)$ are functions, in which case $\mathbb{W}$ is a subset of some function space. Sometimes the values $w(x_1, x_2, \ldots, x_n)$ are elements of some finite set $\mathbb{W}$.
3. A subset $\mathfrak{B}$ of $\mathbb{W}^{\mathbb{I}}$, the set of all functions from $\mathbb{I}$ to $\mathbb{W}$. The aim of the model is to specify which functions $w = (w_1, w_2, \ldots, w_q)$ from $\mathbb{I}$ to $\mathbb{W}$ actually comply with the laws of the dynamical system. The subset $\mathfrak{B}$ of $\mathbb{W}^{\mathbb{I}}$ thus defined is called the *behavior* of the system.

Formalizing this, we come to the following definition of a dynamical system.

Definition 1 *A dynamical system is a triple* $\Sigma = (\mathbb{I}, \mathbb{W}, \mathfrak{B})$*, with* $\mathbb{I} \subseteq \mathbb{R}^n$ *called the* index set*,* $\mathbb{W}$ a set*, called the* signal space*, and* $\mathfrak{B} \subseteq \mathbb{W}^{\mathbb{I}}$ *called* the behavior.

For a trajectory $w : \mathbb{I} \to \mathbb{W}$ we either have $w \in \mathfrak{B}$, which means that the model allows the trajectory w, or $w \notin \mathfrak{B}$, in which case the model forbids the trajectory w.

Example 1 The possible motions of the planets in the solar system are described by *Kepler's laws*:

1. planets move in elliptical orbits, with the sun (assumed in fixed position at the origin of $\mathbb{R}^3$) at one of the foci;
2. the radius vector from the sun to the planet sweeps out equal areas in equal times;

3. the square of the period of revolution is proportional to the third power of the major axis of the ellipse.

The motion of a planet as a function of time defines a dynamical system in the following way. Position as a function of time is given by $w(t) = (w_1(t), w_2(t), w_3(t))$. Thus the index set is given as $\mathbb{I} = \mathbb{R}$, and for the signal space we take $\mathbb{W} = \mathbb{R}^3$. The behavior of the dynamical system is the subset $\mathfrak{B}$ of the set of all functions from $\mathbb{R}$ to $\mathbb{R}^3$ defined by

$$\mathfrak{B} = \{w : \mathbb{R} \to \mathbb{R}^3 \mid w \text{ satisfies Kepler's laws}\}.$$

Example 2 Consider the transverse motion of a homogeneous flexible sheet ('membrane') with surface mass density ρ, and tension τ. Let $w(t, x, y)$ be the displacement from equilibrium of point (x, y) of the membrane at time t. Then w satisfies the partial differential equation:

$$\rho\frac{\partial^2 w}{\partial t^2} - \tau\frac{\partial^2 w}{\partial x^2} - \tau\frac{\partial^2 w}{\partial y^2} = 0. \tag{1}$$

This can be modeled as a dynamical system with index set $\mathbb{I} = \mathbb{R}^3$, signal space $\mathbb{W} = \mathbb{R}$ and behavior defined by

$$\mathfrak{B} = \{w : \mathbb{R}^3 \to \mathbb{R} \mid w \text{ satisfies (1)}\}$$

The following example is an illustration of the fact that often when setting up a model to describe the behavior of a certain set of variables, one has to use auxiliary variables.

Example 3 Consider a linear time-invariant RLC-circuit with N_e external ports with currents $I_1, I_2, \ldots, I_{N_e}$ and voltages $V_1, V_2, \ldots, V_{N_e}$. Denote $I = (I_1, I_2, \ldots, I_{N_e})$ and $V = (V_1, V_2, \ldots, V_{N_e})$. The circuit contains resistors $R_1, R_2, \ldots, R_{N_r}$. The current through and voltage across the k-th resistor are I_{R_k} and V_{R_k}, respectively. Denote by I_R and V_R the vectors of resistor currents and voltages. The network contains N_c capacitors with capacitances $C_1, C_2, \ldots, C_{N_c}$. The current through and voltage across the ℓ-th capacitor are I_{C_ℓ} and V_{C_ℓ}, respectively; the vectors I_C and V_C are defined in the obvious way. Finally, the network contains N_i inductors $L_1, L_2, \ldots, L_{N_i}$. The current through and voltage across the m-th inductor are I_{L_m} and V_{L_m}, respectively; the vectors I_L and V_L are defined in the obvious way.

The network defines a dynamical system in the following way. The index set is $\mathbb{R}$ and the corresponding independent variable is time t. The signal space is $\mathbb{R}^{2N_e}$, the space in which the vectors of external voltages and currents take their values. The behavior $\mathfrak{B}$ is defined by

$$\begin{aligned}\mathfrak{B} = \{(V, I) : \mathbb{R} \to \mathbb{R}^{2N_e} \mid\ & \text{there exists } (V_R, I_R, V_C, I_C, V_L, I_L) \text{ such that} \\ & \text{the constitutive laws of the elements, together} \\ & \text{with Kirchoff's laws are satisfied}\}\end{aligned} \tag{2}$$

In Example 3, in order to describe the time-behavior of the external voltages and currents (the manifest variables), one uses the vectors of voltages and currents of the network elements. These auxiliary variables are called *latent variables*. If one writes down the system equations explicitly, a so called *latent variable representation of* $\mathfrak{B}$ with latent variable $(V_R, I_R, V_C, I_C, V_L, I_L)$ is obtained. A formal definition of this concept is given as follows.

Definition 2 *A* dynamical system with latent variables *is defined as*

$$\Sigma_L = (\mathbb{I}, \mathbb{W}, \mathbb{L}, \mathfrak{B}_{\mathrm{f}}),$$

with $\mathbb{I} \subseteq \mathbb{R}^n$ *the* index set*,* $\mathbb{W}$ *the* manifest signal space*,* $\mathbb{L}$ *the* latent variable space*, and* $\mathfrak{B}_{\mathrm{f}} \subseteq (\mathbb{W} \times \mathbb{L})^{\mathbb{I}}$ *called* the full behavior*.*

The latent variable system Σ_L *defines* a latent variable representation *of the manifest dynamical system* $\Sigma = (\mathbb{I}, \mathbb{W}, \mathfrak{B})$*, with (manifest) behavior* $\mathfrak{B}$ *defined by*

$$\mathfrak{B} := \{w : \mathbb{I} \to \mathbb{W} \mid \textit{there exists } \ell : \mathbb{I} \to \mathbb{W} \textit{ such that } (w, \ell) \in \mathfrak{B}_{\mathrm{f}}\}$$

Example 4 In systems and control we often encounter input/output systems in state space form, given by equations of the form

$$\frac{d}{dt}x = f(x(t), u(t)), \qquad y(t) = g(x(t), u(t)). \tag{3}$$

Here f and g are given functions. The inputs and outputs are denoted by u and y, and take their values in $\mathbb{R}^m$ and $\mathbb{R}^p$, respectively. The manifest variable is (u, y). The latent variable is x. It takes its values in $\mathbb{R}^n$. Equation (3) represents a dynamical system with latent variables, $\Sigma_L = (\mathbb{I}, \mathbb{W}, \mathbb{L}, \mathfrak{B}_{\mathrm{f}})$, with index set $\mathbb{I} = \mathbb{R}$, manifest signal space $\mathbb{W} = \mathbb{R}^m \times \mathbb{R}^p$, latent variable space $\mathbb{L} = \mathbb{R}^n$, and full behavior

$$\mathfrak{B}_{\mathrm{f}} := \{(u, y, x) : \mathbb{R} \to \mathbb{R}^m \times \mathbb{R}^p \times \mathbb{R}^n \mid \text{Equation (3) is satisfied}\}.$$

This latent variable system defines the manifest dynamical system $\Sigma = (\mathbb{R}, \mathbb{R}^m \times \mathbb{R}^p, \mathfrak{B})$ with manifest variable (u, y) and manifest behavior $\mathfrak{B}$ given by

$$\mathfrak{B} := \{(u, y) : \mathbb{R} \to \mathbb{R}^m \times \mathbb{R}^p \mid \text{there exists } x \text{ such that } (u, y, x) \in \mathfrak{B}_{\mathrm{f}}\}.$$

In the above examples and definitions we can recognize some of the central issues in the behavioral approach to dynamical systems. The most important one is that a model is a subset of the set of all manifest variable trajectories, namely that consisting of those trajectories which are possible given the dynamical laws governing the system. This subset is called the behavior of the system, and in general it admits many possible representations. When modeling systems as an interconnection of standard components, as is common practice in computer-aided modeling tools, one invariably encounters (auxiliary) latent variables in addition to the manifest variables that the model aims at describing.

A second important issue is that that all manifest system variables are a priori treated on an equal footing. In principle, the model does not distinguish between inputs and outputs. Of course, after specifying the model, some of the manifest variables might qualify as input variables and others as output variables. However for variables to qualify as inputs, they need to satisfy certain properties; in particular, they need to be *free*, in a sense that will be explained in the sequel.

We now illustrate how the framework put forward in this section applies to systems described by linear, constant-coefficient ordinary differential equations, the so-called *linear differential systems*.

3. LINEAR DIFFERENTIAL SYSTEMS

In order to further illustrate the basic ideas behind the behavioral approach, we now discuss some of the fundamentals of the theory of *linear differential systems*. These are systems $\Sigma = (\mathbb{R}, \mathbb{R}^q, \mathfrak{B})$ with index set equal to $\mathbb{R}$, signal space equal to $\mathbb{R}^q$, and behavior $\mathfrak{B}$ consisting of the space of solutions of a given set of higher order, constant coefficient, linear, ordinary, differential equations. If the manifest variable is $w = (w_1, w_2, \ldots, w_q)$, then one such differential equation (of order n) is of the form

$$\sum_{j=1}^{q} r_j^0 w_j + \sum_{j=1}^{q} r_j^1 \frac{d}{dt} w_j + \cdots + \sum_{j=1}^{q} r_j^n \frac{d^n}{dt^n} w_j = 0.$$

In order to avoid technicalities, we will restrict ourselves here to infinitely often differentiable functions $w : \mathbb{R} \to \mathbb{R}^q$, the space of all such functions being denoted by $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^q)$. In case w has to satisfy, say g, of such differential equations of order at most n, we can arrange the scalar coefficients into real $g \times q$ coefficient matrices R_j, and write down the set of differential equations in terms of one single matrix differential equation

$$R_0 w + R_1 \frac{d}{dt} w + R_2 \frac{d^2}{dt^2} w + \cdots + R_n \frac{d^n}{dt^n} w = 0. \tag{4}$$

A shorthand notation for this type of equation is obtained by defining the $g \times q$ polynomial matrix $R(\xi)$ in the indeterminate ξ by $R(\xi) = R_0 + R_1\xi + R_2\xi^2 + \cdots + R_n\xi^n$. Next, we form the differential operator $R\left(\frac{d}{dt}\right)$ by formally replacing ξ by the differentiation operator $\frac{d}{dt}$. Then Equation (4) is equivalent to $R\left(\frac{d}{dt}\right)w = 0$.

Definition 3 *A dynamical system* $\Sigma = (\mathbb{R}, \mathbb{R}^q, \mathfrak{B})$ *is called* a linear differential system *if there exists a positive integer g and a* $g \times q$ *polynomial matrix* $R(\xi)$ *with real coefficient matrices such that*

$$\mathfrak{B} = \left\{ w : \mathbb{R} \to \mathbb{R}^q \;\middle|\; w \text{ is a solution of } R\left(\frac{d}{dt}\right) w = 0 \right\}. \tag{5}$$

If $\Sigma = (\mathbb{R}, \mathbb{R}^q, \mathfrak{B})$ is a linear differential system, with $\mathfrak{B}$ given by Equation (5), then the equation $R\left(\frac{d}{dt}\right)w = 0$ is called a *kernel representation* of $\mathfrak{B}$. Recall that this is only one possible kernel representation of the behavior $\mathfrak{B}$, because there are always many R's defining the same $\mathfrak{B}$. For example, it can be shown that if U is any unimodular polynomial matrix (i.e., U is square and $\det(U) \in \mathbb{R}$, $\det(U) \neq 0$) such that the product UR makes sense, then R and UR yield the same behavior $\mathfrak{B}$ (see Section 2.5.2 of [6]).

There are many other ways of representing the behavior $\mathfrak{B}$ of given linear differential system. As was mentioned in the previous section, if one sets up a model of a given dynamical system, then often one has to include latent variables in order to specify the manifest behavior. In the context of linear differential systems, a system with latent variables is of the form $\Sigma_L = (\mathbb{R}, \mathbb{R}^q, \mathbb{R}^l, \mathfrak{B}_{\mathrm{f}})$, with full behavior $\mathfrak{B}_{\mathrm{f}}$ equal to the set of all solutions (w, ℓ) of a system of differential equations

$$R\left(\frac{d}{dt}\right)w = M\left(\frac{d}{dt}\right)\ell \tag{6}$$

This system is called a *latent variable representation* of $\mathfrak{B}$ if $\mathfrak{B}$ is equal to the manifest behavior of this latent variable system, that is, if

$$\mathfrak{B} = \{w \mid \exists\, \ell \text{ such that Equation (6) holds}\}. \tag{7}$$

If a linear differential behavior $\mathfrak{B}$ is defined by Equation (7), then we say that $\mathfrak{B}$ is obtained from the latent variable representation Equation (6) by *elimination of the latent variable* ℓ. Of course, it is a fundamental question whether a behavior $\mathfrak{B}$ obtained in this way is a linear differential system, i.e., whether there exists a polynomial matrix $R'(\xi)$ such that $\mathfrak{B} = \left\{w : \mathbb{R} \to \mathbb{R}^q \;\middle|\; w \text{ is a solution of } R'\left(\frac{d}{dt}\right)w = 0\right\}$. The fact that this is indeed the case is known as the *Elimination Theorem*, see Chapter 6 of [6].

We now discuss the concept of free variable, and we illustrate how it connects with the intuitive notion of "input variable" (see also Section 3.3 of [6]). Suppose we have a linear differential system $\Sigma = (\mathbb{R}, \mathbb{R}^q, \mathfrak{B})$, with manifest variable w. The statement '$w \in \mathfrak{B}$' then means that the time trajectory $w = (w_1, w_2, \ldots, w_q)$ complies with the laws of the system, and can indeed occur. The idea behind the concept of *inputs* and *outputs* is that the condition '$w \in \mathfrak{B}$' may leave some of the components $w_1, w_2, \ldots, w_q$ unconstrained: such components can be chosen arbitrarily, and they qualify as *inputs*. After choosing these free components, the remaining components are determined up to initial conditions: these components are the *outputs*. Consider the following example.

Example 5 Let $q(t) \in \mathbb{R}^3$ be the position of a point mass M subject to a force $F(t) \in \mathbb{R}^3$. According to Newton's law this can be modelled as a linear differential system $\Sigma = (\mathbb{R}, \mathbb{R}^6, \mathfrak{B})$, with behavior $\mathfrak{B}$ given by

$$\mathfrak{B} = \left\{ (q, F) \;\middle|\; M \frac{d^2}{dt^2} q - F = 0 \right\}.$$

Note that this is a kernel representation of $\mathfrak{B}$. In effect, it consists of three (differential) equations, and six unknowns. The condition '$(q, F) \in \mathfrak{B}$', expressing that (q, F) complies with Newton's second law, does not put constraints on F: F is allowed to be *any function*. After choosing F, the variable q is determined up to $q(0)$ and $\frac{dq}{dt}(0)$. Also: $(q, F) \in \mathfrak{B}$ does not put constraints on q, so that q is allowed to be any function. After choosing q, the variable F is determined uniquely.

The definition of free variable is as follows.

Definition 4 *Let $\Sigma = (\mathbb{R}, \mathbb{R}^q, \mathfrak{B})$ be a linear differential system, with manifest variable $w = (w_1, w_2, \ldots, w_q)$. For $I = \{i_1, i_2, \ldots, i_k\} \subseteq \{1, 2, \ldots, q\}$, denote by $P_I\mathfrak{B}$ the system obtained by* eliminating *the variables w_j, $j \notin I$.*

1. *The set of variables $\{w_{i_1}, w_{i_2}, \ldots, w_{i_k}\}$ is called* free in $\mathfrak{B}$ *if*

$$P_I\mathfrak{B} = \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{|I|}),$$

   where $|I| = k$, the cardinality of the set I. In other words, the set of variables $\{w_{i_1}, w_{i_2}, \ldots, w_{i_k}\}$ is free in $\mathfrak{B}$ if for any choice of $(w_{i_1}, w_{i_2}, \ldots, w_{i_k}) \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^k)$, there exist w_j, $j \notin I$, such that $(w_1, w_2, \ldots, w_q) \in \mathfrak{B}$.
2. *The set of variables $\{w_{i_1}, w_{i_2}, \ldots, w_{i_k}\}$ is called* maximally free in $\mathfrak{B}$ *if it is free, and if for any $I' \subseteq \{1, 2, \ldots, q\}$ such that $I \subsetneq I'$ we have*

$$P_{I'}\mathfrak{B} \subsetneq \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{|I'|}).$$

   In other words, $\{w_{i_1}, w_{i_2}, \ldots, w_{i_k}\}$ is maximally free, if it is free and if any set of variables obtained by adding to this set one or more of the remaining variables is not free.

The notion of maximally free variable leads to the following definition of input and output variable.

Definition 5 *Let $\Sigma = (\mathbb{R}, \mathbb{R}^q, \mathfrak{B})$ be a linear differential system, with manifest variable $w = (w_1, w_2, \ldots, w_q)$. Possibly after permutation of its components, a partition of w into $w = (w^{(1)}, w^{(2)})$, with $w^{(1)} = (w_1, w_2, \ldots, w_{\mathrm{m}})$ and $w^{(2)} = (w_{\mathrm{m}+1}, w_{\mathrm{m}+2}, \ldots, w_q)$, is called* an input/output partition in $\mathfrak{B}$ *if $\{w_1, w_2, \ldots, w_{\mathrm{m}}\}$ is maximally free.*

In that case, $w^{(1)}$ is called an input of $\mathfrak{B}$, *and $w^{(2)}$ is called an output of $\mathfrak{B}$. Usually, we write u for $w^{(1)}$, and y for $w^{(2)}$.*

Consider now the following example.

Example 6 Consider the linear differential system $\Sigma = (\mathbb{R}, \mathbb{R}^{m+p}, \mathfrak{B})$ with manifest variable (u, y), represented by the latent variable representation

$$\frac{d}{dt}x = Ax + Bu,$$
$$y = Cx + Du,$$
$$w = (u, y)$$

with latent variable x. In $\mathfrak{B}$, u is maximally free: once we choose u, there are no more free components left in y: the only freedom of y is the choice of initial state $x(0)$. The conclusion is that in $\mathfrak{B}$, (u, y) is an input/output partition, with input u and output y.

Note that in the previous example u is input and y is output, not because it has been decided *a priori* to call them as such, or because they are denoted by 'u' and 'y', but because in the behavior $\mathfrak{B}$, the variable u is maximally free.

For a given linear differential system, the manifest variable w in general allows *more than one input/output partition*.

Example 7 Consider the behavior of a resistor R, defined as the set of (voltage, current) pairs compatible with the constitutive relation of the element, namely $V = RI$:

$$\mathfrak{B} = \{(V, I) \mid V = RI\}$$

It is easy to see that in such behavior, V or I can be imposed from the outside, with the remaining variable being bound by the constitutive equation and the value of the other. It follows that V is maximally free, so that V is input and I is then an output. Also, I is maximally free, so that another input/output partition of the external variable has I as input and V as output.

Example 8 Consider the behavior $\mathfrak{B} = \left\{(q, F) \;\middle|\; M\frac{d^2}{dt^2}q - F = 0\right\}$ introduced in Example 5, with external variable $w = (q, F)$. It is easy to see that q is maximally free, so q is input and F is output in $\mathfrak{B}$. However, also F is maximally free, so F is input and q is output in $\mathfrak{B}$.

Although in general a given linear differential system $\Sigma = (\mathbb{R}, \mathbb{R}^q, \mathfrak{B})$ has many input/output partitions $w = (u, y)$, the *number* of input components in any input/output partition of $\mathfrak{B}$ is fixed. This number is denoted by $\mathrm{m}(\mathfrak{B})$ and is called the *input cardinality of* $\mathfrak{B}$:

$$\mathrm{m}(\mathfrak{B}) := \max\{k \in \mathbb{N} \mid \{w_{i_1}, w_{i_2}, \ldots, w_{i_k}\} \text{ is free in } \mathfrak{B}\}.$$

The *output cardinality of* $\mathfrak{B}$, denoted by $\mathrm{p}(\mathfrak{B})$, is the number of output components in any input/output partition of $\mathfrak{B}$. Obviously:

$$\mathrm{p}(\mathfrak{B}) = q - \mathrm{m}(\mathfrak{B}).$$

An important issue in the behavioral approach is the study of how the properties of a given dynamical system are reflected *in the properties of its representations*. As an example, suppose that we have a linear differential system $\Sigma = (\mathbb{R}, \mathbb{R}^q, \mathfrak{B})$ represented in kernel form by $R\left(\frac{d}{dt}\right)w = 0$, where $R(\xi)$ is a given polynomial matrix. How can we compute the input cardinality $\mathrm{m}(\mathfrak{B})$ and the output cardinality $\mathrm{p}(\mathfrak{B})$ of $\mathfrak{B}$ knowing the representing polynomial matrix $R(\xi)$? It turns out that

$$\mathrm{m}(\mathfrak{B}) = q - \mathrm{rank}(R),$$

where $\mathrm{rank}(R)$ denotes the rank of the polynomial matrix R. Suppose now the system is represented by the latent variable representation $R\left(\frac{d}{dt}\right)w = M\left(\frac{d}{dt}\right)\ell$, with latent variable ℓ. How can we compute $\mathrm{m}(\mathfrak{B})$ and $\mathrm{p}(\mathfrak{B})$ in terms of the representing polynomial matrices R and M? It can be shown that

$$\mathrm{m}(\mathfrak{B}) = q - \mathrm{rank}([R \;\; M]) + \mathrm{rank}(M)$$

Example 9 Consider the 'descriptor system' given by the equations

$$E\frac{d}{dt}x = Ax + Bu, \tag{8}$$

$$y = Cx + Du, \tag{9}$$

$$w = (u, y),$$

where u, y and x take their values in $\mathbb{R}^m$, $\mathbb{R}^p$ and $\mathbb{R}^n$, respectively. This is a latent variable representation of the linear differential system $\Sigma = (\mathbb{R}, \mathbb{R}^{m+p}, \mathfrak{B})$, with

$$\mathfrak{B} = \{(u, y) \mid \text{there exists } x \text{ such that Equations (8) and (9) hold}\}$$

By writing the latent variable representation alternatively as

$$\begin{bmatrix} B & 0 \\ -D & I \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix} = \begin{bmatrix} E\frac{d}{dt} - A \\ C \end{bmatrix} x.$$

we get the following expression for the input cardinality of the system:

$$\mathrm{m}(\mathfrak{B}) = m + p - \mathrm{rank}\left(\begin{bmatrix} B & 0 & E\xi - A \\ -D & I & C \end{bmatrix} \right) + \mathrm{rank}\left(\begin{bmatrix} E\xi - A \\ C \end{bmatrix} \right)$$

so

$$\mathrm{m}(\mathfrak{B}) = m - \mathrm{rank}([E\xi - A \;\; B]) + \mathrm{rank}\left(\begin{bmatrix} E\xi - A \\ C \end{bmatrix} \right).$$

Clearly, in the case that $E = I$, the $n \times n$ identity matrix, we have $\mathrm{m}(\mathfrak{B}) = m$, the number of components of u.

Other intrinsic properties of linear differential systems which are reflected in the algebraic properties of their representations are controllability, observability, and stability (see Chapter 7 of [6]). The notion of controllability in the behavioral framework is extensively discussed in the article of S. Shankar appearing in this issue (see also Chapter 5 of [6]). We now briefly discuss the notion of observability.

Observability is a property of systems whose variables are partitioned in two sets, one of which is *observed* while the other is to be *deduced* from the first one. More precisely, let $(\mathbb{R}, \mathbb{R}^{q_1} \times \mathbb{R}^{q_2}, \mathfrak{B})$ be a linear differential system, with the external variable partitioned as $w = (w_1, w_2)$, w_1 being q_1-dimensional and w_2 being q_2-dimensional. We say that w_2 is *observable from* w_1 if

$$(w_1, w_2), (w_1', w_2') \in \mathfrak{B} \quad \text{and} \quad w_1 = w_1' \Longrightarrow w_2 = w_2'$$

It is easy to see that this implies that there exists a map F associating to every portion w_1 of a trajectory in $\mathfrak{B}$ one and only one corresponding portion w_2:

$$(w_1, w_2) \in \mathfrak{B} \Longrightarrow w_2 = F w_1$$

In order to illustrate how such intrinsic property of the behavior is reflected in the algebraic properties of the polynomial matrices describing it, consider the behavior described by

$$R_1\left(\frac{d}{dt}\right) w_1 = R_2\left(\frac{d}{dt}\right) w_2$$

Then it can be shown (see Section 5.3 of [6]) that w_2 is observable from w_1 if and only if the matrix $R_2(\lambda)$ has full column-rank q_2 for all $\lambda \in \mathbb{C}$.

Example 10 Consider a system with two variables, whose behavior is described in kernel form as

$$p\left(\frac{d}{dt}\right) w_1 = q\left(\frac{d}{dt}\right) w_2$$

where p, q are polynomials. It is easy to see that in such system, w_2 is observable from w_1 if and only if $q(\xi)$ is a nonzero constant, that is, $q(\xi) = c \neq 0$, $c \in \mathbb{R}$. Indeed, if such condition is not satisfied then q has at least one root λ, so that if $(\bar{w}_1, \bar{w}_2) \in \mathfrak{B}$, then also $(\bar{w}_1, \bar{w}_2 + \alpha e^{\lambda t}) \in \mathfrak{B}$ for all $\alpha \in \mathbb{R}$, so that by observing $\bar{w}_1$ it is impossible to determine which trajectory in the variables w_2 has occurred.

In order to see the relationship of the behavioral definition of observability with the one known in the state-space setting, consider an input-state-output representation

$$\begin{aligned} \frac{d}{dt} x &= Ax + Bu \\ y &= Cx + Du \end{aligned} \tag{10}$$

of $\mathfrak{B}_\mathrm{f} = \{(u,y,x) |$ (10) is satisfied$\}$, which can be rewritten as

$$\begin{pmatrix} \frac{d}{dt}I_n - A \\ -C \end{pmatrix} x = \begin{pmatrix} B & 0 \\ D & -I_p \end{pmatrix} \begin{pmatrix} u \\ y \end{pmatrix}$$

The variable x is observable (in the behavioral sense) from (u,y) if and only if the matrix

$$\begin{pmatrix} \lambda I_n - A \\ -C \end{pmatrix}$$

has full column rank for all $\lambda \in \mathbb{C}$. This of course is the well-known *Popov–Belevitch–Hautus* test for observability.

4. ABOUT THE CONTENTS OF THIS ISSUE

For this special issue of *Mathematical and Computer Modeling of Dynamical Systems* (*MCMDS*) on Behavioral System Theory we have asked a number of researchers active in this area to provide us with a contribution which would introduce their point of view on the behavioral approach to the community of readers of *MCMDS*, and also give the flavor of the state of the art in their own area of research. A glance at the index will show the range and breadth of the subjects treated over the years in the behavioral framework:

The article by *Tommaso Cotroneo* and *Jacob van Dijk* illustrates the *Behavioral Toolbox*, a Unix-based modeling and simulation package based on the behavioral concept of interconnection of subsystems through terminals, rather than on block-diagram and input-output structures as is common in most such products.

The contribution by *Kiyotsugu Takaba* and *Yutaka Ichihara* concerns the initial value problem for systems of first-order differential-algebraic equations obtained as the result of the interconnection of sub-systems.

The paper by *Shiva Shankar* traces the evolution of the concept of controllability from its introduction by Kalman in the state-space framework, through its formalization in behavioral terms by Willems, to its definition for distributed systems.

The article by *Madhu Belur* and *Harry L. Trentelman* discusses the type of algorithmic issues that arise in the behavioral approach to the synthesis of dissipative systems, and constitutes an illustration of the behavioral point of view on control (see also [1]).

The paper by *Margreta Kuijper* and *Jan Willem Polderman* provides an example of how behavioral ideas about data modeling are applied to coding theory, in this specific instance the list decoding of Reed-Solomon codes (see also [10] for another application of behavioral techniques to coding-theory problems).

The paper by *Thanos Antoulas* and *Andrew Mayo* casts positive-real interpolation problems and the algorithms used for their solution, in the exact data-modeling framework initiated in [7] and further developed in [11].

Finally, our own contribution uses the formalism of bilinear- and quadratic differential forms introduced in [12], in order to study symplectic and variational properties of lumped- and distributed systems.

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