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Hamiltonian and Variational Linear Distributed Systems

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ABSTRACT

We use the formalism of bilinear- and quadratic differential forms in order to study Hamiltonian and variational linear distributed systems. It was shown in [1] that a system described by ordinary linear constant-coefficient differential equations is Hamiltonian if and only if it is variational. In this paper we extend this result to systems described by linear, constant-coefficient partial differential equations. It is shown that any variational system is Hamiltonian, and that any scalar Hamiltonian system is contained (in general, properly) in a particular variational system.

Keywords: Linear Hamiltonian systems, linear variational systems, multi-variable polynomial matrices, bilinear- and quadratic differential forms.

1. INTRODUCTION

The objective of this paper is to present some recent results in the application of quadratic- and bilinear differential forms, introduced in [2], to the modeling and analysis of systems described by linear, constant-coefficient partial differential equations (in the following also called "*nD* systems"). We focus on the relationship between Hamiltonian and variational linear distributed behaviors, which we now define. A linear distributed behavior is called *Hamiltonian* if there exists a non-degenerate, skew-symmetric bilinear functional of the system variables and their partial derivatives up to some finite order, whose divergence is zero along the trajectories of the behavior. A behavior is called *variational* if it consists of all trajectories which are stationary with respect to some quadratic functional of the variables and their partial derivatives up to a given order.

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In [1] Hamiltonian systems described by linear, constant-coefficient ordinary differential equations (in the following "Hamiltonian $1D$ systems") were studied using the formalism of quadratic- and bilinear differential forms, and various structural- and representational properties of this class of behaviors were investigated. In that work it was shown that *a $1D$-system is Hamiltonian if and only if it is variational*. In particular, it was shown that a Hamiltonian $1D$ system admits an interpretation as a *mechanical* system, in that it consists of the set of trajectories stationary with respect to a higher-order "Lagrangian" functional depending on a "generalized position" and a "generalized velocity".

In the present paper we attempt to generalize the representation-free approach of [1] to distributed linear systems: we do not assume any special type of representation of a system as starting point, and concentrate instead on the interplay of system dynamics and bilinear- or quadratic differential forms. By adopting such a point of view, considerable results have been obtained in the investigation of physical properties such as losslessness and dissipativity for systems described by linear, constant-coefficient, partial differential equations (see [3, 4]). The main results presented in this paper can be summarized as follows. In Proposition 14 we prove that every linear, variational nD system is also Hamiltonian; as for the converse, we show how to compute for a Hamiltonian nD-system with one external variable a variational behavior $\mathfrak{B}'$ that contains $\mathfrak{B}$, by means of solving a polynomial equation involving the underlying bilinear differential form.

In writing this paper, we have concentrated primarily on presenting and illustrating the basic concepts of our approach in a manner as simple as possible; consequently we decided to emphasize physical examples and we tried to appeal to the readers' intuition. Also, because of space limitations, we limit our exposition to closed ("autonomous" in behavioral parlance) systems.

The paper is organized as follows: in Section 2 we discuss the basics of multidimensional (nD) behavioral systems. In Section 3 we discuss the basics of bilinear- and quadratic differential forms for multidimensional systems, limiting the exposition to the notions necessary in order to understand the material of this paper. In Section 4 we review the main results of [1] for Hamiltonian $1D$ systems. In Section 5 we present our results on the relationship of Hamiltonian variational systems described by partial differential equations. A final section contains comments on the result presented and some indications of the directions for future research in this area.

2. *ND* BEHAVIORS

The purpose of this section is to introduce the reader to those concepts of multidimensional behavioral system theory which are most relevant for the purposes of the paper; see [4] for a thorough treatment of the subject.

In behavioral system theory, the *behavior* is a subset of the space $\mathbb{W}^{\mathbb{T}}$ consisting of all trajectories from $\mathbb{T}$, the *indexing set*, to $\mathbb{W}$, the *signal space*. In this paper we consider systems with $\mathbb{T} = \mathbb{R}^n$ (from which the terminology "*nD*-system" derives) and $\mathbb{W} = \mathbb{R}^{\mathtt{w}}$. We call $\mathfrak{B}$ a *linear differential nD behavior* if it is the solution set of a system of linear, constant-coefficient partial differential equations (in the following PDEs); more precisely, if $\mathfrak{B}$ is the subset of $\mathfrak{C}^{\infty}(\mathbb{R}^n, \mathbb{R}^{\mathtt{w}})$ consisting of all solutions to

$$R\left(\frac{d}{d\mathbf{x}}\right)w = 0 \tag{1}$$

where R is a polynomial matrix in n indeterminates ξ_i, $i = 1, \ldots, n$, and $\frac{d}{d\mathbf{x}} = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right)$. We call (1) a *kernel representation* of $\mathfrak{B}$.

Obviously, any linear differential *nD* behavior $\mathfrak{B}$ is a linear subspace of $\mathbb{W}^{\mathbb{T}}$. Also, any such behavior is *shift-invariant* in the sense that $\mathfrak{B} = \sigma^{x'}\mathfrak{B}$, where for $x' = (x'_1, \ldots, x'_n)$, the x'-shift $\sigma^{x'}$ is defined as

$$\sigma^{x'} : \left(\mathbb{R}^{\mathtt{w}}\right)^{\mathbb{R}^n} \to \left(\mathbb{R}^{\mathtt{w}}\right)^{\mathbb{R}^n}$$
$$(\sigma^{x'} w)(x_1, \ldots, x_n) := w(x_1 + x'_1, \ldots, x_n + x'_n)$$

We denote the set consisting of all linear, shift-invariant differential *nD*-systems with $\mathtt{w}$ external variables with $\mathcal{L}_n^{\mathtt{w}}$; when $n = 1$, we write simply $\mathcal{L}^{\mathtt{w}}$.

The following are examples of elements of $\mathcal{L}_2^1$ and $\mathcal{L}_3^1$ respectively.

Example 1 Consider the *one-dimensional wave equation* describing the displacement $w(t, x)$ from equilibrium of a homogeneous elastic medium:

$$\rho^2 \frac{\partial^2 w}{\partial t^2} - \tau^2 \frac{\partial^2 w}{\partial x^2} = 0 \tag{2}$$

where ρ and τ are physical constants related to the mass density and the elasticity of the medium, respectively. Such equation defines a linear, shift-invariant, differential system with indexing set $\mathbb{T} = \mathbb{R}^2$, signal space $\mathbb{W} = \mathbb{R}$, and behavior

$$\mathfrak{B} = \{w \in \mathfrak{C}^{\infty}(\mathbb{R}^2, \mathbb{R}) \mid w \text{ satisfies } (2)\}$$

The polynomial associated with the representation (2) is $R(\xi_1, \xi_2) = \rho^2\xi_1^2 - \tau^2\xi_2^2$. In order to stress that the we are dealing with variables t and x, we often write ξ_t instead of ξ_1, and ξ_x instead of ξ_2.

Example 2 Let $w(t, x, y)$ be the displacement of an infinite *vibrating plate* in the position (x, y) at time t; then it can be shown that w satisfies the PDE

$$\rho \frac{\partial^2 w}{\partial t^2} + \frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^2 w}{\partial x \partial y} + \frac{\partial^4 w}{\partial y^4} = 0 \tag{3}$$

where ρ is a constant depending on the physical properties of the plate. Such equation defines a linear, shift-invariant, differential system with indexing set $\mathbb{T} = \mathbb{R}^3$, signal space $\mathbb{W} = \mathbb{R}$, and behavior $\mathfrak{B} = \{w \in \mathfrak{C}^\infty(\mathbb{R}^3, \mathbb{R}) \mid w \text{ satisfies Equation (3)}\}$. The polynomial associated with the representation (3) is $R(\xi_t, \xi_x, \xi_y) = \rho\xi_t^2 + \xi_x^4 + 2\xi_x\xi_y + \xi_y^4$.

Finally, we introduce the notion of (weakly) *autonomous nD*-behavior. Intuitively, an autonomous behavior consists of trajectories which are completely determined by their boundary conditions, that is, systems on which no external influence in the form of "inputs" (more precisely, "free variables") is exerted, see [3]. In order to give a formal definition, we need to define the characteristic ideal and characteristic variety associated with a kernel representation (1). Let $\mathbb{R}^{\mathtt{r}\times\mathtt{w}}[\xi_1, \ldots, \xi_n]$ be the set of all $\mathtt{r} \times \mathtt{w}$ matrices with components in the polynomial ring $\mathbb{R}[\xi_1, \ldots, \xi_n]$ of polynomials in n indeterminates, with real coefficients. For $\mathbb{R} \in \mathbb{R}^{\mathtt{r}\times\mathtt{w}}[\xi_1, \ldots, \xi_n]$, the *characteristic ideal* is the ideal of $\mathbb{R}[\xi_1, \ldots, \xi_n]$ generated by the determinants of all $\mathtt{w} \times \mathtt{w}$ minors of R, and the *characteristic variety* is the set of roots common to all polynomials in the ideal. The behavior $\mathfrak{B}$ represented in kernel form by Equation (1) is said to be (weakly) *autonomous* if its characteristic ideal is not the zero ideal; or equivalently, if its characteristic variety is not all of $\mathbb{C}^n$. Observe that if an *nD*-behavior is represented by a single Equation (1) with a nonzero polynomial, such as the ones considered in Example 1 and Example 2, then it is (weakly) autonomous.

3. BILINEAR- AND QUADRATIC DIFFERENTIAL FORMS

In many modeling and control problems for linear systems it is necessary to study bilinear- and quadratic functionals of the system variables and their derivatives. For finite-dimensional linear systems, an efficient representation for such functionals by means of two-variable polynomial matrices was introduced in [2]; in order to represent bilinear- and quadratic functionals of the variables of *nD*-systems, 2*n*-variable polynomial matrices are used (see [4]).

In order to simplify the notation, define the vector $\mathbf{x} := (x_1, \ldots, x_n)$, the multi-indices $\mathbf{k} := (k_1, \ldots, k_n)$, $\mathbf{l} := (l_1, \ldots, l_n)$, and the notation $\zeta := (\zeta_1, \ldots, \zeta_n)$ and $\eta := (\eta_1, \ldots, \eta_n)$, so that $\zeta^{\mathbf{k}}\eta^{\mathbf{l}} = \zeta_1^{k_1} \cdots \zeta_n^{k_n}\eta_1^{l_1} \cdots \eta_m^{l_n}$.

Let $\mathbb{R}^{\mathtt{w}_1\times\mathtt{w}_2}[\zeta, \eta]$ denote the set of real polynomial $\mathtt{w}_1 \times \mathtt{w}_2$ matrices in the $2n$ indeterminates ζ and η; that is, an element of $\mathbb{R}^{\mathtt{w}_1\times\mathtt{w}_2}[\zeta, \eta]$ is of the form

$$\Phi(\zeta, \eta) = \sum_{\mathbf{k},\mathbf{l}} \Phi_{\mathbf{k},\mathbf{l}}\zeta^{\mathbf{k}}\eta^{\mathbf{l}}$$

where $\Phi_{\mathbf{k},\mathbf{l}} \in \mathbb{R}^{\mathtt{w}_1\times\mathtt{w}_2}$; the sum ranges over the nonnegative multi-indices $\mathbf{k}$ and $\mathbf{l}$, and is assumed to be finite. Such matrix induces a *bilinear differential form* (*BDF* in the following) L_Φ

$$L_\Phi : \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^{\mathtt{w}_1}) \times \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^{\mathtt{w}_2}) \longrightarrow \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R})$$
$$L_\Phi(v, w) := \sum_{\mathbf{k},\mathbf{l}} \left(\frac{d^{\mathbf{k}} v}{d\mathbf{x}^{\mathbf{k}}} \right)^T \Phi_{\mathbf{k},\mathbf{l}} \frac{d^{\mathbf{l}} w}{d\mathbf{x}^{\mathbf{l}}}$$

where the $\mathbf{k}$-th derivative operator $\frac{d^{\mathbf{k}}}{d\mathbf{x}^{\mathbf{k}}}$ is defined as $\frac{d^{\mathbf{k}}}{d\mathbf{x}^{\mathbf{k}}} := \frac{\partial^{k_1}}{\partial x_1^{k_1}} \cdots \frac{\partial^{k_n}}{\partial x_n^{k_n}}$ (similarly for $\frac{d^{\mathbf{l}}}{d\mathbf{x}^{\mathbf{l}}}$).

We call L_Φ *skew-symmetric* if $L_\Phi(w_1, w_2) = -L_\Phi(w_2, w_1)$ for all infinitely differentiable trajectories w_1, w_2. It can be shown that this is the case if and only if Φ is a *skew-symmetric* $2n$-variable polynomial matrix, i.e. if $\mathtt{w}_1 = \mathtt{w}_2$ and $\Phi(\zeta, \eta) = -\Phi(\eta, \zeta)^T$. The $2n$-variable polynomial matrix $\Phi(\zeta, \eta)$ is called *symmetric* if $\mathtt{w}_1 = \mathtt{w}_2 =: \mathtt{w}$ and $\Phi(\zeta, \eta) = \Phi(\eta, \zeta)^T$. In such case, Φ induces also a quadratic functional

$$Q_\Phi : \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^{\mathtt{w}}) \longrightarrow \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R})$$
$$Q_\Phi(w) := L_\Phi(w, w)$$

We will call Q_Φ the *quadratic differential form* (in the following abbreviated with QDF) associated with Φ.

In this paper we also consider vectors $\Psi \in (\mathbb{R}^{\mathtt{w}_1 \times \mathtt{w}_2}[\zeta, \eta])^n$, that is,

$$\Psi(\zeta, \eta) = \begin{pmatrix} \Psi_1(\zeta, \eta) \\ \vdots \\ \Psi_n(\zeta, \eta) \end{pmatrix} =: \operatorname{col}(\Psi_i(\zeta, \eta))_{i=1,\ldots,n}$$

with $\Psi_i \in \mathbb{R}^{\mathtt{w}_1 \times \mathtt{w}_2}[\zeta, \eta]$ and with $\operatorname{col}(A_i)_{i=1,\ldots,n}$ the matrix obtained by stacking the matrices A_i, all with the same number of columns, on top of each other. Such Ψ induces a vector bilinear differential form (in short a *VBDF*), defined as

$$L_\Psi : \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^{\mathtt{w}_1}) \times \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^{\mathtt{w}_2}) \longrightarrow (\mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}))^n$$
$$L_\Psi(v, w) := (\, L_{\Psi_1}(v, w), L_{\Psi_2}(v, w), \ldots, L_{\Psi_n}(v, w)\,)^T.$$

Finally, we introduce the notion of *divergence* of a VBDF. Given a VBDF $L_\Psi = (L_{\Psi_1}, L_{\Psi_2}, \ldots, L_{\Psi_n})^T$, we define its *divergence* as the BDF defined by

$$(\operatorname{div} L_\Psi)(w_1, w_2) := \left(\frac{\partial}{\partial x_1} L_{\Psi_1} \right)(w_1, w_2) + \cdots + \left(\frac{\partial}{\partial x_n} L_{\Psi_n} \right)(w_1, w_2) \qquad (4)$$

for all infinitely differentiable trajectories w_1, w_2. In terms of the $2n$-variable polynomial matrices associated with the BDF's, the relationship between a VBDF and its divergence is expressed as

$$\Phi(\zeta, \eta) = (\zeta_1 + \eta_1)\Psi_1(\zeta, \eta) + \cdots + (\zeta_n + \eta_n)\Psi_n(\zeta, \eta)$$

In order to characterize those BDFs which are the divergence of some VBDF, we need to introduce some notation. The "del" operator is defined as

$$\partial : \mathbb{R}^{w_1 \times w_2}[\zeta_1, \dots, \zeta_n, \eta_1, \dots, \eta_n] \longrightarrow \mathbb{R}^{w_1 \times w_2}[\xi_1, \dots, \xi_n]$$
$$\partial\Phi(\xi_1, \dots, \xi_n) = \Phi(-\xi_1, \dots, -\xi_n, \xi_1, \dots, \xi_n)$$

Observe that if $\Phi(\zeta, \eta)$ is symmetric, then the matrix $\partial\Phi(\xi_1, \dots, \xi_n)$ is *para-Hermitian*, meaning $\partial\Phi(\xi_1, \dots, \xi_n) = \partial\Phi(-\xi_1, \dots, -\xi_n)^T$. Observe also that by means of the "del" operator, a differential operator $\partial\Phi(\frac{d}{d\mathbf{x}})$ can be assigned to a QDF; this has important applications in variational problems, and when considering the problem of which BDFs are the divergence of some VBDF, as we now show. Indeed, it can be shown that L_Φ is the divergence of some VBDF L_Ψ if and only if $\partial\Phi(\xi) = 0$ (see Th. 4, p. 1411 of [4]).

Example 3 Consider the behavior described by Equation (2). On the basis of physical considerations, the total energy of the system trajectory w at time t can be shown to be $\int_{\mathbb{R}} Q_\Phi(w)dx$, where $Q_\Phi(w) = \frac{1}{2}\rho(\frac{\partial w}{\partial t})^2 + \frac{1}{2}\tau(\frac{\partial w}{\partial x})^2$ is associated with the 4-variable polynomial

$$\Phi(\zeta_t, \zeta_x, \eta_t, \eta_x) = \rho\tfrac{1}{2}\zeta_t\eta_t + \tfrac{1}{2}\tau\zeta_x\eta_x$$

Example 4 Consider the behavior defined by the transverse motion of a homogeneous flexible sheet ("membrane") with surface mass density ρ. It can be shown that if $w(t, x, y)$ is the displacement from equilibrium of point (x, y) of the membrane at time t, then w satisfies the PDE (see Section 7.36.3 [5]):

$$\rho\frac{\partial^2 w}{\partial t^2} - \tau\frac{\partial^2 w}{\partial x^2} - \tau\frac{\partial^2 w}{\partial y^2} = 0 \tag{5}$$

This defines the behavior

$$\mathfrak{B} = \{w \in \mathfrak{C}^\infty(\mathbb{R}^3, \mathbb{R}) | \text{ Equation (5) is satisfied}\} \tag{6}$$

On the basis of physical considerations (see [5]) it can be shown that the Lagrangian (i.e. the difference between the kinetic and potential energy) of the membrane at time t is $\int_{\mathbb{R}^2} Q_\Phi(w)dxdy$, where

$$Q_\Phi(w) = \frac{1}{2}\rho\left(\frac{\partial w}{\partial t}\right)^2 - \frac{1}{2}\tau\left[\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right] \tag{7}$$

is associated with the 6-variable polynomial

$$\Phi(\zeta_t, \zeta_x, \zeta_y, \eta_t, \eta_x, \eta_y) := \tfrac{1}{2}\rho\zeta_t\eta_t - \tfrac{1}{2}\tau(\zeta_x\eta_x + \zeta_y\eta_y)$$

4. AUTONOMOUS HAMILTONIAN 1D SYSTEMS

In this section we review some of the results of [1] about autonomous Hamiltonian 1D behaviors; in order to do this, a number of preliminary remarks are in order.

The trajectories of a 1D behavior map $\mathbb{R}$ into a signal space $\mathbb{R}^{\mathtt{w}}$, and normally are considered to be time-signals; for this reason we denote the "independent variable" of which such a trajectory is function with t. It can be shown (see [6]) that an autonomous 1D behavior corresponds to a kernel representation (1) associated with a one-variable polynomial matrix $R(\xi)$ having full column rank $\mathtt{w}$; in particular, such a matrix can be chosen to be square and nonsingular. It can also be shown that the behavior of a 1D autonomous system is finite-dimensional, and consists of vector-polynomial exponential trajectories $w(t) = \sum_{i=1}^{r} \sum_{k=0}^{n_i-1} \alpha_{i,k} t^k e^{\lambda_i t}$, where $\alpha_{i,k} \in \mathbb{R}^{\mathtt{w}}$. The polynomial

$$\chi_{\mathfrak{B}}(\xi) := \Pi_{i=1}^{r}(\xi - \lambda_i)^{n_i}$$

associated with $\mathfrak{B}$ is called the *characteristic polynomial of* $\mathfrak{B}$.

The definition of autonomous Hamiltonian 1D system is as follows.

Definition 5 *Let* $\mathfrak{B} \in \mathcal{L}^{\mathtt{w}}$ be *autonomous.* $\mathfrak{B}$ *is called* Hamiltonian *if there exists a bilinear differential form* L_Ψ*, such that*

(i) $\frac{d}{dt} L_\Psi(w_1, w_2) = 0$ *for all* $w_1, w_2 \in \mathfrak{B}$*;*
(ii) L_Ψ *is skew-symmetric;*
(iii) $L_\Psi(v, w)(0) = 0$ *for all* $v \in \mathfrak{B} \Longleftrightarrow w = 0$ (nondegeneracy).

Observe that in Definition 5 no assumption on the number $\mathtt{w}$ of external variables of $\mathfrak{B}$ is made, in contrast with the usual definition, in which an even number of such variables is assumed.

Example 6 Consider the autonomous behavior $\mathfrak{B}$ represented by the first order differential equation $\frac{d}{dt} w = Aw$, where $A \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}$, and $\mathtt{w}$ is even. Such system is often called Hamiltonian if the matrix A is a *Hamiltonian matrix*, that is, if $A^T J + JA = 0$, where J is equal to the nonsingular skew-symmetric matrix

$$J = \begin{pmatrix} 0 & I_{\mathtt{w}/2} \\ -I_{\mathtt{w}/2} & 0 \end{pmatrix}.$$

This behavior $\mathfrak{B}$ is also Hamiltonian in the sense of Definition 5: the bilinear differential form $L_\Psi(v, w) := v^T J w$ is easily seen to satisfy the conditions (i), (ii) and (iii).

Example 7 Consider two masses m_1 and m_2 attached to springs with constants k_1 and k_2. The first mass is connected to the second one via the first spring, and the second mass is connected to a "wall" with the second spring. Denote the positions of the masses with w_1 and w_2. The system equations are

$$m_1 \frac{d^2}{dt^2} w_1 + k_1 w_1 - k_1 w_2 = 0$$

$$m_2 \frac{d^2}{dt^2} w_2 - k_1 w_1 + (k_1 + k_2) w_2 = 0$$

and it can be shown that such equations describe an autonomous system. Now assume that we are only interested in modeling the position w_1 of the first mass; manipulating the equations we can then eliminate w_2 obtaining a higher-order model for w_1 as

$$m_1 m_2 \frac{d^4}{dt^4} w_1 + (k_1 m_1 + k_2 m_1 + k_1 m_2) \frac{d^2}{dt^2} w_1 + k_1 k_2 w_1 = 0 \tag{8}$$

(see Chapter 6 of [6] for a thorough discussion of the issue of variable elimination). For simplicity, denote $r_0 := k_1 k_2$, $r_2 := k_1 m_1 + k_2 m_1 + k_1 m_2$ and $r_4 := m_1 m_2$. Now define the skew-symmetric two-variable polynomial

$$\Psi(\zeta, \eta) := r_2(\zeta - \eta) + r_4(\zeta^3 - \eta^3) + r_4(\zeta\eta^2 - \zeta^2\eta).$$

The corresponding BDF $L_\Psi(v, w)$ is skew-symmetric and $\frac{d}{dt} L_\Psi(v, w) = 0$ for all $v, w \in \mathfrak{B}$, as can be easily verified. It can also be easily proved that L_Ψ is nondegenerate on $\mathfrak{B}$. Hence the behavior $\mathfrak{B}$ is a Hamiltonian system.

In order to state the main result of this section, that is the equivalence of Hamiltonianity and variationality for $1D$-systems, we need to review the concept of stationarity of a trajectory with respect to a QDF. Let $\Phi \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}[\zeta, \eta]$ be symmetric and consider the corresponding QDF $Q_\Phi(w)$ on $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$. For a given w we define the *cost degradation* of adding the compact-support function $\delta \in \mathfrak{D}(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ to w as

$$J_w(\delta) := \int_{-\infty}^{+\infty} (Q_\Phi(w + \delta) - Q_\Phi(w)) dt = \int_{-\infty}^{+\infty} Q_\Phi(\delta) dt + 2 \int_{-\infty}^{+\infty} L_\Phi(w, \delta) dt$$

The second term on the right in this equation is called the *variation associated with* w: it is a functional associating to δ the real number $2 \int_{-\infty}^{+\infty} L_\Phi(w, \delta) dt$. We call w a *stationary trajectory* of Q_Φ if the variation associated with it is the zero functional. It can be shown (by repeated partial integration of the integral defining the variation) that $w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ is a stationary trajectory with respect to the QDF Q_Φ if and only if w satisfies the differential equation

$$\partial\Phi\left(\frac{d}{dt}\right) w = 0$$

(Recall: $\partial\Phi(\xi)$ is defined as the one-variable polynomial matrix $\Phi(-\xi, \xi)$.) A behavior $\mathfrak{B}$ consisting of all trajectories that are stationary with respect to a given QDF Q_Φ is called *variational* with respect to Q_Φ. It follows from the definition of stationary trajectory that $\mathfrak{B}$ is variational with respect to Q_Φ if and only if $\mathfrak{B} = \ker\, \partial\Phi(\frac{d}{dt})$.

The following theorem is an extension of [1], Theorem 10.

Theorem 8 *Let $\mathfrak{B} \in \mathcal{L}^{\mathtt{w}}$ be autonomous, $\chi_{\mathfrak{B}}(0) \neq 0$. Then $\mathfrak{B}$ is Hamiltonian if and only if there exists a symmetric $\Phi(\zeta,\eta) \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ such that $\mathfrak{B} = \ker\, \partial\Phi(\frac{d}{dt})$. In that case the number $\mathtt{n}(\mathfrak{B})$ of state variables in any minimal state representation of $\mathfrak{B}$ is even, and there exists a full column rank matrix $P(\xi) \in \mathbb{R}^{\mathtt{q}\times\mathtt{w}}[\xi]$, $M = M^T$, $K = K^T \in \mathbb{R}^{\mathtt{q}\times\mathtt{q}}$ nonsingular, with $\mathtt{q} := \frac{\mathtt{n}(\mathfrak{B})}{2}$, such that $\mathfrak{B}$ is equal to* $\ker\, \partial\Phi(\frac{d}{dt})$, *with $\Phi(\zeta,\eta)$ the two variable polynomial matrix*

$$\Phi(\zeta,\eta) := P(\zeta)^T(\zeta\eta M - K)P(\eta), \tag{9}$$

corresponding to the QDF

$$Q_\Phi(w) = \left|\frac{d}{dt}P\left(\frac{d}{dt}\right)w\right|_M^2 - \left|P\left(\frac{d}{dt}\right)w\right|_K^2.$$

If one interprets the latent variable $q = P(\frac{d}{dt})w$ as a *generalized position*, *then* $\dot{q} = \frac{d}{dt}P(\frac{d}{dt})w$ is a *generalized velocity*; consequently the expressions $|\frac{d}{dt}P(\frac{d}{dt})w|_M^2 = |\dot{q}|_M^2$ and $|P(\frac{d}{dt})w|_K^2 = |q|_K^2$ can be interpreted, respectively, as *kinetic* and *potential energy*. From this point of view, the QDF $Q_\Phi(w)$ can be interpreted as a *Lagrangian* of the system. The system of differential equations $\partial\Phi(\frac{d}{dt})w = 0$ (representing the stationary trajectories with respect to this Lagrangian) coincide with the Euler-Poisson equations associated with the Lagrangian.

Remark 9 In [1] algebraic procedures are stated, which compute the "generalized Lagrangian", and the symplectic BDF L_Ψ in Definition 5 starting from a representation of a system.

The following example illustrates the result of Theorem 8.

Example 10 Consider the configuration of Example 7. As shown in that Example, the behavior $\mathfrak{B}$ of the position w_1 of the first mass is represented by Equation (8). Define the latent variable q as $\operatorname{col}(w_1, \frac{d^2}{dt^2}w_1)$. It can be shown that a generalized Lagrangian for $\mathfrak{B}$ is given by $Q_\Phi(w_1) = \dot{q}^T M\dot{q} - q^T Kq$, with

$$M := \begin{pmatrix} r_2 & r_4 \\ r_4 & 0 \end{pmatrix}, \qquad K := \begin{pmatrix} r_0 & 0 \\ 0 & -r_4 \end{pmatrix}.$$

where the r_i are defined as in Example 7. Observe that such generalized Lagrangian does *not* correspond to a difference of energies, as can be readily seen checking the physical dimensions of the terms $\dot{q}^T M\dot{q}$ and $q^T Kq$. By choosing the latent variable q as before, and the matrices

$$\bar{K} := \begin{pmatrix} k_2 & \frac{k_2 m_1}{k_1} \\ \frac{k_2 m_1}{k_1} & \frac{m_1^2(k_1+k_2)}{k_1^2} \end{pmatrix}, \qquad \bar{M} := \begin{pmatrix} m_1 + m_2 & \frac{m_1 m_2}{k_1} \\ \frac{m_1 m_2}{k_1} & \frac{m_1^2 m_2}{k_1^2} \end{pmatrix}, \tag{10}$$

we obtain a new generalized Lagrangian for $\mathfrak{B}$. With this second choice of L, the generalized kinetic energy $\dot{q}^T \bar{M} \dot{q}$ and generalized potential energy $q^T \bar{K} q$ (which are a function of the position and the acceleration of the first mass) coincide with the *physical* kinetic energy and potential energy of the system with the two masses:

$$\begin{aligned} E_{\text{kin}}(w_1, w_2) &= \tfrac{1}{2}\left(m_1\left(\tfrac{d}{dt}w_1\right)^2 + m_2\left(\tfrac{d}{dt}w_2\right)^2\right) \\ E_{\text{pot}}(w_1, w_2) &= \tfrac{1}{2}\left(k_1 w_1^2 - 2k_1 w_1 w_2 + (k_1 + k_2) w_2^2\right) \end{aligned}$$

This can be verified easily, since the generalized position $q = \text{col}(w, \frac{d^2}{dt^2} w_1)$ is related to the actual position (w_1, w_2) by the nonsingular linear map

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & \frac{m_1}{k_1} \end{pmatrix} \begin{pmatrix} w_1 \\ \frac{d^2}{dt^2} w_1 \end{pmatrix}.$$

5. AUTONOMOUS HAMILTONIAN *nD* SYSTEMS

In this section we attempt to generalize the result of Theorem 8 in Section 4 to linear differential *nD* systems, that is, systems described by linear, constant-coefficient PDEs. We first define Hamiltonian *nD* systems, give a couple of examples in order to illustrate the definition, and then we show that every autonomous variational *nD* system is Hamiltonian. Finally, we show that every *scalar* (i.e., $\mathtt{w} = 1$), autonomous Hamiltonian *nD* system is a sub-system of an autonomous variational system that is Hamiltonian with respect to the same VBDF.

The definition of autonomous Hamiltonian *nD* system is as follows.

Definition 11 *Let* $\mathfrak{B} \in \mathcal{L}_n^{\mathtt{w}}$ *be autonomous.* $\mathfrak{B}$ *is called* Hamiltonian *if there exists a VBDF* L_Ψ, *with* $\operatorname{div} L_\Psi \neq 0$, *such that*

(i) $\operatorname{div} L_\Psi(w_1, w_2) = 0$ *for all* $w_1, w_2 \in \mathfrak{B}$;
(ii) L_Ψ *is skew-symmetric;*
(iii) $[L_\Psi(v, w)(0) = 0$ *for all* $v \in \mathfrak{B}] \Longleftrightarrow [w = 0]$ (nondegeneracy)*;*

Note that in this definition it is required that the divergence of L_Ψ is unequal to zero. In the $1D$ case this condition reduces to $\frac{d}{dt} L_\Psi \neq 0$, which in that case is implied by the nondegeneracy condition (iii). In the general nD case, nondegeneracy does not imply $\operatorname{div} L_\Psi \neq 0$. We now consider a couple of examples of Hamiltonian systems.

Example 12 Consider the behavior of the membrane system illustrated in Example 4. Define the 6-variable polynomial vector

$$\Psi(\zeta_t,\zeta_x,\zeta_y,\eta_t,\eta_x,\eta_y) := \begin{pmatrix} \zeta_t-\eta_t \\ -\frac{\tau}{\rho}(\zeta_x-\eta_x) \\ -\frac{\tau}{\rho}(\zeta_y-\eta_y) \end{pmatrix}$$

and observe that $L_\Psi(v,w) = -L_\Psi(w,v)$ for every pair of trajectories v,w, so that L_Ψ is skew-symmetric. It is also easily seen that $\operatorname{div} L_\Psi \neq 0$. Moreover, along every pair of trajectories v,w belonging to the behavior $\mathfrak{B}$ defined in Equation (6), we have

$$\begin{aligned}
&(\operatorname{div} L_\Psi)(v,w) \\
&= \frac{\partial}{\partial t}\left(\frac{\partial v}{\partial t}w - v\frac{\partial w}{\partial t}\right) + \frac{\partial}{\partial x}\left[-\frac{\tau}{\rho}\left(\frac{\partial v}{\partial x}w - v\frac{\partial w}{\partial x}\right)\right] + \frac{\partial}{\partial y}\left[-\frac{\tau}{\rho}\left(\frac{\partial v}{\partial y}w - v\frac{\partial w}{\partial y}\right)\right] \\
&= \left(\frac{\partial^2 v}{\partial t^2} - \frac{\tau}{\rho}\frac{\partial^2 v}{\partial x^2} - \frac{\tau}{\rho}\frac{\partial^2 v}{\partial y^2}\right)w - v\left(\frac{\partial^2 w}{\partial t^2} - \frac{\tau}{\rho}\frac{\partial^2 w}{\partial x^2} - \frac{\tau}{\rho}\frac{\partial^2 w}{\partial y^2}\right) = 0
\end{aligned}$$

so that L_Ψ also satisfies property (i) of Definition 11. It can also be shown that L_Ψ is nondegenerate, and consequently that $\mathfrak{B}$ defined in Equation (6) is a symplectic behavior.

Example 13 Let $w(t,x)$ be the position at time t of point x of an infinitely long stiff beam subject to vibrations. It can be shown (see, e.g., [5]) that the behavior consisting of all possible motions $w(\cdot,\cdot)$ is described by

$$\mathfrak{B} = \left\{ w \in \mathfrak{C}(\mathbb{R}^2,\mathbb{R}) \,\middle|\, \frac{\partial^2 w}{\partial t^2} + a^2\frac{\partial^4 w}{\partial x^4} = 0 \right\} \tag{11}$$

where a is a constant which depends on the physical properties of the beam. We claim that the system is symplectic. Indeed, the two-dimensional VBDF associated with

$$\Psi(\zeta_t,\zeta_x,\eta_t,\eta_x) := \begin{pmatrix} \zeta_t-\eta_t \\ a^2(\zeta_x^3-\eta_x^3-\zeta_x^2\eta_x+\zeta_x\eta_x^2) \end{pmatrix} \tag{12}$$

that is,

$$L_\Psi(v,w) = \begin{pmatrix} \frac{\partial v}{\partial t}w - \frac{\partial w}{\partial t}v \\ a^2\left(\frac{\partial^3 v}{\partial x^3}w - v\frac{\partial^3 w}{\partial x^3} - \frac{\partial^2 v}{\partial x^2}\frac{\partial w}{\partial x} + \frac{\partial v}{\partial x}\frac{\partial^2 w}{\partial x^2}\right) \end{pmatrix}$$

is skew-symmetric, as it is easy to verify. Also, $\operatorname{div} L_\Psi \neq 0$. Moreover, since

$$\begin{aligned}
&(\operatorname{div} L_\Psi)(v,w)\\
&= \frac{\partial}{\partial t}\left(\frac{\partial v}{\partial t}w - \frac{\partial w}{\partial t}v\right) + \frac{\partial}{\partial x}\left[a^2\left(\frac{\partial^3 v}{\partial x^3}w - v\frac{\partial^3 w}{\partial x^3} - \frac{\partial^2 v}{\partial x^2}\frac{\partial w}{\partial x} + \frac{\partial v}{\partial x}\frac{\partial^2 w}{\partial x^2}\right)\right]\\
&= \frac{\partial^2 v}{\partial t^2}w - \frac{\partial^2 w}{\partial t^2}v + a^2\left(\frac{\partial^4 v}{\partial x^4}w - v\frac{\partial^4 w}{\partial x^4}\right)\\
&= \left(\frac{\partial^2 v}{\partial t^2} + a^2\frac{\partial^4 v}{\partial x^4}\right)w + \left(\frac{\partial^2 w}{\partial t^2} + a^2\frac{\partial^4 w}{\partial x^4}\right)v
\end{aligned}$$

we conclude that $(\operatorname{div} L_\Psi)(v,w) = 0$ for all v, $w \in \mathfrak{B}$. It is a matter of tedious verification to show that L_Ψ is also non-degenerate, and consequently that $\mathfrak{B}$ defined in Equation (11) is Hamiltonian.

Similar as in the $1D$ case, for systems described by partial differential equations we have the notion of stationarity with respect to a given QDF. Let $\Phi(\zeta,\eta)$ be a symmetric $2n$-variable polynomial matrix, and consider the corresponding QDF $Q_\Phi(w)$ on $\mathfrak{C}^\infty(\mathbb{R}^n,\mathbb{R}^{\mathtt{w}})$. For a given w the *cost degradation* of adding the compact-support function $\delta \in \mathfrak{D}(\mathbb{R}^n,\mathbb{R}^{\mathtt{w}})$ to w is defined as

$$J_w(\delta) := \int_{\mathbb{R}^n} Q_\Phi(w+\delta) - Q_\Phi(w)d\mathbf{x} = \int_{\mathbb{R}^n} Q_\Phi(\delta)d\mathbf{x} + 2\int_{\mathbb{R}^n} L_\Phi(w,\delta)d\mathbf{x}$$

The second term on the right in this equation is called the *variation associated with w*: it is a functional associating to δ the real number $2\int_{\mathbb{R}^n} L_\Phi(w,\delta)d\mathbf{x}$. We call w a *stationary trajectory* of Q_Φ if the variation associated with it is the zero functional. It can be shown (by repeated application of the Gauss divergence theorem) that $w \in \mathfrak{C}^\infty(\mathbb{R}^n,\mathbb{R}^{\mathtt{w}})$ is a stationary trajectory with respect to the QDF Q_Φ if and only if w satisfies the system of linear partial differential equations

$$\partial\Phi\left(\frac{d}{d\mathbf{x}}\right)w = 0$$

(Recall: $\partial\Phi(\xi)$ is defined as the n-variable polynomial matrix $\Phi(-\xi,\xi)$.) As in the $1D$ case, a behavior $\mathfrak{B} \in \mathcal{L}_n^{\mathtt{w}}$ consisting of all trajectories that are stationary with respect to a given QDF Q_Φ is called *variational* with respect to Q_Φ. It follows from the definition of stationary trajectory that $\mathfrak{B}$ is variational with respect to Q_Φ if and only if $\mathfrak{B} = \ker\, \partial\Phi(\frac{d}{d\mathbf{x}})$.

We now show that every nD variational behavior is Hamiltonian.

Proposition 14 *Let $\mathfrak{B} \in \mathcal{L}_n^{\mathtt{w}}$ be an autonomous nD behavior. Assume that $\mathfrak{B}$ is variational with respect to some QDF Q_Φ, i.e. $\mathfrak{B} = \ker \partial\Phi(\frac{d}{d\mathbf{x}})$; then $\mathfrak{B}$ is Hamiltonian.*

Proof Define a $2n$-variable $\mathtt{w} \times \mathtt{w}$ polynomial matrix $\Phi'(\zeta, \eta) := \partial\Phi(-\zeta) - \partial\Phi(\eta)$. Since $\Phi(\zeta, \eta)$ is symmetric, $\partial\Phi$ is para-Hermitian, i.e., $\partial\Phi(\xi) = \partial\Phi(-\xi)^T$. From this we conclude that for all v, w we have

$$L_{\Phi'}(v, w) = \left(\partial\Phi\left(\frac{d}{d\mathbf{x}}\right)v\right)^T w - v^T\left(\partial\Phi\left(\frac{d}{d\mathbf{x}}\right)w\right).$$

As a consequence, since $B = \ker \partial\Phi(\frac{d}{d\mathbf{x}})$, for all $v, w \in \mathfrak{B}$ we have $L_{\Phi'}(v, w) = 0$. Since also

$$\partial\Phi'(\xi) = \partial\Phi(\xi) - \partial\Phi(\xi) = 0$$

we conclude (see Th. 4 of [4]) that there exists a $\Psi = \mathrm{col}(\Psi_1, \Psi_2, \ldots, \Psi_n) \in (\mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta_1, \ldots, \zeta_n, \eta_1, \ldots, \eta_n])^n$ such that $\mathrm{div}\, L_\Psi = L_{\Phi'}$. We will prove that Ψ can be choosen to be skew-symmetric, that is, $\Psi_i(\zeta, \eta)^T = -\Psi_i(\eta, \zeta), i = 1, 2, \ldots, n$. Indeed, we have

$$\Phi'(\zeta, \eta) = \sum_{i=1}^{n}(\zeta_i + \eta_i)\Psi_i(\zeta, \eta)$$

so

$$\Phi'(\eta, \zeta)^T = \sum_{i=1}^{n}(\zeta_i + \eta_i)\Psi_i(\eta, \zeta)^T$$

Note that from the definition of $\Phi'(\zeta, \eta)$ we have $\Phi'(\eta, \zeta)^T = -\Phi'(\zeta, \eta)$. Thus we get

$$\begin{aligned}\Phi'(\zeta, \eta) &= \tfrac{1}{2}(\Phi'(\zeta, \eta) - \Phi'(\eta, \zeta)^T) \\ &= \sum_{i=1}^{n}(\zeta_i + \eta_i)\frac{1}{2}(\Psi_i(\zeta, \eta) - \Psi_i(\eta, \zeta)^T)\end{aligned}$$

By redefining $\Psi_i(\zeta, \eta)$ as $\frac{1}{2}(\Psi_i(\zeta, \eta) - \Psi_i(\eta, \zeta)^T)$ we thus get a skew-symmetric VBDF. Observe that $\mathrm{div}\, L_\Psi(v, w) = L_{\Phi'}(v, w) \neq 0$, but that $\mathrm{div}\, L_\Psi(v, w) = 0$ for all $v, w \in \mathfrak{B}$. The proof of the nondegeneracy of L_Ψ is rather technical and laborious, and is omitted.

We illustrate the content of Proposition 14 with an example.

Example 15 Consider the behavior $\mathfrak{B}$ of the vibrating membrane of Example 4. It can be shown that $\mathfrak{B}$ defined in Equation (6) is stationary with respect to the quadratic functional defined in Equation (7) and associated with the 6-variable polynomial

$$\Phi(\zeta_t, \zeta_x, \zeta_y, \eta_t, \eta_x, \eta_y) = \rho\zeta_t\eta_t - \tau\zeta_x\eta_x - \tau\zeta_y\eta_y$$

Observe that $\partial\Phi(\xi_t,\xi_x,\xi_y) = -\rho\xi_t^2 + \tau\xi_x^2 + \tau\xi_y^2$. We proceed as in the proof of Proposition 14, and define the 6-variable polynomial

$$\begin{aligned}\Phi'(\zeta_t,\zeta_x,\zeta_y,\eta_t,\eta_x,\eta_y) &= \partial\Phi(-\zeta_t,-\zeta_x,-\zeta_y) - \partial\Phi(\eta_t,\eta_x,\eta_y)\\ &= (\zeta_t+\eta_t)\rho(\eta_t-\zeta_t) + (\zeta_x+\eta_x)\tau(\zeta_x-\eta_x)\\ &\quad + (\zeta_y+\eta_y)\tau(\zeta_y-\eta_y)\end{aligned}$$

Observe that $L_{\Phi'} = \operatorname{div} L_\Psi$, where

$$\Psi(\zeta_t,\zeta_x,\zeta_y,\eta_t,\eta_x,\eta_y) = \begin{pmatrix}\rho(\eta_t-\zeta_t)\\ \tau(\zeta_x-\eta_x)\\ \tau(\zeta_y-\eta_y)\end{pmatrix}$$

Such 6-variable polynomial vector is proportional to that considered in Example 12.

We proceed to show that in the scalar case, that is, when the number $\mathtt{w}$ of external variables equals 1, every Hamiltonian system is contained in a particular variational one.

In order to do this, we need to characterize the property of a skew-symmetric BDF being zero along a behavior, in terms of properties of the polynomial matrices involved in the representation of the BDF and of the system itself. The following result can be proven using Gröbner basis techniques (see [7]) and the results of [8].

Lemma 16 *Let $\mathfrak{B} \in \mathcal{L}_n^1$ be represented in kernel form by a polynomial $R(\xi_1,\ldots,\xi_n)$, and let Φ be a $2n$-variable skew-symmetric polynomial. Then $L_\Phi(v,w) = 0$ for all $v,w \in \mathfrak{B}$ if and only if there exists an n-variable polynomial X such that*

$$\Phi(\zeta,\eta) = R(\zeta)X(\eta) - X(\zeta)R(\eta) \tag{13}$$

Consider now the system $\mathfrak{B} \in \mathcal{L}_n^1$, represented in kernel form by $R(\frac{\partial}{\partial \mathbf{x}})w = 0$. Assume that $\mathfrak{B}$ is Hamiltonian with respect to the VBDF $L_\Psi = \operatorname{col}(L_{\Psi_1}, L_{\Psi_2},\ldots, L_{\Psi_n})$. Consider the equation

$$\sum_{i=1}^{n}(\zeta_i+\eta_i)\Psi_i(\zeta,\eta) = R(\zeta)X(\eta) - X(\zeta)R(\eta) \tag{14}$$

in the unknown n-variable polynomial $X(\xi)$. According to the above lemma, the Equation (14) has a solution $X \neq 0$. Now substitute ζ_i and η_i with $-\xi_i$ and ξ_i respectively, obtaining $R(-\xi)X(\xi) - X(-\xi)R(\xi) = 0$. Conclude from this that the n-variable polynomial $R'(\xi) := X(-\xi)R(\xi)$ satisfies $R'(\xi) = R'(-\xi)$. Define $\mathfrak{B}' := \ker X(-\frac{d}{d\mathbf{x}})R(\frac{d}{d\mathbf{x}})$. Then $\mathfrak{B}'$ is autonomous, and obviously $\mathfrak{B} \subseteq \mathfrak{B}'$. Define a $2n$-variable polynomial $\Phi(\zeta,\eta)$ by

$$\Phi(\zeta,\eta) := \tfrac{1}{2}R'(\zeta) + \tfrac{1}{2}R'(\eta).$$

Then $\Phi(\zeta,\eta)$ satisfies $\partial\Phi = R'$ so $\mathfrak{B}' = \ker\partial\Phi(\frac{d}{d\mathbf{x}})$ Consequently, $\mathfrak{B}'$ is stationary w.r.t. Q_Φ.

We illustrate this procedure with an example.

Example 17 Consider the model for the vibrations of an infinitely long beam illustrated in Example 13. We proceed to solve the Equation (14) and observe that with $\Psi(\zeta,\eta)$ defined as in Equation (12), the following holds:

$$\sum_{i=1}^{n}(\zeta_i+\eta_i)\Psi_i(\zeta,\eta)=(\zeta_t^2+a^2\zeta_x^2)-(\eta_t^2+a^2\eta_x^2)=R(\zeta)-R(\eta)$$

that is, $X(\xi)=1$ in Equation (14). Conclude from this, substituting $-\xi$ in place of ζ and ξ in place of η, that $R(-\xi)=R(\xi)$. Now define

$$\Phi(\zeta,\eta):=\tfrac{1}{2}(R(\zeta)+R(\eta))=\tfrac{1}{2}(\zeta_t^2+a^2\zeta_x^2+\eta_t^2+a^2\eta_x^2)$$

and observe that since $\partial\Phi=R$, $\mathfrak{B}$ is the set of stationary trajectories with respect to Q_Φ.

Example 18 Consider the behavior of an infinite vibrating plate illustrated in Example 2. It is a matter of straightforward (though tedious) verification to see that the behavior described by Equation (3) is Hamiltonian with respect to the VBDF induced by

$$\Psi(\zeta,\eta)=\begin{pmatrix}\rho(\zeta_t-\eta_t)\\ \zeta_x^3-\eta_x^3-\zeta_x^2\eta_x+\zeta_x\eta_x^2+\zeta_x\zeta_y^2-\eta_x\eta_y^2-\zeta_y^2\eta_x+\zeta_x\eta_y^2\\ \zeta_y^3-\eta_y^3-\eta_x^2\eta_y-\eta_y\zeta_x^2+\eta_x^2\zeta_y+\eta_y^2\zeta_y+\zeta_x^2\zeta_y-\eta_y\zeta_y^2\end{pmatrix}$$

With easy calculations it can be shown that

$$\begin{aligned}\sum_{i=1}^{n}(\zeta_i+\eta_i)\Psi_i(\zeta,\eta)&=(\rho\zeta_t^2+\zeta_x^2+2\zeta_x^2\zeta_y^2+\zeta_y^4)-(\rho\eta_t^2+\eta_x^2+2\eta_x^2\eta_y^2+\eta_y^4)\\&=R(\zeta)-R(\eta)\end{aligned}$$

Substituting $-\xi$ in place of ζ and ξ in place of η_i, we obtain that $R(-\xi)=R(\xi)$. Now define

$$\begin{aligned}\Phi(\zeta,\eta)&:=\tfrac{1}{2}(R(\zeta)+R(\eta))\\&=\tfrac{1}{2}\rho(\zeta_t^2+\eta_t^2)+\tfrac{1}{2}(\zeta_x^2+\eta_x^2)+(\zeta_x^2\zeta_y^2+\eta_x^2\eta_y^2)+\tfrac{1}{2}(\zeta_y^4+\eta_y^4)\end{aligned}$$

and observe that since $\partial\Phi=R$, $\mathfrak{B}$ is the set of stationary trajectories with respect to Q_Φ.

6. CONCLUSIONS AND FURTHER WORK

We have used the formalism of bilinear- and quadratic differential forms in order to study Hamiltonian and variational linear systems. We have shown that for systems described by ordinary linear constant-coefficient differential equations, a system is Hamiltonian if and only if it is variational (see Proposition 8). The main results regarding systems described by linear, constant-coefficient PDEs are Proposition 14, in which we state that a variational system is also Hamiltonian. We also showed how to construct, for the scalar case, a variational system that contains a given Hamiltonian one, by solving a polynomial equation involving the underlying skew-symmetric bilinear differential form.

Due to space limitations and to the need to concentrate on those aspect of our work which are more closely related to modeling, we have been forced to omit several interesting results which we briefly consider in the following.

Most notable among the results of [1] are the definition of controllable Hamiltonian system and the characterization of Hamiltonianity for autonomous- and controllable systems in terms of various representations (kernel, image, state-space). Other important results regarding $1D$ systems will be treated elsewhere: for example, the relationship of our definition of Hamiltonian system with LQ-optimal control, which leads to a generalization of the Hamiltonian system as it is commonly intended in the state-space setting, is the subject of a forthcoming paper.

A great deal of work remains to be done for the construction of a representation-free theory of Hamiltonian and variational systems described by linear, constant-coefficient PDEs. In particular, issues such as the equivalence- or lack of it- of Hamiltonian and variational system with more than one external variable; the characterization of the Hamiltonian and the variational property of a behavior in terms of properties of its representations; and the design of effective algorithms which, starting from a representation of the system, test the system for Hamiltonianity, variationality, construct the quadratic functionals with respect to which the system is stationary, etcetera, need to be addressed.

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