# PICK MATRIX CONDITIONS FOR SIGN-DEFINITE SOLUTIONS OF THE ALGEBRAIC RICCATI EQUATION*

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**Abstract.** We study the existence of positive and negative semidefinite solutions of algebraic Riccati equations (ARE) corresponding to linear quadratic problems with an indefinite cost functional. The formulation of reasonable necessary and sufficient conditions for the existence of such solutions is a long-standing open problem. A central role is played by certain two-variable polynomial matrices associated with the ARE. Our main result characterizes all unmixed solutions of the ARE in terms of the Pick matrices associated with these two-variable polynomial matrices. As a corollary of this result, we find that the signatures of the extremal solutions of the ARE are determined by the signatures of particular Pick matrices.

**Key words.** algebraic Riccati equation, existence of semidefinite solutions, two-variable polynomial matrices, Pick matrices, dissipative systems

**AMS subject classifications.** 93C05, 93C15, 49N05, 49N10

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**1. Introduction and problem statement.** Let $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$ be such that $(A, B)$ is a controllable pair. Let $Q \in \mathbb{R}^{n \times n}$ be symmetric and let $R \in \mathbb{R}^{m \times m}$ be nonsingular and symmetric. Finally, let $S \in \mathbb{R}^{m \times n}$. The quadratic equation

$$(1) \qquad A^T K + KA + Q - (KB + S^T)R^{-1}(B^T K + S) = 0$$

in the unknown $n \times n$ matrix $K$ is called the (continuous-time) algebraic Riccati equation (the ARE). Since its introduction in control theory at the beginning of the sixties, the ARE has been studied extensively because of its prominent role in linear quadratic optimal control and filtering, $H_\infty$-optimal control, differential games, and stochastic filtering and control. We refer to the papers collected in [2] for a discussion of the ARE and its applications and for an overview of the existing literature.

In this paper, we restrict ourselves to the case in which $R$ is positive definite. However, the weighting matrix

$$M := \left( \begin{array}{cc} Q & S^T \\ S & R \end{array} \right)$$

is allowed to be indefinite. Our aim is to address a long-standing open problem concerning the ARE, namely, the problem of formulating reasonable necessary and sufficient conditions for the existence of at least one real positive semidefinite solution or of at least one real negative semidefinite solution. We want to stress that the main difficulty is the *indefiniteness of M*. For the case in which $M$ is positive semidefinite, the problem is already well understood. For this case, necessary and sufficient conditions for the existence of at least one real positive semidefinite solution were obtained

in [5] and [6]. Basically, these necessary and sufficient conditions can be formulated as follows: factor $M = (C \ D)^T(C \ D)$. Then the ARE (1) has at least one real positive semidefinite solution if and only if the system $(A, B, C, D)$ is output stabilizable (see also [17], [23], or [24]).

For *indefinite* weighting matrices $M$, the problem was listed in [13] among a series of open problems in the field of systems and control. Partial results for this problem were obtained in [19, 20, 1, 7, 8]. For an overview and a discussion of these results, as well as their relation to the classical problem of the existence of nonnegative storage functions for dissipative systems, we refer to [13].

In the present paper we will present a solution to this open problem, under the assumption that the pair $(A, B)$ is controllable. It will be proven that the signs of the smallest and largest real symmetric solution, respectively, depend on the signs of certain constant $n \times n$ matrices (so called *Pick matrices*, associated with the ARE), which are easily constructed from the parameters appearing in the ARE. A necessary and sufficient condition for the existence of a real symmetric positive semidefinite solution of the ARE (1) will turn out to be that (i) it has at least one real symmetric solution, and (ii) a suitable Pick matrix is negative semidefinite. Likewise, the existence of at least one negative semidefinite solution is determined by the positive semidefiniteness of a suitable Pick matrix. In the process of establishing these conditions we obtain a number of intermediate results, among which are a new characterization of all unmixed real symmetric solutions of the ARE, and a new characterization of the supremal and infimal real symmetric solutions, all in terms of the Pick matrices associated with the ARE.

A few words on notation are required at this point. In this paper we adopt the usual symbols $\mathbb{R}$ and $\mathbb{C}$ in order to denote the real and complex numbers, respectively. The open and closed right half-planes of $\mathbb{C}$ are denoted, respectively, by $\mathbb{C}_+^0$ and $\mathbb{C}_+$. Given $\lambda \in \mathbb{C}$, its complex conjugate is denoted by $\bar{\lambda}$. The space of $n$-dimensional real, respectively complex, vectors is denoted by $\mathbb{R}^n$, respectively $\mathbb{C}^n$, and the space of $m \times n$ real, respectively complex, matrices, by $\mathbb{R}^{m \times n}$, respectively $\mathbb{C}^{m \times n}$.

The symbol $\mathbb{R}^{\bullet \times n}$ denotes the space of real matrices with $n$ columns, and $\mathbb{R}^{m \times \bullet}$ denotes the space of real matrices with $m$ rows. Given two column vectors $x$ and $y$, we denote with $\mathrm{col}(x, y)$ the vector obtained by stacking $x$ over $y$. If $A \in \mathbb{R}^{m \times n}$, then $A^T \in \mathbb{R}^{n \times m}$ denotes its transpose, and if $A \in \mathbb{C}^{m \times n}$, then $A^* \in \mathbb{C}^{n \times m}$ denotes its conjugate transpose $\bar{A}^T$. If $A \in \mathbb{C}^{n \times n}$ is Hermitian, i.e., $A^* = A$, then we define the signature of $A$ as the triple $\mathrm{sign}(A) = (n_-, n_0, n_+)$, where $n_-$ is the number of negative eigenvalues of $A$, $n_0$ the algebraic multiplicity of 0 as an eigenvalue of $A$, and $n_+$ the number of positive eigenvalues of $A$.

The ring of polynomials with real coefficients in the indeterminate $\xi$ is denoted by $\mathbb{R}[\xi]$; analogously, the ring of two-variable polynomials with real coefficients in the indeterminates $\zeta$ and $\eta$ is denoted by $\mathbb{R}[\zeta, \eta]$. The space of all $n \times m$ polynomial matrices in the indeterminate $\xi$ is denoted by $\mathbb{R}^{n \times m}[\xi]$, and that consisting of all $n \times m$ polynomial matrices in the indeterminates $\zeta$ and $\eta$ by $\mathbb{R}^{n \times m}[\zeta, \eta]$. The space of polynomial matrices with real coefficients in the indeterminate $\xi$ with $n$ columns is denoted by $\mathbb{R}^{\bullet \times n}[\xi]$, and $\mathbb{R}^{m \times \bullet}[\xi]$ is the space of polynomial matrices with $m$ rows. Given a matrix $R \in \mathbb{R}^{n \times m}[\xi]$, we define $R^{\sim}(\xi) := R^T(-\xi) \in \mathbb{R}^{m \times n}[\xi]$. If $F \in \mathbb{R}^{m \times n}[\xi]$, then $F$ can be written as $F(\xi) = F_0 + F_1\xi + \cdots + F_L\xi^L$, where $F_j \in \mathbb{R}^{m \times n}$ for $j = 0, 1, \ldots, L$. We call the $m \times (L+1)n$ matrix $\tilde{F} := \begin{pmatrix} F_0 & F_1 & \ldots & F_L \end{pmatrix}$ the *coefficient matrix* of $F$. It is easy to see that

$$F(\xi) = \tilde{F} \begin{pmatrix} I_n \\ I_n\xi \\ \vdots \\ I_n\xi^L \end{pmatrix}.$$

For a given finite-dimensional Euclidean space $X$, we denote by $\mathfrak{C}^\infty(\mathbb{R}, X)$ the set of all infinitely differentiable functions from $\mathbb{R}$ to $X$, and by $\mathfrak{D}(\mathbb{R}, X)$ the subset of $\mathfrak{C}^\infty(\mathbb{R}, X)$ consisting of those functions having compact support. Finally, if $K$ is a symmetric $n \times n$ matrix, the quadratic form on $\mathbb{R}^n$ defined by $x \mapsto x^T K x$ is denoted by $\mid x \mid_K^2$.

**2. Linear differential systems and quadratic differential forms.** In this section we give a brief review of the notion of linear differential systems. The reader is referred to the textbook [9] or to [21] for a thorough exposition. A linear differential system is a linear subspace $\mathcal{B}$ of $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^q)$ of all solutions $w$ of a given system of linear, constant coefficient, higher order differential equations. Such a system of differential equations can always be represented as a single equation

$$(2) \qquad R\left(\frac{d}{dt}\right)w = 0,$$

where $R \in \mathbb{R}^{\bullet \times q}[\xi]$ is a real polynomial matrix with $q$ columns. The linear space $\mathcal{B}$ is called the behavior of the linear differential system, and (2) is called a kernel representation of $\mathcal{B}$. The variable $w$ is called the manifest variable of $\mathcal{B}$. An alternative way to represent the behavior of a linear differential system is as an image representation. If $M \in \mathbb{R}^{q \times d}[\xi]$ and $\mathcal{B} = \{w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^q) \mid \exists\, l \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^d)$ such that $w = M(\frac{d}{dt})l\}$, then we call

$$(3) \qquad w = M\left(\frac{d}{dt}\right)l$$

an image representation of $\mathcal{B}$. Not all behaviors admit an image representation; indeed, a behavior can be represented in image form if and only if every one of its kernel representations is associated with a polynomial matrix $R \in \mathbb{R}^{\bullet \times q}[\xi]$ such that $\mathrm{rank}(R(\lambda))$ is constant for all $\lambda \in \mathbb{C}$, or equivalently, such that $\mathcal{B}$ is controllable. The image representation (3) of $\mathcal{B}$ is called *observable* if $(M(\frac{d}{dt})l = 0) \implies (l = 0)$. It can be shown that this is the case if and only if the matrix $M(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}$.

We proceed to review the notion of state maps introduced in [12]. We will consider only the case of image representations in this paper. Let (3) be an image representation of the behavior $\mathcal{B}$. A polynomial matrix $X \in \mathbb{R}^{n \times d}[\xi]$ is said to induce a *state map* for $\mathcal{B}$ (or, simply, for $M$) if the latent variable $x := X(\frac{d}{dt})l$ satisfies the *axiom of state*. This means that if we define the *full behavior* as

$$\mathcal{B}_{\mathrm{full}} = \Big\{ (w, x) \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^q \times \mathbb{R}^n) \mid \text{there exists}$$

$$l \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^d) \text{ such that } w = M\left(\frac{d}{dt}\right)l,\ x = X\left(\frac{d}{dt}\right)l \Big\},$$

then $(w_1, x_1), (w_2, x_2) \in \mathcal{B}_{\mathrm{full}}$ and $x_1(0) = x_2(0)$ imply that $(w_1, x_1) \wedge (w_2, x_2)$, the concatenation of $(w_1, x_1)$ and $(w_2, x_2)$ at $t = 0$, belongs to the closure in the topology

of $\mathfrak{L}^1_{\text{loc}}$ of $\mathcal{B}_{\text{full}}$ (see [12]). Now assume that the image representation (3) is observable. Then a state map for the system can be computed as follows. If necessary, permute the components of $w$ so that

$$(4) \qquad\qquad M = \begin{pmatrix} U \\ Y \end{pmatrix}$$

with $U \in \mathbb{R}^{d \times d}[\xi]$, $\det(U) \neq 0$, and $YU^{-1}$ is a proper rational matrix (it can be shown that such a permutation always exists). Now consider the set

$$(5) \qquad\qquad \{r \in \mathbb{R}^{1 \times d}[\xi] \mid rU^{-1} \text{ is strictly proper}\}.$$

It is not difficult to show that this set is a vector space over $\mathbb{R}$. It has been proved in [12] that $X$ is a state map for (3) if and only if its rows (interpreted as elements of the vector space $\mathbb{R}^{1 \times d}[\xi]$ over $\mathbb{R}$) span the vector space (5), and is a *minimal state map* (i.e., inducing a state variable of minimal possible dimension) if and only if its rows form a basis for (5). If this holds true, the number of rows of $X$ is called the *McMillan degree* of $M$, denoted $n(M)$, or, referring to the behavior being represented in image form, *the McMillan degree of $\mathcal{B}$*, denoted $n(\mathcal{B})$. It can be shown (see Proposition 3.5.5 of [12]) that $n(M) = \deg(\det(U))$.

In many modeling and control problems it is necessary to study certain functionals of the system variables and their derivatives. In the context of linear systems these functionals are often taken to be quadratic. An efficient representation for such quadratic functionals by means of two-variable polynomial matrices has been proposed in [18]. In this section we review the definitions and results of such a two-variable polynomial framework, which are used in the rest of the paper.

Let $\Phi \in \mathbb{R}^{q_1 \times q_2}[\zeta, \eta]$; then $\Phi$ can be written in the form

$$\Phi(\zeta, \eta) = \sum_{h,k=0}^{N} \Phi_{h,k} \zeta^h \eta^k,$$

where $\Phi_{h,k} \in \mathbb{R}^{q_1 \times q_2}$ and $N$ is an integer. The two-variable polynomial matrix $\Phi$ induces a bilinear functional acting on infinitely differentiable trajectories as follows:

$$L_\Phi : \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{q_1}) \times \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{q_2}) \longrightarrow \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}),$$

$$L_\Phi(w_1, w_2) = \sum_{h,k=0}^{N} \left( \frac{d^h w_1}{dt^h} \right)^T \Phi_{h,k} \frac{d^k w_2}{dt^k}.$$

If $\Phi$ is a *symmetric two-variable polynomial matrix*, i.e., if $q_1 = q_2$ and $\Phi_{h,k} = \Phi_{k,h}^T$ for all $h$, $k$, then it induces also a quadratic functional $Q_\Phi : \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^q) \to \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R})$ defined by $Q_\Phi(w) := L_\Phi(w, w)$. We will call $Q_\Phi$ the *quadratic differential form* (QDF) associated with $\Phi$. We denote the set of all symmetric $q \times q$ two-variable polynomial matrices by $\mathbb{R}^{q \times q}_s[\zeta, \eta]$. The QDF $Q_\Phi$ is called nonnegative, denoted $Q_\Phi \geq 0$, if $Q_\Phi(w) \geq 0$ for all $w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^q)$.

With every $\Phi \in \mathbb{R}^{q \times q}_s[\zeta, \eta]$ we associate its *coefficient matrix*, which is defined as the infinite symmetric matrix with a finite number of nonzero elements, given by

$$\tilde{\Phi} := \begin{pmatrix} \Phi_{0,0} & \Phi_{0,1} & \cdots & \Phi_{0,N} & \cdots \\ \Phi_{1,0} & \Phi_{1,1} & \cdots & \Phi_{1,N} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \Phi_{N,0} & \Phi_{N,1} & \cdots & \Phi_{N,N} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

Clearly, $Q_\Phi \geq 0$ if and only if $\tilde{\Phi} \geq 0$.

The association of two-variable polynomial matrices with QDFs allows us to develop a calculus that has applications in stability theory, optimal control, and $H_\infty$-control (see [18], [16] and [22]). We restrict our attention to a couple of concepts that are used extensively in this paper. One of them is the map $\partial : \mathbb{R}_s^{q \times q}[\zeta, \eta] \longrightarrow \mathbb{R}^{q \times q}[\xi]$, defined by

$$\partial \Phi(\xi) := \Phi(-\xi, \xi).$$

Observe that for every $\Phi \in \mathbb{R}_s^{q \times q}[\zeta, \eta]$, $\partial \Phi$ is para-Hermitian, i.e., $\partial \Phi = (\partial \Phi)^\sim$. Another feature of the calculus of QDFs that is used in this paper is the derivative of a QDF. Given a QDF $Q_\Phi$ we define its derivative as the QDF $\frac{d}{dt} Q_\Phi$ defined by $(\frac{d}{dt} Q_\Phi)(w) := \frac{d}{dt}(Q_\Phi(w))$. $Q_\Phi$ is called the derivative of $Q_\Psi$ if $\frac{d}{dt} Q_\Psi = Q_\Phi$. In terms of the two-variable polynomial matrices associated with the QDFs, this relationship is expressed equivalently as $(\zeta + \eta)\Psi(\zeta, \eta) = \Phi(\zeta, \eta)$.

In this paper, we also use integrals of QDFs. In order to make sure that the integrals exist, we assume that the trajectories on which the QDF acts are of compact support, that is, they belong to $\mathfrak{D}(\mathbb{R}, \mathbb{R}^q)$. Given a QDF $Q_\Phi$, we define its integral as the functional

$$\int Q_\Phi : \mathfrak{D}(\mathbb{R}, \mathbb{R}^q) \longrightarrow \mathbb{R},$$

$$\int Q_\Phi(w) = \int_{-\infty}^{+\infty} Q_\Phi(w) dt.$$

Questions such as when the integral of a QDF is a positive semidefinite operator arise naturally in the study of dissipativity. We call a QDF $Q_\Phi$ *average nonnegative* if $\int Q_\Phi \geq 0$, i.e., $\int_{-\infty}^{\infty} Q_\Phi(w) dt \geq 0$ for all $w \in \mathfrak{D}(\mathbb{R}, \mathbb{R}^q)$. A QDF can be tested for average nonnegativity by analyzing the behavior of the para-Hermitian matrix $\partial \Phi$ on the imaginary axis. Indeed, it is proven in [18] that

$$(6) \qquad \int Q_\Phi \geq 0 \iff \partial \Phi(i\omega) \geq 0 \quad \forall \, \omega \in \mathbb{R}.$$

**3. Storage functions and polynomial spectral factorization.** In the context of dissipative systems, a QDF measures the power going into a system: its integral over the real line then measures the net flow of energy going into the system. The concept of storage function emerges in the framework of QDFs as follows. Let $\Phi \in \mathbb{R}_s^{q \times q}[\zeta, \eta]$; the QDF $Q_\Psi$ is said to be a *storage function* for $Q_\Phi$ (or $\Psi$ is a storage function for $\Phi$) if the following *dissipation inequality* holds:

$$\frac{d}{dt} Q_\Psi \leq Q_\Phi.$$

Storage functions are related to dissipation functions, which we now define. A QDF $Q_\Delta$ is a *dissipation function* for $Q_\Phi$ (or $\Delta$ is a dissipation function for $\Phi$) if $Q_\Delta \geq 0$ and $\int Q_\Phi = \int Q_\Delta$. There is a close relationship between storage functions, average nonnegativity, and dissipation functions.

PROPOSITION 1. *Let $\Phi \in \mathbb{R}_s^{q \times q}[\zeta, \eta]$. The following conditions are equivalent:*
1. $\int Q_\Phi \geq 0$;
2. $\Phi$ *admits a storage function;*
3. $\Phi$ *admits a dissipation function.*

*Moreover, there exists a one-to-one relation between storage functions $\Psi$ and dissipation functions $\Delta$ for $\Phi$, defined by*

$$\frac{d}{dt}Q_\Psi = Q_\Phi - Q_\Delta$$

*or, equivalently,*

(7) $$(\zeta + \eta)\Psi(\zeta, \eta) = \Phi(\zeta, \eta) - \Delta(\zeta, \eta).$$

Since storage functions measure the energy stored inside a system, it is to be expected that they are related to the memory, to the state, of the system. This intuition has been formalized in [15] in more general terms than those needed in the rest of this paper. For our purposes, the following result from [15] will do.

PROPOSITION 2. *Let $\mathcal{B}$ be represented by $w = M(\frac{d}{dt})l$, and let $X \in \mathbb{R}^{n \times d}[\xi]$ induce a state map for $\mathcal{B}$. Let $P$ be a symmetric $q \times q$ matrix, and define the two-variable polynomial matrix $\Phi(\zeta, \eta) = M^T(\zeta)PM(\eta)$. Let $Q_\Psi$ be a storage function for $Q_\Phi$. Then $Q_\Psi$ is a quadratic function of the state, i.e., there exists a symmetric $n \times n$ matrix $K$ such that $Q_\Psi(l) = | X(\frac{d}{dt})l |_K^2$ for all $l \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^d)$; equivalently, $\Psi(\zeta, \eta) = X^T(\zeta)KX(\eta)$.*

Given an average nonnegative QDF, in general there exist an infinite number of storage functions. It turns out that they all lie between two extremal storage functions.

PROPOSITION 3. *Let $\int Q_\Phi \geq 0$. Then there exist storage functions $\Psi_-$ and $\Psi_+$ such that any other storage function $\Psi$ for $\Phi$ satisfies*

$$Q_{\Psi_-} \leq Q_\Psi \leq Q_{\Psi_+}.$$

In the following we call $Q_{\Psi_-}$ the *smallest* and $Q_{\Psi_+}$ the *largest storage function* of $Q_\Phi$.

In many cases it is of interest to compute explicitly a storage function for a given QDF. We review here a procedure to compute the extremal storage functions $Q_{\Psi_-}$ and $Q_{\Psi_+}$ introduced in Proposition 3. For this we need to introduce the notion of polynomial spectral factorization of a para-Hermitian polynomial matrix. Let $P$ be a para-Hermitian polynomial matrix. A factorization $P = F^\sim F$, with $F$ a real polynomial matrix, is called a *polynomial spectral factorization* of $P$, and $F$ is called a *spectral factor* of $P$. The factorization is called *Hurwitz* if $F$ is square and the roots of $\det(F)$ lie in $\mathbb{C}_-$. It is called *semi-Hurwitz* if the roots of $\det(F)$ lie in $\mathbb{C}_-^0$. The factorization is called *(semi-)anti-Hurwitz* if $F$ is square and the roots of $\det(F)$ lie in $\mathbb{C}_+$ (respectively, in $\mathbb{C}_+^0$). It is well known (see, for example, [10]) that $P$ has a semi-Hurwitz and a (semi-)anti-Hurwitz spectral factorization if and only if $P(i\omega) \geq 0$ for all $\omega \in \mathbb{R}$, and a Hurwitz and an anti-Hurwitz spectral factorization if and only if $P(i\omega) > 0$ for all $\omega \in \mathbb{R}$. The following result shows how to use semi-Hurwitz and semi-anti-Hurwitz polynomial spectral factorizations of $\partial \Phi$ to compute the extremal storage functions of $\Phi$.

PROPOSITION 4. *Let $\Phi(\zeta, \eta) \in \mathbb{R}_s^{\bullet \times \bullet}[\zeta, \eta]$. Assume $\det(\partial \Phi) \neq 0$ and $\partial \Phi(i\omega) \geq 0$ for all $\omega \in \mathbb{R}$. Then the smallest and the largest storage functions $\Psi_-$ and $\Psi_+$ of $\Phi$ can be constructed as follows. Let $H$ and $A$ be semi-Hurwitz, respectively semi-anti-*

*Hurwitz, polynomial spectral factors of $\partial\Phi$. Then*

$$\Psi_+(\zeta,\eta) = \frac{\Phi(\zeta,\eta) - A^T(\zeta)A(\eta)}{\zeta+\eta},$$

$$\Psi_-(\zeta,\eta) = \frac{\Phi(\zeta,\eta) - H^T(\zeta)H(\eta)}{\zeta+\eta}.$$

It also turns out that if $P$ is para-Hermitian, and if $P(i\omega) > 0$ for all $\omega \in \mathbb{R}$, then for every factorization of the scalar polynomial $\det(P)$ as $\det(P) = f^\sim f$, where $f$ and $f^\sim$ have no common roots, there exists a polynomial spectral factorization of $P$ as $P = F^\sim F$, with $\det(F) = f$. This result is taken from [3].

PROPOSITION 5. *Let $P \in \mathbb{R}^{m\times m}[\xi]$ be para-Hermitian. Assume that $P(i\omega) > 0$ for all $\omega \in \mathbb{R}$. Then for every factorization $\det(P) = f^\sim f$, with $f \in \mathbb{R}[\xi]$ such that $f$ and $f^\sim$ are coprime, there exists $F \in \mathbb{R}^{m\times m}[\xi]$ such that $P = F^\sim F$ and $\det(F) = f$.*

**4. Pick matrices.** In this section we discuss Pick matrices associated with average nonnegative quadratic differential forms. In the following, let $\Phi(\zeta,\eta) \in \mathbb{R}^{q\times q}[\zeta,\eta]$. Assume that $\partial\Phi(i\omega) > 0$ for all $\omega \in \mathbb{R}$. Since $\partial\Phi$ is para-Hermitian, the degree of the polynomial $\det(\partial\Phi)$ is even, say $2n$. Also, $\det(\partial\Phi(i\omega)) > 0$ for all $\omega \in \mathbb{R}$, so $\det(\Phi)$ can be factored as $f^\sim f$ with $f \in \mathbb{R}[\xi]$ such that $f$ and $f^\sim$ are coprime. Of course, for a given $\Phi$ there are many $f$'s that satisfy these properties. With any such $f$, we associate a *Pick matrix*, denoted by $T_f$.

Pick matrices are most easily introduced in the special case in which the singularities of $\partial\Phi$ are semisimple, i.e., every singularity $\lambda$ of $\partial\Phi$ has the property that its algebraic multiplicity (its multiplicity as a root of $\det(\partial\Phi)$) is equal to its geometric multiplicity (i.e., $q - \operatorname{rank}(\partial\Phi(\lambda))$, the rank deficiency at $\lambda$). For the moment, assume this to be the case.

Let $f \in \mathbb{R}[\xi]$ be such that $\det(\partial\Phi) = f^\sim f$ and $(f, f^\sim)$ coprime. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the roots of $f$. We use the convention that if the algebraic multiplicity of $\lambda_i$ is $m_i$, then it appears in this list $m_i$ times, and we have ordered the roots in such a way that $\lambda_1, \lambda_2, \ldots, \lambda_{m_1}$ are equal, that $\lambda_{m_1+1}, \lambda_{m_1+2}, \ldots, \lambda_{m_1+m_2}$ are equal, etc. Clearly, the other singularities of $\partial\Phi$ are then $-\lambda_1, -\lambda_2, \ldots, -\lambda_n$, the roots of $f^\sim$. Now for $i = 1, 2, \ldots, n$, let $v_i \in \mathbb{C}^q$ be such that

$$\partial\Phi(\lambda_i)v_i = 0,$$

and such that $v_1, v_2, \ldots, v_n$ are linearly independent. The Pick matrix associated with $f$ is now defined as the matrix

$$(8) \qquad T_f := \begin{pmatrix} \frac{v_1^*\Phi(\bar\lambda_1,\lambda_1)v_1}{\bar\lambda_1+\lambda_1} & \frac{v_1^*\Phi(\bar\lambda_1,\lambda_2)v_2}{\bar\lambda_1+\lambda_2} & \cdots & \frac{v_1^*\Phi(\bar\lambda_1,\lambda_k)v_n}{\bar\lambda_1+\lambda_n} \\ \frac{v_2^*\Phi(\bar\lambda_2,\lambda_1)v_1}{\bar\lambda_2+\lambda_1} & \frac{v_2^*\Phi(\bar\lambda_2,\lambda_2)v_2}{\bar\lambda_2+\lambda_2} & \cdots & \frac{v_2^*\Phi(\bar\lambda_2,\lambda_k)v_n}{\bar\lambda_2+\lambda_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{v_n^*\Phi(\bar\lambda_n,\lambda_1)v_1}{\bar\lambda_n+\lambda_1} & \frac{v_n^*\Phi(\bar\lambda_n,\lambda_2)v_2}{\bar\lambda_n+\lambda_2} & \cdots & \frac{v_n^*\Phi(\bar\lambda_n,\lambda_n)v_n}{\bar\lambda_n+\lambda_n} \end{pmatrix}.$$

Note that $T_f = T_f^* \in \mathbb{C}^{n\times n}$, where $2n$ is the degree of $\det(\partial\Phi)$. Note that the $n$ functions $e^{\lambda_1 t}v_1, e^{\lambda_2 t}v_2, \ldots, e^{\lambda_n t}v_n$ span an $n$-dimensional subspace of the $2n$-dimensional complex linear space of solutions of the system of differential equations

$$(9) \qquad\qquad (\partial\Phi)\left(\frac{d}{dt}\right)w = 0.$$

In the general case in which the singularities of $\partial\Phi$ are not all semisimple, the definition of $T_f$ is also straightforward but notationally more involved. We will introduce the Pick matrix in the general case now.

The definition is most easily understood against the background of computing solutions to the system of differential equations (9). In general, a basis for the linear space of solutions of (9) is obtained by analyzing the structure of the singularities of the polynomial matrix $\partial\Phi$. Again let $\det(\partial\Phi) = f^\sim f$ be a given factorization, with $\deg(f) = n$. Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be the roots of $f$. Again, this list of roots does not necessarily consist of distinct complex numbers. In fact, we use the convention that if a given root $\lambda_i$ has geometric multiplicity $n_i$, then we include it $n_i$ times in our list of roots. Hence, $\lambda_1, \lambda_2, \ldots, \lambda_{n_1}$ are equal, $\lambda_{n_1+1}, \lambda_{n_1+2}, \ldots, \lambda_{n_1+n_2}$ are equal, etc. It is well known that there exist integers $d_1, d_2, \ldots, d_k \geq 1$ such that $d_1 + d_2 + \cdots + d_{n_1} = m_1$, the algebraic multiplicity of $\lambda_1$, $d_{n_1+1} + d_{n_1+2} + \cdots + d_{n_1+n_2} = m_2$, the algebraic multiplicity of $\lambda_{n_1+1}$, etc. The sum $\sum_i m_i$ of the algebraic multiplicities is equal to $n$, the degree of $f$.

The $n$-dimensional subspace of solutions of (9) with exponents in $\{\lambda_1, \lambda_2, \ldots, \lambda_k\}$ is then computed as follows. Let $\partial\Phi^{(i)}$ be the $i$th derivative of $\partial\Phi$. For each $i = 1, 2, \ldots, k$ there exist $d_i$ complex vectors $a_{i,0}, a_{i,1}, \ldots, a_{i,d_i-1} \in \mathbb{C}^q$ such that

(10)
$$
\begin{pmatrix}
\binom{0}{0}\partial\Phi^{(0)}(\lambda_i) & \binom{1}{0}\partial\Phi^{(1)}(\lambda_i) & \cdots & \cdots & \binom{d_i-1}{0}\partial\Phi^{(d_i-1)}(\lambda_i) \\
0 & \binom{1}{1}\partial\Phi^{(0)}(\lambda_i) & \cdots & \cdots & \binom{d_i-1}{1}\partial\Phi^{(d_i-2)}(\lambda_i) \\
0 & 0 & \ddots & & \vdots \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & \binom{d_i-1}{d_i-1}\partial\Phi^{(0)}(\lambda_i)
\end{pmatrix}
\begin{pmatrix}
a_{i,0} \\
a_{i,1} \\
\vdots \\
\vdots \\
a_{i,d_i-1}
\end{pmatrix} = 0
$$

and such that the $n$ vectors $a_{i,j}$ are linearly independent. Using these vectors we form the matrices $V_i \in \mathbb{C}^{d_i q \times d_i}$ defined by

(11)
$$
V_i := \begin{pmatrix}
\binom{0}{0}a_{i,0} & \binom{1}{1}a_{i,1} & \cdots & \binom{d_i-2}{d_i-2}a_{i,d_i-2} & \binom{d_i-1}{d_i-1}a_{i,d_i-1} \\
\binom{1}{0}a_{i,1} & \binom{2}{1}a_{i,2} & \cdots & \binom{d_i-1}{d_i-2}a_{i,d_i-1} & 0 \\
\vdots & \vdots & & 0 & 0 \\
\binom{d_i-2}{0}a_{i,d_i-2} & \binom{d_i-1}{1}a_{i,d_i-1} & & \vdots & \vdots \\
\binom{d_i-1}{0}a_{i,d_i-1} & 0 & \cdots & 0 & 0
\end{pmatrix}.
$$

For $i = 1, 2, \ldots, k$, define the matrix function $W_i : \mathbb{R} \to \mathbb{R}^{q \times d_i}$ by

$$
W_i(t) := e^{\lambda_i t}\begin{pmatrix} I_{q \times q} & t I_{q \times q} & \cdots & t^{d_i-1}I_{q \times q} \end{pmatrix} V_i,
$$

and the matrix function $W : \mathbb{R} \to \mathbb{R}^{q \times n}$ by

$$
W(t) := \begin{pmatrix} W_1(t) & W_2(t) & \cdots & W_k(t) \end{pmatrix}.
$$

Then the columns of $W$ form a basis for the $n$-dimensional subspace of solutions of (9) with exponents in $\{\lambda_1, \lambda_2, \ldots, \lambda_k\}$.

We now introduce the Pick matrix $T_f$ associated with $\Phi$ and the factorization

$\det(\partial\Phi) = f^\sim f$. For $i, j = 1, 2, \dots, k$, define the nonsingular $d_j \times d_j$ matrix $\Lambda_{i,j}$ by

$$(12) \qquad \Lambda_{i,j} = \begin{pmatrix} 1 & 0 & 0 & \cdots & \cdots & 0 \\ \frac{-1}{\bar{\lambda}_i + \lambda_j} & 1 & 0 & \cdots & \cdots & 0 \\ \frac{2!}{(\bar{\lambda}_i + \lambda_j)^2} & \frac{-2!}{\bar{\lambda}_i + \lambda_j} & 1 & 0 & \cdots & 0 \\ \frac{-3!}{(\bar{\lambda}_i + \lambda_j)^3} & \frac{3!}{(\bar{\lambda}_i + \lambda_j)^2} & \frac{-3!}{\bar{\lambda}_i + \lambda_j} & 1 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ \frac{(-1)^{d_j-1}(d_j-1)!}{(\bar{\lambda}_i + \lambda_j)^{d_j-1}} & \cdots & \cdots & \frac{(d_j-1)!}{(\bar{\lambda}_i + \lambda_j)^2} & \frac{-(d_j-1)!}{\bar{\lambda}_i + \lambda_j} & 1 \end{pmatrix}.$$

Also, for $i, j = 1, 2, \dots, k$ we define $\Theta_{i,j} \in \mathbb{C}^{d_i q \times d_j q}$ by

$$(13) \qquad \Theta_{i,j} := \begin{pmatrix} \Phi(\bar{\lambda}_i, \lambda_j) & \frac{\partial\Phi}{\partial\eta}(\bar{\lambda}_i, \lambda_j) & \cdots & \frac{\partial^{d_j-1}\Phi}{\partial\eta^{d_j-1}}(\bar{\lambda}_i, \lambda_j) \\ \frac{\partial\Phi}{\partial\zeta}(\bar{\lambda}_i, \lambda_j) & \frac{\partial^2\Phi}{\partial\zeta\partial\eta}(\bar{\lambda}_i, \lambda_j) & \cdots & \frac{\partial^{d_j}\Phi}{\partial\zeta\partial\eta^{d_j-1}}(\bar{\lambda}_i, \lambda_j) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^{d_i-1}\Phi}{\partial\zeta^{d_i-1}}(\bar{\lambda}_i, \lambda_j) & \frac{\partial^{d_i}\Phi}{\partial\zeta^{d_i-1}\partial\eta}(\bar{\lambda}_i, \lambda_j) & \cdots & \frac{\partial^{d_i+d_j-2}\Phi}{\partial\zeta^{d_i-1}\partial\eta^{d_j-1}}(\bar{\lambda}_i, \lambda_j) \end{pmatrix}.$$

Here, $\frac{\partial^{i+j}\Phi}{\partial\zeta^i\partial\eta^j}(\zeta, \eta)$ denotes the $(i,j)$th partial derivative with respect to $\zeta$ and $\eta$ of $\Phi(\zeta, \eta)$. We define the shift operator $\sigma : \mathbb{C}^{d_i q \times d_j q} \to \mathbb{C}^{d_i q \times d_j q}$ acting on matrices $M$ that are partitioned into $q \times q$ blocks as follows: if

$$M = \begin{pmatrix} M_{1,1} & M_{1,2} & \cdots & M_{1,d_j} \\ M_{2,1} & M_{2,2} & \cdots & M_{2,d_j} \\ \vdots & \vdots & \ddots & \vdots \\ M_{d_i,1} & M_{d_i,2} & \cdots & M_{d_i,d_j}, \end{pmatrix},$$

then

$$\sigma(M) := \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & M_{1,1} & \cdots & M_{1,d_j-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & M_{d_i-1,1} & \cdots & M_{d_i-1,d_j-1} \end{pmatrix}.$$

In terms of $\Theta_{i,j}$ and the shift-operator $\sigma$, for $i, j = 1, 2, \dots, k$, we define the matrices $\Sigma_{i,j} \in \mathbb{C}^{d_i q \times d_j q}$ by

$$\Sigma_{i,j} := \frac{1}{\bar{\lambda}_i + \lambda_j}\Theta_{i,j} + \frac{1}{(\bar{\lambda}_i + \lambda_j)^2}\sigma(\Theta_{i,j}) + \frac{1}{(\bar{\lambda}_i + \lambda_j)^3}\sigma^2(\Theta_{i,j}) + \cdots$$

$$(14) \qquad\qquad + \frac{1}{(\bar{\lambda}_i + \lambda_j)^{\max(d_i,d_j)-1}}\sigma^{\max(d_i,d_j)-1}(\Theta_{i,j}).$$

Here, for a given $M$, $\sigma^2(M)$ is defined as $\sigma(\sigma(M))$, etc. The Pick matrix associated with $\Phi$ and the factorization $\det(\partial\Phi) = f^\sim f$ is now defined as the matrix $T_f \in \mathbb{C}^{n \times n}$, $T_f = (T_{i,j})_{i,j=1,2,\dots,k}$, where the $(i,j)$th block is the complex $d_i \times d_j$ matrix given by

$$(15) \qquad\qquad T_{i,j} := \Lambda_{j,i}^* V_i^* \Sigma_{i,j} V_j \Lambda_{i,j}.$$

Note that $T_f$ is a Hermitian matrix.

For related material on Pick matrices, their application in interpolation problems, and connections with systems and control, see [4], [25].

**5. The Riccati equation, linear matrix inequalities, and storage functions.** In this section we study the connection between the existence of real symmetric solutions of the ARE and average nonnegativity of a given QDF associated with the ARE.

We associate with the ARE (1) the system with manifest variable $w = \mathrm{col}(x, u)$ represented by $\frac{d}{dt}x = Ax + Bu$, or equivalently

$$(16) \qquad \left( \ \frac{d}{dt}I_n - A \quad -B \ \right) \left( \begin{array}{c} x \\ u \end{array} \right) = 0.$$

Equation (16) constitutes a kernel representation of the behavior

$$(17) \qquad \mathfrak{B} = \{\mathrm{col}(x, u) \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^n) \times \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^m) \mid \ (16) \text{ is satisfied}\}.$$

A standing assumption in the remainder of this paper is that the pair $(A, B)$ is controllable. Under this assumption, $\mathfrak{B}$ can be represented in image form. One such representation can be computed as follows. Let $X \in \mathbb{R}^{n \times m}[\xi]$ and $U \in \mathbb{R}^{m \times m}[\xi]$ induce a right coprime factorization of the rational matrix $(\xi I_n - A)^{-1}B$, i.e., $(\xi I_n - A)^{-1}B = X(\xi)U(\xi)^{-1}$ and

$$\mathrm{rank} \left( \begin{array}{c} X(\lambda) \\ U(\lambda) \end{array} \right) = m$$

for all $\lambda \in \mathbb{C}$. Then $\mathfrak{B}$ is represented in observable image form as

$$(18) \qquad \left( \begin{array}{c} x \\ u \end{array} \right) = \left( \begin{array}{c} X(\frac{d}{dt}) \\ U(\frac{d}{dt}) \end{array} \right) l.$$

Observe that any such $X$ yields a minimal state map $X(\frac{d}{dt})$ for $\mathfrak{B}$.

Given the matrices $Q = Q^T \in \mathbb{R}^{n \times n}$, $R = R^T \in \mathbb{R}^{m \times m}$, and $S \in \mathbb{R}^{m \times n}$, and the polynomial matrices $X$ and $U$, we define the symmetric $m \times m$ two-variable polynomial matrix $\Phi$ by

$$(19) \qquad \Phi(\zeta, \eta) = \left( \ X(\zeta)^T \quad U(\zeta)^T \ \right) \left( \begin{array}{cc} Q & S^T \\ S & R \end{array} \right) \left( \begin{array}{c} X(\eta) \\ U(\eta) \end{array} \right).$$

Note that if $l$ and $\mathrm{col}(x, u)$ are related by (18), then the QDF $Q_\Phi$ associated with $\Phi$ satisfies

$$Q_\Phi(l) = \left( \ x^T \quad u^T \ \right) \left( \begin{array}{cc} Q & S^T \\ S & R \end{array} \right) \left( \begin{array}{c} x \\ u \end{array} \right).$$

Of course, $(\xi I_n - A)^{-1}B$ admits many right coprime factorizations. If $X_1 U_1^{-1} = X_2 U_2^{-1}$ are two right coprime factorizations, then they are related by a unimodular transformation: there exists a unimodular $V$ such that $X_2 = X_1 V$ and $U_2 = U_1 V$. Hence the associated two-variable polynomial matrices are related by $\Phi_1(\zeta, \eta) = V^T(\zeta)\Phi_2(\zeta, \eta)V(\eta)$.

*Example* 6. In the Riccati equation (1), let $A = \left( \begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix} \right)$, $B = \left( \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right)$, $Q = \left( \begin{smallmatrix} 1 & a \\ a & 3 \end{smallmatrix} \right)$, $R = \left( \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right)$, and $S = \left( \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} \right)$. Here $a$ is a parameter taking values in $\mathbb{R}$. Clearly, $(\xi I - A)^{-1}B = X(\xi)U(\xi)^{-1}$, with $X(\xi) = \left( \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right)$, and $U(\xi) = \left( \begin{smallmatrix} \xi & 0 \\ 0 & \xi-1 \end{smallmatrix} \right)$. The corresponding two-variable polynomial matrix is $\Phi(\zeta, \eta) = \left( \begin{smallmatrix} 1+\zeta\eta & a \\ a & 3+(\zeta-1)(\eta-1) \end{smallmatrix} \right)$.

The next result connects the average nonnegativity of the QDF associated with (19) with the existence of real symmetric solutions to the linear matrix inequality associated with the ARE (1) and with the existence of storage functions for $Q_\Phi$.

THEOREM 7.  *Let $\Phi(\zeta, \eta)$ be defined by* (19), *where $X$ and $U$ are such that $X(\xi)U(\xi)^{-1}$ is a right coprime factorization of $(\xi I_n - A)^{-1}B$. Then the following statements are equivalent:*

1. $\int Q_\Phi \geq 0$;
2. *there exists $K = K^T \in \mathbb{R}^{n \times n}$ such that $\mid X(\frac{d}{dt})l \mid_K^2$ is a storage function for $Q_\Phi$;*
3. *there exists $K = K^T \in \mathbb{R}^{n \times n}$ such that the $(n+m) \times (n+m)$ symmetric matrix*

$$L(K) := \left( \begin{array}{cc} Q - A^T K - KA & -KB + S^T \\ -B^T K + S & R \end{array} \right)$$

*satisfies $L(K) \geq 0$.*
*In fact, for every $K = K^T \in \mathbb{R}^{n \times n}$ there holds*

(20)
$$\frac{d}{dt} \left| X\left(\frac{d}{dt}\right)l \right|_K^2 = Q_\Phi(l) - \left| \left( \begin{array}{c} X(\frac{d}{dt})l \\ U(\frac{d}{dt})l \end{array} \right) \right|_{L(K)}^2$$

*for all $l \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^m)$; equivalently,*

$$(\zeta + \eta)X^T(\zeta)KX(\eta) = \Phi(\zeta, \eta) - \left( \begin{array}{cc} X(\zeta)^T & U(\zeta)^T \end{array} \right) L(K) \left( \begin{array}{c} X(\eta) \\ U(\eta) \end{array} \right).$$

*Consequently, for $K = K^T \in \mathbb{R}^{n \times n}$ the following statements are equivalent:*

(i) $L(K) \geq 0$;
(ii) $\mid X(\frac{d}{dt})l \mid_K^2$ *is a storage function for $Q_\Phi$;*
(iii)

$$\left| \left( \begin{array}{c} X(\frac{d}{dt})l \\ U(\frac{d}{dt})l \end{array} \right) \right|_{L(K)}^2$$

*is a dissipation function for $Q_\Phi$.*

*Proof.* We prove the equivalence of (i),(ii), and (iii). The first part of the theorem follows easily from this and from Proposition 1. We need the following lemma.

LEMMA 8.  *Let $X \in \mathbb{R}^{n \times m}[\xi]$ and $U \in \mathbb{R}^{m \times m}[\xi]$ be such that $X(\xi)U(\xi)^{-1}$ is a right coprime factorization of $(\xi I - A)^{-1}B$. Then the mapping*

$$\mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^m) \rightarrow \mathbb{R}^n \times \mathbb{R}^m,$$

$$l \mapsto \left( \begin{array}{c} (X(\frac{d}{dt})l)(0) \\ (U(\frac{d}{dt})l)(0) \end{array} \right)$$

*is surjective.*

*Proof of Lemma 8.* Let $(x_0, u_0) \in \mathbb{R}^n \times \mathbb{R}^m$. Let $\tilde{u} \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^m)$ be such that $\tilde{u}(0) = u_0$. Consider the differential equation $\dot{x} = Ax + B\tilde{u}$, $x(0) = x_0$, and let $\tilde{x} \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^n)$ be its solution. Evidently,

$$\mathrm{col}(\tilde{x}, \tilde{u}) \in \mathrm{im} \left( \begin{array}{c} X(\frac{d}{dt}) \\ U(\frac{d}{dt}) \end{array} \right),$$

so there exists $l \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^m)$ such that

$$\left( \begin{array}{c} \tilde{x} \\ \tilde{u} \end{array} \right) = \left( \begin{array}{c} X(\frac{d}{dt}) \\ U(\frac{d}{dt}) \end{array} \right) l.$$

Consequently,

$$\left( \begin{array}{c} x_0 \\ u_0 \end{array} \right) = \left( \begin{array}{c} \tilde{x}(0) \\ \tilde{u}(0) \end{array} \right) = \left( \begin{array}{c} (X(\frac{d}{dt})l)(0) \\ (U(\frac{d}{dt})l)(0) \end{array} \right).$$

This concludes the proof of the lemma.

We resume the proof of Theorem 7. Let $K = K^T \in \mathbb{R}^{n \times n}$. We first prove that, for all $l$, (20) holds; equivalently,

$$(21) \qquad (\zeta + \eta)X(\zeta)^T K X(\eta) = \Phi(\zeta, \eta) - \left( \begin{array}{cc} X(\zeta)^T & U(\zeta)^T \end{array} \right) L(K) \left( \begin{array}{c} X(\eta) \\ U(\eta) \end{array} \right).$$

Indeed, from the fact that $X(\xi)U(\xi)^{-1} = (\xi I - A)^{-1}B$ it follows that $\xi X(\xi) = AX(\xi) + BU(\xi)$. Consequently,

$$(\zeta + \eta)X(\zeta)^T K X(\eta) = X(\zeta)^T A^T K X(\eta) + U(\zeta)^T B^T K X(\eta)$$
$$+ X(\zeta)^T K A X(\eta) + X(\zeta)^T K B U(\eta),$$

which can be rewritten as

$$\left( \begin{array}{cc} X(\zeta)^T & U(\zeta)^T \end{array} \right) \left( \begin{array}{cc} A^T K + K A & K B \\ B^T K & 0 \end{array} \right) \left( \begin{array}{c} X(\eta) \\ U(\eta) \end{array} \right).$$

With $\Phi(\zeta, \eta)$ defined by (19), equation (21) then follows immediately. Since, by Lemma 8, the map

$$l \mapsto \left( \begin{array}{c} (X(\frac{d}{dt})l)(0) \\ (U(\frac{d}{dt})l)(0) \end{array} \right)$$

is surjective, we have

$$\left| \left( \begin{array}{c} X(\frac{d}{dt})l \\ U(\frac{d}{dt})l \end{array} \right) \right|^2_{L(K)} \geq 0 \quad \forall\, l \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^m)$$

if and only if $L(K) \geq 0$. Thus the equivalence of (i), (ii), and (iii) follows immediately from (20).

The equivalence of statements 1. and 2. follows from Propositions 1 and 2. The equivalence of 2. and 3. is an immediate consequence of the equivalence of (i) and (ii).    □

If we assume that the matrix $R$ is positive definite, the result of Theorem 7 can be sharpened, and a connection can be established between the QDF $\Phi(\zeta, \eta)$ defined in (19) and the ARE (1).

THEOREM 9.   *Let $\Phi(\zeta, \eta)$ be defined by (19), where $X$ and $U$ are such that $X(\xi)U(\xi)^{-1}$ is a right coprime factorization of $(\xi I_n - A)^{-1}B$. Assume $R > 0$. Then the following statements are equivalent:*

1. $\int Q_\Phi \geq 0$;
2. *There exists a real symmetric solution to the* ARE.
*In fact, for every $K = K^T \in \mathbb{R}^{n \times n}$ the following conditions are equivalent:*
  (i) *$-K$ satisfies the* ARE;
  (ii) *$\mid X(\frac{d}{dt})l \mid_K^2$ is a storage function for $Q_\Phi$ with associated dissipation function*

$$\Delta(\zeta, \eta) = \left(\begin{array}{cc} X(\zeta)^T & U(\zeta)^T \end{array}\right) L(K) \left(\begin{array}{c} X(\eta) \\ U(\eta) \end{array}\right) = F(\zeta)^T F(\eta),$$

*where*

$$F(\xi) := R^{-\frac{1}{2}}(-B^T K + S)X(\xi) + R^{\frac{1}{2}}U(\xi);$$

  (iii) *$\mid X(\frac{d}{dt})l \mid_K^2$ is a storage function for $Q_\Phi$, and the rank of the coefficient matrix of the* QDF *$Q_\Phi(l) - \frac{d}{dt} \mid X(\frac{d}{dt})l \mid_K^2$ is equal to $m$.*

  *Proof.* We begin by proving the implication 1. $\Rightarrow$ 2. From condition 1. and the equivalence in (6), we obtain $\partial\Phi(i\omega) \geq 0$ for all $\omega \in \mathbb{R}$. Since $\det(\partial\Phi) \neq 0$, there exists a semi-Hurwitz factorization $\partial\Phi = H^\sim H$, with $\det(H) \neq 0$. According to Proposition 4, this yields the smallest storage function as induced by the two-variable polynomial matrix

$$\Psi_-(\zeta, \eta) = \frac{\Phi(\zeta, \eta) - H^T(\zeta)H(\eta)}{\zeta + \eta}.$$

By Proposition 2, there exists $K = K^T \in \mathbb{R}^{n \times n}$ such that $\Psi_-(\zeta, \eta) = X^T(\zeta)KX(\eta)$. We claim that $-K$ satisfies the ARE. Indeed, as in the proof of Theorem 7, we have

$$(22) \qquad H^T(\zeta)H(\eta) = \left(\begin{array}{cc} X(\zeta)^T & U(\zeta)^T \end{array}\right) L(K) \left(\begin{array}{c} X(\eta) \\ U(\eta) \end{array}\right).$$

Since $\det(H) \neq 0$, the coefficient matrix $\tilde{H}$ of $H$ has full row rank $m$. Since by Lemma 8 the mapping $l \mapsto \text{col}((X(\frac{d}{dt})l)(0), (U(\frac{d}{dt})l)(0))$ is surjective, the coefficient matrix of $\text{col}(X(\eta), U(\eta))$ has full row rank. Consequently, $L(K)$ has rank $m$. Since $R > 0$, $\text{rank}(L(K)) = m$ if and only if the Schur complement of $R$ in $L(K)$ is zero, that is, if and only if

$$Q - A^T K - KA - (-KB + S^T)R^{-1}(-B^T K + S) = 0,$$

in other words, if and only if $-K$ satisfies the ARE. This concludes the proof of the implication 1. $\Rightarrow$ 2. The implication 2. $\Rightarrow$ 1. follows from the implication (i) $\Rightarrow$ (ii) below.

Next, we prove the equivalence of (i), (ii), and (iii) of Theorem 9.

(i) $\Rightarrow$ (ii). Assume $-K$ satisfies the ARE. Then it is easily seen that

$$L(K) = \left(\begin{array}{cc} R^{-1/2}(-B^T K + S) & R^{1/2} \end{array}\right)^T \left(\begin{array}{cc} R^{-1/2}(-B^T K + S) & R^{1/2} \end{array}\right) \geq 0.$$

From Theorem 7 it then follows that $\mid X(\frac{d}{dt})l \mid_K^2$ is a storage function for $Q_\Phi$, with associated dissipation function $F^T(\zeta)F(\eta)$, where $F(\xi) = R^{-1/2}(-B^T K + S)X(\xi) + R^{1/2}U(\xi)$.

(ii) $\Rightarrow$ (iii). According to (ii), $\text{rank}(L(K)) = m$. The coefficient matrix of the QDF $Q_\Phi(l) - \frac{d}{dt} \mid X(\frac{d}{dt})l \mid_K^2$ is equal to

$$\left(\begin{array}{c} \tilde{X} \\ \tilde{U} \end{array}\right)^T L(K) \left(\begin{array}{c} \tilde{X} \\ \tilde{U} \end{array}\right),$$

with $\mathrm{col}(\tilde{X}, \tilde{U})$ the coefficient matrix of $\mathrm{col}(X, U)$. By Lemma 8 this coefficient matrix has full row rank. This proves the implication.

(iii) $\Rightarrow$ (i). Assume $L(K)$ has rank $m$. Since $\mathrm{rank}(R) = m$, this implies that the Schur complement of $R$ is equal to zero, equivalently that $-K$ satisfies the ARE. □

*Example 6, continued.* For the Riccati equation of Example 6 we have $\partial \Phi(\xi) = \begin{pmatrix} 1-\xi^2 & a \\ a & 4-\xi^2 \end{pmatrix}$, so $\partial \Phi(i\omega) = \begin{pmatrix} 1+\omega^2 & a \\ a & 4+\omega^2 \end{pmatrix}$. By (6), the Riccati equation has a real symmetric solution if and only if $\partial \Phi(i\omega) \geq 0$ for all $\omega \in \mathbb{R}$. This holds if and only if $-2 \leq a \leq 2$.

Connections between the ARE and the linear matrix inequality in statement 3. of Theorem 7 are well-known. See, for example, Chap. 8 of [2], where solutions $K$ of the linear matrix inequality such that $\mathrm{rank}(L(K)) = m$ are called *rank minimizing*. In a behavioral framework, connections between such concepts and storage functions were established in [14]; see also Chapter 5 of [11].

**6. Pick matrices and the algebraic Riccati equation.** In this section we derive the main result of this paper, a characterization of all unmixed real symmetric solutions of the ARE in terms of the Pick matrices associated with the two-variable polynomial matrix (19). As a corollary of this result, we obtain necessary and sufficient conditions for the existence of sign-definite solutions of the ARE. These conditions are in terms of the Pick matrices associated with the Hurwitz and anti-Hurwitz factorizations of $\det(\partial \Phi)$.

In this section, let $X(\xi)U(\xi)^{-1}$ be an arbitrary right coprime factorization of $(\xi I_n - A)^{-1}B$, and let the two-variable polynomial matrix $\Phi$ associated with the ARE be given by (19). From the fact that $\partial \Phi$ is para-Hermitian, we know that $\det(\partial \Phi)$ has even degree. In fact, the degree of $\det(\partial \Phi)$ is twice the dimension of the underlying state space system (17).

LEMMA 10. *Let $\Phi(\zeta, \eta)$ be defined as in (19), and assume $R > 0$. Then the degree of $\det(\partial \Phi)$ is $2n$.*

*Proof.* Observe that $\partial \Phi = X^\sim QX + X^\sim S^T U + U^\sim SX + U^\sim RU$. Multiplying this equality on the right by $U^{-1}$ and on the left by $(U^\sim)^{-1}$ yields $(U^\sim)^{-1}\partial \Phi U^{-1} = (U^\sim)^{-1}X^\sim QXU^{-1} + (U^\sim)^{-1}X^\sim S^T + SXU^{-1} + R$. Now observe that $X(\xi)U^{-1}(\xi) = (\xi I_n - A)^{-1}B$ is a matrix of strictly proper rational functions. It follows that $(U^\sim)^{-1}\partial \Phi U^{-1}$ is a matrix of proper rational functions and consequently $\deg(\det(\partial \Phi)) \leq \deg(\det(U)) + \deg(\det(U^\sim)) = 2n$. We now show that $\deg(\det(\partial \Phi)) = 2n$. Indeed, since $\lim_{|\lambda| \longrightarrow \infty}(U^\sim(\lambda))^{-1}\partial \Phi(\lambda)U(\lambda) = R > 0$, it follows that $(U^\sim)^{-1}\partial \Phi U$ has an inverse whose entries are also proper rational functions. Consequently $\deg(\det(\partial \Phi)) = 2n$. □

Assume now that $\int Q_\Phi \geq 0$, equivalently $\partial \Phi(i\omega) \geq 0$, for all $\omega \in \mathbb{R}$ (see (6)). According to Theorem 9 this is equivalent to the existence of a real symmetric solution of the ARE. Observe that every polynomial spectral factorization of $\partial \Phi$ as $\partial \Phi = F^\sim F$ with $F \in \mathbb{R}^{m \times m}[\xi]$ yields a factorization of $\det(\partial \Phi)$ as $\det(\partial \Phi) = f^\sim f$, with $f = \det(F)$ and $\deg(f) = n$. Let $\mathcal{F}$ be the set of all polynomials of degree $n$, with positive highest degree coefficient, that can occur as the determinant of a polynomial spectral factor of $\partial \Phi$:

$$\mathcal{F} := \{f \in \mathbb{R}[\xi] \mid f(\xi) = f_0 + f_1\xi + \cdots + f_n\xi^n, \quad f_n > 0,$$

(23) $\quad$ and there exists $F \in \mathbb{R}^{m \times m}[\xi]$ such that $\partial \Phi = F^\sim F$ and $\det(F) = f\}.$

Also, let $\mathcal{S}$ be the set of all real symmetric solutions of the ARE:

$$\mathcal{S} := \{K \in \mathbb{R}^{n \times n} \mid K = K^T \text{ and } K \text{ satisfies the ARE}\}.$$

For any $K \in \mathcal{S}$, denote $A_K := A - BR^{-1}(B^T K + S)$ and let $\chi_{A_K}$ be the characteristic polynomial of $A_K$. Our basic result states that there is a one-to-one correspondence between $\mathcal{F}$ and $\mathcal{S}$.

THEOREM 11. $\mathcal{S} \neq \emptyset$ if and only if $\partial\Phi(i\omega) \geq 0$ for all $\omega \in \mathbb{R}$. In that case there exists a bijection between $\mathcal{F}$ and $\mathcal{S}$. Such bijection $\mathrm{Ric} : \mathcal{F} \to \mathcal{S}$ is defined as follows. For any $f \in \mathcal{F}$, let $F \in \mathbb{R}^{m \times m}[\xi]$ be such that $f = \det(F)$ and $\partial\Phi = F^{\sim}F$. Then define $\mathrm{Ric}(f) = K$, where $K = K^T \in \mathbb{R}^{n \times n}$ is the unique solution of

(24)
$$\frac{\Phi(\zeta, \eta) - F^T(\zeta)F(\eta)}{\zeta + \eta} = X^T(\zeta)(-K)X(\eta).$$

For any $K \in \mathcal{S}$ we have $\partial\Phi = (F_K)^{\sim}F_K$, where

$$F_K(\xi) := R^{-1/2}(B^T K + S)X(\xi) + R^{1/2}U(\xi).$$

Furthermore, for any $K \in \mathcal{S}$ we have $\det(F_K) = \sqrt{\det(R)} \, \chi_{A_K}$, whence $\det(\partial\Phi) = \det(R) \, (\chi_{A_K})^{\sim}\chi_{A_K}$ and

$$K = \mathrm{Ric}(\sqrt{\det(R)} \, \chi_{A_K}).$$

*Proof.* We begin by showing that the map $\mathrm{Ric} : \mathcal{F} \to \mathcal{S}$ is well defined. Let $f_1$ and $f_2$ be two elements of $\mathcal{F}$, and let $F_1, F_2 \in \mathbb{R}^{m \times m}[\xi]$ be such that $\partial\Phi = F_1^{\sim}F_1 = F_2^{\sim}F_2$ and $\det(F_i) = f_i$, $i = 1, 2$. It is well known (see, for example, Theorem 5.3 of [10]) that there exists an orthogonal $m \times m$ matrix $L$ such that $F_2 = LF_1$. Now let $K_1$ and $K_2$ be $n \times n$ symmetric matrices such that

$$\frac{\Phi(\zeta, \eta) - F_i^T(\zeta)F_i(\eta)}{\zeta + \eta} = X^T(\zeta)(-K_i)X(\eta),$$

$i = 1, 2$. (Such matrices exist because of Theorem 2.) Then necessarily

$$X^T(\zeta)(-K_1)X(\eta) = X^T(\zeta)(-K_2)X(\eta).$$

It follows from the fact that $X(\frac{d}{dt})$ is a minimal state map that the map $l \to (X(\frac{d}{dt})l)(0)$ is surjective. Hence for all $x_0 \in \mathbb{R}^n$ there holds $x_0^T(-K_1)x_0 = x_0^T(-K_2)x_0$, which implies $K_1 = K_2$. This shows that $\mathrm{Ric}$ is well defined.

We proceed to show that $\mathrm{Ric}$ is bijective. We first prove that it is injective. Assume that $\mathrm{Ric}(f_1) = K_1 = \mathrm{Ric}(f_2) = K_2$. Let $F_1$ and $F_2$ be $m \times m$ polynomial matrices such that $\partial\Phi = F_1^{\sim}F_1 = F_2^{\sim}F_2$ and $\det(F_i) = f_i$, $i = 1, 2$. From the fact that $K_1 = K_2$ and from (24) it follows that $F_1^T(\zeta)F_1(\eta) = F_2^T(\zeta)F_2(\eta)$. This implies that

$$\det(F_1(\zeta))\det(F_1(\eta)) = \det(F_2(\zeta))\det(F_2(\eta)),$$

so that $f_1(\zeta)f_1(\eta) = f_2(\zeta)f_2(\eta)$. Given that the highest degree coefficient of $f_1$ and $f_2$ is positive (see (23)), we conclude that $f_1 = f_2$. This concludes the proof of the injectivity of $\mathrm{Ric}$. In order to prove that $\mathrm{Ric}$ is surjective, let $K = K^T$ be a solution to the ARE. According to Theorem 9 there holds

$$(\zeta + \eta)X^T(\zeta)(-K)X(\eta) = \Phi(\zeta, \eta) - F_K(\zeta)^T F_K(\eta),$$

where $F_K \in \mathbb{R}^{m \times m}[\xi]$ is defined by

$$F_K(\xi) = R^{-\frac{1}{2}}(B^T K + S)X(\xi) + R^{\frac{1}{2}}U(\xi).$$

Note that $\partial \Phi = (F_K)^\sim F_K$. Define now $f := \det(F_K)$. Then $K = \mathrm{Ric}(f)$. This also proves the second statement of the theorem.

Next we prove that for all $K \in \mathcal{S}$ we have $\det(F_K) = \det(R^{1/2})\chi_{A_K}$. Consider the $(n+m) \times (n+m)$ polynomial matrix

$$P(\xi) := \begin{pmatrix} \xi I - A & B \\ -R^{-1/2}(B^T K + S) & R^{1/2} \end{pmatrix}.$$

Computing the determinant of $P$ yields

$$\begin{aligned}
\det(P(\xi)) &= \det(\xi I - A)\det(R^{1/2} + R^{-1/2}(B^T K + S)(\xi I - A)^{-1}B) \\
&= \det(R^{1/2})\det(\xi I - A + BR^{-1}(B^T K + S)).
\end{aligned}$$

Using the fact that $X(\xi)U(\xi)^{-1}$ is a right coprime factorization of $(\xi I - A)^{-1}B$ and that $(A, B)$ is a controllable pair, we have $\det(U(\xi)) = \det(\xi I - A)$, so we obtain $\det(R^{1/2})\chi_{A_K} = \det(F_K)$. The remaining statements of the theorem follow immediately from this. $\quad \square$

In the above, we have assumed that $\partial \Phi(i\omega) \geq 0$ for all $\omega \in \mathbb{R}$. In the case that, in addition, $\partial \Phi$ is nonsingular along the imaginary axis, equivalently $\partial \Phi(i\omega) > 0$ for all $\omega \in \mathbb{R}$, the one-to-one correspondence between polynomials and the set of real symmetric solutions of the ARE can be made even more explicit. This will be explained next.

Define $\mathcal{F}_{\mathrm{cop}}$ as the set of all real polynomials $f$ such that the *determinant* of $\partial \Phi$ admits a factorization $f^\sim f$ such that $f$ and $f^\sim$ are coprime:

$$\begin{aligned}
\mathcal{F}_{\mathrm{cop}} = \{f \in \mathbb{R}[\xi] \mid f(\xi) = f_0 + f_1\xi + \cdots + f_n\xi^n, \quad &f_n > 0, \ (f, f^\sim) \text{ coprime} \\
&\text{and } \det(\partial \Phi) = f^\sim f\}.
\end{aligned}$$

It is easily seen that if $\partial \Phi(i\omega) \geq 0$ for all $\omega \in \mathbb{R}$, then $\mathcal{F}_{\mathrm{cop}} \neq \emptyset$ if and only if $\partial \Phi(i\omega) > 0$ for all $\omega \in \mathbb{R}$. Hence it follows from Proposition 5 that $\mathcal{F}_{\mathrm{cop}} \subset \mathcal{F}$. In the remainder of this section we assume that $\partial \Phi(i\omega) > 0$ for all $\omega \in \mathbb{R}$.

Note that if $f \in \mathcal{F}_{\mathrm{cop}}$ and $K = \mathrm{Ric}(f)$, then, according to Theorem 11, $f = \sqrt{\det(R)}\chi_{A_K}$, so $\chi_{A_K}$ and $(\chi_{A_K})^\sim$ are coprime; equivalently, $\sigma(A_K) \cap \sigma(-A_K) = \emptyset$. If a solution $K$ of the ARE satisfies this property, we call it *unmixed*. The set of all unmixed solutions of the ARE is denoted by $\mathcal{S}_{\mathrm{unm}}$. It follows immediately from Theorem 11 that Ric defines a bijection between $\mathcal{F}_{\mathrm{cop}}$ and $\mathcal{S}_{\mathrm{unm}}$.

We now explain the connection between the bijection Ric and the Pick matrices $T_f$ associated with $\Phi$. Recall that the bijection Ric between $\mathcal{F}_{\mathrm{cop}}$ and $\mathcal{S}_{\mathrm{unm}}$ is defined as follows. For a given $f \in \mathcal{F}_{\mathrm{cop}}$, let $F \in \mathbb{R}^{m \times m}[\xi]$ be such that $\partial \Phi = F^\sim F$ and $\det(F) = f$, and take $K = \mathrm{Ric}(f)$ to be the unique solution of (24). For the sake of exposition, assume for the moment that the singularities of $\partial \Phi$ are semisimple. We show how to compute, for $f \in \mathcal{F}_{\mathrm{cop}}$, the corresponding unmixed solution $K = \mathrm{Ric}(f)$, using the Pick matrix $T_f$.

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the roots of $f$, with the convention that if a root has algebraic multiplicity $m_i$, then it appears in this list $m_i$ times, and $\lambda_1, \lambda_2, \ldots, \lambda_{m_1}$ are equal, $\lambda_{m_1+1}, \lambda_{m_1+2}, \ldots, \lambda_{m_1+m_2}$ are equal, etc. Let $v_i \in \mathbb{C}^m$ be such that $\partial \Phi(\lambda_i)v_i =$

0 and $v_1, v_2, \dots, v_n$ are linearly independent. Evaluating (24) at $(\zeta, \eta) = (\bar{\lambda}_i, \lambda_j)$, premultiplying the result by $v_i^*$ and postmultiplying it by $v_j$, we get

$$\frac{v_i^* \Phi(\bar{\lambda}_i, \lambda_j) v_j}{\bar{\lambda}_i + \lambda_j} - \frac{v_i^* F^T(\bar{\lambda}_i) F(\lambda_j) v_j}{\bar{\lambda}_i + \lambda_j} = -v_i^* X^T(\bar{\lambda}_i) K X(\lambda_j) v_j.$$

Note that, by coprimeness of $f$ and $f^\sim$, $\bar{\lambda}_i + \lambda_j \neq 0$ for all $(i, j)$. Now make the crucial observation that for all $j$

$$F(\lambda_j) v_j = 0.$$

Indeed, by definition of $v_j$ we have $F^T(-\lambda_j) F(\lambda_j) v_j = 0$. Since, however, $F^T(-\lambda_j)$ is nonsingular (by coprimeness of $f$ and $f^\sim$), the claim follows. Thus we immediately obtain

$$\frac{v_i^* \Phi(\bar{\lambda}_i, \lambda_j) v_j}{\bar{\lambda}_i + \lambda_j} = -v_i^* X^T(\bar{\lambda}_i) K X(\lambda_j) v_j,$$

which is equivalent to

$$T_f = -(S_f)^* K S_f,$$

where $S_f$ is the *zero state matrix* associated with $f$, defined by

$$S_f := \begin{pmatrix} X(\lambda_1) v_1 & \dots & X(\lambda_n) v_n \end{pmatrix}.$$

For a motivation of the terminology zero state matrix, we refer to the proof of Theorem 12 below. Note that $S_f \in \mathbb{C}^{n \times n}$. In Theorem 12 we will prove that for any $f \in \mathcal{F}_{\text{cop}}$ the zero state matrix $S_f$ is nonsingular. This immediately implies that the solution $K = \text{Ric}(f)$ is given by

(25) $$K = \text{Ric}(f) = -(S_f^*)^{-1} T_f (S_f)^{-1}.$$

The above argument can be generalized to the case in which not all singularities of $\partial \Phi$ are semisimple. In the general case, the zero state matrix $S_f$ associated with the polynomial factor $f$ is defined in the following way. Let $\lambda_1, \lambda_2, \dots, \lambda_k$ be the roots of $f$. As in section 4, we use the convention that if a given root $\lambda_i$ has geometric multiplicity $n_i$, then we include it $n_i$ times in our list of roots. For $i = 1, 2, \dots, k$, let $V_i \in \mathbb{C}^{d_i m \times d_i}$ be defined by (10) and (11) (with $q = m$). Furthermore, define the $n \times d_i$ matrix $S_i$ by

$$S_i := \begin{pmatrix} X(\lambda_i) & X^{(1)}(\lambda_i) & \dots & X^{(d_i - 1)}(\lambda_i) \end{pmatrix} V_i,$$

where $X^{(j)}$ denotes the $j$th derivative of $X$. The zero state matrix in the general case is then defined by

(26) $$S_f := \begin{pmatrix} S_1 & S_2 & \dots & S_k \end{pmatrix}.$$

Again, $S_f \in \mathbb{C}^{n \times n}$.

The following theorem is the main result of this paper. It yields the representation (25) of the bijection Ric in the general, not necessarily semisimple, case.

THEOREM 12. *Assume $\partial \Phi(i\omega) \geq 0$ for all $\omega \in \mathbb{R}$. Then the following three statements are equivalent:*

(i) $\partial\Phi(i\omega) > 0$ *for all* $\omega \in \mathbb{R}$;

(ii) $\mathcal{F}_{\text{cop}} \neq \emptyset$;

(iii) $\mathcal{S}_{\text{unm}} \neq \emptyset$.

*Assume that this holds. Then* $\text{Ric} : \mathcal{F}_{\text{cop}} \to \mathcal{S}_{\text{unm}}$ *is a bijection. For all* $f \in \mathcal{F}_{\text{cop}}$ *the zero state matrix* $S_f$ *defined by* (26) *is nonsingular. Furthermore, for any* $f \in \mathcal{F}_{\text{cop}}$, *the corresponding solution* $\text{Ric}(f) \in \mathcal{S}_{\text{unm}}$ *is given by*

$$\text{Ric}(f) = -(S_f^*)^{-1} T_f S_f^{-1}. \tag{27}$$

*Proof.* The claim that conditions (i), (ii), and (iii) of Theorem 12 are equivalent, and the claim that under this condition Ric defines a bijection between $\mathcal{F}_{\text{cop}}$ and $\mathcal{S}_{\text{unm}}$, follow from Theorem 11.

We prove that the zero state matrix (26) is nonsingular. Let $F \in \mathbb{R}^{m \times m}[\xi]$ be such that $\det(\partial\Phi) = f^\sim f$ and $\det(F) = f$. Let $\xi = \text{col}(\xi_1, \xi_2, \dots, \xi_k)$, with $\xi_i = \text{col}(\xi_{i,1}, \xi_{i,2}, \dots, \xi_{i,d_i}) \in \mathbb{C}^{d_i}$, satisfy $S_f \xi = 0$, equivalently,

$$\sum_{i=1}^{k} \begin{pmatrix} X(\lambda_i) & X^{(1)}(\lambda_i) & \dots & X^{(d_i - 1)}(\lambda_i) \end{pmatrix} V_i \xi_i = 0.$$

We will show that $\xi_i = 0$ for $i = 1, 2, \dots, k$.

Recall that the system $\frac{d}{dt}x = Ax + Bu$ has an observable image representation

$$\begin{pmatrix} x \\ u \end{pmatrix} = \begin{pmatrix} X(\frac{d}{dt}) \\ U(\frac{d}{dt}) \end{pmatrix} l, \tag{28}$$

and that $X(\frac{d}{dt})$ is a minimal state map for this system. Consider the extended system $\mathcal{B}_{\text{ext}}$, obtained by including $f = F(\frac{d}{dt})l$ as a manifest variable, represented by the image representation

$$\begin{pmatrix} x \\ u \\ f \end{pmatrix} = \begin{pmatrix} X(\frac{d}{dt}) \\ U(\frac{d}{dt}) \\ F(\frac{d}{dt}) \end{pmatrix} l.$$

We claim that in the system $\mathcal{B}_{\text{ext}}$, $\text{col}(x, u)$ is output and $f$ is input, and that $X(\frac{d}{dt})$ is a minimal state map also for $\mathcal{B}_{\text{ext}}$.

To prove this, first note that

$$F^\sim F = X^\sim Q X + X^\sim S^T U + U^\sim S X + U^\sim R U.$$

Multiplying this equality on the right by $U^{-1}$ and on the left by $(U^\sim)^{-1}$ yields

$$(U^\sim)^{-1} F^\sim F U^{-1} = (U^\sim)^{-1} X^\sim Q X U^{-1} + (U^\sim)^{-1} X^\sim S^T + S X U^{-1} + R.$$

Since $X U^{-1}$ is strictly proper and $R > 0$, this implies that $F U^{-1}$ is a proper rational matrix with nonsingular feedthrough term. This implies that also its inverse, $U F^{-1}$, is proper, and $X F^{-1} = X U^{-1} U F^{-1}$ is strictly proper. Since, therefore,

$$\begin{pmatrix} X \\ U \end{pmatrix} F^{-1}$$

is a proper rational matrix, in the system $\mathcal{B}_{\text{ext}}$, $\text{col}(x, u)$ is output and $f$ is input.

Next we prove that $X(\frac{d}{dt})$ is a minimal state map for $\mathcal{B}_{\text{ext}}$. To prove this, we show that the rows of $X$ form a basis for the real linear space $\mathcal{S}_1 = \{r \in \mathbb{R}^{1 \times m}[\xi] \mid rF^{-1} \text{ is strictly proper}\}$. Since $X$ induces a minimal state map for our original system (28), the rows of $X$ form a basis for the real linear space $\mathcal{S}_2 = \{r \in \mathbb{R}^{1 \times m}[\xi] \mid rU^{-1} \text{ is strictly proper}\}$. Since $UF^{-1}$ and $FU^{-1}$ are proper, $rF^{-1}$ is strictly proper if and only if $rU^{-1}$ is strictly proper. Hence the two linear spaces $\mathcal{S}_1$ and $\mathcal{S}_2$ coincide, so the rows of $X$ indeed form a basis for $\mathcal{S}_1$.

Define a particular latent variable trajectory for $\mathcal{B}_{\text{ext}}$ by

$$\tilde{l}(t) = \sum_{i=1}^{k} e^{\lambda_i t} \begin{pmatrix} I_{m \times m} & tI_{m \times m} & \cdots & t^{d_i - 1} I_{m \times m} \end{pmatrix} V_i \xi_i.$$

Then we clearly have $\partial \Phi(\frac{d}{dt})\tilde{l} = 0$. Using the fact that none of the $\lambda_i$'s is a singularity of $F^{\sim}$, this implies that $F(\frac{d}{dt})\tilde{l} = 0$. Our aim is to prove that $\tilde{l} = 0$. Indeed, look at the trajectory of the system $\mathcal{B}_{\text{ext}}$ corresponding to the choice of latent variable $\tilde{l}$. The input $f = F(\frac{d}{dt})\tilde{l}$ is equal to zero. Furthermore, a straightforward calculation shows that the value of the corresponding state trajectory at time $t = 0$ equals

$$\left( X\left(\frac{d}{dt}\right)\tilde{l} \right)(0) = \sum_{i=1}^{k} \begin{pmatrix} X(\lambda_i) & X^{(1)}(\lambda_i) & \cdots & X^{(d_i - 1)}(\lambda_i) \end{pmatrix} V_i \xi_i = 0.$$

Hence the output $(x, u) = (X(\frac{d}{dt})\tilde{l}, U(\frac{d}{dt})\tilde{l}$ of $\mathcal{B}_{\text{ext}}$ is zero. By observability of the image representation (28), this implies $\tilde{l} = 0$, as claimed.

Next, we prove that this implies $\xi_i = 0$ for all $i$. Indeed, since $\tilde{l} = 0$ we have

$$\tilde{l}(0) = \sum_{i=1}^{k} \begin{pmatrix} I_{m \times m} & 0 & \cdots & 0 \end{pmatrix} V_i \xi_i = 0.$$

Consequently, $\tilde{l}(0) = \sum_{i=1}^{k} \sum_{j=0}^{d_i - 1} a_{i,j} \xi_{i,j} = 0$. Since the vectors $a_{i,j}$ are linearly independent, this yields $\xi_{i,j} = 0$ for all $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, d_i$. This proves that the zero state matrix $S_f$ is nonsingular.

To prove (27) we use that $K_f = \text{Ric}(f)$ is uniquely defined by

$$(29) \qquad \Phi(\zeta, \eta) - F^T(\zeta)F(\eta) = -(\zeta + \eta)X^T(\zeta)K_f X(\eta),$$

with $F \in \mathbb{R}^{m \times m}[\xi]$ such that $\partial \Phi = F^{\sim} F$ and $\det(F) = f$. The idea is to evaluate (29) and its partial derivatives with respect to $\zeta$ and $\eta$ at the points $(\bar{\lambda}_i, \lambda_j)$. For all indices $(r, s)$ we have

$$\frac{\partial^{r+s} \Phi}{\partial \eta^r \partial \zeta^s}(\zeta, \eta) - F^{(s)T}(\zeta)F^{(r)}(\eta)$$
$$= sX^{(s-1)T}(\zeta)K_f X^{(r)}(\eta) + rX^{(s)T}(\zeta)K_f X^{(r-1)}(\eta)$$
$$+ (\zeta + \eta)X^{(s)T}(\zeta)K_f X^{(r)}(\eta).$$

Using this, for $i, j = 1, 2, \ldots, k$, we form the matrices $\Phi_{i,j}$ defined by (13). Next, with $\Sigma_{i,j}$ defined by (14) and $\Lambda_{i,j}$ defined by (12), a straightforward calculation shows that

$$\Lambda_{j,i}^* V_i^* \Sigma_{i,j} V_j \Lambda_{i,j}$$

$$(30) \qquad = -V_i^* \begin{pmatrix} X^T(\bar{\lambda}_i) \\ X^{(1)T}(\bar{\lambda}_i) \\ \vdots \\ X^{(d_i - 1)T}(\bar{\lambda}_i) \end{pmatrix} K_f \begin{pmatrix} X(\lambda_j) & X^{(1)}(\lambda_j) & \cdots & X^{(d_j - 1)}(\lambda_j) \end{pmatrix} V_j.$$

The crucial point here is that the terms involving $F^{(r)T}(\bar{\lambda}_i)F^{(s)}(\lambda_j)$ vanish, since for $i = 1, 2, \ldots, k$ we have

(31)
$$\begin{pmatrix} \binom{0}{0}\partial F^{(0)}(\lambda_i) & \binom{1}{0}\partial F^{(1)}(\lambda_i) & \cdots & \cdots & \binom{d_i-1}{0}\partial F^{(d_i-1)}(\lambda_i) \\ 0 & \binom{1}{1}\partial F^{(0)}(\lambda_i) & \cdots & \cdots & \binom{d_i-1}{1}\partial F^{(d_i-2)}(\lambda_i) \\ 0 & 0 & \ddots & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \binom{d_i-1}{d_i-1}\partial F^{(0)}(\lambda_i) \end{pmatrix} V_i = 0.$$

The latter follows from (10), combined with the fact that for $i = 1, 2, \ldots, k$ the matrices $F^T(-\lambda_i)$ are nonsingular. Since (30) holds for all $i, j = 1, 2, \ldots, k$, we obtain $T_f = S_f^* K_f S_f$. This completes the proof. □

This result yields a procedure for computing all unmixed solutions of the ARE (1). We sum up the steps that are required here.

1. Compute a right coprime factorization $X(\xi)U(\xi)^{-1}$ of $(\xi I - A)^{-1}B$.
2. Form the corresponding two-variable polynomial matrix $\Phi$ given by (19).
3. Check whether $\partial\Phi(i\omega) > 0$ for all $\omega \in \mathbb{R}$.
4. Factor $\det(\partial\Phi) = f^\sim f$ with $f$ and $f^\sim$ coprime.

The following then computes the unique solution $K = K^T$ of the ARE such that its "closed loop characteristic polynomial" $\chi_{A_K}$ equals $\sqrt{\det(R)} \, f$.

5. Compute the zero state matrix $S_f$.
6. Compute the Pick matrix $T_f$.
7. Solve the equation $T_f = -S_f^* K S_f$.
8. Set $K = \text{Ric}(f)$.

It is worthwhile to observe that similar results have been obtained in Chapter 5 of [11] for QDFs not necessarily associated with a state space representation (16). Note that the procedure circumvents the need to do a polynomial spectral factorization of $\partial\Phi$.

We now go back to the problem of establishing necessary and sufficient conditions for the existence of sign-definite solutions to the ARE. Our main result here is an immediate consequence of Theorem 12 and is based on the result of Proposition 3, namely, that the largest (smallest) storage function for $\Phi$ is associated with an anti-Hurwitz (Hurwitz) factorization of $\partial\Phi$. Let $K_-$ and $K_+$ be the smallest, respectively the largest, real symmetric solution of the ARE.

COROLLARY 13. *Let $\Phi(\zeta, \eta)$ be defined as in (19). Assume that $\partial\Phi(i\omega) > 0$ for all $\omega \in \mathbb{R}$. Factor $\det(\partial\Phi) = (f_A)^\sim f_A = (f_H)^\sim f_H$, where $f_A$ and $f_H$ have their roots in the open right half plane and open left half plane, respectively. Then we have*

$$K_- = -(S_{f_A}^*)^{-1}T_{f_A}S_{f_A}^{-1},$$

$$K_+ = -(S_{f_H}^*)^{-1}T_{f_H}S_{f_H}^{-1}.$$

*Consequently, $\text{sign}(K_-) = -\text{sign}(T_{f_H})$ and $\text{sign}(K_+) = -\text{sign}(T_{f_A})$. In particular, the ARE (1) has a negative semidefinite (negative definite) solution if and only if the Pick matrix $T_{f_A}$ is positive semidefinite (respectively, positive definite). It has a positive semidefinite (positive definite) solution if and only if the Pick matrix $T_{f_H}$ is negative semidefinite (respectively, negative definite).*

*Example 6, continued.* For the Riccati equation of Example 6 we have $\partial\Phi(\xi) = \begin{pmatrix} 1-\xi^2 & a \\ a & 4-\xi^2 \end{pmatrix}$, and we have $\partial\Phi(i\omega) > 0$ for all $\omega \in \mathbb{R}$ if and only if $-2 < a < 2$. Assume

this to be the case. We have $\det(\partial\Phi(\xi)) = (1-\xi^2)(4-\xi^2) - a^2$. Set $k = 3 + \sqrt{9 + 4a^2}$. The singularities of $\partial\Phi$ are then equal to $\lambda_1 = -\sqrt{1 + \frac{1}{2}k}$, $\lambda_2 = -\sqrt{4 - \frac{1}{2}k}$, $-\lambda_1$, and $-\lambda_2$. Clearly, $\det(\partial\Phi)$ can be factored as $f^\sim f$ with $(f^\sim, f)$ coprime in four different ways, and the Riccati equation has four real symmetric solutions, all of them unmixed. Here we compute the largest real symmetric solution, i.e., the solution $K$ satisfying $\chi_{A_K} = f_H$, with $f_H(\xi) = (\xi + \sqrt{1 + \frac{1}{2}k})(\xi + \sqrt{4 - \frac{1}{2}k})$. Note that we are in the semisimple situation, i.e., the algebraic multiplicity of each singularity equals its corresponding rank deficiency. Solving $\partial\Phi(\lambda_1)v_1 = 0$ and $\partial\Phi(\lambda_2)v_2 = 0$ yields $v_1 = \left(\begin{smallmatrix} 2a/k \\ 1 \end{smallmatrix}\right)$ and $v_2 = \left(\begin{smallmatrix} 1 \\ -2a/k \end{smallmatrix}\right)$. The zero state matrix $S_{f_H}$ is hence given by $S_{f_H} = \left(\begin{smallmatrix} 2a/k & 1 \\ 1 & -2a/k \end{smallmatrix}\right)$. Next we compute the Pick matrix corresponding to $f_H$. Clearly,

$$T_{f_H} = \begin{pmatrix} \dfrac{v_1^* \Phi(\lambda_1, \lambda_1) v_1}{2\lambda_1} & \dfrac{v_1^* \Phi(\lambda_1, \lambda_2) v_2}{\lambda_1 + \lambda_2} \\ \dfrac{v_2^* \Phi(\lambda_2, \lambda_1) v_1}{\lambda_2 + \lambda_1} & \dfrac{v_2^* \Phi(\lambda_2, \lambda_2) v_2}{2\lambda_2} \end{pmatrix},$$

which is equal to

(32)
$$T_{f_H} =$$

$$\begin{pmatrix} \frac{1}{2\lambda_1}\left(\frac{4a^2}{k^2}(2+k) + \frac{2a^2}{k} + \frac{k}{2} + 5 - 2\lambda_1\right) & \frac{1}{\lambda_1+\lambda_2}\left(\frac{2a}{k}(1 + \lambda_1\lambda_2) + a - \frac{4a^3}{k^2} - \frac{8a}{k} + \frac{2a}{k}(\lambda_1 + \lambda_2)\right) \\ \frac{1}{\lambda_1+\lambda_2}\left(\frac{2a}{k}(1 + \lambda_1\lambda_2) + a - \frac{4a^3}{k^2} - \frac{8a}{k} + \frac{2a}{k}(\lambda_1 + \lambda_2)\right) & \frac{1}{2\lambda_2}\left(\frac{4a^2}{k^2}(8 + 2\lambda_2) - \frac{6a^2}{k} - \frac{k}{2} + 5\right) \end{pmatrix}.$$

This yields $K^+ = -(S_{f_H}^T)^{-1} T_{f_H} S_{f_H}$ as the solution corresponding to $f_H$. Note that this gives the largest real symmetric solution for each value of $a$ between $-2$ and 2. For example, if $a = 0$, then $k = 6$, so $\lambda_1 = -2$ and $\lambda_2 = -1$. This yields $K^+ = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 3 & 0 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 3 \end{smallmatrix}\right)$. Recall that $Q = \left(\begin{smallmatrix} 1 & a \\ a & 3 \end{smallmatrix}\right)$, so for $a = 0$ we have $Q > 0$. In this particular case it follows immediately that the ARE has a positive semidefinite solution (the corresponding linear quadratic problem is positive semidefinite). For values of $a$ satisfying $-2 < a < -\sqrt{3}$ or $\sqrt{3} < a < 2$, $Q$ is indefinite, so for this case it is a nontrivial matter to check whether the ARE has a positive (semi-) definite solution. According to Corollary 13, for a given $a \in (-2, 2)$ the ARE has a positive (semi-) definite solution if and only if for that value of $a$ the Pick matrix (32) is negative (semi-) definite. As an example, take $a = 1.8$. In this case $Q$ is indefinite. The Pick matrix corresponding to this value of $a$ is computed as $T_{f_H} = \left(\begin{smallmatrix} -3.6835 & 0.3111 \\ 0.3111 & -0.2637 \end{smallmatrix}\right)$. The eigenvalues of $T_{f_H}$ are computed as $-3.7116$ and $-0.2356$, so we conclude that for $a = 1.8$ our ARE has a positive definite solution. For $a = 1.98$ we compute $T_{f_H} = \left(\begin{smallmatrix} -3.7839 & 0.4375 \\ 0.4375 & 0.0894 \end{smallmatrix}\right)$, which has eigenvalues $-3.8327$ and $0.1382$. For this value of $a$ our ARE does not have a positive semidefinite solution.

In order to check whether for a given $a$ the ARE of this example has at least one negative (semi-) definite solution, one should compute the Pick matrix $T_{f_A}$ associated with the polynomial $f_A(\xi) = (\xi - \sqrt{1 + \frac{1}{2}k})(\xi - \sqrt{4 - \frac{1}{2}k})$, and check whether it is positive (semi-) definite.

**7. Conclusions.** In this paper we applied ideas from the calculus of two-variable polynomial matrices to the problem of characterizing all umixed solutions of the algebraic Riccati equation and formulating necessary and sufficient conditions for the existence of (semi) definite solutions.

We started from the two-variable polynomial matrix corresponding to the underlying quadratic functional, and associated with this a nonsingular one-variable

polynomial matrix. Then we showed that there is a bijection between the set of all scalar polynomial spectral factors of the determinant of this one-variable polynomial matrix and the set of all unmixed solutions of the ARE. For every such scalar polynomial spectral factor we defined a constant Hermitian matrix, called the Pick matrix, and we expressed the unmixed solution corresponding to this polynomial spectral factor in terms of its Pick matrix. This enabled us to conclude that the signatures of the extremal solutions of the ARE are determined by the Pick matrices corresponding to these solutions.

In this paper, we have restricted ourselves to the case in which $(A, B)$ is a controllable pair, mainly in order to be able to use image representations. As a possible direction for future research, we mention the extension of our results to the noncontrollable case. Another interesting problem would be to generalize our results to the discrete-time algebraic Riccati equation.

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