# Labels from Reductions

## the theory of relative pushouts

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# Motivations

Reductions model computing systems by expressing their evolution as internal 'reactions' between components:

Church's beta-reduction rule:

$$(\lambda x.M)N \longrightarrow M\{x := N\}$$

Milner's mechanics of $\pi$-calculus interaction:

$$\overline{a}\langle n\rangle.P \mid a(x).Q \longrightarrow P \mid Q\{x := n\}$$

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- Cons: contextual semantics, lack of compositionality, ...

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# The role of labels

Labelled transitions systems model evolution indirectly, splitting actions in atomic components.

$\overline{a}\langle n\rangle.P \xrightarrow{\overline{a}n} P$: ready to evolve to $P$ by engaging in action $\overline{a}$;

$a(x).Q \xrightarrow{a(n)} Q\{x := n\}$: dual rule for partners in interactions;

Finally, an inference rule dictates

$$\frac{A \xrightarrow{\overline{a}n} A' \qquad B \xrightarrow{a(n)} B'}{A \mid B \longrightarrow A' \mid B'}$$

Advantages:

- SOS rules & compositionality
- Bisimulation / coinduction techniques
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# Labels from reductions: Objectives

1. Bridge the world of reductions and that of labels
2. Equip rewrite systems with compositional semantics, systematically

Technically: from a reduction system $\longrightarrow$, extract a LTS $\xrightarrow{lbl}$ such that:

1. $a \longrightarrow b$ *iff* $a \xrightarrow{} b$
2. *bisimulation on LTS is a congruence for all contexts.*

This talk: Not overly technical, but not simply a pointer to the literature.
I'd like to illustrate specific results, simply, and point to open questions.

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# Plan of the talk

1. Basic Theory: Slice Pushouts
2. Finer Control: G-Categories
3. Examples: Graphs and Petri Nets
4. Towards a General Theory: Luxes

# The basic idea

Labels characterise terms by reflect the “outwards” actions:

- $\overline{a}\langle n\rangle.P$ is inert by itself, but willing to offer $a$ in the right context:

  e.g., $\mathscr{C}[-] = 1 \mid a(x).Q$

## Deriving labels

$$a \xrightarrow{c[-]} b \qquad \text{when} \qquad c[a] \longrightarrow b$$

So,

$$\overline{a}\langle n\rangle.P \xrightarrow{1|a(x).Q} P \mid Q\{x := n\}$$

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# Are there bad contexts too?

Contexts must be lean! They must be:

- Small (no junk). Redundant labels mean wasting work:

$$\lambda x.x \xrightarrow{(-)y} y, \qquad \text{but not} \qquad \lambda x.x \xrightarrow{(-)yz} yz,$$

as $(-)yz$ arises as the composition of $(-)z$ and $(-)y$.

- Relevant. Irrelevant labels clutter the bisimulation:

terms: $P ::= 0 \mid a \mid \overline{a} \mid P \mid P$ rule: $a \mid \overline{a} \longrightarrow 0$

Then $\mathscr{C}[a \mid \overline{a}] \longrightarrow \mathscr{C}[0]$, for all $\mathscr{C}[-]$ and $a$. So, $a \mid \overline{a} \sim b \mid \overline{b} \sim \ldots$

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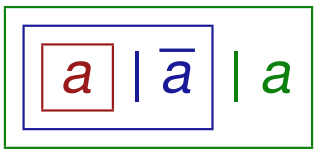

# Formalising ‘smallness’ and ‘relevance’

## Breakthrough (Leifer & Milner)

A context $c[-]$ is the ‘smallest’ to create redex $l$ in $a$, if all contexts $c'[-]$ that create $l$ in $a$ factor as $c'[-] = e[c[-]]$, for a unique context $e[-]$.

Rephrase ‘smallness’ as a problem of unique arrow factorisation in categories!




Can express minimality of $c$ as existence of a universal factorisation of all equations of the kind $c'\,a = d'\,l$.

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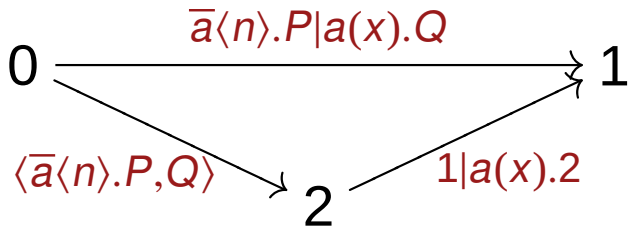
# A categorical framework

## PROP category (MacLane)

Arrow $f : \underline{n} \to \underline{m}$: a $m$-tuple of contexts containing altogether $n$ 'holes.'

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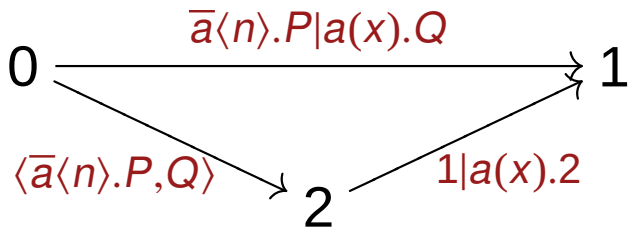


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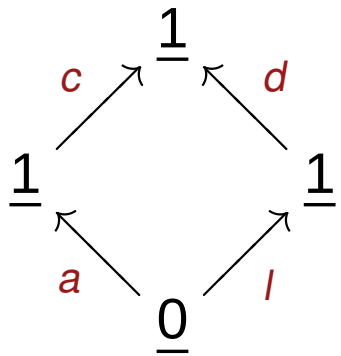

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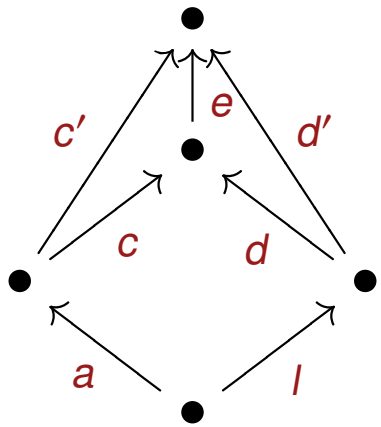


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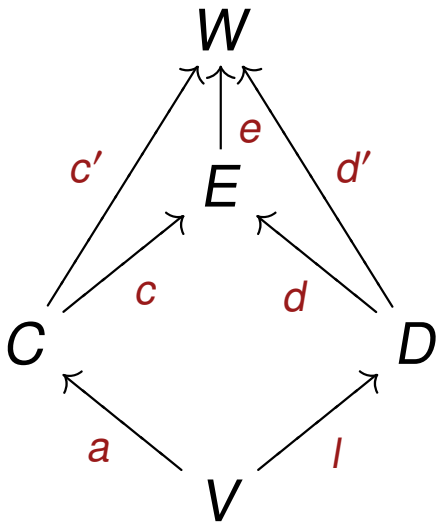
# Slice pushouts




## Slice Pushout (a.k.a. Relative Pushout)

These properties spell out that the diagram above is a so-called pushout in the slice category $\mathbf{C}/W$, a slice pushout for short.

Good labels are all and only the $c$ in some slice pushout.

# The theory of slice pushouts

A reactive system $\mathbb{C}$ is a triple $\langle \mathbf{C}, \mathbf{D}, \mathcal{R} \rangle$ where:

- $\mathbf{C}$ is a category with a distinguished object $\underline{0}$;
- $\mathbf{D}$ is a subcategory of 'reactive contexts;'
- $\mathcal{R}$ is a set of reaction rules $\langle l, r \rangle$, for $l, r \in \mathbf{C}(\underline{0}, W)$.

The reaction relation is defined as

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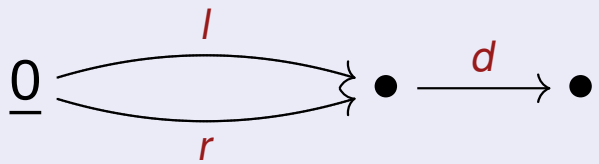
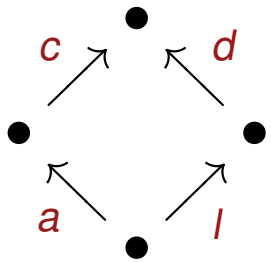


# Deriving bisimulation congruences

$LTS(\mathbb{C})$, the LTS associated to $\mathbb{C}$ has

- Nodes: $a : \underline{0} \to W$
- Transitions: $a \xrightarrow{c} dr$ for $\langle l, r \rangle \in \mathcal{R}$, $d \in \mathbb{D}$, and a slice pushout




**Theorem**

*Bisimulation on $LTS(\mathbb{C})$ is a congruence, provided* **C** *has all slice pushouts.*

Proof: simple (based on two factorisation properties of pushouts).
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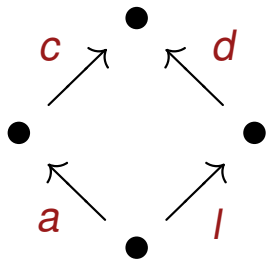


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# What's the matter with structural congruence?

This works well with 'rigid' terms, it cracks with 'fluid' ones!

Recall that context $1 \mid \overline{a}$ is never an acceptable label for $a \mid \overline{a}$.

But this is wrong for a commutative parellel composition! Because of structural congruence, $a \mid \overline{a}$ can offer $\overline{a}$ to the environment, as well as internally.




Only the left one could possibly be an SPO!

The derived LTS can only account for a $a \mid \overline{a} \longrightarrow 0$ transition.
Beware of confusion! Must eliminate ambiguity as to where the redex is.

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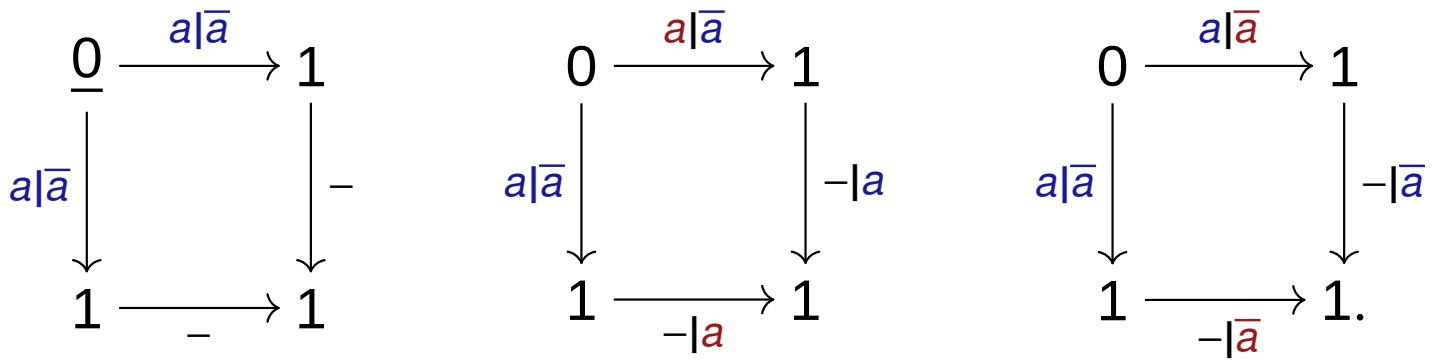


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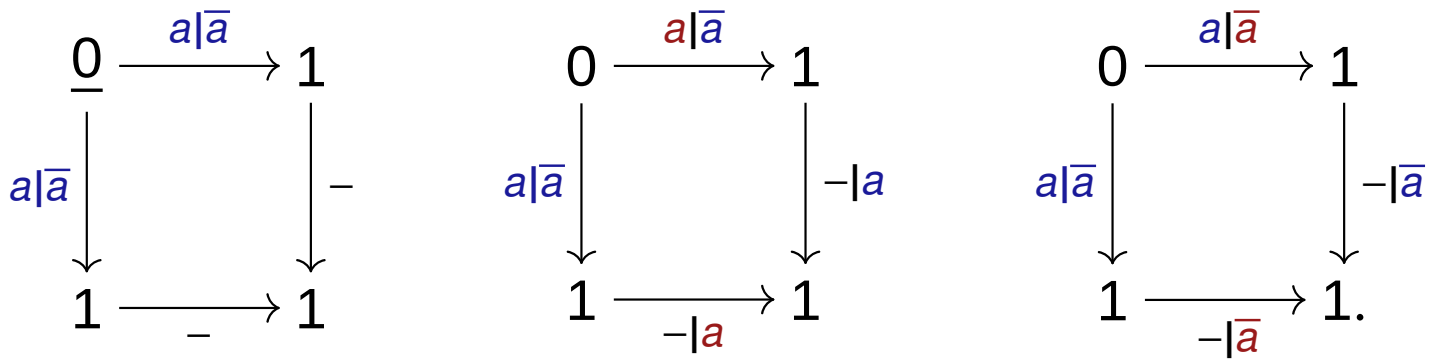


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# Two ‘competing’ approaches to ‘term colouring’

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- Sassone/Sobocinski: G-categories – work with several examples, notably very general graph-like structures




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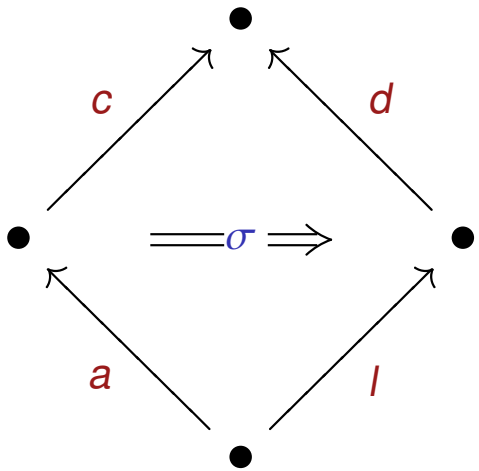


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# Plan of the talk

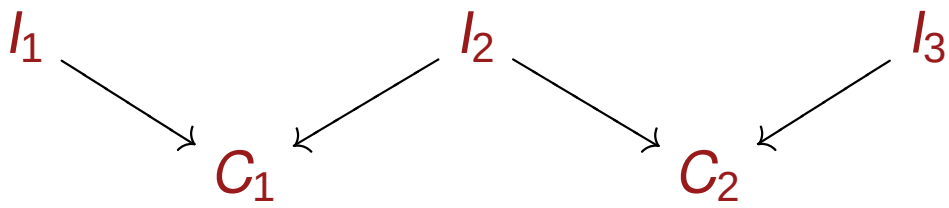
# Cospan categories

**Theorem**

*The bicategory* **Cospan(C)** *has (G-)slice pushouts, for a general class of categories* **C** *(viz., the adhesive ones).*

**Cospan(C)**:

- Objects: as in **C**;
- Arrows: $I_1 \longrightarrow C \longleftarrow I_2$, and composition: by pushout

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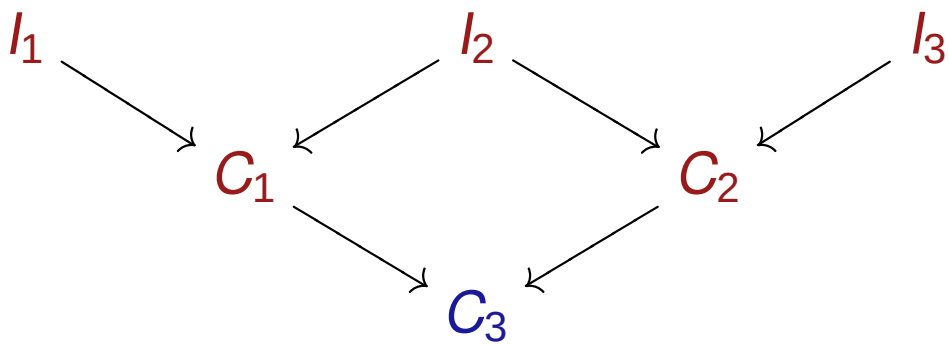


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# Graph rewriting systems

A graph rewrite system, *GR*, determines a reactive system on **Cospan(Graph)** whose reductions correspond to the standard 'double-pushout' semantics of graph rewriting.

$L \xleftarrow{l} K \xrightarrow{r} R$ translates to $\langle 0 \to L \xleftarrow{l} K, 0 \to R \xleftarrow{r} K \rangle \in \mathcal{R}$

As **Cospan(Graph)** has slice pushouts, we systematically construct a LTS *LTS(GR)* such that:

- nodes are graphs (up-to-iso) with output interfaces
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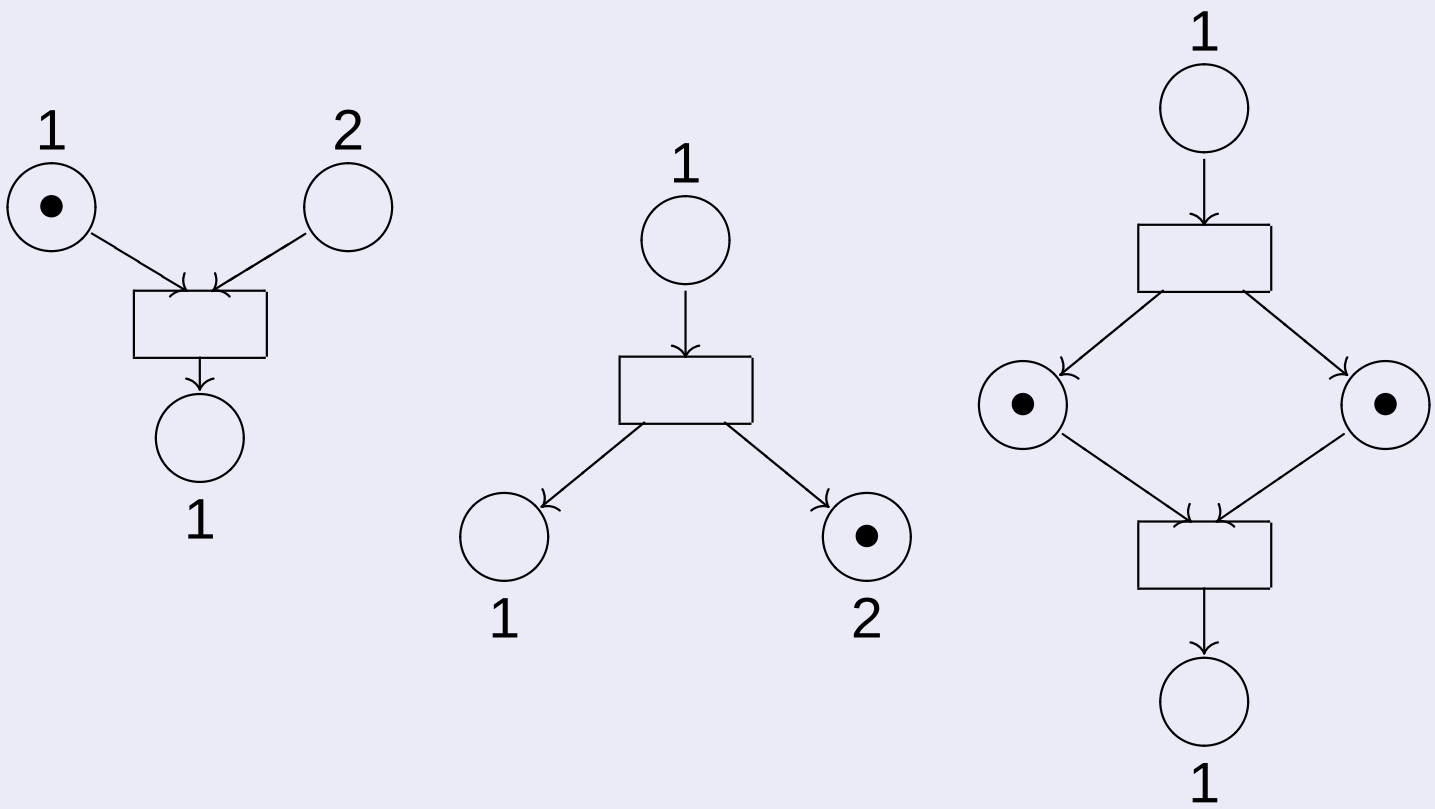
# The case of Petri nets

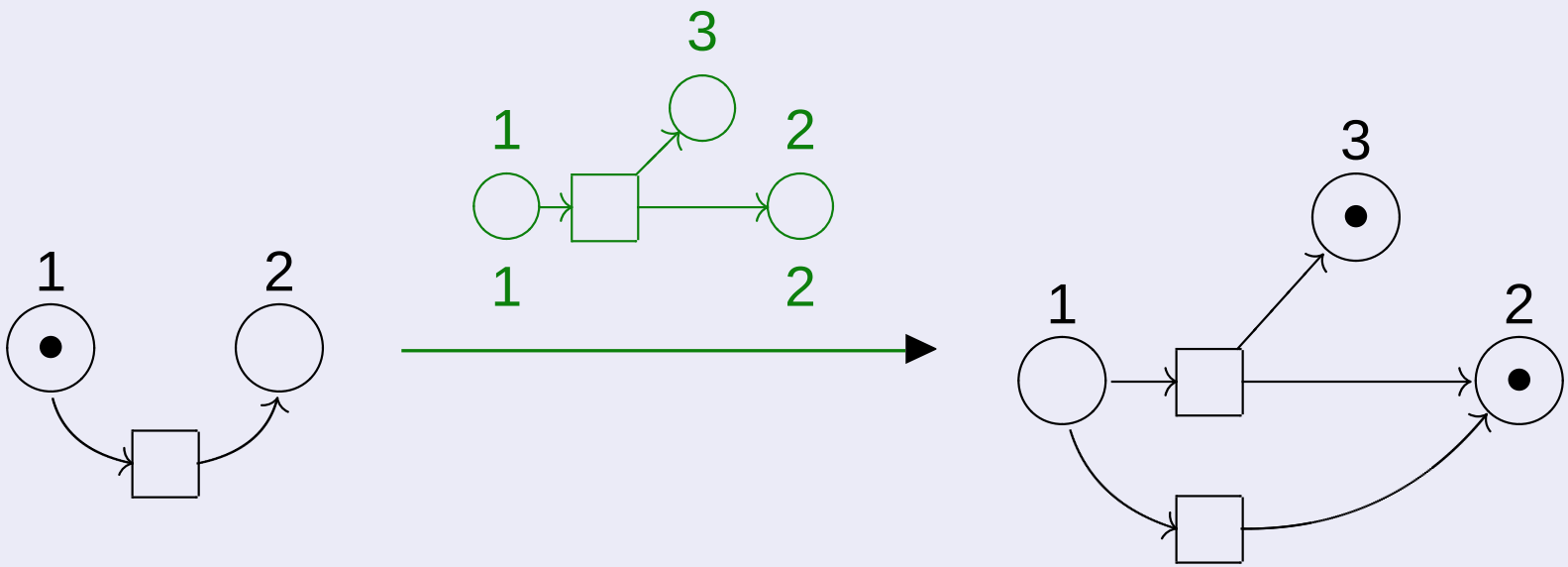
## Petri Nets with Interfaces: inner and outer faces/places



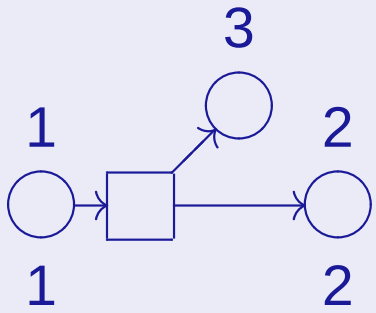

The token game can easily be decribed as a graph rewrite system and, therefore, the model fits the **Cospan(C)** framework.

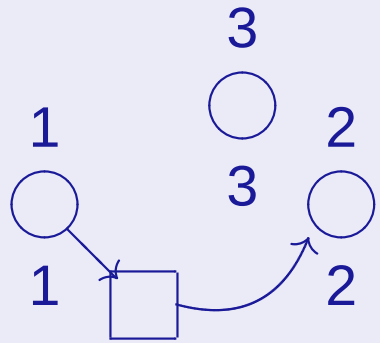
# The derived LTS

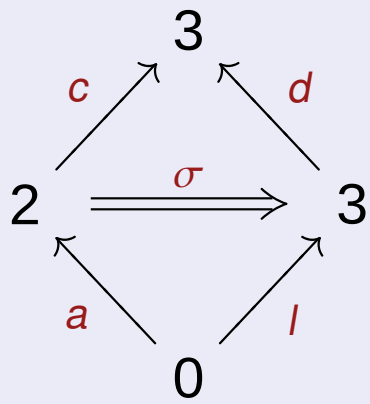
We therefore benefit ‘for free’ of a derived LTS where bisimulation is a congruence. It is instructive to look at one label.

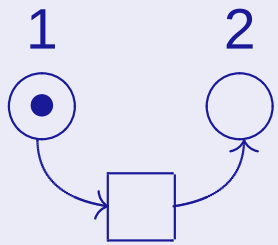


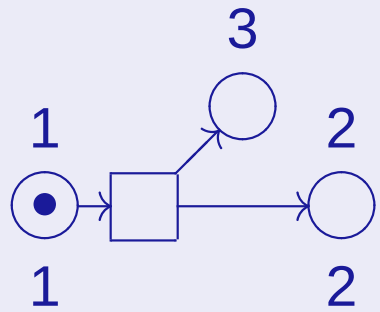
# How the slice pushout is arrived at

$c =$ 


$d =$ 


$a =$ 


$l =$ 

# A correspondence theorem

LTS $T_{\pm}$:

- Nodes: Petri terms modulo isomorphism;
- Transitions: of three types
  - $p \xrightarrow{+i} p'$ if $p'$ results from $p$ by adding a token at its $i$th outer place;
  - $p \xrightarrow{-i} p'$ if $p'$ results from $p$ by eating a token from its $i$th outer place;
  - $p \xrightarrow{\tau} p'$ if $p'$ results from $p$ through the firing of one transition.

**Theorem**

*The bisimulation on the derived LTS coincide with that on $T_{\pm}$*

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# Plan of the talk

# Open terms and parametric rules

A collection of ground rules $\overline{a}.P \mid a.Q \longrightarrow P \mid Q$ is not satisfactory:
it gives infinitely many rules, infinitely many higher-order labels.

We would like a calculus like:

$$\textit{terms}: \quad P ::= 0 \mid a.P \mid \overline{a}.P \mid P \mid P, \qquad \textit{rule}: \quad a.1 \mid \overline{a}.2 \longrightarrow 1 \mid 2$$

where the labels are derivable for open terms via universal properties imposed both on the contexts and the parameters.

$$\langle a.P, 1\rangle \xrightarrow{1\mid\overline{a}.2} P \mid 1 \qquad \text{and} \qquad a.P \mid 1 \xrightarrow[\overline{a}.1]{} P \mid 1,$$

- Labels: smallest context and the most general parameter that make a reduction possible.

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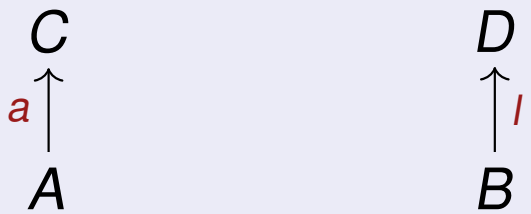
# Locally universal hexagons

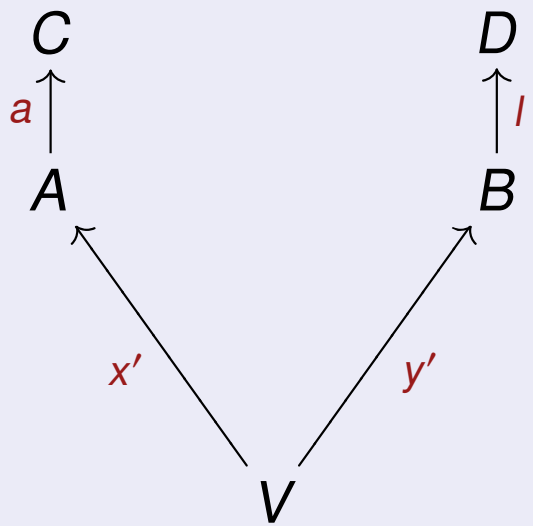
## Luxes




Hmmm!?!?

# Locally universal hexagons

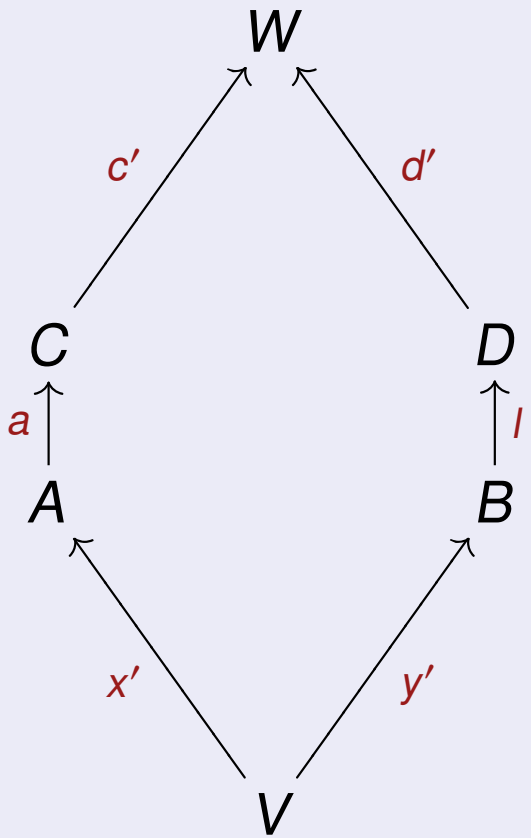
## Luxes




Get span

# Locally universal hexagons

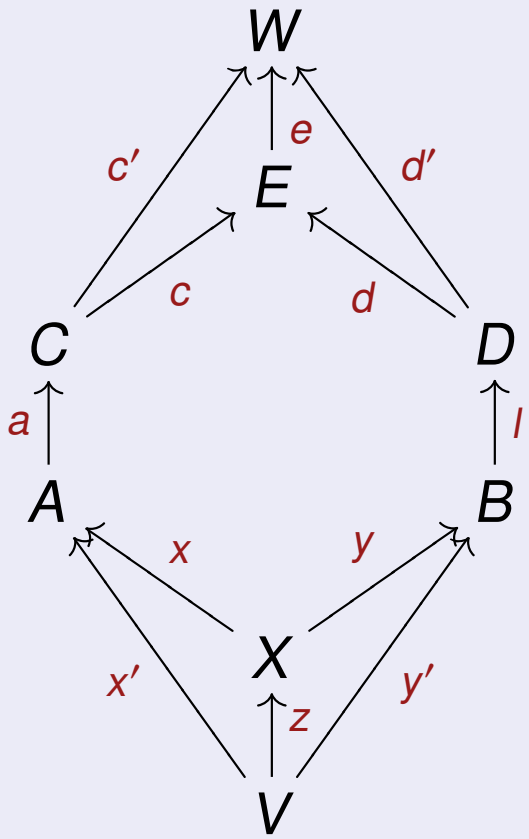
## Luxes




Get a square

# Locally universal hexagons

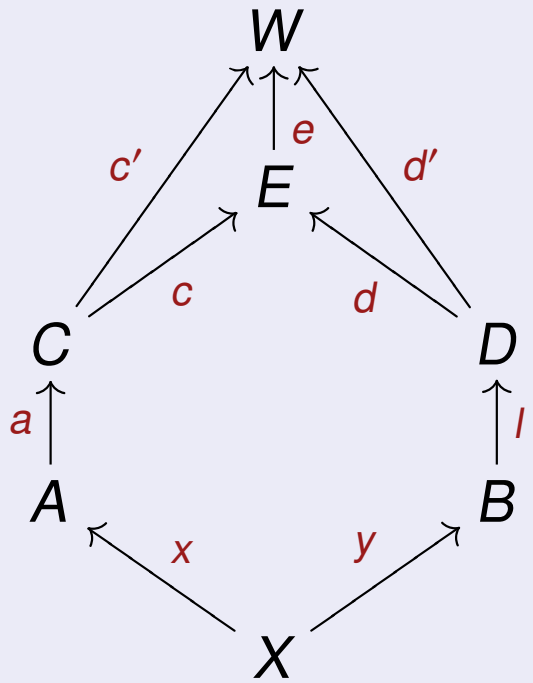
## Luxes




It's a lux!

# Locally universal hexagons

## Luxes




## A Slice Pushout?

# Locally universal hexagons

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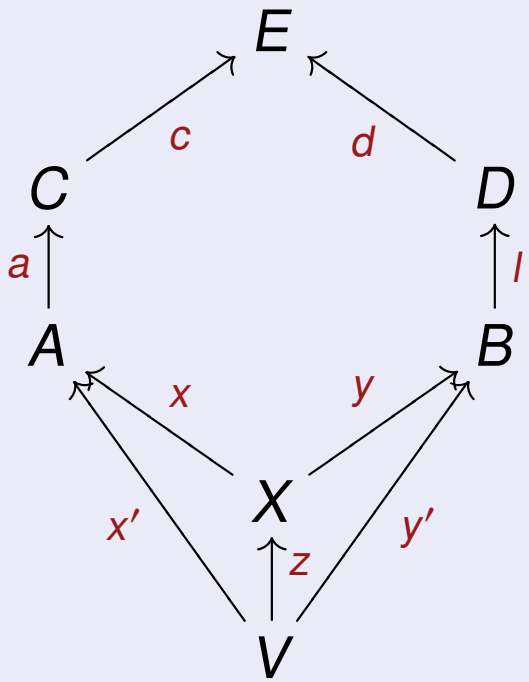
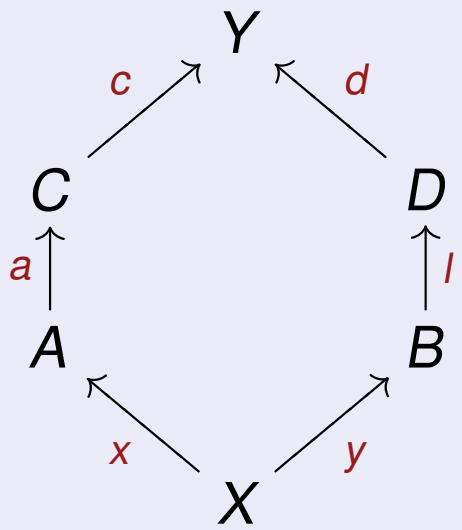



## A Coslice Pullback?

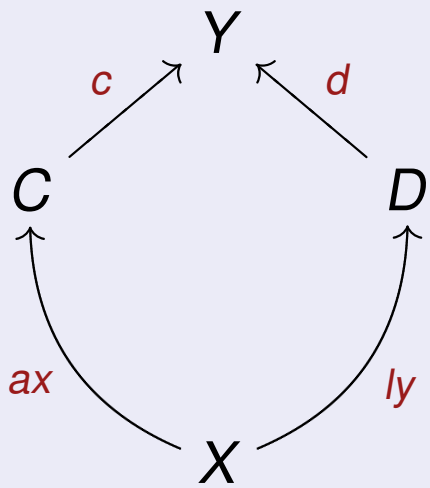
# Characterising categories with luxes

## Theorem

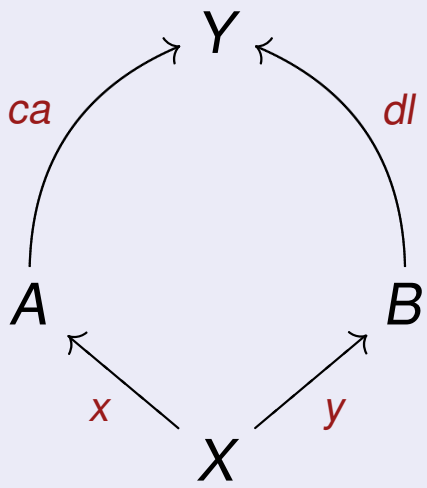
*A commutative diagram (i) is a lux iff diagram (ii) is a SPO and diagram (iii) is an CPB.*




(i)




(ii)




(iii)

# Characterising categories with luxes (ctd)

**Theorem**

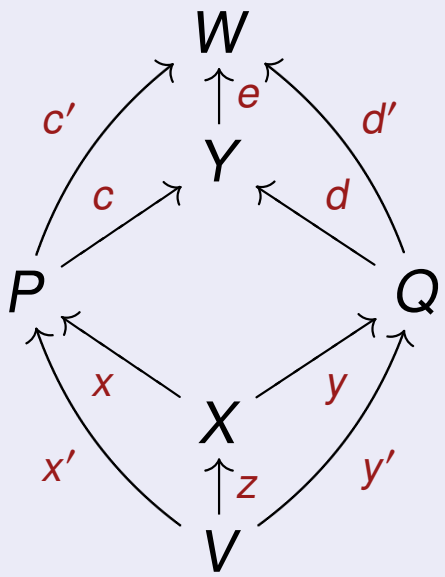
*The following conditions are both sufficient for the existence of luxes in* **C**.

1. **C** *has SPO, CPB and all arrows are mono;*
2. **C** *has SPO, CPB and all arrows are epi.*

# Characterising categories with luxes (ctd)

## Theorem

*A category has luxes if it has SPO and CPB, and these commute*



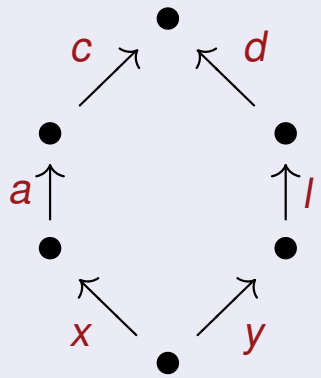

Commutativity of slice pushouts and coslice pullbacks

# Open LTS

## Definition (LTS from an Open Reactive System)

Given an open reactive system $\mathbb{C}$, $LTS(\mathbb{C})$ is:

- Nodes: arbitrary arrows $a : A \to C$ – i.e., terms are possibly open;
- Transitions: $a \xrightarrow[x]{c} b$ whenever there exist $\langle l, r \rangle \in \mathcal{R}$ and a lux




such that $b = dry$.

# The congruence theorem for luxes

**Theorem (Congruence I)**

*Let $\mathbb{C}$ be an open reactive system over* **C**.
*If* **C** *has luxes and all arrows of* **C** *are mono, then $p \sim q$ implies $cp \sim cq$ for all contexts $c$ in* **C**.

**Theorem (Congruence II)**

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# The parametric CCS case

**Theorem**

*All free PROPs have luxes*

Application to the example of parametric CCS nicely gives expected labels

$$\langle a.P, 1\rangle \xrightarrow{1|\overline{a}.2} P \mid 1$$

$$a.P \mid 1 \xrightarrow[\overline{a}.1]{} P \mid 1$$

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But problems too: we have $a.P \xrightarrow{1|\overline{a}.Q} P \mid Q$, where $Q$ is irrelevant.

Even worse, $a.1 \xrightarrow[X]{1|a.P|\overline{a}.Q} a.X \mid P \mid Q$, where the term is not involved in the reduction.

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- But there can still be unwanted overlaps between parameters and contexts.

Refine the notion of lux to rule out overlap between SPO and CPB.




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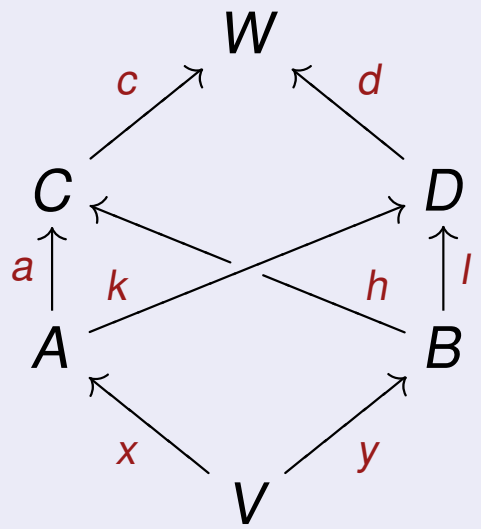


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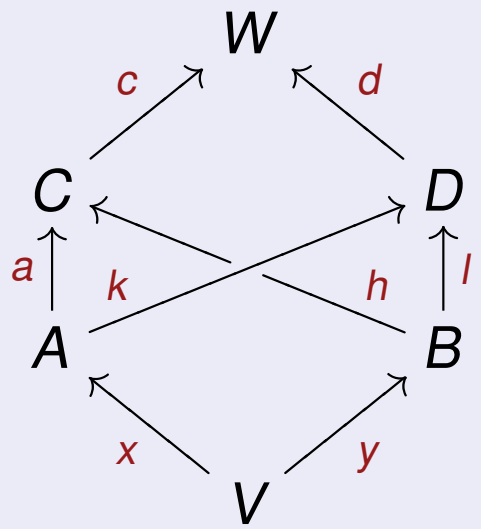


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- Stage 2: understand structural congruence ✓
- Stage 3: test and adapt theory on complex structures ✓
- Stage 4: metatheorems for wide classes of models ✓, open-ended
- Stage 5: parameters and open terms ×, open problem

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