Congruences for Contextual Graph-Rewriting

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Plan of the talk

1. Deriving bisimulation congruences
2. Cospans as generalised contexts
3. Bisimulation for graph rewriting

Deriving Congruences

- Many syntactic formalisms for concurrency and mobility
- Unification efforts:

1. Milner et al ‘90s-now: action calculi, bigraphs
2. Montanari et al ‘90s-now: tile systems.
3. Sewell, Leifer, Milner, Sassone and Sobocinski: meta theory of process calculi

Labels in LTS

- *Slogan:* Labels should be smallest contexts which allow reaction/interaction
  - eg. simple CCS-style calculus $a.P + P' \xrightarrow{-|\bar{a}} P$
  - *Sewell (1998)*: Detailed **syntactic analysis** of simplified process calculi
  - *Leifer and Milner (2000)*: General notion of smallest context - the **relative pushout**.
  - *Sassone and Sobocinski (2002)*: 2-categorical generalisation to allow handling of **structural congruences**.

Reactive Systems

- A reactive system
  - objects = typed *"holes"*
  - arrows = contexts
  - 2-cells = *"structural congruence"*

$$a \longrightarrow\!\!\!\!\triangleright\, b$$

if there exists

$$\langle l, r \rangle,\ d \in \mathbb{D}$$

and

$$\rho : dl \Rightarrow a,\ \rho' : b \Rightarrow dr$$

GRPOs

- Given $\alpha\colon ca \Rightarrow db$

$$\begin{tikzcd}
 & I_4 & \\
I_2 \arrow[ur, "c"] \arrow[rr, phantom, "\alpha"] & & I_3 \arrow[ul, "d"'] \\
 & I_1 \arrow[ul, "a"] \arrow[ur, "b"'] &
\end{tikzcd}$$

$$\langle I_5, e, f, g, \beta, \gamma, \delta \rangle$$

$$\delta b \bullet g\beta \bullet \gamma a = \alpha$$

$$\begin{tikzcd}
 & I_4 & \\
 & \arrow[dl, phantom, "\gamma"] \arrow[dr, phantom, "\delta"] & \\
I_2 \arrow[uur, "c"] \arrow[r, "e"] & I_5 \arrow[uu, "g"] & I_3 \arrow[uul, "d"'] \arrow[l, "f"'] \\
 & \arrow[u, phantom, "\beta"] & \\
 & I_1 \arrow[uul, "a"] \arrow[uur, "b"'] &
\end{tikzcd}$$

LTS

- Nodes: $[a] : 0 \to I_1$
- Labels: $[a] \xrightarrow{[f]} [a']$

$$\exists \langle l, r \rangle \in \mathcal{R} \quad \exists f \in \mathbb{C} \quad \exists d \in \mathbb{D} \quad \exists \alpha : fa \Rightarrow dl \quad \exists \alpha' : dr \Rightarrow a'$$

and
$$\begin{tikzcd}
& I_4 & \\
I_2 \arrow[ur, "f"] \arrow[rr, phantom, "\alpha"] & & I_3 \arrow[ul, "d"'] \\
& 0 \arrow[ul, "a"] \arrow[ur, "l"'] &
\end{tikzcd}$$
is a GRPO wrt itself

Properties of LTS

- *Bisimulation* is a congruence
- *Trace equivalence* is a congruence
- *Failures equivalence* is a congruence

What's the point?

- Why am I telling you all this??

Cospan Bicategories

- Given **C**, **Cospan**(**C**) has
  - Objects: those of **C**
  - Arrows: cospans $I_1 \xrightarrow{f} C \xleftarrow{g} I_2$
  - 2-cells: cospan "homorphisms"
  - Composition by pushout along common interfaces.
  - intuitively: category of contexts over **C**.

Composition

- Identities: $I_1 \xrightarrow{id} I_1 \xleftarrow{id} I_1$
- Composition by pushout

$$\begin{tikzcd}
 & & C +_{I_2} D & & \\
I_1 \arrow[r, "f"] & C \arrow[ur, bend left, "i_1"] & I_2 \arrow[l, "g"'] \arrow[r, "f'"] & D \arrow[ul, bend right, "i_2"'] & I_3 \arrow[l, "g'"']
\end{tikzcd}$$

$$a : (C +_{I_2} D) +_{I_3} E \to C +_{I_2} (D +_{I_3} E)$$

$$e_l : (I_1 +_{I_1} C) \to C$$

$$e_r : (C +_{I_2} I_2) \to C$$

satisfying coherence

Cospans on Graphs

- What is this when **C** is **Graphs**?

Desiderata

- For a suitable, general class of categories **C**, **Cospan**(**C**) has redex-GRPOs.

- Would allow to derive a coinduction principle for each “category of contexts” over a suitable **C**.

Adhesive Categories

- What is an adhesive category?

Adhesive Categories

- A category $\mathbf{C}$ is adhesive when

1. It has pushouts along monos
2. It has pullbacks
3. pushouts along monos are VK squares

Van Kampen Square

- Given a cube with back faces pullbacks:
- top face pushout iff front faces pullbacks

$$
\begin{tikzcd}
 & & C' \arrow[dll, "m'"'] \arrow[drr, "f'"] \arrow[ddd, "c"] & & \\
A' \arrow[dr, "g'"'] \arrow[ddd, "a"'] & & & & B' \arrow[dlll, "n'"] \arrow[ddd, "b"] \\
 & D' \arrow[ddd, "d"] & & & \\
 & & C \arrow[dll, "m"'] \arrow[drr, "f"] & & \\
A \arrow[dr, "g"'] & & & & B \arrow[dlll, "n"] \\
 & D & & &
\end{tikzcd}
$$

Graphs is Adhesive

- You didn't expect otherwise, did you??

Left-Linear Cospans

- When **C** is adhesive **LLC**(**C**) is the bicategory
  - objects as in **C**
  - arrows cospans $I_1 \stackrel{m}{\rightarrowtail} C \xleftarrow{g} I_2$

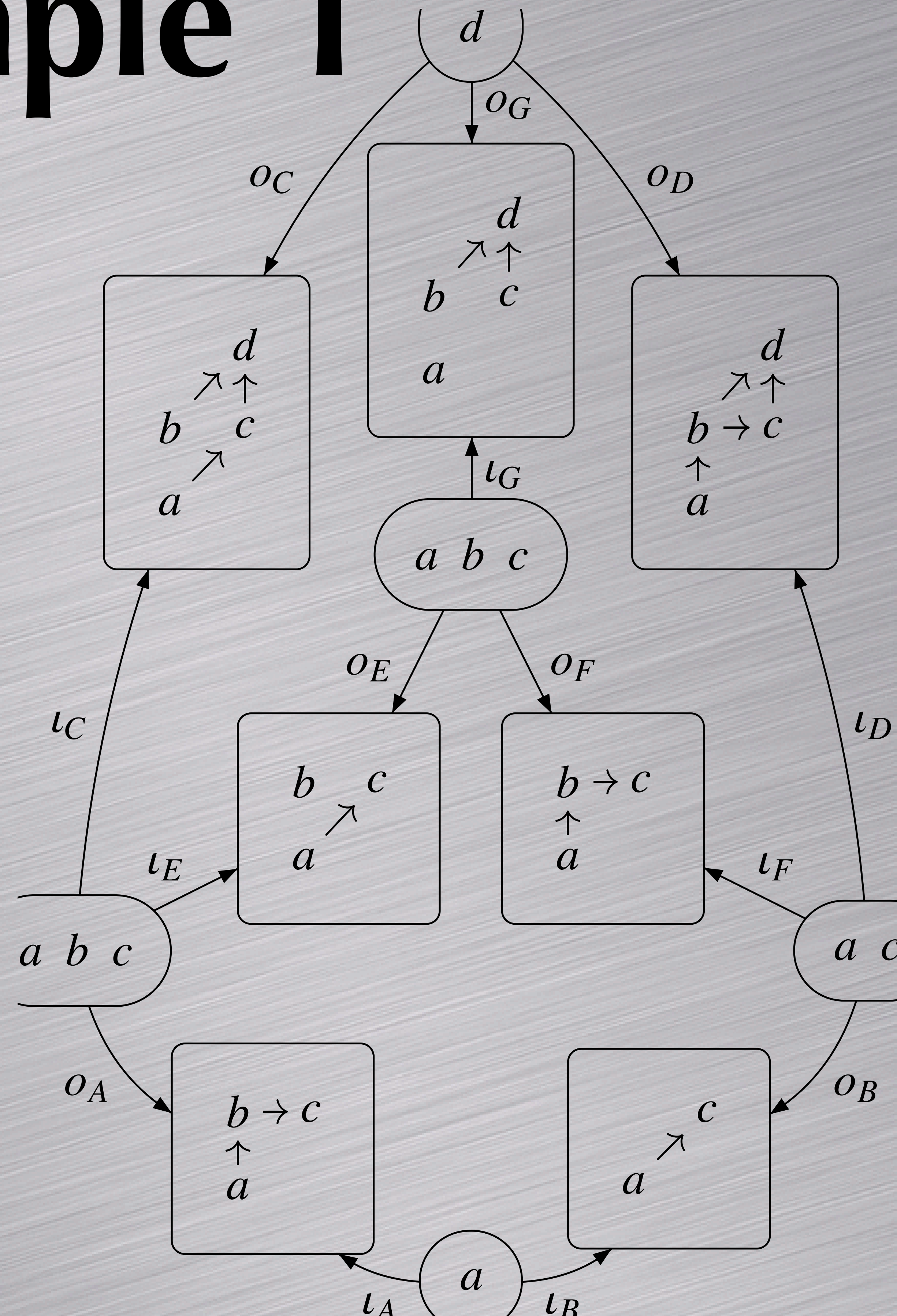
GRPOs for cospans

- ***Theorem***: Suppose that **C** is an adhesive category.
- Then, **LLC**(**C**) has redex-GRPOs.

Example 1

- All morphisms mono

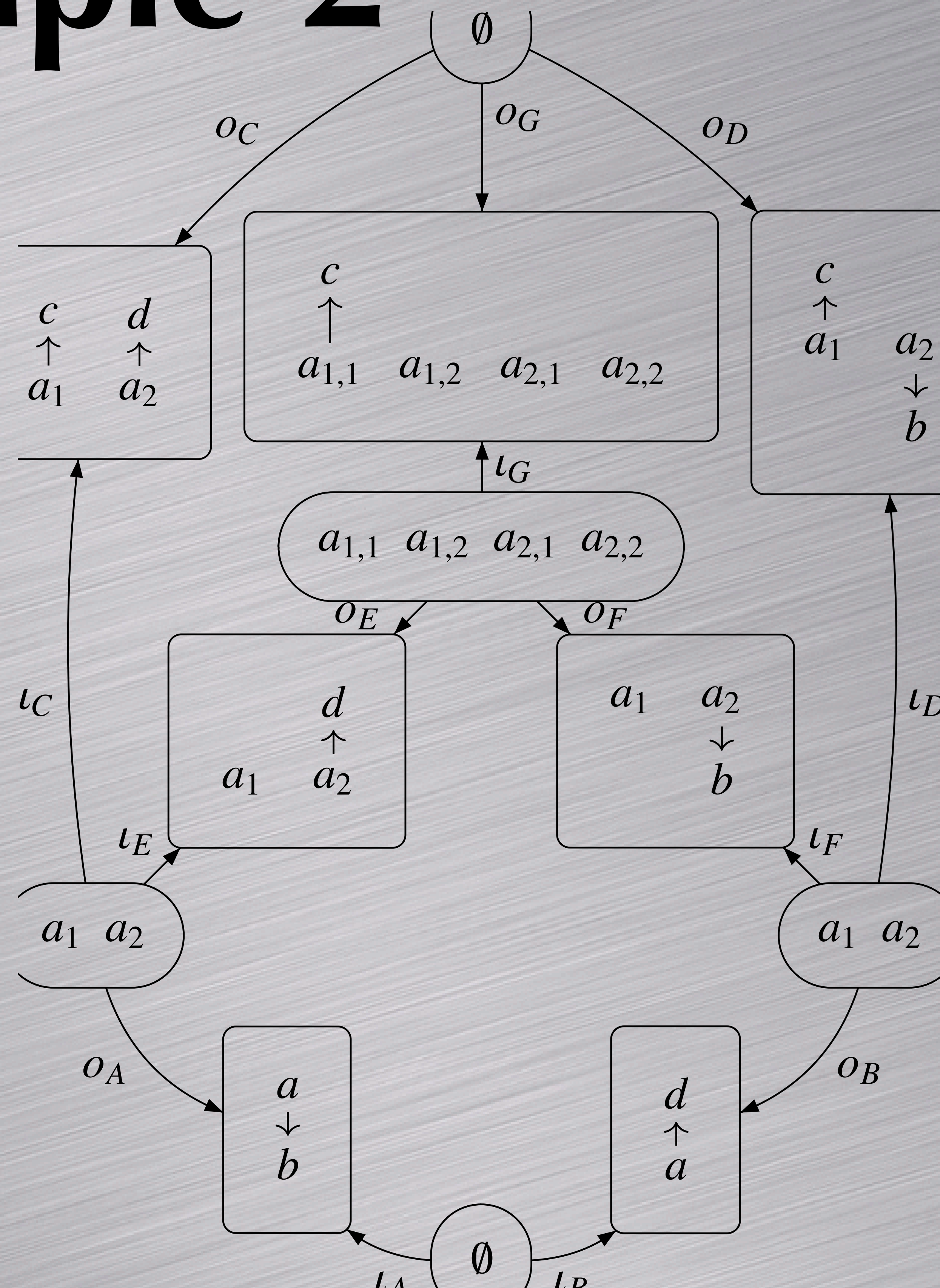
$$\begin{tikzcd}
 & d \\
b \arrow[r] \arrow[ur] & c \arrow[u] \\
a \arrow[u] \arrow[ur] &
\end{tikzcd}$$



Example 2

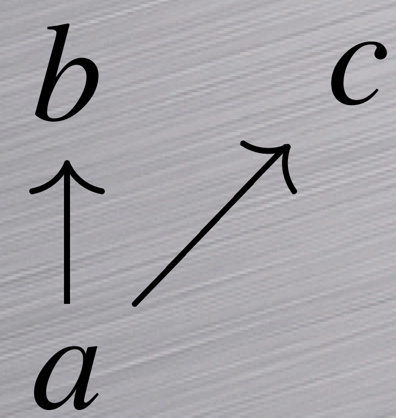
- o_A and o_L not mono

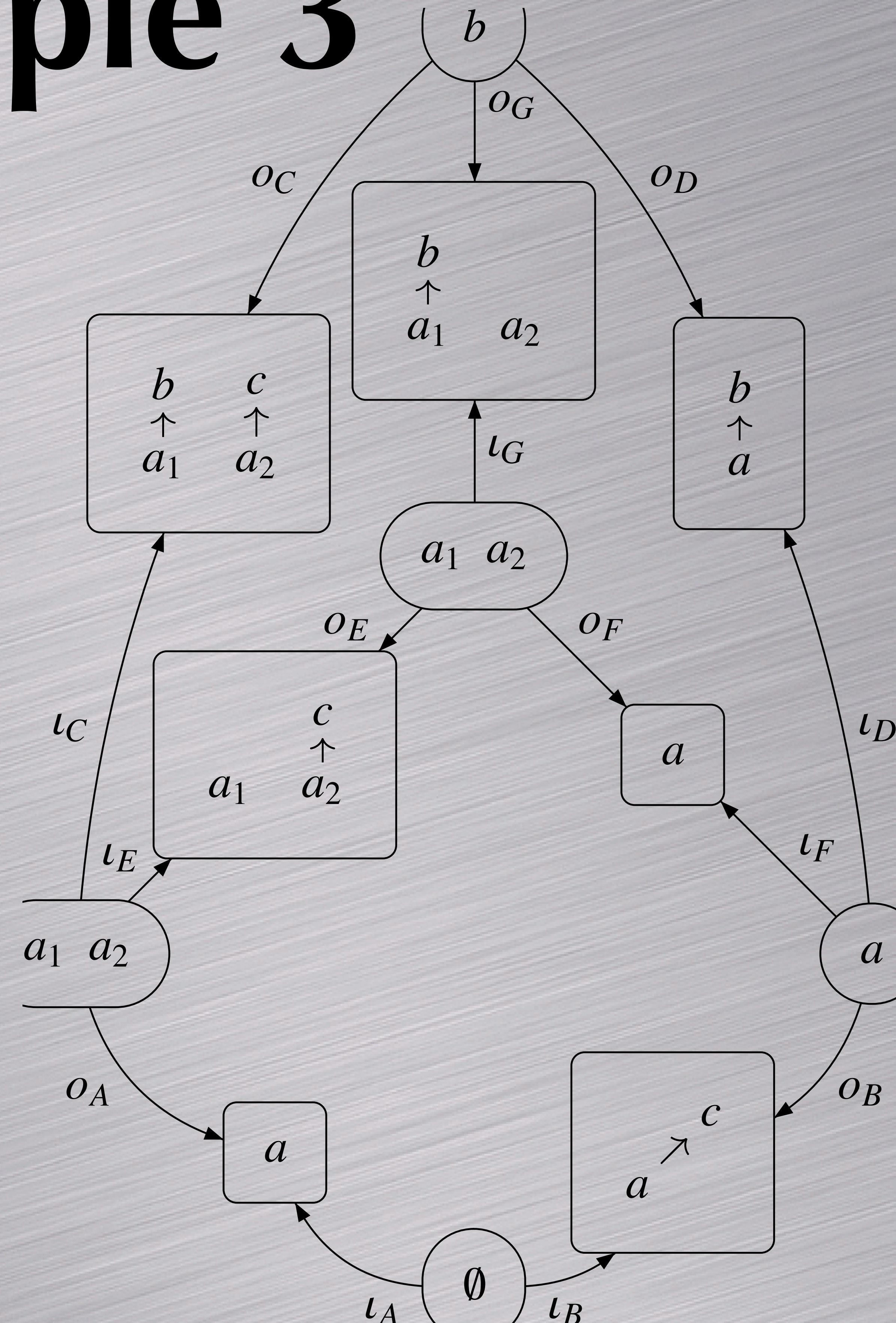
$$
\begin{tikzcd}
c & d \\
& a \arrow[ul] \arrow[u] \arrow[d] \\
& b
\end{tikzcd}
$$



Example 3

- o_A not mono





GRPOs in LLC(C)

- Given redex square...

$$
\begin{tikzcd}
& I_4 & \\
I_2 \arrow[ur, "C"] \arrow[rr, phantom, "\alpha"] & & I_3 \arrow[ul, "D"'] \\
& I_1 \arrow[ul, "A"] \arrow[ur, "B"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
& C \arrow[d, "i_2"] & I_4 \arrow[l, "o_C"'] \arrow[r, "o_D"] & D \arrow[d, "i_2"'] & \\
I_2 \arrow[ur, "\iota_C"] \arrow[dr, "o_A"'] & A +_{I_2} C \arrow[rr, squiggly, "\alpha"] & & B +_{I_3} D & I_3 \arrow[ul, "\iota_D"'] \arrow[dl, "o_B"] \\
& A \arrow[u, "i_1"'] & I_1 \arrow[l, "\iota_A"] \arrow[r, "\iota_B"'] & B \arrow[u, "i_1"] &
\end{tikzcd}
$$

GRPOs of Cospans

- ... find minimal factorisation

$$\begin{tikzcd}
 & & I_4 & & \\
 & \gamma & & \delta & \\
I_2 \arrow[uurr, "C"] \arrow[rr, "E"] & & I_5 \arrow[uu, "G"] & & I_3 \arrow[uull, "D"'] \arrow[ll, "F"'] \\
 & & \beta & & \\
 & & I_1 \arrow[uull, "A"] \arrow[uurr, "B"'] & &
\end{tikzcd}$$

$$\begin{tikzcd}
 & & I_4 \arrow[dl, bend right, "o_C"'] \arrow[d, "o_G"] \arrow[dr, bend left, "o_D"] & & \\
 & C & G \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & D & \\
I_2 \arrow[ur, bend left, "\iota_C"] \arrow[r, "\iota_E"'] & E \arrow[u, "\rho_1"] & I_5 \arrow[l, "o_E"] \arrow[u, "\iota_G"'] \arrow[r, "o_F"'] & F \arrow[u, "\rho_2"'] & I_3 \arrow[l, "\iota_F"] \arrow[ul, bend right, "\iota_D"']
\end{tikzcd}$$

Construction

$$
\begin{tikzcd}
 & C \arrow[dr, "i_2"] & I_4 \arrow[l, "o_C"'] \arrow[r, "o_D"] & D \arrow[dl, "i_2"'] & \\
I_2 \arrow[ur, tail, "\iota_C"] \arrow[dr, "o_A"'] & & X & & I_3 \arrow[ul, tail, "\iota_D"'] \arrow[dl, "o_B"] \\
 & A \arrow[ur, tail, "i_1"] & I_1 \arrow[l, tail, "\iota_A"] \arrow[r, tail, "\iota_B"'] & B \arrow[ul, tail, "i_1"'] &
\end{tikzcd}
$$

$$Y = A \cup_X B$$

$$
\begin{tikzcd}
 & & I_5 \arrow[dll, "o_E"'] \arrow[ddd, "o_F"] \arrow[dr, tail, "\iota_G"] & \\
E \arrow[dr, tail, "\rho_1"] \arrow[ddd, "\sigma_1"'] & & & G \arrow[dll, "\pi_1"] \arrow[ddd, "\pi_2"] \\
 & C \arrow[ddd, "i_2"] & & \\
 & & F \arrow[dll, "\sigma_2"'] \arrow[dr, tail, "\rho_2"] & \\
Y \arrow[dr, tail, "\mu"'] & & & D \arrow[dll, "i_2"] \\
 & X & &
\end{tikzcd}
$$

Graph Rewriting as Reactive System

- For every span $L \xleftarrow{l} K \xrightarrow{r} R$
  let $\left\langle 0 \to L \xleftarrow{l} K,\ 0 \to R \xleftarrow{r} K \right\rangle \in \mathcal{R}$
- ***Lemma***:

  $\longrightarrow$ double-pushout rewrite

  $\longrightarrow\!\!\!\triangleright$ reaction relation in reactive system

$$C \longrightarrow D \qquad \text{iff} \qquad C^0_0 \longrightarrow\!\!\!\triangleright D^0_0$$

LTS for graph rewriting

- The resulting LTS has:
  - **Nodes**: graphs (up-to-iso) with output interface (possibly non-mono)
  - **Labels**: smallest graph contexts (up-to-iso) which allow reaction
- ***Theorem***: Bisimulation, trace equivalence, failures equivalence are congruences

Advantages of LTS

- Transfer of concepts from process algebra to graph rewriting
- Labelled, compositional semantics
  - the class of adhesive categories covers many categories with "*graph-like*" objects

And what's this for?

- What's missing here??

Special Cases

- Rewriting with borrowed contexts [*Ehrig and Koenig (2004)*]
  - LTS for graph rewriting, up-to-iso not taken into account, all interfaces mono
  - ***Theorem****:* when restricting our approach to linear cospans we derive *the same* LTS
  - ***Corollary****:* their congruence theorem
- Bigraphs...

The case of Bigraphs

- **Bigraphs** can be seen as **LLC**(**dpl-grph**).
- It follows from the theorem that **Bigraphs** has **GRPOs**.
- Main difference with Milner's original bigraphs: input-lineary and name aliasing.
- The case of **Trigraphs** ... as above
- ...

Conclusion

- Construction of labels for an interesting class of reactive systems
- Two applications so far, more in the future?

Minimality

$$\begin{tikzcd}
 & I_4 & \\
I_2 \arrow[ur, "c"] \arrow[r, "e"] & I_5 \arrow[u, "g"] \arrow[ul, phantom, "\gamma"] \arrow[ur, phantom, "\delta"] & I_3 \arrow[ul, "d"'] \arrow[l, "f"'] \\
 & I_1 \arrow[ul, "a"] \arrow[ur, "b"'] \arrow[u, phantom, "\beta"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & I_4 & \\
I_2 \arrow[ur, "c"] \arrow[r, "e'"] & I_6 \arrow[u, "g'"] \arrow[ul, phantom, "\gamma'"] \arrow[ur, phantom, "\delta'"] & I_3 \arrow[ul, "d"'] \arrow[l, "f'"'] \\
 & I_1 \arrow[ul, "a"] \arrow[ur, "b"'] \arrow[u, phantom, "\beta'"] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & I_6 & \\
I_2 \arrow[ur, "e'"] \arrow[r, "e"'] & I_5 \arrow[u, "h"] \arrow[ul, phantom, "\varphi"] \arrow[ur, phantom, "\psi"] & I_3 \arrow[ul, "f'"'] \arrow[l, "f"]
\end{tikzcd}
\qquad
\varphi\colon e' \Rightarrow he \qquad \psi\colon hf \Rightarrow f'
\qquad
\begin{tikzcd}
I_4 & \\
I_6 \arrow[u, "g'"] & I_5 \arrow[ul, "g"'] \arrow[l, "h"] \arrow[ul, phantom, "\tau"]
\end{tikzcd}
\qquad
\tau\colon g'h \Rightarrow g$$

$$\tau e \bullet g'\varphi \bullet \gamma' = \gamma$$

$$\delta' \bullet g'\psi \bullet \tau^{-1} f = \delta$$

$$\psi b \bullet h\beta \bullet \varphi a = \beta'$$

Essential Uniqueness

$$
\begin{tikzcd}
 & I_6 & \\
I_2 \arrow[ur, "e'"] \arrow[r, "e"'] \arrow[ur, phantom, "\varphi" near end, shift left=6] & I_5 \arrow[u, "h"] & I_3 \arrow[ul, "f'"'] \arrow[l, "f"] \arrow[ul, phantom, "\psi" near end, shift right=6]
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
I_4 & \\
I_6 \arrow[u, "g'"] & I_5 \arrow[ul, "g"'] \arrow[l, "h"] \arrow[ul, phantom, "\tau"]
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & I_6 & \\
I_2 \arrow[ur, "e'"] \arrow[r, "e"'] \arrow[ur, phantom, "\varphi'" near end, shift left=6] & I_5 \arrow[u, "h'"] & I_3 \arrow[ul, "f'"'] \arrow[l, "f"] \arrow[ul, phantom, "\psi'" near end, shift right=6]
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
I_4 & \\
I_6 \arrow[u, "g'"] & I_5 \arrow[ul, "g"'] \arrow[l, "h'"] \arrow[ul, phantom, "\tau'"]
\end{tikzcd}
$$

$$\exists!\, \xi : h \to h'$$

$$\xi e \bullet \varphi = \varphi'$$

$$\psi \bullet \xi^{-1} f = \psi'$$

$$\tau' \bullet g' \xi = \tau$$