Mellin Transform Based Performance Analysis of Fast Frequency Hopping Using Product Combining

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Abstract— **In this contribution, we analyze the bit error rate (BER) performance of fast frequency hopping (FFH) assisted M-ary frequency shift keying (MFSK) using product combining. Product combining constitutes an efficient yet low-complexity scheme that may be employed in FFH-MFSK receiver to combat the detrimental effects of interference or jamming. We propose a novel approach to the analysis of this receiver system, which is based on the Mellin transform. Using this approach, the probability density function (PDF) of the product combiner output is expressed in a closed form. Based on the resultant PDF, the BER of the FFH-MFSK product combining receiver operating in Rayleigh fading channel is evaluated analytically. It is shown that the Mellin transform simplifies the analysis of the product combining receiver.**

I. INTRODUCTION

The fast frequency hopping (FFH) product combining (PC) receiver has been shown to combat both partial band noise jamming (PBNJ) and multitone jamming (MTJ) efficiently [1]–[4] In [1]–[4], the authors have analyzed the product combining based FFH binary frequency shift keying (BFSK) receiver under various fading and jamming conditions, employing the characteristic function [5] and using natural logarithm to convert the product into summation, for the sake of deriving the probability density function (PDF) of the product combiner output. The problem associated with the characteristic function based analysis is that closed form expressions for PDF of the product combiner output cannot be readily obtained. Consequently, the symbol error probability is expressed using a double integral when BFSK is considered. For M-ary FSK (MFSK), the corresponding expression is expected to be more complicated, involving multiple integrals. In [6], the employment of Fox's H-functions was proposed to derive the PDF of the diversity combiner output and the BER of FFH-BFSK PC receiver. In their analysis, the authors of [6] have exploited the fact that a product of H-functions is also an H-function [7]. The authors [6] have also employed another technique which consists of generalized F-variates for the sake of deriving the corresponding BER expressions. However, this later method is more computationally demanding. The methods proposed in [6] have been shown to work well for BFSK system, when Rayleigh fading channels are considered. However, they might lead to more computationally cumbersome formulae when applied to M-ary systems. Moreover, as reported by the authors of [6], the H-function may not be readily invoked for the sake of deriving the PDF of the product combiner output, when more generalized forms of fading such as Rician and Nakagami model are considered.

In this paper, we employ the Mellin transform [7], [8] to analyze PC aided FFH-MFSK. The Mellin transform is an integral transform, similar to Laplace and Fourier transform. The Mellin transform of a random variable is related to its PDF and the Mellin transform of a product of random variables is the product of the Mellin transforms of the individual random variables [7], [8]. This fact allows us to derive the PDF of the product combiner output which can in turn be used to determine an expression characterising the bit error rate (BER) of the system. It will be shown that the proposed Mellin transform based technique substantially simplifies the BER analysis of the PC aided FFH-MFSK system and hence facilitates for the first time the analysis of M-ary FSK based FFH PC assisted receiver.

The remainder of this paper is structured as follows. In Section II, the system under consideration is briefly described. In Section III, the proposed Mellin transform based technique is discussed and both the relevant statistics as well as the corresponding BER expression are derived. In Section IV our numerical results are discussed and compared with simulation results. Finally, in Section V, our conclusions are presented.

II. SYSTEM DESCRIPTION

The system under consideration is similar to that considered in [2] and [6], except that we consider an M-ary FSK system, where we have $M \geq 2$. In the FFH-MFSK transmitter the MFSK signal modulates a carrier generated by a frequency synthesizer, which is controlled by an L-tuple address output by a pseudo-noise (PN) generator in order to implement frequency hopping, where L is the number of frequency hops per symbol. Hence, the FFH frequency is changed L times within each symbol duration. Thus, the hop or chip interval T_h is related to the symbol interval T_s by the relation $T_h = T_s/L$. Correspondingly, the bandwidth occupied by the signal transmitted during each FFH chip interval is approximated by that

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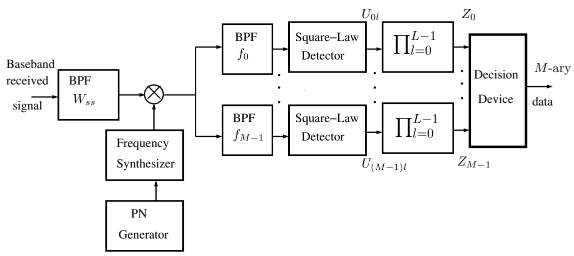

Fig. 1. Receiver schematic of the FFH-MFSK system using PC

of its main spectral lobe occupying $R_h = 1/T_h$. The frequency separation between adjacent frequency tones is assumed to be R_h. Thus, the orthogonality of the FFH frequencies is maintained. The channel is modeled as flat Rayleigh fading channel for each of the transmitted frequencies. We assume that the frequency separation between adjacent signalling frequencies is higher than the coherence bandwidth of the channel. Therefore, all FH tones conveying the same symbol experience independent fading. The transmitted signal is also corrupted by additive white Gaussian noise (AWGN) of one-side power spectral density N_0.

The block diagram of the FFH M-ary FSK receiver is shown in Fig. 1 which is constituted by a bandpass filter, a frequency de-hopper, M branches of the noncoherent MFSK demodulator and finally a decision device. The bandpass filter (BPF) of Fig. 1 removes any frequencies that fall outside the spread spectrum bandwidth W_{ss}. The de-hopper consists of a PN generator, which is identical to and aligned with the PN generator of the transmitter, as well as a frequency synthesizer and a multiplier. The de-hopper de-spreads the recieved signal by exploiting the knowledge of the transmitter's unique FFH address. Each of the M demodulator branches corresponds to an MFSK tone and consists of a BPF, a square-law detector and a product combiner, as shown in Fig. 1.

III. BER Analysis

We assume without loss of generality that the first of the M tones is activated by the transmitter. It can be shown that the PDF of the detector output corresponding to the signal tone normalized by the noise variance is given by [5], [6]

$$f_{U_{0l}}(x) = \frac{1}{1+\gamma_h} e^{\frac{-x}{1+\gamma_h}} = ae^{-ax}, \qquad x \geq 0, \tag{1}$$

where $\gamma_h = \gamma_c/L$ is the signal to noise power ratio (SNR) per hop, γ_c is the SNR per symbol and $a = 1/(1+\gamma_h)$. Similarly, corresponding to all the undesired tones $m = 1, 2, \ldots, M-1$, the PDF of the the square-law detector output, which is normalized by the noise variance, is given by

$$f_{U_{ml}}(x) = e^{-x}, \qquad x \geq 0, m > 1. \tag{2}$$

A. Mellin Transform

The Mellin transform of a function $f(x)$ is defined as [9]

$$M[f(x), z] = \int_0^\infty x^{z-1} f(x) dx. \tag{3}$$

In terms of probability theory, the Mellin transform of a random variable is defined based on its PDF. Specifically, the Mellin transforms of the PDFs given by (1) and (2) characterising the variables U_0 and $U_m, m = 1, 2, \ldots, M-1$ respectively, may be expressed as

$$M[f_{U_{0l}}(x), z] = \int_0^\infty x^{z-1} f_{U_{0l}}(x) dx = a^{1-z}\Gamma(z) \tag{4}$$

and

$$M[f_{U_{ml}}(y), z] = \Gamma(z), \quad m = 1, 2, \ldots, M-1, \tag{5}$$

where the results of (4) and (5) have been obtained using the transform tables [10].

Now, according to the properties of the Mellin transform, the transform of the product of random variables is equal to the product of the Mellin transforms of the individual random variables [7], [8]. Thus, if $Z_m = \prod_{l=0}^{L-1} U_{ml}$, $m = 0, 1, \ldots, M-1$, represents the output of the mth product combiner shown in Fig. 1, we have

$$M[f_{Z_m}(y), z] = \prod_{l=0}^{L-1} M[f_{U_{ml}}(x), z], \quad m = 0, 1, \ldots, M-1. \tag{6}$$

Since the faded random variables in all hops are independent and identically distributed, from (6) we have

$$M[f_{Z_m}(y), z] = \Big(M[f_{U_{ml}}(x), z]\Big)^L, \quad m = 0, 1, \ldots, M-1. \tag{7}$$

Hence, with the aid of (4), for the signal tone we have

$$M[f_{Z_0}(y), z] = \Big(M[f_{U_{0l}}(x), z]\Big)^L = a^{L(1-z)}\Gamma^L(z), \tag{8}$$

while, from (5) for $m = 1, 2, \ldots, M-1$, we have

$$M[f_{Z_m}(y), z] = \Big(M[f_{U_{ml}}(x), z]\Big)^L = \Gamma^L(z). \tag{9}$$

B. Inverse Mellin Transform

The PDF of $Z_m, m = 0, 1, \ldots, M-1$ can be obtained as the inverse Mellin transform of the expressions given in (8) and (9). The inverse Mellin transform is defined as [7]

$$f(x) = \int_{c-i\infty}^{c+i\infty} x^{-z} M[f(x), z] dz, \tag{10}$$

where $i = \sqrt{-1}$ and the integration is along any path $Re(z) = c$, such that $M[f(x), z]$ exists and is an analytic function of the complex variable z for $c_1 \leq Re(z) \leq c_2$ and c lies between the two real points c_1 and c_2. Thus, from (8) for the signal tone

$$f_{Z_0}(y) = \int_{c-i\infty}^{c+i\infty} y^{-z} a^{L(1-z)} \Gamma^L(z) dz. \tag{11}$$

From the Residue Theorem [7], [9], [11], we know that the complex integral at the right-hand side of the above equation can be computed by summing the residues of the integrand associated with all its poles. Thus, we have

$$f_{Z_0}(y) = \sum_j Res\Big[a^{L(1-z)} y^{-z} \Gamma^L(z)\Big]_{(z=-j)}, \tag{12}$$

where $Res[.]_j$ represents the residue at the jth pole of the integrand and the summation is carried out over all possible values of j. The PDF of Z_0 can be determined numerically from (12) by using symbolic mathematics based softwares such as Maple or Mathematica, employing the appropriate function for finding residues of an integrable expression. However, it is insightful to derive analytical expressions for (12). It can be shown with the aid of [11], [12] that the function $\Gamma(z)$ has an infinite number of poles at $z = -j$ for $j = 1, 2, \ldots$. The residue of $\Gamma(z)$ at $z = -j$ is given by [11], [12]

$$\Gamma(z)(z+j)|_{(z=-j)} = \frac{(-1)^j}{j!}. \tag{13}$$

Now, $\Gamma^L(z)$ in (12) has an Lth-order pole at each integer value of $z = -j$. Consequently, using the corresponding relationship characterising the residues of multiple poles [7], [11], [12] and the Leibnitz' rule [12] for higher order derivatives of a product of functions, the PDF of Z_0 may be expressed as

$$\begin{aligned} f_{Z_0}(y) &= \frac{a^L}{(L-1)!}\sum_{j=0}^{\infty}\sum_{r=0}^{L-1}\binom{L-1}{r} \\ &\times \Big[\mathcal{U}^{(r)}(z)\mathcal{V}^{(L-1-r)}(z)\Big]_{(z=-j)}, \end{aligned} \tag{14}$$

where $\mathcal{U}(z) = (a^L y)^{-z}$ and $\mathcal{V}(z) = \Gamma^L(z)(z+j)^L$, while $\mathcal{U}^{(r)}(z)$ and $\mathcal{V}^{(r)}(z)$ denote the rth derivatives of $\mathcal{U}(z)$ and $\mathcal{V}(z)$, respectively.
The rth derivative of $\mathcal{U}$, when evaluated at $z = -j$, can be readily expressed as

$$\mathcal{U}^{(r)}(z)|_{(z=-j)} = [-\ln(a^L y)]^r (a^L y)^j. \tag{15}$$

In order to derive an expression for the rth derivative of $\mathcal{V}(z)$, we first express $\mathcal{V}(z)$ as [6]

$$\mathcal{V}(z) = \Gamma^L(z)(z+j)^L = \frac{\Gamma^L(z+j+1)}{\prod_{k=1}^{j}(z+j-k)^L}. \tag{16}$$

When evaluated at $z = -j$, $\mathcal{V}(z)$ can be expressed as

$$\mathcal{V}(z)|_{(z=-j)} = \frac{(-1)^{jL}}{(j!)^L}. \tag{17}$$

Next, we note that we have [6]

$$\frac{d\mathcal{V}(z)}{dz} = \mathcal{V}(z)\frac{d[\ln \mathcal{V}(z)]}{dz}. \tag{18}$$

Moreover, $d[\ln \Gamma(z)]/dz = \psi(z)$ holds, where $\psi(.)$ is the Psi function [12]. Thus, it can be shown that

$$\mathcal{V}^{(1)}(z)|_{(z=-j)} = L\frac{(-1)^{jL}}{(j!)^L}\Big[\psi(1) - \sum_{k=1}^{j}\frac{1}{(-k)}\Big], \tag{19}$$

where $\psi(1) \approx -0.5772156649$, which equals the negative of the Euler's constant [12]. The higher order derivatives of $\mathcal{V}(z)$ can be derived from (19). Thus, we have

$$\mathcal{V}^{(r)}(z) = \frac{d^{r-1}}{dz^{r-1}}[\mathcal{V}^{(1)}(z)]. \tag{20}$$

Using Leibnitz' rule [12] in the context of (20), we arrive at

$$\begin{aligned} \mathcal{V}^{(r)}(z)|_{(z=-j)} &= L\Bigg\{\sum_{t=0}^{r-1}\binom{r-1}{t}\mathcal{V}^{(t)}(z)|_{(z=-j)}(-1)^{r-t} \\ &\times \quad (r-1-t)!\sum_{k=0}^{\infty}\Big[\frac{1}{(1+k)^{r-t}}\Big] \\ &+ \quad \sum_{t=0}^{r-1}\binom{r-1}{t}\mathcal{V}^{(t)}(z)|_{(z=-j)}\sum_{k=1}^{j}\frac{1}{(k)^{r-t}}\Bigg\}, \end{aligned} \tag{21}$$

where we have used the rth derivative of $\psi(x)$ given by [12]

$$\psi^{(r)}(x) = (-1)^{r+1} r! \sum_{k=0}^{\infty}\Big[\frac{1}{(x+k)^{r+1}}\Big]. \tag{22}$$

We note that the expression given by (16) does not hold at $z = j = 0$. Thus, for $z = j = 0$ we denote $\mathcal{V}(z)$ by $\mathcal{V}_0(z) = z^L\Gamma^L(z) = \Gamma^L(z+1)$. Then using the method employed to derive (21), it can be shown that

$$\begin{aligned} \mathcal{V}_0^{(r)}(z)|_{z=0} &= L\sum_{t=0}^{r-1}\binom{r-1}{t}\mathcal{V}_0^{(t)}(z)|_{(z=0)}(-1)^{r-t} \\ &\times \quad (r-1-t)!\sum_{k=0}^{\infty}\Big[\frac{1}{(1+k)^{r-t}}\Big]. \end{aligned} \tag{23}$$

Having derived the expressions for all the derivatives of $\mathcal{V}(z)$ given by (21) for $j > 0$ and by (23) for $j = 0$, as well as those of $\mathcal{U}(z)$ given by (15), the PDF of Z_0 can be determined from (14). The PDF of Z_m, $m > 0$ can also be determined from (14), in conjunction with $a = 1$ in (15). Finally, the probability of symbol error can be evaluated as [5]

$$P_s = 1 - \int_0^{\infty} f_{Z_0}(y_0)\Big[\int_0^{y_0} f_{Z_m}(y_m)dy_m\Big]^{M-1} dy_0. \tag{24}$$

Correspondingly, the BER can be determined using the relation [5] $P_b = \frac{M/2}{M-1}P_s$. Alternatively, we can express the probability of symbol error as [5]

$$P_s = 1 - \int_0^{\infty} f_{Z_0}(y_0)\Big[F_{Z_m}(y_0)\Big]^{M-1} dy_0, \tag{25}$$

where $F_{Z_m}(y_0)$ is the cumulative distribution function (CDF) of $Z_m, m > 0$, which is given by

$$F_{Z_m}(y_0) = \int_0^{y_0} f_{Z_m}(y_m)dy_m. \tag{26}$$

By using the Mellin transform of the integral of a function [9], [10], we have

$$M[F_{Z_m}(y_0), z] = -\frac{1}{z}\Gamma^L(z+1). \tag{27}$$

Upon taking the inverse Mellin transform according to (10), the CDF of $Z_m, m > 0$ may be expressed employing the technique used to derive the expression for the PDF of Z_0 given by (14). The corresponding expression has not been included here owing to lack of space. Therefore, P_s can also be computed using (25), which involves only one integration rather than two integrations, as seen in (24).

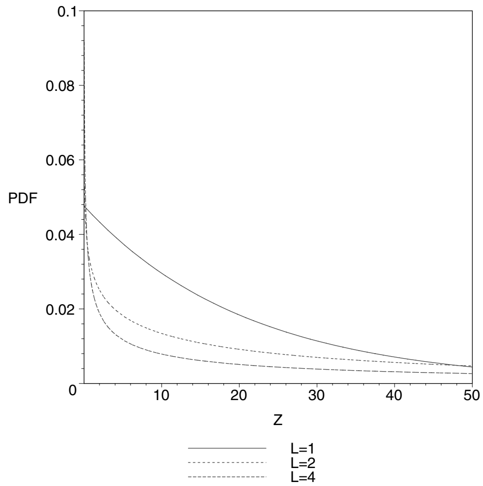

Fig. 2. The PDF of the product combiner output corresponding to the desired signal tone for FFH 4-ary FSK communicating in Rayleigh fading channel, assuming $E_b/N_0 = 10dB$ and various values of L.

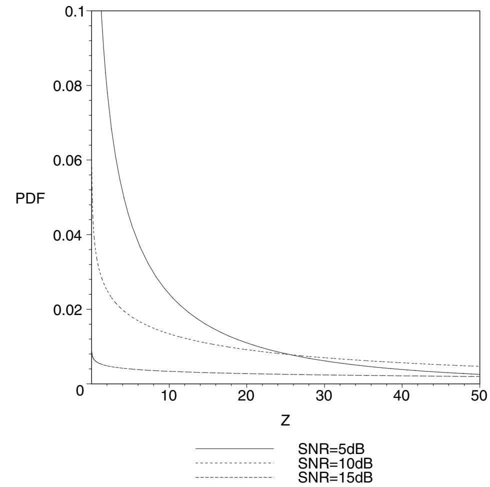

Fig. 3. The PDF of the product combiner output corresponding to the desired signal tone, for FFH 4-ary FSK communicating over a Rayleigh fading channel, assuming $L = 2$ and various values of E_b/N_0.

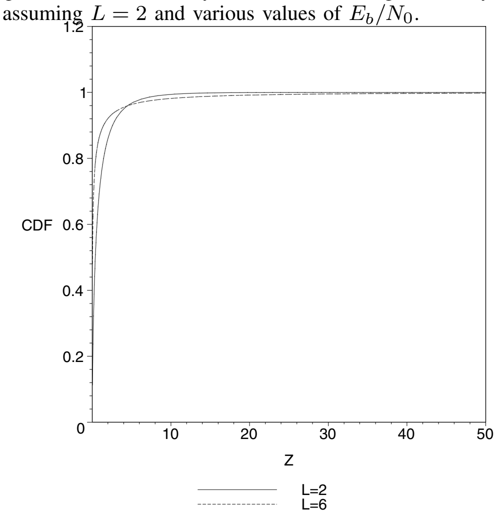

Fig. 4. The CDF of the product combiner output corresponding to the non-signal tone, for FFH 4-ary FSK communicating over a Rayleigh fading channel, assuming $E_b/N_0 = 10dB$ and $L = 2$ and 6.

IV. Numerical Results and Discussion

The PDF and the CDF given by (14) and (26) respectively have to be evaluated numerically. Note that the infinite series seen in (14) should be convergent in order to allow the computation of the PDF from a finite number of terms. It has been found that residues for $j \leq 20$ are sufficient for computing the PDF and the BER sufficiently accurately. In order to perform the infinite integration seen in (24), plotting the PDF $f_{Z_0}(y_0)$ can assist us in finding the value of Z_0 at which the PDF converges. In Fig. 2, we have plotted the PDF of the product combiner output corresponding to the desired signal tone for FFH-4FSK, assuming $E_b/N_0 = 10dB$ and various values of the diversity order. It can be seen from Fig. 2 that the PDF curve becomes flatter and its tail gets longer upon increasing the values of L. This is expected, because owing to the multiplication invoked in the combiner, there is a non-zero probability of Z_0 attaining high values, if L is high. Thus for high values of L, the PDF converges slowly. The effects of the SNR on the PDF of the product combiner output corresponding to the desired signal tone is shown in Fig. 3. As expected, the results of Fig. 3 show that the convergence of the PDF is slow when the SNR is high, because it may result in high symbol energy and, consequently, high expected combiner output value. Increase of modulation order M has similar effect (not shown) on the PDF curves. However, the effect of increasing the modulation order on the shape of the PDF is not as significant as that of the SNR.

Thus, the evaluation of the PDF from (14) is computationally cumbersome for large values of M, L and SNR. An influential factor that facilitates the numerical evaluation of the BER is the fact that the CDF of the combiner output for non-signal tones rapidly approaches unity, as seen in Fig. 4, which is valid for a wide range of L values. Thus, while using (25), $F_{Z_0}(y_m)$ may be replaced by unity for the sake of approximate computation of the symbol error rate. Specifically, we may modify (25) as

$$P_s \approx 1 - \left[\int_0^{Z_{0i}} f_{Z_0}(y_0) \Big[F_{Z_m}(y_0) \Big]^{M-1} dy_0 + \int_{Z_{0i}}^{Z_{0t}} f_{Z_0}(y_0) dy_0 \right], \quad (28)$$

where Z_{0i} is a suitably chosen value of Z_0, such that $\Big[F_{Z_m}(y_0) \Big]^{M-1} \approx 1$ at $Z_0 > Z_{0i}$, and Z_{0t} is an appropriate substitute for $+\infty$.

Using the approximation seen in (28), we have evaluated the BER and plotted it in Fig. 5 for $M = 4$, assuming various values of L. It can be seen that our analytic technique is fairly accurate and the results obtained match the simulation results for most values of the SNR in the range below 15 dB. In Fig. 6, both analytical and simulation results are provided for the BER of the system assuming $L = 4$ and various values of M. We observe that except for a slight difference for $M = 64$ at $E_b/N_0 = 15dB$, all analytical results tally with the corresponding simulation results. This inaccuracy accrues from the approximation invoked in (28), as explained above. In general, for $E_b/N_0 \geq 20dB$, accurate computation of BER is difficult and some precision has to be sacrifised. Some values of the BER at $E_b/N_0 = 20dB$, assuming various modulation and divesity orders are listed in Table I. As the results suggest, the BER values obtained analytically differ from the simulation results, for high M and L values, for example, when we have $M = 2$ and $L = 5$ or $M = 4$ and $L = 4$. Nonetheless, the results are fairly reliable for most values of E_b/N_0, M and L of practical interest.

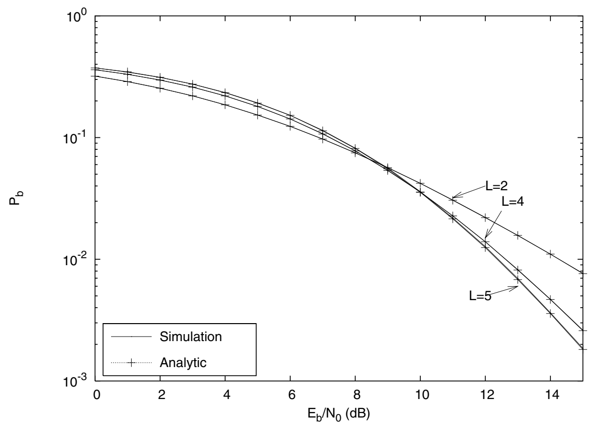

Fig. 5. Comparison of the analytical and simulation results of the BER versus the SNR performance of **FFH** 4**-ary FSK** PC receiver communicating over a Rayleigh fading channel, assuming **various** L **values**.

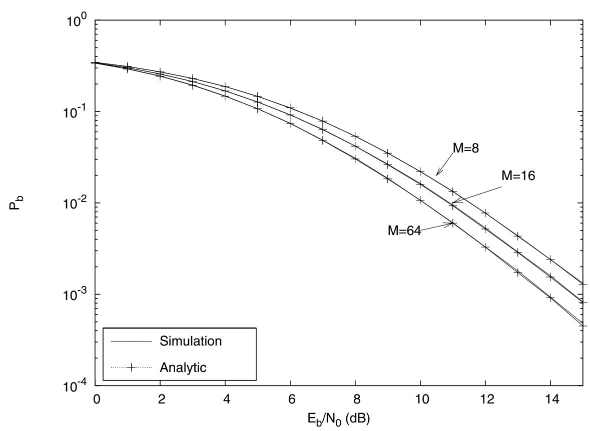

Fig. 6. Comparison of the analytical and simulation results of the BER versus the SNR performance of **FFH** M**-ary FSK** PC receiver communicating over a Rayleigh fading channel, assuming $L = 4$ **and for various values of** M.

V. Conclusion

We have used the Mellin transform to analyze the PC aided receiver used in a FFH-MFSK system, operating in a Rayleigh fading channel. Employing the proposed Mellin transform based technique, the PDF of the product combiner output was determined in a closed form, as seen in (14). With the aid of the PDF, the BER of the system was evaluated numerically. The proposed technique has been shown to be accurate and the analytical results obtained using (14) and (28) match the simulation results for most practical values of the modulation order, the diversity order $L \leq 5$ and $E_b/N_0 \leq 15dB$. For large values of L and E_b/N_0, the PDF of the combiner ouput corresponding to the desired signal tone becomes flatter and converges slowly, resulting in less accurate computation of the BER. The proposed method can readily be applied to a scenario where the channel is interferred by PBNJ or MTJ. Furthermore, the analysis of the PC aided receiver in more generalized fading conditions, e.g. Rician or Nakagami-m channels, may also be undertaken, since the relevant Mellin transforms exist [10].

TABLE I

Comparison of the analytical and simulation BER results for the FFH-MFSK PC receiver in a Rayleigh fading channel, for $E_b/N_0 = 20dB$ and various values of M and L.

M	L	BER	
		Analytic	Simulation
2	2	0.00266	0.00266
2	3	0.001018	0.001013
2	4	0.0004747	0.004628
2	5	0.000331	0.000259
4	2	0.001121	0.0010895
4	3	0.0003013	0.00028983
4	4	0.0001042	9.5066e-5

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