An Algebra-Coalgebra Framework for System Specification

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Abstract

We present an abstract equational framework for the specification of systems having both observational and computational features. Our approach is based on a clear separation between the two categories of features, and uses algebra, respectively coalgebra to formalise them. This yields a coalgebraically defined notion of observational indistinguishability, as well as an algebraically defined notion of reachability under computations. The relationship between the computations yielding new system states and the observations that can be made about these states is specified using liftings of the coalgebraic structure of state spaces to a coalgebraic structure on computations over these state spaces. Also, correctness properties of system behaviour are formalised using equational sentences, with the associated notions of satisfaction abstracting away observationally indistinguishable, respectively unreachable states, and with the resulting proof techniques employing coinduction, respectively induction. Suitably instantiating the approach yields a formalism for the specification and verification of objects.

1 Introduction

Existing approaches to system specification usually fall within one of two categories: approaches primarily concerned with computations yielding new system states [8], and respectively approaches concerned with observations that can be made about existing system states [10,9,4]. Depending on whether the emphasis is on computations or on observations, the underlying formalisms typically employ algebraic techniques with a semantics based on initial models, and respectively coalgebraic techniques with a semantics based on final models. These formalisms also attempt to accommodate basic observational,

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respectively computational features, either in their syntax or in their semantics. (Examples include the use of observational equivalence relations in algebraic approaches, see e.g. [8], and the use of initial states in coalgebraic approaches, see e.g. [9].) In addition, some of these formalisms identify the need for a certain compatibility between the two categories of features (see e.g. recent extensions of [8], including [11]). However, a recurring problem in this work is the failure to fully and naturally capture either the observational or the computational aspect of systems, due to the choice to employ either an algebraic or a coalgebraic syntax. Moreover, such a choice usually results in the expressiveness of the resulting formalism w.r.t. both computational and observational features being substantially diminished.

The present paper aims to fully exploit the expressive power of algebra and coalgebra when specifying purely computational, respectively purely observational structures, and to combine their complementary contributions when specifying structures having both computational and observational features. A first step towards achieving the aim was taken in [3], where a coalgebraic, equational formalism for the specification of observational structures allowing for a choice in the result type of observations was developed. The duality between the structures considered in [3] and those specifiable in many-sorted algebra is reflected in the resulting formalism, which employs notions of covariable, coterm and coequation (dual to the standard ones of variable, term and equation) for specification and reasoning. The approach in [3] is here generalised to an abstract setting, with the resulting framework also subsuming other existing equational approaches to system specification, including [9,4] and (a restricted version of) [8]. Furthermore, dualising this approach yields an abstract framework for the specification of structures that involve computation. The two approaches are then integrated in order to obtain a specification framework for systems having both computational and observational features. This integration builds on earlier work in [14] on relating operational and denotational semantics. Following [14], liftings of the coalgebraic structure of state spaces to computations over these state spaces are used to interpret computations on given state spaces. (A similar approach to integrating algebraic and coalgebraic features is presented in [5], where liftings of algebraic structures to transition systems over these structures are used to define transition relations on one-sorted algebras. The approach presented here generalises the one in [5] by considering arbitrary algebraic and respectively coalgebraic structures.) Equational sentences are then used to formalise correctness properties of system behaviour, and techniques for proving the satisfaction of such properties are investigated. (Correctness properties refer either to the equivalence of computations under observations, or to the preservation of state invariants by computations.) Suitably instantiating the resulting approach yields a formalism for the specification and verification of objects.

The paper is structured as follows. The coalgebraic framework for the specification of observational structures is described in Section 2. Next, the combined framework is presented in Section 3. Finally, an instance of the resulting approach is outlined in Section 4.

2 Specifying Observational Structures Coalgebraically

[12] presents a general coalgebraic framework for the specification of state-based, dynamical systems, with arbitrary endofunctors on Set being used to specify system behaviour, and with coalgebras of such endofunctors providing (abstractions of) particular implementations of the specified behaviours. The approach in [12] is here specialised in order to give a categorical account of *equational* coalgebraic approaches to system specification [2]. A framework which unifies some of the existing equational approaches to system specification (including [8,9,4,3]) is introduced, and the existence of suitable denotations for the specification techniques supported by this framework is investigated. In particular, a generalisation of a result in [12] regarding the existence of cofree coalgebras is formulated, and an instance of this generalisation is used to obtain a cofreeness result for coalgebraic equational specification.

We begin by noting that the Set-theoretic notions of bisimulation, subcoalgebra and homomorphic image generalise to endofunctors on arbitrary categories as follows.

Definition 2.1 Let $\mathsf{F} : \mathsf{C} \to \mathsf{C}$ denote an arbitrary endofunctor. An **F-bisimulation** between F-coalgebras $\langle C, \gamma \rangle$ and $\langle D, \delta \rangle$ is a *relation* (see [1, p. 101]) $\langle \langle R, \rho \rangle, r_1, r_2 \rangle$ between $\langle C, \gamma \rangle$ and $\langle D, \delta \rangle$ in $\mathsf{Coalg(G)}$. The greatest [3] G-bisimulation between $\langle C, \gamma \rangle$ and $\langle D, \delta \rangle$, if it exists, is called **F-bisimilarity**. An F-coalgebra $\langle C, \gamma \rangle$ is **extensional** if $\langle \langle R, \rho \rangle, r_1, r_2 \rangle \leq \langle \langle C, \gamma \rangle, 1_C, 1_C \rangle$ for any F-bisimulation $\langle \langle R, \rho \rangle, r_1, r_2 \rangle$ on $\langle C, \gamma \rangle$.

Definition 2.2 Let $\mathsf{F} : \mathsf{C} \to \mathsf{C}$ denote an arbitrary endofunctor and let $\langle C, \gamma \rangle$ denote an F-coalgebra. An **F-subcoalgebra** of $\langle C, \gamma \rangle$ is an object $\langle \delta, d \rangle$ of the slice category $\mathsf{Coalg(F)}/\gamma$, such that $d : D \to C$ is a C-monomorphism. Dually, a **homomorphic image** of $\langle C, \gamma \rangle$ is an object $\langle \delta, e \rangle$ of the category $\mathsf{Coalg(F)}\backslash\gamma$, such that $e : C \to D$ is a C-epimorphism. A class $\mathcal{C}$ of F-coalgebras is a **covariety** if and only if it is closed under subcoalgebras, homomorphic images and arbitrary coproducts.

In the following we shall make extensive use of the fact that the functor $\mathsf{U_C} : \mathsf{Coalg(F)} \to \mathsf{C}$ taking F-coalgebras to their carrier creates colimits, and that $\mathsf{U_C}$ creates certain limits whenever these limits are preserved by F.

We now introduce an abstract syntax for specifying observational structures.

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[2] Viewed from a different angle, our framework also generalises the one in [12], as it considers endofunctors on arbitrary categories.

[3] w.r.t. $\leq$, see [1]

Definition 2.3 An **(abstract) cosignature** is a pair (C, F), with C a category with final object, pullbacks [4] and limits of ω^{op}-chains, and with $\mathsf{F} : \mathsf{C} \to \mathsf{C}$ an endofunctor which preserves pullbacks and limits of ω^{op}-chains.

Abstract cosignatures specify the type of information that can be observed about particular systems, while the coalgebras of the endofunctors involved provide (abstractions of) specific system implementations.

Definition 2.4 Let (C, F) denote an abstract cosignature. A (C, F)**-coalgebra** (**coalgebra homomorphism**) is an F-coalgebra (coalgebra homomorphism).

The category of (C, F)-coalgebras and (C, F)-coalgebra homomorphisms is denoted $\mathsf{Coalg}(\mathsf{C}, \mathsf{F})$. Also, the functor taking (C, F)-coalgebras to their carrier is denoted $\mathsf{U}_\mathsf{C} : \mathsf{Coalg}(\mathsf{C}, \mathsf{F}) \to \mathsf{C}$.

Example 2.5 *Many-sorted cosignatures* are defined in [3] as pairs (S, Δ) with S a set of sorts and Δ an $S \times S^+$-sorted set of operation symbols (with S^+ denoting the set of finite, non-empty sequences of sorts). One writes $\delta : s \to s_1 \ldots s_n$ for $\delta \in \Delta_{s,s_1 \ldots s_n}$. For a many-sorted cosignature (S, Δ), a Δ-*coalgebra* is given by an S-sorted set C together with, for each $\delta : s \to s_1 \ldots s_n$ in Δ, a function $\delta_C : C_s \to C_{s_1} + \ldots + C_{s_n}$, while a Δ-*homomorphism* between Δ-coalgebras A and C is given by an S-sorted function $f : A \to C$ additionally satisfying: $[\iota_1 \circ f_{s_1}, \ldots , \iota_n \circ f_{s_n}](\delta_A(a)) = \delta_C(f_s(a))$ for each $\delta : s \to s_1 \ldots s_n$ in Δ and each $a \in A_s$, with $s, s_1, \ldots , s_n \in S$ (where $\iota_j : C_{s_j} \to C_{s_1} + \ldots + C_{s_n}$ for $j = 1, \ldots , n$ denote the coproduct injections).

Then, the category $\mathsf{Coalg}(S, \Delta)$ of Δ-coalgebras and Δ-homomorphisms coincides with $\mathsf{Coalg}(\mathsf{Set}^S, \mathsf{G}_\Delta)$, with Set^S denoting the category of S-sorted sets and S-sorted functions, and with $\mathsf{G}_\Delta : \mathsf{Set}^S \to \mathsf{Set}^S$ being given by:

$$(\mathsf{G}_\Delta X)_s = \prod_{\delta \in \Delta_{s,s_1 \ldots s_n}} (X_{s_1} + \ldots + X_{s_n}), \quad X \in |\mathsf{Set}^S|,\ s \in S.$$

Example 2.6 *Destructor hidden signatures* (see [2]) are an instance of the notion of abstract cosignature, and so are the *co-signatures* of [4].

The existence of final objects and of limits of ω^{op}-chains in C together with the preservation of limits of ω^{op}-chains by F results in the existence of final (C, F)-coalgebras (see e.g. [13]).

Theorem 2.7 *Let* (C, F) *denote an abstract cosignature. Then,* $\mathsf{Coalg}(\mathsf{C}, \mathsf{F})$ *has a final object.*

Also, for an abstract cosignature (C, F), existence of pullbacks in C together with preservation of pullbacks by F results in the existence of pullbacks in $\mathsf{Coalg}(\mathsf{C}, \mathsf{F})$.

Proposition 2.8 *Let* (C, F) *denote an abstract cosignature. Then,* $\mathsf{Coalg}(\mathsf{C}, \mathsf{F})$ *has pullbacks and* U_C *preserves them.*

[4] The underlying category of an abstract cosignature therefore has all finite limits.

Corollary 2.9 *Let* (C, F) *denote an abstract cosignature. Then,* U_C *preserves and reflects monomorphisms.*

Also, Theorem 2.7 together with Proposition 2.8 yield the following.

Corollary 2.10 *For an abstract cosignature* (C, F)*,* $\mathsf{Coalg}(\mathsf{C}, \mathsf{F})$ *has finite limits.*

Kernel pairs in $\mathsf{Coalg}(\mathsf{C}, \mathsf{F})$ define (C, F)-bisimulations. Also, since $\mathsf{Coalg}(\mathsf{C}, \mathsf{F})$ has a final object, greatest bisimulations on (C, F)-coalgebras exist and are given by the kernel pairs of the unique (C, F)-coalgebra homomorphisms into the final (C, F)-coalgebra. Moreover, one can give the following characterisation of extensionality of (C, F)-coalgebras [5] .

Proposition 2.11 *Let* (C, F) *denote an abstract cosignature. Then, a* (C, F)*-coalgebra is extensional if and only if its unique homomorphism into the final* (C, F)*-coalgebra is a monomorphism* [6] *.*

If, in addition, C has quotients, then so does $\mathsf{Coalg}(\mathsf{C}, \mathsf{F})$.

Proposition 2.12 *Let* (C, F) *denote an abstract cosignature, such that* C *has quotients. Then,* $\mathsf{Coalg}(\mathsf{C}, \mathsf{F})$ *has quotients, and* $\mathsf{U}_\mathsf{C} : \mathsf{Coalg}(\mathsf{C}, \mathsf{F}) \to \mathsf{C}$ *creates (and therefore preserves) them.*

Proof (Sketch) The fact that U_C creates kernel pairs (since F preserves kernel pairs) and coequalisers is used. □

Since U_C preserves quotients, the quotient of a (C, F)-coalgebra homomorphism defines a homomorphic image of the domain of this homomorphism. However, in general, this does not yield a subcoalgebra of the codomain of the given homomorphism. For this, *regularity* of C (see [1, p. 90]) is required.

Proposition 2.13 *Let* (C, F) *denote an abstract cosignature, such that* C *is regular. Then, if* $e : \langle C, \gamma \rangle \to \langle E, \eta \rangle$ *denotes the quotient of a* (C, F)*-coalgebra homomorphism* $f : \langle C, \gamma \rangle \to \langle D, \delta \rangle$ *and* $m : \langle E, \eta \rangle \to \langle D, \delta \rangle$ *denotes the unique* (C, F)*-coalgebra homomorphism resulting from the universality of* e*, then* m *is a* C*-monomorphism.*

Proof. Since U_C preserves quotients, it follows that e defines a quotient of f in C. Then, letting $f = m' \circ e$ denote the unique epi-mono factorisation of f in C yields $m = m'$ (as e is a C-epimorphism). Hence, m is a C-monomorphism. □

On the other hand, if C has cokernel pairs (and hence images) and F preserves equalisers, then so does $\mathsf{Coalg}(\mathsf{C}, \mathsf{F})$.

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[5] This holds for arbitrary endofunctors F, provided that a final F-coalgebra exists.

[6] This is equivalent to (C, F)-bisimilarity on the given coalgebra being given by the equality relation.

Proposition 2.14 *Let* (C, F) *denote an abstract cosignature, such that* C *has cokernel pairs and* F *preserves equalisers. Then,* $\mathsf{Coalg}(\mathsf{C}, \mathsf{F})$ *has images, and* $\mathsf{U}_\mathsf{C} : \mathsf{Coalg}(\mathsf{C}, \mathsf{F}) \to \mathsf{C}$ *creates (and therefore preserves) them.*

Proof (Sketch) The fact that U_C creates cokernel pairs and equalisers (since F preserves equalisers) is used. □

Since U_C preserves images, the image of a (C, F)-coalgebra homomorphism defines a subcoalgebra of the codomain of that homomorphism. However, in general this does not yield a homomorphic image of the domain of the given homomorphism. For this, coregularity of C is required.

Proposition 2.15 *Let* (C, F) *denote an abstract cosignature, such that* C *is coregular and* F *preserves equalisers. Then, if* $\iota : \langle I, \xi \rangle \to \langle D, \delta \rangle$ *denotes the image of a* (C, F)*-coalgebra homomorphism* $f : \langle C, \gamma \rangle \to \langle D, \delta \rangle$ *and* $s : \langle C, \gamma \rangle \to \langle I, \xi \rangle$ *denotes the unique* (C, F)*-coalgebra homomorphism resulting from the couniversality of* ι*, then* s *is a* C*-epimorphism.*

Proof. Similar to that of Proposition 2.13. □

Abstract cosignatures specify basic ways of observing a particular system. More complex observations can then be defined in terms of the basic ones, e.g. by considering (finitary) iterations of the endofunctor defining the cosignature. The next definition formally captures such complex observations, by exploiting the fact that the result of an observation depends solely on the coalgebra structure.

Definition 2.16 Let (C, F) denote an abstract cosignature. A (C, F)**-observer** is a pair (K, c), with $\mathsf{K} : \mathsf{C} \to \mathsf{C}$ an arbitrary endofunctor and with $c : \mathsf{U}_\mathsf{C} \Rightarrow \mathsf{K} \circ \mathsf{U}_\mathsf{C}$ a natural transformation. (K is called the **type** of the observer.)

That is, (C, F)-observers are parameterised by (C, F)-coalgebras. A (C, F)-observer (K, c) specifies, for each (C, F)-coalgebra $\langle C, \gamma \rangle$, a C-arrow $c_\gamma : C \to \mathsf{K}C$, extracting information of type K from γ. In addition, (C, F)-observers are well-behaved w.r.t. coalgebra homomorphisms, in that if $f : \langle C, \gamma \rangle \to \langle D, \delta \rangle$ is a (C, F)-coalgebra homomorphism, then $c_\delta \circ f = \mathsf{K}f \circ c_\gamma$. That is, the extraction of K-information from coalgebras commutes with coalgebra homomorphisms.

Pairs of observers are then used to constrain system implementations, by requiring that the two observers yield similar results.

Definition 2.17 Let (C, F) denote an abstract cosignature. A (C, F)**-coequation** is a tuple (K, l, r), with (K, l) and (K, r) denoting (C, F)-observers. A (C, F)-coalgebra $\langle C, \gamma \rangle$ **satisfies** a (C, F)-coequation (K, l, r) (written $\langle C, \gamma \rangle \models_{(\mathsf{C}, \mathsf{F})} (\mathsf{K}, l, r)$) if and only if $l_\gamma = r_\gamma$.

For an abstract cosignature (C, F) and a set E of (C, F)-coequations, the full subcategory of $\mathsf{Coalg}(\mathsf{C}, \mathsf{F})$ whose objects satisfy the coequations in E is denoted $\mathsf{Coalg}(\mathsf{C}, \mathsf{F}, E)$.

Example 2.18 For a many-sorted cosignature (S, Δ) and an S-sorted set $\mathcal{C}$ (of *covariables*), the (S-sorted) set $T_\Delta[\mathcal{C}]$ of Δ-*coterms* with covariables from $\mathcal{C}$ is defined in [3] as the least S-sorted set satisfying:

(i) $Z \in T_\Delta[\mathcal{C}]_s$ for $Z \in \mathcal{C}_s$ and $s \in S$

(ii) $[t_1, \ldots, t_n]\delta \in T_\Delta[\mathcal{C}]_s$ for $\delta \in \Delta_{s,s_1 \ldots s_n}$ and $t_i \in T_\Delta[\mathcal{C}]_{s_i}$, $i = 1, \ldots, n$

Also, a notion of *substitution* of coterms for covariables is defined similarly to that of substitution of terms for variables in many-sorted algebra. Specifically, if $t \in T_\Delta[\{Z_1, \ldots, Z_n\}]_s$ with $Z_i : s_i$ for $i = 1, \ldots, n$, and if $t_i \in T_\Delta[\mathcal{C}]_{s_i}$ for $i = 1, \ldots, n$, then the coterm obtained by substituting $t_1, \ldots, t_n$ for $Z_1, \ldots, Z_n$ in t, denoted $[t_1/Z_1, \ldots, t_n/Z_n]t$ (or $[\overline{t}/\overline{Z}]t$ for short) is defined inductively on the structure of t as follows:

(i) $[\overline{t}/\overline{Z}]Z_i = t_i, \quad i \in \{1, \ldots, n\}$

(ii) $[\overline{t}/\overline{Z}]([t'_1, \ldots, t'_m]\delta) = [[\overline{t}/\overline{Z}]t'_1, \ldots, [\overline{t}/\overline{Z}]t'_m]\delta, \quad \delta \in \Delta_{s,s'_1 \ldots s'_m}$.

Given a Δ-coalgebra C, a set $\{Z_1, \ldots, Z_n\}$ of covariables with $Z_i : s_i$ for $i = 1, \ldots, n$ and a covariable $Z \in \{Z_1, \ldots, Z_n\}$ with $Z : s$, one writes $\iota_Z : C_s \to C_{s_1} + \ldots + C_{s_n}$ for the corresponding coproduct injection. Then, the interpretation provided by coalgebras to the operation symbols in many-sorted cosignatures extends naturally to an interpretation of coterms over these operation symbols. Specifically, the interpretation of a Δ-coterm $t \in T_\Delta[\mathcal{C}]$ in a Δ-coalgebra C, denoted t_C, is defined by:

(i) $Z_C = \iota_Z, \quad Z \in \mathcal{C}_s,\ s \in S$

(ii) $([t_1, \ldots, t_n]\delta)_C = [(t_1)_C, \ldots, (t_n)_C] \circ \delta_C, \quad \delta \in \Delta_{s,s_1 \ldots s_n}, \quad t_i \in T_\Delta[\mathcal{C}]_{s_i}$, $i = 1, \ldots, n$

with $[(t_1)_C, \ldots, (t_n)_C] : C_{s_1} + \ldots + C_{s_n} \to \coprod_{s \in S} \coprod_{Z \in \mathcal{C}_s} C_s$ denoting the unique Set-arrow induced by $(t_i)_C : C_{s_i} \to \coprod_{s \in S} \coprod_{Z \in \mathcal{C}_s} C_s$ with $i = 1, \ldots, n$.

Equational sentences are used in [3] to constrain the interpretations of coterms by coalgebras. If Δ denotes a many-sorted cosignature, a Δ-*coequation* is given by a tuple $((l, r), (t_1, \mathcal{C}'_1), \ldots, (t_n, \mathcal{C}'_n))$ (alternatively denoted $l = r$ if $(t_1, \mathcal{C}'_1), \ldots, (t_n, \mathcal{C}'_n)$), with $l, r \in T_\Delta[\mathcal{C}]_s$ and $t_i \in T_\Delta[\mathcal{C}_i]_s$, $\mathcal{C}'_i \subseteq \mathcal{C}_i$ for $i = 1, \ldots, n$, for some $s \in S$. A Δ-coalgebra C *satisfies* a Δ-coequation e of the above form if and only if $l_C(c) = r_C(c)$ holds whenever $c \in C_s$ is such that $(t_i)_C(c) \in \iota_{Z_i}(C_{s_i})$ for some $Z_i \in (\mathcal{C}'_i)_{s_i}$, for $i = 1, \ldots, n$ (case in which c is said to satisfy the conditions $(t_1, \mathcal{C}'_1), \ldots, (t_n, \mathcal{C}'_n)$).

Then, many-sorted Δ-coterms and respectively Δ-coequations are an instance of the abstract notions of observer, respectively coequation. Given a Δ-coterm $t \in T_\Delta[\mathcal{C}]_s$ with $\mathcal{C}$ a finite (S-sorted) set of covariables, together with some conditions C of form $(t_1, \mathcal{C}_1), \ldots, (t_n, \mathcal{C}_n)$ for sort s, one can define a G_Δ-observer (K, c) with $\mathsf{K} : \mathsf{Set}^S \to \mathsf{Set}^S$ being given by:

$$(\mathsf{K}X)_s = 1 + \coprod_{s' \in S} \coprod_{Z \in \mathcal{C}_{s'}} X_{s'}$$
$$(\mathsf{K}X)_{s'} = 1 \quad \text{if} \quad s' \in S \setminus \{s\}$$

and with $c : \mathsf{U} \Rightarrow \mathsf{K} \circ \mathsf{U}$ (where $\mathsf{U} : \mathsf{Coalg}(S, \Delta) \to \mathsf{Set}^S$ denotes the functor taking a Δ-coalgebra to its carrier) given by:

$$(c_A)_s(a) = \begin{cases} t_A(a) & \text{if } C \text{ holds in } a \\ * & \text{otherwise} \end{cases}$$
$$(c_A)_{s'}(a') = * \quad \text{if} \quad s' \in S \setminus \{s\}$$

for each Δ-coalgebra A. Then, a Δ-coalgebra A satisfies a Δ-coequation of form $l = r$ if C if and only if, when viewed as a G_Δ-coalgebra, it satisfies the G_Δ-coequation (K, c_l, c_r), with (K, c_l) and (K, c_r) being the G_Δ-observers induced by the Δ-coterms l and respectively r together with the conditions C. (Note that K only depends on the set of covariables appearing in the coequation.)

Example 2.19 Many-sorted equations in one hidden variable over destructor hidden signatures (see [2]) are an instance of the abstract notion of coequation, and so are the equations used in [4].

The next result generalises a property of the hidden algebraic notion of satisfaction of equations in one hidden variable.

Proposition 2.20 *Let* (C, F) *denote an abstract cosignature, let* $f : \langle C, \gamma \rangle \to \langle D, \delta \rangle$ *denote a* (C, F)*-coalgebra homomorphism, and let* (K, l, r) *denote a* (C, F)*-coequation. Then, the following hold:*

(i) *If* f *is a* C*-epimorphism, then* $\langle C, \gamma \rangle \models (\mathsf{K}, l, r)$ *implies* $\langle D, \delta \rangle \models (\mathsf{K}, l, r)$.

(ii) *If* $\mathsf{K}f$ *is a* C*-monomorphism, then* $\langle D, \delta \rangle \models (\mathsf{K}, l, r)$ *implies* $\langle C, \gamma \rangle \models (\mathsf{K}, l, r)$.

Proof. $\langle C, \gamma \rangle \models (\mathsf{K}, l, r)$ gives $l_\gamma = r_\gamma$. Postcomposing with $\mathsf{K}f$ then yields $l_\gamma; \mathsf{K}f = r_\gamma; \mathsf{K}f$, or equivalently (using the naturality of l and r), $f; l_\delta = f; r_\delta$. The fact that f is a C-epimorphism then yields $l_\delta = r_\delta$, that is, $\langle D, \delta \rangle \models (\mathsf{K}, l, r)$. Also, $\langle D, \delta \rangle \models (\mathsf{K}, l, r)$ gives $l_\delta = r_\delta$. Precomposing with f yields $f; l_\delta = f : r_\delta$, i.e. $l_\gamma; \mathsf{K}f = r_\gamma; \mathsf{K}f$. The fact that $\mathsf{K}f$ is a C-monomorphism then yields $l_\gamma = r_\gamma$, that is, $\langle C, \gamma \rangle \models (\mathsf{K}, l, r)$. □

If, in addition, C has coproducts (and therefore so does $\mathsf{Coalg}(\mathsf{C}, \mathsf{F})$, as U_C creates coproducts), then coproducts in $\mathsf{Coalg}(\mathsf{C}, \mathsf{F})$ preserve the satisfaction of coequations.

Proposition 2.21 *Let* (C, F) *denote an abstract cosignature such that* C *has coproducts, let* $(\langle C_i, \gamma_i \rangle)_{i \in I}$ *denote a family of* (C, F)*-coalgebras, and let* (K, l, r)

denote a (C,F)*-coequation. Then,* $\langle C_i,\gamma_i\rangle \models_{(\mathsf{C},\mathsf{F})} (\mathsf{K},l,r)$ *for* $i\in I$ *implies* $\coprod_{i\in I}\langle C_i,\gamma_i\rangle \models_{(\mathsf{C},\mathsf{F})} (\mathsf{K},l,r)$.

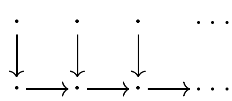
Proof. If $\langle C,\gamma\rangle = \coprod_{i\in I}\langle C_i,\gamma_i\rangle$, and if $\iota_i : \langle C_i,\gamma_i\rangle \to \langle C,\gamma\rangle$ with $i\in I$ denote the coproduct injections, then the conclusion follows from $l_\gamma\circ\iota_i = \mathsf{K}\iota_i\circ l_{\gamma_i} = \mathsf{K}\iota_i\circ r_{\gamma_i} = r_\gamma\circ\iota_i$ for $i\in I$. □

Corollary 2.22 *Let* (C,F) *denote an abstract cosignature such that* C *has coproducts, and let* E *denote a set of* (C,F)*-coequations of form* (K,l,r) *with* K *preserving monomorphisms. Then,* $|\mathsf{Coalg}(\mathsf{C},\mathsf{F},\mathsf{E})|$ *is a covariety*[7].

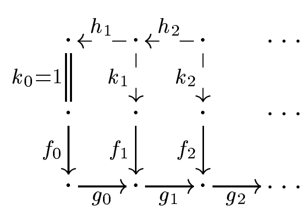
Proof. The conclusion follows directly from Propositions 2.20 and 2.21. □

Given $\mathsf{F}:\mathsf{C}\to\mathsf{C}$ together with an F-coalgebra $\langle C,\gamma\rangle$ and a C-monomorphism $\iota : X \rightarrowtail C$, we let $(\mathsf{Coalg}(\mathsf{F})/\gamma)^\iota$ denote the full subcategory of $\mathsf{Coalg}(\mathsf{F})/\gamma$ whose objects $\langle\delta,d\rangle$ are such that $d:D\to C$ factors through ι. The following lemma will be used to prove the existence of final objects in $(\mathsf{Coalg}(\mathsf{F})/\gamma)^\iota$.

Lemma 2.23 *Let* C *denote a category with pullbacks and limits of* ω^{op}*-chains, and let* $\mathsf{F}:\mathsf{C}\to\mathsf{C}$ *denote an endofunctor which preserves pullbacks and limits of* ω^{op}*-chains. Then,* C *has, and* F *preserves limits of diagrams of the following shape:*



Proof (Sketch) Given a diagram of the above shape, with its arrows labelled by $f_0,f_1,\ldots$ and respectively $g_0,g_1,\ldots$, let h_i and k_i define the pullback of $g_{i-1}\circ f_{i-1}\circ k_{i-1}$ along f_i.



Then, a limit of the ω^{op}-chain defined by $h_1,h_2,\ldots$ yields a limit for the initial diagram (by postcomposing the limiting arrows with $k_0,k_1,\ldots$). Moreover, preservation by F of pullbacks and of limits of ω^{op}-chains results in the preservation by F of limits of diagrams of the above shape. □

Proposition 2.24 *Let* C *denote a category with pullbacks and limits of* ω^{op}*-chains, let* $\mathsf{F}:\mathsf{C}\to\mathsf{C}$ *denote an endofunctor which preserves pullbacks and limits of* ω^{op}*-chains, let* $\langle C,\gamma\rangle$ *denote an* F*-coalgebra, and let* $\iota : X\rightarrowtail C$ *denote*

[7] However, not any covariety is coequationally specifiable, see [4].

a C*-monomorphism. Then,* $(\mathsf{Coalg}(\mathsf{F})/\gamma)^{\iota}$ *has a final object. Furthermore, the final object defines an* F*-subcoalgebra of* γ*.*

Proof (Sketch) We begin by noting that, since F preserves pullbacks, F also preserves monomorphisms. We now consider the following diagram:

$$\begin{array}{ccccccc}
X & & \mathsf{F}X & & \mathsf{F}^2X & & \cdots \\
\downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \mathsf{F}\iota} & & \downarrow{\scriptstyle \mathsf{F}^2\iota} & & \\
C & \xrightarrow{\gamma} & \mathsf{F}C & \xrightarrow{\mathsf{F}\gamma} & \mathsf{F}^2C & \xrightarrow{\mathsf{F}^2\gamma} & \cdots
\end{array}$$

and let $(L, (l_n : L \to \mathsf{F}^n X)_{n=0,1,\ldots})$ denote its limit in C. Then, $(\mathsf{F}L, (\mathsf{F}l_n)_{n=0,1,\ldots})$ defines a limit for:

$$\begin{array}{ccccccc}
\mathsf{F}X & & \mathsf{F}^2X & & \mathsf{F}^3X & & \cdots \\
\downarrow{\scriptstyle \mathsf{F}\iota} & & \downarrow{\scriptstyle \mathsf{F}^2\iota} & & \downarrow{\scriptstyle \mathsf{F}^3\iota} & & \\
\mathsf{F}C & \xrightarrow{\mathsf{F}\gamma} & \mathsf{F}^2C & \xrightarrow{\mathsf{F}^2\gamma} & \mathsf{F}^3C & \xrightarrow{\mathsf{F}^3\gamma} & \cdots
\end{array}$$

while $(L, (l_n)_{n=1,2,\ldots})$ defines a cone for the above diagram. This yields a unique mediating arrow $\lambda : L \to \mathsf{F}L$. Then, $(\lambda, \iota \circ l_0)$ defines a final object in $(\mathsf{Coalg}(\mathsf{F})/\gamma)^{\iota}$. (The fact that l_0, and hence $\iota \circ l_0$, is a monomorphism follows from the couniversality of L together with standard properties of coequalisers. Also, the fact that $\iota \circ l_0$ defines an F-coalgebra homomorphism from λ to γ follows from: $\gamma \circ \iota \circ l_0 = \mathsf{F}\iota \circ l_1 = \mathsf{F}\iota \circ \mathsf{F}l_0 \circ \lambda$, while finality of $(\lambda, \iota \circ l_0)$ in $(\mathsf{Coalg}(\mathsf{F})/\gamma)^{\iota}$ follows by couniversality of L.) □

Proposition 2.24 together with the existence of equalisers in C results in the existence of final objects in $(\mathsf{Coalg}(\mathsf{C},\mathsf{F})/\gamma)^E$, with $(\mathsf{Coalg}(\mathsf{C},\mathsf{F})/\gamma)^E$ denoting the full subcategory of $\mathsf{Coalg}(\mathsf{C},\mathsf{F})/\gamma$ whose objects satisfy a set E of (C,F)-coequations.

Proposition 2.25 *Let* (C,F) *denote an abstract cosignature, let* $\langle C, \gamma \rangle$ *denote a* (C,F)*-coalgebra, and let* E *denote an enumerable set of* (C,F)*-coequations. Then,* $(\mathsf{Coalg}(\mathsf{C},\mathsf{F})/\gamma)^E$ *has a final object. Furthermore, the final object defines a* (C,F)*-subcoalgebra of* γ*.*

Proof. Say $E = \{e_1, e_2, \ldots\}$. For $i = 1, 2, \ldots$, if e_i is of form (K_i, l_i, r_i) with $\mathsf{K}_i : \mathsf{C} \to \mathsf{C}$ and $l_i, r_i : \mathsf{U_C} \Rightarrow \mathsf{K}_i \circ \mathsf{U_C}$, let $\iota_i : X_i \rightarrowtail C$ denote an equaliser for $C \underset{r_{i,\gamma}}{\overset{l_{i,\gamma}}{\rightrightarrows}} \mathsf{K}_i C$. Also, let $(X, (\iota'_i : X \rightarrowtail X_i)_{i \in \{1,2,\ldots\}})$ denote the limit of the following diagram:

$$\begin{array}{ccccccc}
X_1 & & X_2 & & X_3 & & \cdots \\
\downarrow{\scriptstyle \iota_1} & & \downarrow{\scriptstyle \iota_2} & & \downarrow{\scriptstyle \iota_3} & & \\
C & = & C & = & C & = & \cdots
\end{array}$$

and let $\iota = \iota_i \circ \iota'_i$. Then $(\mathsf{Coalg}(\mathsf{C},\mathsf{F})/\gamma)^E$ coincides with $(\mathsf{Coalg}(\mathsf{F})/\gamma)^{\iota}$, and the conclusion follows by Proposition 2.24. □

Corollary 2.26 *Let* (C,F) *denote an abstract cosignature and let* E *denote an enumerable set of* (C,F)*-coequations. Then,* $\mathsf{Coalg}(\mathsf{C},\mathsf{F},E)$ *has a final object.*

Proof (Sketch) Theorem 2.7 and Proposition 2.25 are used. □

A notion of satisfaction of coequations up to bisimulation can also be defined.

Definition 2.27 Let (C,F) denote an abstract cosignature. A (C,F)-coalgebra $\langle C,\gamma\rangle$ **satisfies** a (C,F)-coequation (K,l,r) **up to bisimulation** (written $\langle C,\gamma\rangle \models^b_{(\mathsf{C},\mathsf{F})} (\mathsf{K},l,r)$) if and only if $\langle l_\gamma,r_\gamma\rangle : C \to \mathsf{K}C \times \mathsf{K}C$ factors through $\langle \mathsf{K}r_1,\mathsf{K}r_2\rangle : \mathsf{K}R \to \mathsf{K}C \times \mathsf{K}C$, with $\langle\langle R,\rho\rangle, r_1, r_2\rangle$ denoting (C,F)-bisimilarity on $\langle C,\gamma\rangle$:

$$\begin{array}{ccc} C & \xrightarrow{\langle l_\gamma,r_\gamma\rangle} & \mathsf{K}C \times \mathsf{K}C \\ & \searrow^{c} & \uparrow{\scriptstyle \langle \mathsf{K}r_1,\mathsf{K}r_2\rangle} \\ & & \mathsf{K}R \end{array}$$

It follows immediately that standard satisfaction of (K,l,r) by $\langle C,\gamma\rangle$ implies its satisfaction up to bisimulation by $\langle C,\gamma\rangle$. If, in addition, $\langle C,\gamma\rangle$ is extensional (and hence $r_1 = r_2 = 1_{\langle C,\gamma\rangle}$), then the converse also holds.

Remark 2.28 If K preserves kernel pairs, then $\langle C,\gamma\rangle \models^b (\mathsf{K},l,r)$ is equivalent to $\mathsf{K}! \circ l_\gamma = \mathsf{K}! \circ r_\gamma$. For, if $\langle C,\gamma\rangle \models^b (\mathsf{K},l,r)$, that is, $\langle l_\gamma,r_\gamma\rangle = \langle \mathsf{K}r_1,\mathsf{K}r_2\rangle \circ c$, postcomposing with $\mathsf{K}! \times \mathsf{K}!$ yields $\langle \mathsf{K}! \circ l_\gamma, \mathsf{K}! \circ r_\gamma\rangle = \langle \mathsf{K}(! \circ r_1) \circ c, \mathsf{K}(! \circ r_2) \circ c\rangle$, and therefore, since $! \circ r_1 =! \circ r_2$, $\mathsf{K}! \circ l_\gamma = \mathsf{K}! \circ r_\gamma$. Also, $\mathsf{K}! \circ l_\gamma = \mathsf{K}! \circ r_\gamma$ together with the fact that $\mathsf{K}r_1, \mathsf{K}r_2$ define a kernel pair for $\mathsf{K}!$ yield a unique C-arrow $c : C \to \mathsf{K}R$ such that $\langle l_\gamma,r_\gamma\rangle = \langle \mathsf{K}r_1,\mathsf{K}r_2\rangle \circ c$.

If, in addition, C (and therefore $\mathsf{Coalg}(\mathsf{C},\mathsf{F})$, see Proposition 2.12) has quotients, satisfaction up to bisimulation can be expressed in terms of standard satisfaction by the codomains of the quotients of the unique homomorphisms into the final coalgebra.

Proposition 2.29 *Let* (C,F) *denote an abstract cosignature, and let* (K,l,r) *denote a* (C,F)*-coequation. If* C *has quotients and* K *preserves kernel pairs, then* $C,\gamma\rangle \models^b \langle \mathsf{K},l,r\rangle$ *if and only if* $\langle E,\eta\rangle \models \langle \mathsf{K},l,r\rangle$*, for any* (C,F)*-coalgebra* $\langle C,\gamma\rangle$ *(where* $e : \langle C,\gamma\rangle \to \langle E,\eta\rangle$ *denotes the quotient of the unique* (C,F)*-coalgebra homomorphism from* $\langle C,\gamma\rangle$ *to the final* (C,F)*-coalgebra).*

Proof (Sketch) Remark 2.28 together with naturality of l, r and the fact that U_C preserves epimorphisms (see Corollary 2.36) are used. □

The notion of satisfaction of coequations up to bisimulation satisfies properties similar to those of standard satisfaction. In particular, Propositions 2.20 and 2.21 hold; moreover, no restriction on the homomorphism f is required by ii of Proposition 2.20.

Proofs of satisfaction of coequations up to bisimulation can benefit from the use of *generic bisimulations*, as defined below.

Definition 2.30 Let (C, F) denote an abstract cosignature. A **generic (C, F)-bisimulation** is given by a tuple $\langle \mathsf{R}, \pi_1, \pi_2 \rangle$ with $\mathsf{R} : \mathsf{Coalg}(\mathsf{C}, \mathsf{F}) \to \mathsf{Coalg}(\mathsf{C}, \mathsf{F})$ and $\pi_1, \pi_2 : \mathsf{R} \Rightarrow \mathsf{Id}_{\mathsf{Coalg}(\mathsf{C},\mathsf{F})}$, such that for any (C, F)-coalgebra $\langle C, \gamma \rangle$, $\langle \pi_{1,\gamma}, \pi_{2,\gamma} \rangle$ defines a (C, F)-bisimulation on $\langle C, \gamma \rangle$.

Then, proving that a (C, F)-coequation (K, l, r), with K preserving kernel pairs, holds, up to bisimulation, in a subclass $\mathcal{C}$ of $\mathsf{Coalg}(\mathsf{C}, \mathsf{F})$ [8] can be reduced to exhibiting a generic (C, F)-bisimulation $\langle \mathsf{R}, \pi_1, \pi_2 \rangle$ such that $\langle l_\gamma, r_\gamma \rangle$ factors through $\langle \mathsf{K}\pi_{1,\gamma}, \mathsf{K}\pi_{2,\gamma} \rangle$ whenever $\langle C, \gamma \rangle$ belongs to $\mathcal{C}$:

$$\begin{array}{ccc} C & \xrightarrow{\langle l_\gamma, r_\gamma \rangle} & \mathsf{K}C \times \mathsf{K}C \\ & {\scriptstyle c}\searrow & \uparrow {\scriptstyle \langle \mathsf{K}\pi_{1,\gamma}, \mathsf{K}\pi_{2,\gamma} \rangle} \\ & & \mathsf{K}R_\gamma \end{array}$$

(where $\mathsf{R}\langle C, \gamma \rangle = \langle R_\gamma, \rho_\gamma \rangle$).

Translations between abstract cosignatures, specifying a change in the type of information that can be observed about a system, are captured by *abstract cosignature morphisms*.

Definition 2.31 An **(abstract) cosignature morphism** between cosignatures (C, F) and (D, G) is a pair (U, η), with $\mathsf{U} : \mathsf{D} \to \mathsf{C}$ a functor with right adjoint, right inverse, and with $\eta : \mathsf{U} \circ \mathsf{G} \Rightarrow \mathsf{F} \circ \mathsf{U}$ a natural transformation.

Abstract cosignature morphisms $(\mathsf{U}, \eta) : (\mathsf{C}, \mathsf{F}) \to (\mathsf{D}, \mathsf{G})$ induce *reduct functors* $\mathsf{U}_\eta : \mathsf{Coalg}(\mathsf{D}, \mathsf{G}) \to \mathsf{Coalg}(\mathsf{C}, \mathsf{F})$, with U_η taking a (D, G)-coalgebra $\langle C, \gamma \rangle$ to the (C, F)-coalgebra $\langle \mathsf{U}C, \eta_C \circ \mathsf{U}\gamma \rangle$. (Intuitively, the action of U_η can be regarded as extracting a (C, F)-system from a given (D, G)-system.)

Example 2.32 A many-sorted cosignature morphism between many-sorted cosignatures (S, Δ) and (S', Δ') is defined in [3] as being given by a function $\phi : S \to S'$ together with an $S \times S^+$-sorted function $(\phi_{s,w})_{s \in S, w \in S^+}$, with $\phi_{s,w} : \Delta_{s,w} \to \Delta'_{\phi(s), \phi^+(w)}$ for $s \in S$ and $w \in S^+$ (where ϕ^+ denotes the pointwise extension of $\phi : S \to S'$ to a function from S^+ to S'^+). Cosignature morphisms $\phi : (S, \Delta) \to (S', \Delta')$ induce *reduct functors* [9] $\mathsf{U}_\phi : \mathsf{Coalg}(S', \Delta') \to \mathsf{Coalg}(S, \Delta)$, taking (S', Δ')-coalgebras C' to the (S, Δ)-coalgebras having carrier $C' \upharpoonright_\Delta = (C'_{\phi(s)})_{s \in S}$ and operations given by $\delta_{C \upharpoonright_\Delta} = \delta_{C'}$.

Then, $\mathsf{U}_\phi : \mathsf{Coalg}(S', \Delta') \to \mathsf{Coalg}(S, \Delta)$ is naturally isomorphic to U_{η_ϕ}, where $\eta_\phi : \mathsf{U} \circ \mathsf{G}_{\Delta'} \Rightarrow \mathsf{G}_\Delta \circ \mathsf{U}$ is given by:

$$(\eta_{\phi,X})_s((x_{\delta'})_{\delta' \in \Delta'_{\phi(s)}}) = (x_{\phi(\delta)})_{\delta \in \Delta_s}, \quad s \in S,\ X \in |\mathsf{Set}^{S'}|$$

and where $\mathsf{U} : \mathsf{Set}^{S'} \to \mathsf{Set}^S$ denotes the functor taking $A \in |\mathsf{Set}^{S'}|$ to $(A_{\phi(s)})_{s \in S} \in |\mathsf{Set}^S|$.

[8] For instance, $\mathcal{C}$ may consist of all (C, F)-coalgebras satisfying, possibly only up to bisimulation, a given set of (C, F)-coequations.

[9] The terminology is borrowed from the theory of *institutions*, see e.g. [6].

An abstract cosignature morphism $(\mathsf{U}, \eta) : (\mathsf{C}, \mathsf{F}) \to (\mathsf{D}, \mathsf{G})$ determines a translation, itself denoted η, of (C, F)-observers to (D, G)-observers. This translation takes a (C, F)-observer (K, c) to the (D, G)-observer $(\mathsf{R} \circ \mathsf{K} \circ \mathsf{U}, c')$ with, for each (D, G)-coalgebra $\langle D, \delta \rangle$, $c'_\delta : D \to \mathsf{RKU}D$ being given by $c^\flat_{\eta_D \circ \mathsf{U}\delta}$ (where $c^\flat_{\eta_D \circ \mathsf{U}\delta} : D \to \mathsf{RKU}D$ denotes the unique D-arrow satisfying $\mathsf{U}c^\flat_{\eta_D \circ \mathsf{U}\delta} = c_{\eta_D \circ \mathsf{U}\delta} : \mathsf{U}D \to \mathsf{KU}D$). (Naturality of c' follows from the naturality of c using standard properties of adjunctions.)

The translation of (C, F)-observers into (D, G)-observers extends naturally to a translation of (C, F)-coequations into (D, G)-coequations. As one would expect, this translation has the property that a (C, F)-coequation holds in the (C, F)-reduct of a (D, G)-coalgebra if and only if its translation along η holds in that (D, G)-coalgebra. That is, the *satisfaction condition* of institutions (see [6]) holds.

Proposition 2.33 *Let* $(\mathsf{U}, \eta) : (\mathsf{C}, \mathsf{F}) \to (\mathsf{D}, \mathsf{G})$ *denote an abstract cosignature morphism, let* $\langle D, \delta \rangle$ *denote a* (D, G)*-coalgebra, and let* e *denote a* (C, F)*-coequation. Then,* $\mathsf{U}_\eta \delta \models_{(\mathsf{C}, \mathsf{F})} e$ *if and only if* $\delta \models_{(\mathsf{D}, \mathsf{G})} \eta(e)$*.*

Proof. If e is of form (K, l, r), then $\mathsf{U}_\eta \delta \models_{(\mathsf{C}, \mathsf{F})} e$ rewrites to $l_{\eta_D \circ \mathsf{U}\delta} = r_{\eta_D \circ \mathsf{U}\delta}$, while $\delta \models_{(\mathsf{D}, \mathsf{G})} \eta(e)$ rewrites to $l^\flat_{\eta_D \circ \mathsf{U}\delta} = r^\flat_{\eta_D \circ \mathsf{U}\delta}$. The conclusion then follows from standard properties of adjunctions. □

The specifications (respectively specification morphisms) of the resulting institution will be referred to as *abstract coalgebraic specifications* (*specification morphisms*).

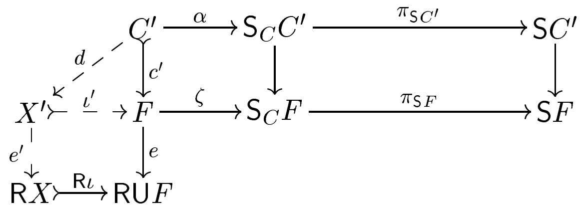
We now investigate the existence of cofree constructions along abstract cosignature morphisms (i.e. the existence of right adjoints to the reduct functors induced by such cosignature morphisms). Our starting point is a result of [12]. There, categories of coalgebras of endofunctors $\mathsf{T}, \mathsf{S} : \mathsf{Set} \to \mathsf{Set}$, together with functors $\mathsf{U}_\eta : \mathsf{Coalg}(\mathsf{S}) \to \mathsf{Coalg}(\mathsf{T})$ induced by natural transformations $\eta : \mathsf{S} \Rightarrow \mathsf{T}$ were considered (with U_η taking an S-coalgebra $\langle C, \gamma \rangle$ to the T-coalgebra $\langle C, \eta_C \circ \gamma \rangle$), and the existence of cofree coalgebras w.r.t. U_η was proved under the assumption that, for any set C, the endofunctor $\mathsf{S} \times C : \mathsf{Set} \to \mathsf{Set}$ (taking a set X to the set $\mathsf{S}X \times C$) has a final coalgebra. The result in [12] is here generalised to arbitrary categories, with the generalisation accounting for a possible change in their underlying category when moving from one category of coalgebras to another [10].

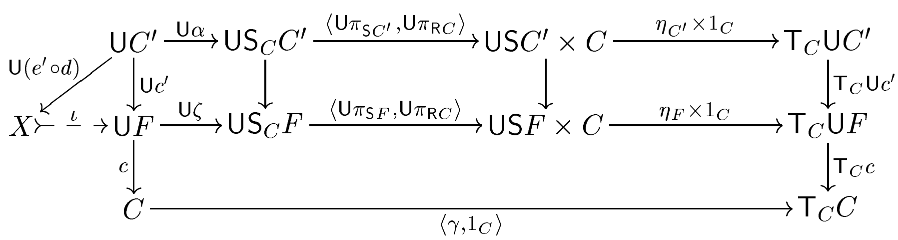
Theorem 2.34 *Let* C *and* D *denote categories with pullbacks and binary products, and let* $\mathsf{U} : \mathsf{D} \to \mathsf{C}$ *denote a functor having a right adjoint, right inverse* R*. Also let* $\mathsf{T} : \mathsf{C} \to \mathsf{C}$ *and* $\mathsf{S} : \mathsf{D} \to \mathsf{D}$ *denote arbitrary endofunctors, and let* $\eta : \mathsf{U} \circ \mathsf{S} \Rightarrow \mathsf{T} \circ \mathsf{U}$ *denote a natural transformation. If, for any* C*-object* C*, the functor* $\mathsf{S} \times \mathsf{R}C$ *has a final coalgebra, and if* D *has limits of* ω^{op}*-chains and* $\mathsf{S} \times \mathsf{R}C$ *preserves pullbacks and limits of* ω^{op}*-chains, then the functor*

[10] This result was first stated in [2]. A complete proof is given in the following.

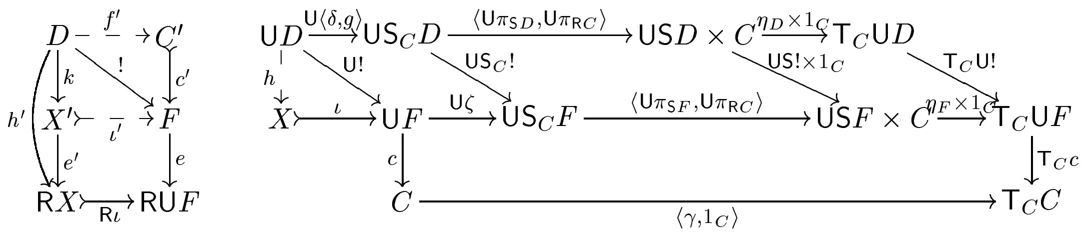
$\mathsf{U}_\eta : \mathsf{Coalg}(\mathsf{S}) \to \mathsf{Coalg}(\mathsf{T})$, *taking an* S*-coalgebra* $\langle D, \gamma\rangle$ *to the* T*-coalgebra* $\langle \mathsf{U}D, \eta_D \circ \mathsf{U}\gamma\rangle$ *and an* S*-coalgebra homomorphism* $f : \gamma \to \gamma'$ *to the* T*-coalgebra homomorphism* $\mathsf{U}f : \mathsf{U}_\eta\gamma \to \mathsf{U}_\eta\gamma'$ *has a right adjoint* C_η.

Proof. Let $\langle C, \gamma\rangle$ denote a T-coalgebra, let $\mathsf{T}_C : \mathsf{C} \to \mathsf{C}$ and $\mathsf{S}_C : \mathsf{D} \to \mathsf{D}$ be given by $\mathsf{T}_C = \mathsf{T} \times C$ and $\mathsf{S}_C = \mathsf{S} \times \mathsf{R}C$, and let $\zeta : F \to \mathsf{S}_C F$ denote a final S_C-coalgebra. Then, both $\xi = (\eta_F \times 1_C) \circ \langle \mathsf{U}\pi_{\mathsf{S}F}, \mathsf{U}\pi_{\mathsf{R}C}\rangle \circ \mathsf{U}\zeta : \mathsf{U}F \to \mathsf{T}_C\mathsf{U}F$ and $\langle \gamma, 1_C\rangle : C \to \mathsf{T}_C C$ define T_C-coalgebras. (The fact that $\mathsf{U} \circ \mathsf{R} = \mathsf{Id}_\mathsf{C}$ is used here.) Now let $c : \mathsf{U}F \to C$ be given by $\pi_C \circ \langle \mathsf{U}\pi_{\mathsf{S}F}, \mathsf{U}\pi_{\mathsf{R}C}\rangle \circ \mathsf{U}\zeta$, let $(X, \iota : X \rightarrowtail \mathsf{U}F)$ denote the equaliser of $\mathsf{T}_C c \circ \xi, \langle\gamma, 1_C\rangle \circ c : \mathsf{U}F \to \mathsf{T}_C C$, and let $(X', \iota' : X' \rightarrowtail F, e' : X' \to \mathsf{R}X)$ denote the pullback of the universal arrow $e : F \to \mathsf{RU}F$ induced by the unit of the adjunction $\mathsf{U} \dashv \mathsf{R}$ along $\mathsf{R}\iota : \mathsf{R}X \rightarrowtail \mathsf{RU}F$. (Existence of equalisers in C is a consequence of the existence of products and pullbacks. Also, each of ι, $\mathsf{R}\iota$ and ι' are monomorphisms.) Now let (α, c') denote a final object in $(\mathsf{Coalg}(\mathsf{S}_C)/\zeta)^{\iota'}$, with $\alpha : C' \to \mathsf{S}_C C'$ and $c' : C' \rightarrowtail F$, let d denote the unique factorisation of c' through ι', and let $\gamma' : C' \to \mathsf{S}C'$ be given by $\pi_{\mathsf{S}C'} \circ \alpha$. (Existence of a final $(\mathsf{Coalg}(\mathsf{S}_C)/\zeta)^{\iota'}$-object follows by Proposition 2.24.) Finally, let $\epsilon_\gamma : \mathsf{U}C' \to C$, $\epsilon_\gamma = c \circ \mathsf{U}c'$.





Then, ϵ_γ defines a T-coalgebra homomorphism from $\mathsf{U}_\eta\gamma'$ to γ. This follows from the fact that $\mathsf{U}c'$ factors through ι, together with the fact that (X, ι) is an equaliser. Moreover, ϵ_γ defines a couniversal arrow from U_η to γ. To show this, let $\langle D, \delta\rangle$ denote an arbitrary S-coalgebra and let $f : \mathsf{U}_\eta\delta \to \gamma$ denote a T-coalgebra homomorphism. The adjunction $\mathsf{U} \dashv \mathsf{R}$ immediately yields a unique D-arrow $g : D \to \mathsf{R}C$ such that $\mathsf{U}g = f$. Then, $\langle \delta, g\rangle : D \to \mathsf{S}_C D$ defines an S_C-coalgebra, with $! : \langle\delta, g\rangle \to \zeta$ as unique S_C-coalgebra homomorphism into the final S_C-coalgebra. Also, $\tau = (\eta_D \times 1_C) \circ \langle \mathsf{U}\pi_{\mathsf{S}D}, \mathsf{U}\pi_{\mathsf{R}C}\rangle \circ \mathsf{U}\langle\delta, g\rangle : \mathsf{U}D \to \mathsf{T}_C\mathsf{U}D$ defines a T_C-coalgebra, and $f : \mathsf{U}D \to C$ defines a T_C-coalgebra homomorphism from τ to $\langle\gamma, 1_C\rangle$. (This follows from f being a T-coalgebra homomorphism, together with $\mathsf{U}g = f$.) In addition, the following holds: $(\mathsf{US}! \times 1_C) \circ \langle \mathsf{U}\pi_{\mathsf{S}D}, \mathsf{U}\pi_{\mathsf{R}C}\rangle \circ \mathsf{U}\langle\delta, g\rangle = \langle \mathsf{U}\pi_{\mathsf{S}F}, \mathsf{U}\pi_{\mathsf{R}C}\rangle \circ \mathsf{U}\zeta \circ \mathsf{U}!$ (by ! defining an S_C-coalgebra homomorphism, together with the naturality of π). Postcomposing this with π_C gives $\mathsf{U}g = c \circ \mathsf{U}!$, that is, $c \circ \mathsf{U}! = f$.



The following also holds: $\mathsf{T}_C c \circ \xi \circ \mathsf{U}! = \mathsf{T}_C c \circ \mathsf{T}_C \mathsf{U}! \circ \tau = \mathsf{T}_C f \circ \tau = \langle \gamma, 1_C \rangle \circ f = \langle \gamma, 1_C \rangle \circ c \circ \mathsf{U}!$. Then, the fact that (X, ι) is an equaliser for $\mathsf{T}_C c \circ \xi$ and $\langle \gamma, 1_C \rangle \circ c$ yields a unique C-arrow $h : \mathsf{U}D \to X$ such that $\mathsf{U}! = \iota \circ h$. This, in turn, yields a unique D-arrow $h' : D \to \mathsf{R}X$ such that $\mathsf{U}h' = h$. Furthermore, $\mathsf{U}(e \circ !) = \mathsf{U}e \circ \mathsf{U}! = \mathsf{U}! = \iota \circ h = \mathsf{UR}\iota \circ \mathsf{U}h' = \mathsf{U}(\mathsf{R}\iota \circ h')$ together with $\mathsf{U} \dashv \mathsf{R}$ and $\mathsf{U} \circ \mathsf{R} = \mathsf{Id}_\mathsf{C}$ give $e \circ ! = \mathsf{R}\iota \circ h'$, which, in turn, yields a unique mediating arrow $k : D \to X'$. Since $\iota' \circ k = !$, it follows that $(\langle \delta, g \rangle, !) \in |(\mathsf{Coalg}(\mathsf{S}_C)/\zeta)^{\iota'}|$. Finality of α then yields a unique S_C-coalgebra homomorphism $f' : \langle \delta, g \rangle \to \alpha$, which also defines an S-coalgebra homomorphism $f' : \delta \to \gamma'$, with $\epsilon_\gamma \circ \mathsf{U}f' = c \circ \mathsf{U}c' \circ \mathsf{U}f' = c \circ \mathsf{U}! = f$. Uniqueness of such a homomorphism follows from the uniqueness of S_C-coalgebra homomorphisms into ζ, together with c' being a monomorphism.

The correspondence $\gamma \mapsto \gamma'$ extends to a functor $\mathsf{C}_\eta : \mathsf{Coalg}(\mathsf{T}) \to \mathsf{Coalg}(\mathsf{S})$ which is right adjoint to U_η. □

Corollary 2.35 *Let* (C, F) *and* (D, G) *denote abstract cosignatures and let* $(\mathsf{U}, \eta) : (\mathsf{C}, \mathsf{F}) \to (\mathsf{D}, \mathsf{G})$ *denote an abstract cosignature morphism. Then,* $\mathsf{U}_\eta : \mathsf{Coalg}(\mathsf{D}, \mathsf{G}) \to \mathsf{Coalg}(\mathsf{C}, \mathsf{F})$ *has a right adjoint.*

Proof. It suffices to check that the C, D, U, F and G satisfy the conditions in the hypothesis of Theorem 2.34. First, the requirements on C, D and U are guaranteed by Definitions 2.3 and 2.31. Also, for a C-object C, the existence of a final $\mathsf{G} \times \mathsf{R}C$-coalgebra follows by D having a final object and limits of ω^{op}-chains, together with the observation that the functor $\mathsf{G} \times \mathsf{R}C$ is ω^{op}-continuous whenever G is. Finally, for a C-object C, preservation by $\mathsf{G} \times \mathsf{R}C$ of pullbacks and of limits of ω^{op}-chains follows from the preservation by G of pullbacks and of limits of ω^{op}-chains by a similar argument. □

The right adjoint C_η to U_η provides a canonical way of constructing a most general (D, G)-system over a given (C, F)-(sub)system. This makes C_η a suitable denotation for the cosignature morphism η.

Taking (C, F) in Theorem 2.35 to be the trivial cosignature over C (i.e. $\mathsf{F} = 1$) results in the existence of cofree (C, F)-coalgebras w.r.t. $\mathsf{U}_\mathsf{C} : \mathsf{Coalg}(\mathsf{C}, \mathsf{F}) \to \mathsf{C}$.

Corollary 2.36 *Let* (C, F) *denote an abstract cosignature. Then,* U_C *has a right adjoint.*

Proof. The conclusion follows by taking (C, F), (D, G) and (U, η) in Theorem 2.35 to be $(\mathsf{C}, 1)$, (C, F) and $(\mathsf{Id}_\mathsf{C}, !)$ respectively, with $! : \mathsf{F} \Rightarrow 1$ denoting the natural transformation whose C-component is given by the unique C-arrow of C into 1, for $C \in |\mathsf{C}|$. □

If attention is restricted to coalgebraic specifications whose sets of coequations are enumerable, then Corollary 2.35 generalises to specification morphisms.

Theorem 2.37 *Let* $(\mathsf{U}, \eta) : (\mathsf{C}, \mathsf{F}, E) \to (\mathsf{D}, \mathsf{G}, E')$ *denote an abstract coalgebraic specification morphism. If* E' *is enumerable, then* $\mathsf{U}_\eta \restriction_{\mathsf{Coalg}(\mathsf{D},\mathsf{G},E')}$: $\mathsf{Coalg}(\mathsf{D}, \mathsf{G}, E') \to \mathsf{Coalg}(\mathsf{C}, \mathsf{F}, E)$ *has a right adjoint.*

Proof. Let $\langle C, \gamma \rangle$ denote a $(\mathsf{C}, \mathsf{F}, E)$-coalgebra and let $\epsilon_\gamma : \mathsf{U}_\eta \langle D, \delta \rangle \to \langle C, \gamma \rangle$ denote a couniversal arrow from U_η to $\langle C, \gamma \rangle$. Also, let $\langle \delta', d \rangle$, with $\delta' : D' \to \mathsf{G}D'$ and $d : D' \rightarrowtail D$ denote a final object in $(\mathsf{Coalg}(\mathsf{D}, \mathsf{G})/\delta)^{E'}$. Then, $\mathsf{U}_\mathsf{C}\epsilon_\gamma \circ \mathsf{U}d : \mathsf{U}D' \to C$ defines a (C, F)-coalgebra homomorphism $\mathsf{U}_\mathsf{C}\epsilon_\gamma \circ \mathsf{U}d : \mathsf{U}_\eta \delta' \to \gamma$, which is also a couniversal arrow from $\mathsf{U}_\eta \restriction_{\mathsf{Coalg}(\mathsf{D},\mathsf{G},E')}$ δ' to γ. (Its couniversality is a consequence of the couniversality of ϵ_γ and of the finality of $\langle \delta', d \rangle$.) The correspondence $\gamma \mapsto \delta'$ extends to a right adjoint to $\mathsf{U}_\eta \restriction_{\mathsf{Coalg}(\mathsf{D},\mathsf{G},E')}$. □

We conclude this section by noting that dualising the approach described here yields notions of *abstract signature*, *algebra* of an abstract signature, *reachable algebra*, *constructor* and *equation* (dual to those of abstract cosignature, coalgebra of abstract cosignature, extensional coalgebra, observer and respectively coequation), with results similar to the ones previously formulated holding for categories of algebras.

As one would expect, the many-sorted algebraic notions of signature, algebra, term and equation are instances of the abstract concepts thus obtained [11]. Specifically, if (S, Σ) denotes a many-sorted signature and $\mathsf{F}_\Sigma : \mathsf{Set}^S \to \mathsf{Set}^S$ denotes the endofunctor given by:

$$(\mathsf{F}_\Sigma X)_s = \coprod_{\sigma \in \Sigma_{s_1 \ldots s_n, s}} (X_{s_1} \times \ldots \times X_{s_n}), \quad X \in |\mathsf{Set}^S|, \ s \in S$$

then the category $\mathsf{Alg}(S, \Sigma)$ coincides with $\mathsf{Alg}(\mathsf{Set}^S, \mathsf{F}_\Sigma)$. Also, Σ-terms $t \in T_\Sigma(\mathcal{V})_s$ with $\mathcal{V} = \{V_1, \ldots, V_m\}$ a set of variables, with $V_i : s_i$ for $i = 1, \ldots, m$ induce $(\mathsf{Set}^S, \mathsf{F}_\Sigma)$-constructors (K, c), with $\mathsf{K} : \mathsf{Set}^S \to \mathsf{Set}^S$ being given by:

$$(\mathsf{K}X)_{s'} = \begin{cases} X_{s_1} \times \ldots \times X_{s_m} & \text{if } s' = s \\ \emptyset & \text{otherwise} \end{cases}$$

[11] A different approach to generalising algebraic equations was taken in [7], where abstract equations were captured by relations on the carriers of particular algebras. What distinguishes our approach from the one in [7] is the absence of quantifiers: here, quantifiers are abstracted away, whereas in [7] they are made explicit.

and with $c : \mathsf{K} \circ \mathsf{U}_{\mathsf{Set}^S} \Rightarrow \mathsf{U}_{\mathsf{Set}^S}$ being given by:

$$(c_A)_{s'} = \begin{cases} t_A & \text{if } s' = s \\ ! : \emptyset \to A_{s'} & \text{otherwise} \end{cases}$$

Finally, unconditional Σ-equations e of form $(\forall\, \mathcal{V})\ l = r$ induce $(\mathsf{Set}^S, \mathsf{F}_\Sigma)$-equations (K, c_l, c_r), with $\mathsf{K} : \mathsf{Set}^\mathsf{S} \to \mathsf{Set}^\mathsf{S}$ as before, and with $c_l, c_r : \mathsf{K} \circ \mathsf{U}_{\mathsf{Set}^S} \Rightarrow \mathsf{U}_{\mathsf{Set}^S}$ being the $(\mathsf{Set}^S, \mathsf{F}_\Sigma)$-constructors associated to l and r respectively, and moreover, this correspondence has the property that e holds in a Σ-algebra A if and only if (K, c_l, c_r) holds in the associated F_Σ-algebra.

3 Integrating Computations and Observations in System Specification

Section 2 has illustrated how algebra, respectively coalgebra can be used to specify and reason about the computational, respectively observational aspect of systems. This section presents an abstract specification framework concerned with relating these two aspects.

Our approach builds on the functorial approach to operational semantics of [14]. Specifically, we consider lifting the coalgebraic structure of semantic domains (of states) to a coalgebraic structure on computations over these semantic domains. This yields endofunctors on categories of coalgebras, with the algebras of such endofunctors interpreting computations on particular semantic domains. Abstract equations (respectively coequations) are then used to formalise correctness properties of system behaviour, with the associated notion of satisfaction abstracting away bisimilar (respectively unreachable) states.

We begin by noting that an abstract signature (C, F) induces a monad (T, η, μ) on C, with T being obtained as the colimit object of the following ω-chain:

$$\mathsf{F}_0 = \mathsf{Id}_\mathsf{C} \xrightarrow{\iota_1} \mathsf{F}_1 = \mathsf{Id}_\mathsf{C} + \mathsf{F}\mathsf{F}_0 \xrightarrow{1_{\mathsf{Id}_\mathsf{C}} + \mathsf{F}\iota_1} \mathsf{F}_2 = \mathsf{Id}_\mathsf{C} + \mathsf{F}\mathsf{F}_1 \xrightarrow{1_{\mathsf{Id}_\mathsf{C}} + \mathsf{F}(1_{\mathsf{Id}_\mathsf{C}} + \mathsf{F}\iota_1)} \cdots$$

If $q_i : \mathsf{F}_i \Rightarrow \mathsf{T}$ for $i = 0, 1, \ldots$ denote the colimit arrows, then the unit η of T is given by q_0, while the multiplication μ of T is obtained by first noting that the colimit preservation property of F yields a unique natural transformation $\alpha : \mathsf{FT} \Rightarrow \mathsf{T}$ satisfying $q_{i+1} \circ \iota_2 = \alpha \circ \mathsf{F}q_i$ for $i = 0, 1, \ldots$, and then using the colimiting property of T to define a unique natural transformation $\mu : \mathsf{T}^2 \Rightarrow \mathsf{T}$ satisfying: $(q_0)_\mathsf{T}; \mu = 1_\mathsf{T}, (q_1)_\mathsf{T}; \mu = [1_\mathsf{T}, \alpha], (q_2)_\mathsf{T}; \mu = [1_\mathsf{T}, \mathsf{F}[1_\mathsf{T}, \alpha]; \alpha], \ldots$.

Also, an abstract signature morphism $(\mathsf{U}, \xi) : (\mathsf{C}, \mathsf{F}) \to (\mathsf{C}', \mathsf{F}')$ induces a natural transformation $\nu : \mathsf{TU} \Rightarrow \mathsf{UT}'$, with ν arising from the observation that TU is a colimit object for the following ω-chain:

$$\mathsf{U} \xrightarrow{(\iota_1)_\mathsf{U}} \mathsf{U} + \mathsf{FU} \xrightarrow{1_\mathsf{U} + \mathsf{F}(\iota_1)_\mathsf{U}} \mathsf{U} + \mathsf{F}(\mathsf{U} + \mathsf{FU}) \xrightarrow{1_\mathsf{U} + \mathsf{F}(1_\mathsf{U} + \mathsf{F}(\iota_1)_\mathsf{U})} \cdots$$

whereas UT' together with $\xi_0; \mathsf{U}q_0', \xi_1; \mathsf{U}q_1', \ldots$ define a cocone for this ω-chain, with $\xi_0, \xi_1, \ldots$ being given by: $\xi_0 = 1_{\mathsf{U}}$ and $\xi_{i+1} = 1_{\mathsf{U}} + \mathsf{F}\xi_i; \xi_{\mathsf{F}_i'}$ for $i = 0, 1, \ldots$. (The fact that U preserves coproducts as well as colimits of ω-chains is used here.) Moreover, if ξ is a natural isomorphism, then so is ν. (In this case, UT' is itself a colimit object for the above ω-chain.)

Particular liftings of the monads induced by abstract signatures to categories of coalgebras of abstract cosignatures will now be specified using the notion of *lifted signature*.

Definition 3.1 An **(abstract) lifted signature** is a tuple $(\mathsf{C}, \mathsf{G}, \mathsf{F}, \sigma)$, with (C, G) an abstract cosignature, (C, F) an abstract signature, and $\sigma : \mathsf{TU_C} \Rightarrow \mathsf{GTU_C}$ (where $\mathsf{U_C} : \mathsf{Coalg}(\mathsf{C}, \mathsf{G}) \to \mathsf{C}$ denotes the functor taking (C, G)-coalgebras to their carrier, and where (T, η, μ) denotes the monad induced by F), such that the following diagram commutes:

$$\begin{array}{ccc}
\mathsf{T}^2\mathsf{U_C} & \overset{\sigma_{\mathsf{T}_\sigma}}{\Longrightarrow} & \mathsf{GT}^2\mathsf{U_C} \\
{\scriptstyle \mu_{\mathsf{U_C}}}\Downarrow & & \Downarrow{\scriptstyle \mathsf{G}\mu_{\mathsf{U_C}}} \\
\mathsf{TU_C} & \overset{\sigma}{\Longrightarrow} & \mathsf{GTU_C} \\
{\scriptstyle \eta_{\mathsf{U_C}}}\Uparrow & & \Uparrow{\scriptstyle \mathsf{G}\eta_{\mathsf{U_C}}} \\
\mathsf{U_C} & \underset{\lambda}{\Longrightarrow} & \mathsf{GU_C}
\end{array}$$

where $\lambda : \mathsf{U_C} \Rightarrow \mathsf{GU_C}$ is given by: $\lambda_{\langle C, \gamma \rangle} = \gamma$ for $\langle C, \gamma \rangle \in |\mathsf{Coalg}(\mathsf{G})|$, while $\mathsf{T}_\sigma : \mathsf{Coalg}(\mathsf{G}) \to \mathsf{Coalg}(\mathsf{G})$ is given by:

- $\mathsf{T}_\sigma \langle C, \gamma \rangle = \langle \mathsf{T}C, \sigma_\gamma \rangle$ for $\langle C, \gamma \rangle \in |\mathsf{Coalg}(\mathsf{G})|$
- $\mathsf{T}_\sigma f = \mathsf{T} f$ for $f \in \| \mathsf{Coalg}(\mathsf{G}) \|$

(and consequently $\mathsf{U_C T}_\sigma = \mathsf{TU_C}$).

An **(abstract) lifted signature morphism** from $(\mathsf{C}, \mathsf{G}, \mathsf{F}, \sigma)$ to $(\mathsf{C}', \mathsf{G}', \mathsf{F}', \sigma')$ is a tuple (U, τ, ξ) with $(\mathsf{U}, \tau) : (\mathsf{C}, \mathsf{G}) \to (\mathsf{C}', \mathsf{G}')$ an abstract cosignature morphism and $(\mathsf{U}, \xi) : (\mathsf{C}, \mathsf{F}) \to (\mathsf{C}', \mathsf{F}')$ an abstract signature morphism, such that $\xi : \mathsf{FU} \Rightarrow \mathsf{UF}'$ is a natural isomorphism, and such that the following diagram commutes:

$$\begin{array}{ccccc}
\mathsf{TU_C U}_\tau & = & \mathsf{TUU_{C'}} & \overset{\nu_{\mathsf{U_{C'}}}}{\Longrightarrow} & \mathsf{UT'U_{C'}} \\
{\scriptstyle \sigma_{\mathsf{U}_\tau}}\Downarrow & & & & \Downarrow{\scriptstyle \mathsf{U}\sigma'} \\
\mathsf{GTU_C U}_\tau & & & & \mathsf{UG'T'U_{C'}} \\
\| & & & & \Downarrow{\scriptstyle \tau_{\mathsf{T'U_{C'}}}} \\
\mathsf{GTUU_{C'}} & & \underset{\mathsf{G}\nu_{\mathsf{U_{C'}}}}{\Longrightarrow} & & \mathsf{GUT'U_{C'}}
\end{array}$$

where $\nu : \mathsf{TU} \Rightarrow \mathsf{UT}'$ is the natural isomorphism induced by the natural isomorphism ξ.

Remark 3.2 The definition of λ immediately yields $\lambda_{\mathsf{T}_\sigma} = \sigma$.

The constraints in Definition 3.1 result in the tuple $(\mathsf{T}_\sigma, \eta, \mu)$ defining a monad over $\mathsf{Coalg}(\mathsf{G})$. An algebra of this monad is given by a G-coalgebra

$\langle C, \gamma \rangle$ together with a G-coalgebra homomorphism $\alpha : \mathsf{T}_\sigma \langle C, \gamma \rangle \to \langle C, \gamma \rangle$, additionally satisfying: $\alpha \circ \eta_C = 1_C$ and $\alpha \circ \mu_C = \alpha \circ \mathsf{T}\alpha$. Equivalently, a T_σ-algebra is given by a C-object C carrying both a G-coalgebra structure γ and a T-algebra structure α, such that α defines a G-coalgebra homomorphism from σ_γ to γ, and such that $\alpha \circ \eta_C = 1_C$ and $\alpha \circ \mu_C = \alpha \circ \mathsf{T}\alpha$. Also, a T_σ-algebra homomorphism from $\langle \langle C, \gamma \rangle, \alpha \rangle$ to $\langle \langle D, \delta \rangle, \beta \rangle$ is given by a C-arrow $f : C \to D$ defining both a G-coalgebra homomorphism from $\langle C, \gamma \rangle$ to $\langle D, \delta \rangle$ and a T-algebra homomorphism from $\langle C, \alpha \rangle$ to $\langle D, \beta \rangle$.

Lifted signature morphisms $(\mathsf{U}, \tau, \xi) : (\mathsf{C}, \mathsf{G}, \mathsf{F}, \sigma) \to (\mathsf{C}', \mathsf{G}', \mathsf{F}', \sigma')$ induce reduct functors $\mathsf{U}_{(\tau,\xi)} : \mathsf{Alg}(\mathsf{T}'_{\sigma'}) \to \mathsf{Alg}(\mathsf{T}_\sigma)$, taking a $(\mathsf{T}'_{\sigma'})$-algebra $\langle \langle C', \gamma' \rangle, \alpha' \rangle$ to the T_σ-algebra $\langle \langle \mathsf{U}C', \tau_{C'} \circ \mathsf{U}\gamma' \rangle, \mathsf{U}\alpha' \circ \nu_{C'} \rangle$.

For a lifted signature $(\mathsf{C}, \mathsf{G}, \mathsf{F}, \sigma)$, existence of finite limits in $\mathsf{Coalg}(\mathsf{C}, \mathsf{G})$ results in the existence of finite limits in $\mathsf{Alg}(\mathsf{T}_\sigma)$.

Proposition 3.3 *Let* $(\mathsf{C}, \mathsf{G}, \mathsf{T}, \sigma)$ *denote a lifted signature. Then,* $\mathsf{Alg}(\mathsf{T}_\sigma)$ *has finite limits, and the functor* $\mathsf{U}_{\mathsf{Coalg}(\mathsf{C},\mathsf{G})} : \mathsf{Alg}(\mathsf{T}_\sigma) \to \mathsf{Coalg}(\mathsf{C}, \mathsf{G})$ *taking* T_σ*-algebras to their underlying* (C, G)*-coalgebras preserves them.*

Proof. The conclusion follows from Corollary 2.10 together with the fact that $\mathsf{U}_{\mathsf{Coalg}(\mathsf{C},\mathsf{G})}$ creates limits. □

In particular, $\mathsf{Alg}(\mathsf{T}_\sigma)$ has a final object, given by the T_σ-algebra $\langle \langle F, \zeta \rangle, !_{\sigma_\zeta} \rangle$, with $\langle F, \zeta \rangle$ denoting a final G-coalgebra, and with $!_{\sigma_\zeta} : \langle \mathsf{T}F, \sigma_\zeta \rangle \to \langle F, \zeta \rangle$ denoting the unique G-coalgebra homomorphism of $\langle \mathsf{T}F, \sigma_\zeta \rangle$ into $\langle F, \zeta \rangle$. The final T_σ-algebra provides an interpretation of arbitrary computations on abstract states.

The fact that $\mathsf{U}_{\mathsf{Coalg}(\mathsf{C},\mathsf{G})}$ creates limits yields a T_σ-algebra structure on (C, G)-bisimilarity on the underlying coalgebra of a T_σ-algebra, such that the coalgebra homomorphisms defining the bisimilarity relation constitute T_σ-algebra homomorphisms.

Also, existence of an initial object in $\mathsf{Coalg}(\mathsf{C}, \mathsf{G})$ (following from the existence of an initial object in C) results in the existence of an initial T_σ-algebra.

Proposition 3.4 *Let* $(\mathsf{C}, \mathsf{G}, \mathsf{F}, \sigma)$ *denote a lifted signature. Then,* $\mathsf{Alg}(\mathsf{T}_\sigma)$ *has an initial object.*

Proof. The functor taking T_σ-algebras to their carrier has a left adjoint. Hence, an initial object in $\mathsf{Alg}(\mathsf{T}_\sigma)$ is given by the T_σ-algebra $\langle \mathsf{T}_\sigma(\langle 0, !_{\mathsf{G}0}), \mu_0 \rangle = \langle \langle \mathsf{T}0, \sigma_{!_{\mathsf{G}0}} \rangle, \mu_0 \rangle$, where 0 denotes an initial C-object (and consequently $!_{\mathsf{G}0} : 0 \to \mathsf{G}0$ defines an initial G-coalgebra). □

The initial T_σ-algebra provides an observational structure on ground computations.

Proposition 3.5 *For a lifted signature* $(\mathsf{C}, \mathsf{G}, \mathsf{F}, \sigma)$*,* $\mathsf{Alg}(\mathsf{T}_\sigma)$ *has pushouts and* $\mathsf{U}_{\mathsf{Coalg}(\mathsf{C},\mathsf{G})} : \mathsf{Alg}(\mathsf{T}_\sigma) \to \mathsf{Coalg}(\mathsf{C}, \mathsf{G})$ *preserves them.*

Proof. Existence of pushouts in C results in the existence of pushouts in

$\mathsf{Coalg}(\mathsf{C}, \mathsf{G})$ and in their preservation by $\mathsf{U_C} : \mathsf{Coalg}(\mathsf{C}, \mathsf{G}) \to \mathsf{C}$. Then, existence of pushouts in $\mathsf{Coalg}(\mathsf{C}, \mathsf{G})$ together with preservation of pushouts by T_σ (following from the preservation of pushouts by F and therefore by T, together with pushouts in $\mathsf{Coalg}(\mathsf{C}, \mathsf{G})$ being created by $\mathsf{U_C}$) results in the existence of pushouts in $\mathsf{Alg}(\mathsf{T}_\sigma)$, and in their preservation by $\mathsf{U}_{\mathsf{Coalg}(\mathsf{C},\mathsf{G})}$. □

Corollary 3.6 $\mathsf{Alg}(\mathsf{T}_\sigma)$ *has finite colimits.*

Remark 3.7 If G preserves equalisers, then $\mathsf{U_C}$ preserves images (see Proposition 2.14). Also, the dual of Proposition 2.12 results in $\mathsf{U}_{\mathsf{Coalg}(\mathsf{C},\mathsf{G})}$ preserving images. Finally, $\mathsf{U_C}$ reflects epimorphisms (as it creates pushouts), and so does $\mathsf{U}_{\mathsf{Coalg}(\mathsf{C},\mathsf{G})}$ (by the dual of Corollary 2.9). If, in addition, C is coregular, it follows that each T_σ-algebra homomorphism f has a factorisation of form $f = \iota \circ e$, with ι a T_σ-algebra monomorphism (given by the image of f in $\mathsf{Alg}(\mathsf{T}_\sigma)$) and e a T_σ-algebra epimorphism (given by the unique T_σ-algebra homomorphism resulting from the couniversality of ι). That is, the domain of ι defines a reachable [12] T_σ-algebra.

Once the relationship between computations and observations has been specified (by means of a lifted signature), abstract equations and respectively coequations can be used to formalise correctness properties of the specified behaviour. Specifically, high-level requirements involving the equivalence of computations can be captured by equations, whereas low-level requirements involving system invariants can be captured by coequations. Since the interest is in the *observable* result of *ground* computations, the associated notions of satisfaction abstract away bisimilar, respectively unreachable behaviours.

Definition 3.8 Let $(\mathsf{C}, \mathsf{G}, \mathsf{F}, \sigma)$ denote a lifted signature. A T_σ-algebra $\langle\langle C, \gamma\rangle, \alpha\rangle$ **satisfies** a (C, G)-coequation of form (K, l, r) **up to reachability** (written $\langle\langle C, \gamma\rangle, \alpha\rangle \models^r (\mathsf{K}, l, r)$) if and only if there exists $a : D \to \mathsf{K}C$ in C such that $a \circ [e_1, e_2] = [l_\gamma, r_\gamma]$

$$\begin{array}{ccc} C + C & \xrightarrow{[l_\gamma, r_\gamma]} & \mathsf{K}C \\ {\scriptstyle [e_1,e_2]}\downarrow & \nearrow_{a} & \\ D & & \end{array}$$

where $e_1, e_2 : \langle\langle C, \gamma\rangle, \alpha\rangle \to \langle\langle D, \delta\rangle, \beta\rangle$ define the cokernel pair of the unique T_σ-algebra homomorphism from the initial T_σ-algebra to $\langle\langle C, \gamma\rangle, \alpha\rangle$.

Also, $\langle\langle C, \gamma\rangle, \alpha\rangle$ **satisfies** a (C, F)-equation of form (K', l', r') **up to bisimulation** (written $\langle\langle C, \gamma\rangle, \alpha\rangle \models^b (\mathsf{K}', l', r')$) if and only if $\langle l'_{\alpha'}, r'_{\alpha'}\rangle$ factors through $\langle r_1, r_2\rangle$ in C:

$$\begin{array}{ccc} \mathsf{K}C & \xrightarrow{\langle l'_{\alpha'}, r'_{\alpha'}\rangle} & C \times C \\ & \searrow_{c} & \uparrow{\scriptstyle \langle r_1, r_2\rangle} \\ & & R \end{array}$$

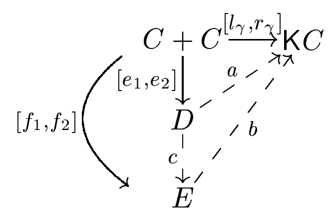
[12] A T_σ-algebra $\langle\langle C, \gamma\rangle, \alpha\rangle$ is **reachable** if and only if $[e_1, e_2] \leq [1_A, 1_A]$ for any co-relation $[e_1, e_2]$ on $\langle\langle C, \gamma\rangle, \alpha\rangle$.

where $r_1, r_2 : \langle\langle R, \rho\rangle, \xi\rangle \to \langle\langle C, \gamma\rangle, \alpha\rangle$ define the kernel pair of the unique T_σ-algebra homomorphism of $\langle\langle C, \gamma\rangle, \alpha\rangle$ into the final T_σ-algebra, and where $\langle C, \alpha'\rangle$ denotes the (C, F)-algebra induced by the T-algebra $\langle C, \alpha\rangle$.

Remark 3.9 $\langle C, \gamma\rangle \models (\mathsf{K}, l, r)$ implies $\langle\langle C, \gamma\rangle, \alpha\rangle \models^r (\mathsf{K}, l, r)$. (This follows by taking $a = l_\gamma \circ e = r_\gamma \circ e$, with $e : \langle\langle D, \delta\rangle, \beta\rangle \to \langle\langle C, \gamma\rangle, \alpha\rangle$ denoting the unique T_σ-algebra homomorphism satisfying $e \circ [e_1, e_2] = [1_C, 1_C]$.) Moreover, if $\langle C, \alpha\rangle$ is reachable, the converse also holds. (Since the notions of reachability in $\mathsf{Alg}(\mathsf{T})$ and $\mathsf{Alg}(\mathsf{T}_\sigma)$ coincide, reachability of $\langle C, \alpha\rangle$ is equivalent to $e_1 = e_2 = 1_C$.) Similarly, $\langle C, \alpha'\rangle \models (\mathsf{K}', l', r')$ implies $\langle\langle C, \gamma\rangle, \alpha\rangle \models^b (\mathsf{K}', l', r')$. Moreover, if $\langle C, \gamma\rangle$ is extensional, the converse also holds. (Since $\mathsf{U}_{\mathsf{Coalg(C,G)}}$ preserves cokernel pairs, extensionality of $\langle C, \gamma\rangle$ is equivalent to $r_1 = r_2 = 1_C$.)

Since $\mathsf{U_C} \circ \mathsf{U}_{\mathsf{Coalg(C,G)}}$ preserves cokernel pairs, $\langle\langle C, \gamma\rangle, \alpha\rangle \models^r (\mathsf{K}, l, r)$ is equivalent to $l_\gamma \circ ! = r_\gamma \circ !$, with $!$ denoting the unique T_σ-algebra homomorphism of the initial T_σ-algebra into $\langle\langle C, \gamma\rangle, \alpha\rangle$. Also, since $\mathsf{U_C} \circ \mathsf{U}_{\mathsf{Coalg(C,G)}}$ preserves kernel pairs, $\langle\langle C, \gamma\rangle, \alpha\rangle \models^b (\mathsf{K}', l', r')$ is equivalent to $!' \circ l'_{\alpha'} = !' \circ r'_{\alpha'}$, with $!'$ denoting the unique T_σ-algebra homomorphism of $\langle\langle C, \gamma\rangle, \alpha\rangle$ into the final T_σ-algebra.

The minimality of the image of the unique homomorphism from the initial algebra among the subalgebras of a given algebra of a lifted signature yields an inductive technique for proving the satisfaction of coequations up to reachability. Specifically, proving that a (C, G)-coequation holds, up to reachability, in a subclass $\mathcal{A}$ of $\mathsf{Alg}(\mathsf{T}_\sigma)$ can be reduced to exhibiting a natural monomorphism $\iota : \mathsf{S} \Rightarrow \mathsf{Id}_{\mathsf{Alg}(\mathsf{T}_\sigma)}$, for some endofunctor $\mathsf{S} : \mathsf{Alg}(\mathsf{T}_\sigma) \to \mathsf{Alg}(\mathsf{T}_\sigma)$, such that $l_\gamma \circ \iota_{\langle\langle C,\gamma\rangle,\alpha\rangle} = r_\gamma \circ \iota_{\langle\langle C,\gamma\rangle,\alpha\rangle}$ whenever $\langle\langle C, \gamma\rangle, \alpha\rangle$ is in $\mathcal{A}$.



For this, in turn, yields a C-arrow $b : E \to \mathsf{K}C$ such that $b \circ [f_1, f_2] = [l_\gamma, r_\gamma]$ (where $f_1, f_2 : \langle\langle C, \gamma\rangle, \alpha\rangle \Rightarrow \langle\langle E, \eta\rangle, \xi\rangle$ define the cokernel pair of $\iota_{\langle\langle C,\gamma\rangle,\alpha\rangle}$. On the other hand, the universality of $[e_1, e_2]$ yields a T_σ-algebra homomorphism $c : \langle\langle D, \delta\rangle, \beta\rangle \to \langle\langle E, \eta\rangle, \xi\rangle$ such that $c \circ [e_1, e_2] = [f_1, f_2]$. Then, a is taken to be $b \circ c$.

Similarly, the maximality of bisimilarity among the bisimulations on a given coalgebra yields a coinductive technique for proving the satisfaction of equations up to bisimulation. Specifically, proving that a (C, F)-equation holds, up to bisimulation, in a subclass $\mathcal{A}$ of $\mathsf{Alg}(\mathsf{T}_\sigma)$ can be reduced to exhibiting a generic (C, G)-bisimulation $\langle \mathsf{R}, \pi_1, \pi_2\rangle$, such that $\langle l'_{\alpha'}, r'_{\alpha'}\rangle$ factors through $\langle \pi_{1,\gamma}, \pi_{2,\gamma}\rangle$ whenever $\langle\langle C, \gamma\rangle, \alpha\rangle$ belongs to $\mathcal{A}$.

The following result further justifies the use of inductive and coinductive techniques in proving the satisfaction of coequations up to reachability and respectively of equations up to bisimulation by T_σ-algebras.

Proposition 3.10 *Let* $(\mathsf{C},\mathsf{G},\mathsf{F},\sigma)$ *denote a lifted signature. Then, the following hold:*

(i) *A* (C,G)*-coequation is satisfied up to reachability by the initial* T_σ*-algebra precisely when it is satisfied up to reachability by any* T_σ*-algebra.*

(ii) *A* (C,F)*-equation is satisfied up to bisimulation by the final* T_σ*-algebra precisely when it is satisfied up to bisimulation by any* T_σ*-algebra.*

Proof. i follows from the observation that, for a T_σ-algebra $\langle\langle C,\gamma\rangle,\alpha\rangle$ and a (C,G)-observer (K,c), $c_\gamma \circ ! = \mathsf{K}! \circ c_{\sigma_{!_{\mathsf{G}0}}}$, where $! : \langle\langle \mathsf{T}0, \sigma_{!_{\mathsf{G}0}}\rangle,\mu_0\rangle \to \langle\langle C,\gamma\rangle,\alpha\rangle$ denotes the unique T_σ-algebra homomorphism of the initial T_σ-algebra into $\langle\langle C,\gamma\rangle,\alpha\rangle$. ii follows by a similar argument. □

In assigning suitable denotations to lifted signatures, neither the initial nor the final algebra seem appropriate – the former does not identify bisimilar states, whereas the latter contains unreachable states. Existence of an algebra which is reachable and whose underlying coalgebra is extensional requires further constraints on the lifted signature involved. Because of the operational nature of our approach, coregularity of the underlying category of the lifted signature together with preservation of equalisers by the functor defining its underlying cosignature are particularly suitable in this respect, as they ensure the existence of a reachable subalgebra of the final algebra (see Remark 3.7). Moreover, this algebra has the property that it satisfies (in the standard sense) precisely those equations which are satisfied up to bisimulation by the initial algebra, and precisely those coequations which are satisfied up to reachability by the final coalgebra.

Proposition 3.11 *Let* $(\mathsf{C},\mathsf{G},\mathsf{F},\sigma)$ *denote a lifted signature. Also, let* $\iota : \langle\langle R,\gamma\rangle,\alpha\rangle \to \langle\langle F,\zeta\rangle,!_{\sigma_\zeta}\rangle$ *denote the image of the unique* T_σ*-algebra homomorphism from the initial* T_σ*-algebra* $\langle\langle \mathsf{T}0,\sigma_{!_{\mathsf{G}0}}\rangle,\mu_0\rangle$ *to the final* T_σ*-algebra* $\langle\langle F,\zeta\rangle,!_{\sigma_\zeta}\rangle$*. Then, the following hold:*

(i) *If* (K,l,r) *denotes a* (C,F)*-equation such that* K *preserves epimorphisms, then* $\langle\langle \mathsf{T}0,\sigma_{!_{\mathsf{G}0}}\rangle,\mu_0\rangle \models^b (\mathsf{K},l,r)$ *if and only if* $\langle R,\alpha\rangle \models (\mathsf{K},l,r)$*.*

(ii) *If* C *is coregular and* G *preserves equalisers, and if* (K,l,r) *denotes a* (C,G)*-coequation such that* K *preserves monomorphisms, then* $\langle\langle F,\zeta\rangle,!_{\sigma_\zeta}\rangle \models^r (\mathsf{K},l,r)$ *if and only if* $\langle R,\gamma\rangle \models (\mathsf{K},l,r)$*.*

Proof (Sketch) i follows from $\langle\langle \mathsf{T}0,\sigma_{!_{\mathsf{G}0}}\rangle,\mu_0\rangle \models^b (\mathsf{K},l,r)$ being equivalent to $\langle\langle R,\gamma\rangle,\alpha\rangle \models^b (\mathsf{K},l,r)$ (as K preserves epimorphisms), which, in turn is equivalent to $\langle R,\alpha\rangle \models (\mathsf{K},l,r)$ (as $\langle R,\gamma\rangle$ is extensional, see Remark 3.9). Also, ii follows from $\langle\langle F,\zeta\rangle,!_{\sigma_\zeta}\rangle \models^r (\mathsf{K},l,r)$ being equivalent to $\langle\langle R,\gamma\rangle,\alpha\rangle \models^r (\mathsf{K},l,r)$ (as K preserves monomorphisms), which, in turn is equivalent to $\langle R,\gamma\rangle \models (\mathsf{K},l,r)$. (The last equivalence follows by Remark 3.9 after noting that, if C is coregular and G preserves equalisers, then Remark 3.7 results in $\langle\langle R,\gamma\rangle,\alpha\rangle$, and therefore $\langle R,\alpha\rangle$ being reachable). □

The triviality of the algebraic component of lifted signature morphisms results in the notions of reachability associated to the source and target of such morphisms being essentially the same. This, in turn, yields an institution w.r.t. the satisfaction of coequations by algebras of lifted signatures.

Theorem 3.12 *Let* $(\mathsf{U}, \tau, \xi) : (\mathsf{C}, \mathsf{G}, \mathsf{F}, \sigma) \to (\mathsf{C}', \mathsf{G}', \mathsf{F}', \sigma')$ *denote a lifted signature morphism, let* $\langle\langle C, \gamma\rangle, \alpha\rangle$ *denote a* $\mathsf{T}'_{\sigma'}$*-algebra, and let* (K, l, r) *denote a* (C, G)*-coequation. Then,* $\langle\langle C, \gamma\rangle, \alpha\rangle \models \tau(\mathsf{K}, l, r)$ *if and only if* $\mathsf{U}_{(\tau,\xi)}\langle\langle C, \gamma\rangle, \alpha\rangle \models (\mathsf{K}, l, r)$.

Proof. The conclusion follows from the observation that $\mathsf{U}!' \circ \nu_{0'} = !$, with $! : \langle\langle \mathsf{T}0, \sigma_{!_{\mathsf{G}0}}\rangle, \mu_0\rangle \to \langle\langle \mathsf{U}C, \tau_C \circ \mathsf{U}\gamma\rangle, \mathsf{U}\alpha \circ \nu_C\rangle$ and $!' : \langle\langle \mathsf{T}'0', \sigma'_{!_{\mathsf{G}'0'}}\rangle, \mu'_0\rangle \to \langle\langle C, \gamma\rangle, \alpha\rangle$ denoting the unique T_σ- and respectively $\mathsf{T}'_{\sigma'}$-algebra homomorphisms from the initial T_σ- and respectively $\mathsf{T}'_{\sigma'}$-algebras to $\mathsf{U}_{(\tau,\xi)}\langle\langle C, \gamma\rangle, \alpha\rangle$ and $\langle\langle C, \gamma\rangle, \alpha\rangle$. (The fact that $\mathsf{U}0' = 0$, following from the colimit preservation property of left adjoints, is used here.) □

The conditions in the definition of lifted signature morphisms are not, however, sufficient to yield an institution w.r.t. the satisfaction of equations by algebras of lifted signatures. For this, additional constraints are required.

Definition 3.13 A lifted signature morphism (U, τ, ξ) is called **horizontal** if and only if τ is a natural isomorphism.

The notions of bisimilarity associated to the source and target signatures of horizontal lifted signature morphisms coincide. This, in turn, yields an institution.

Theorem 3.14 *Let* $(\mathsf{U}, \tau, \xi) : (\mathsf{C}, \mathsf{G}, \mathsf{F}, \sigma) \to (\mathsf{C}', \mathsf{G}', \mathsf{F}', \sigma')$ *denote a lifted signature morphism, let* $\langle\langle C, \gamma\rangle, \alpha\rangle$ *denote a* $\mathsf{T}'_{\sigma'}$*-algebra, and let* (K, l, r) *denote a* (C, F)*-equation. Then,* $\langle\langle C, \gamma\rangle, \alpha\rangle \models \tau(\mathsf{K}, l, r)$ *if and only if* $\mathsf{U}_{(\tau,\xi)}\langle\langle C, \gamma\rangle, \alpha\rangle \models (\mathsf{K}, l, r)$.

Proof (Sketch) The proof uses the fact that $\mathsf{U}F' \simeq F$ (with $\langle F, \zeta\rangle$ and $\langle F', \zeta'\rangle$ denoting the final G- and respectively G'-coalgebras), and is similar to that of Theorem 3.12. □

The conditions defining horizontal lifted signature morphisms might appear restrictive at first, as such morphisms are not able to capture development steps which specialise the observational features of an existing component. However, these conditions merely state that a notion of observation must be established *before* beginning to reason about the observational equivalence of computations, and that changing the notion of observation leads to different equivalences between computations.

A consequence of the definition of lifted signature morphisms (and indeed, the reason for this particular definition) is the existence of cofree constructions along such morphisms. As one would expect, the cofree construction induced by a lifted signature morphism builds on the cofree construction induced by its underlying cosignature morphism.

Theorem 3.15 *Let* $(\mathsf{U}, \tau, \xi) : (\mathsf{C}, \mathsf{G}, \mathsf{F}, \sigma) \to (\mathsf{C}', \mathsf{G}', \mathsf{F}', \sigma')$ *denote a lifted signature morphism. Then,* $\mathsf{U}_{(\tau,\xi)} : \mathsf{Alg}(\mathsf{T}'_{\sigma'}) \to \mathsf{Alg}(\mathsf{T}_\sigma)$ *has a right adjoint.*

Proof. Let $\langle\langle C, \gamma\rangle, \alpha\rangle \in |\mathsf{Alg}(\mathsf{T}_\sigma)|$, let $\epsilon_\gamma : \langle \mathsf{U}C', \tau_{C'} \circ \mathsf{U}\gamma'\rangle \to \langle C, \gamma\rangle$ denote a couniversal arrow from $\mathsf{U}_\tau : \mathsf{Coalg}(\mathsf{G}') \to \mathsf{Coalg}(\mathsf{G})$ to $\langle C, \gamma\rangle$, and let $\alpha' : \mathsf{T}C' \to C'$ denote the unique extension of $\alpha \circ \mathsf{T}\epsilon_\gamma \circ \nu_{C'}^{-1} : \langle \mathsf{U}\mathsf{T}'C', \tau_{\mathsf{T}'C'} \circ \mathsf{U}\sigma'_{\gamma'}\rangle \to \langle C, \gamma\rangle$ to a G'-coalgebra homomorphism.

$$
\begin{array}{ccc}
\mathsf{UT}'C' & \xrightarrow{\tau_{\mathsf{T}'C'} \circ \mathsf{U}\sigma'_{\gamma'}} & \mathsf{GUT}'C' \\
\nu_{C'}^{-1}\downarrow\ \uparrow\nu_{C'} & & \mathsf{G}\nu_{C'}\uparrow\ \downarrow\mathsf{G}\nu_{C'}^{-1} \\
\mathsf{TU}C' & \xrightarrow{\sigma_{\mathsf{U}_\tau\gamma'}} & \mathsf{GTU}C' \\
\mathsf{T}\epsilon_\gamma\downarrow & & \downarrow\mathsf{GT}\epsilon_\gamma \\
\mathsf{T}C & \xrightarrow{\sigma_\gamma} & \mathsf{GT}C \\
\alpha\downarrow & & \downarrow\mathsf{G}\alpha \\
C & \xrightarrow[\gamma]{} & \mathsf{G}C
\end{array}
\qquad\qquad
\begin{array}{ccc}
\mathsf{T}'C' & \xrightarrow{\sigma'_{\gamma'}} & \mathsf{G}'\mathsf{T}'C' \\
\alpha'\downarrow & & \downarrow\mathsf{G}'\alpha' \\
C' & \xrightarrow[\gamma']{} & \mathsf{G}'C'
\end{array}
$$

(Commutativity of the inner, and therefore also outer, upper-left square of the above diagram is a consequence of the definition of lifted signature morphisms.) Then, $\langle\langle C', \gamma'\rangle, \alpha'\rangle$ defines a $\mathsf{T}'_{\sigma'}$-algebra.

Moreover, $\langle\langle C', \gamma'\rangle, \alpha'\rangle$ is cofree over $\langle\langle C, \gamma\rangle, \alpha\rangle$ w.r.t. $\mathsf{U}_{(\tau,\xi)}$. For, given an arbitrary $\mathsf{T}'_{\sigma'}$-algebra $\langle\langle D, \delta\rangle, \beta\rangle$ together with a T_σ-homomorphism $f : \mathsf{U}_{(\tau,\xi)}\langle\langle D, \delta\rangle, \beta\rangle \to \langle\langle C, \gamma\rangle, \alpha\rangle$, the unique extension of f to a G'-coalgebra homomorphism $\bar{f} : \langle D, \delta\rangle \to \langle C', \gamma'\rangle$ defines a $\mathsf{T}'_{\sigma'}$-homomorphism from $\langle\langle D, \delta\rangle, \beta\rangle$ to $\langle\langle C', \gamma'\rangle, \alpha'\rangle$. This follows from the couniversality of ϵ_γ, after noting that $\epsilon_\gamma \circ \mathsf{U}(\bar{f} \circ \beta) = \epsilon_\gamma \circ \mathsf{U}(\alpha' \circ \mathsf{T}'\bar{f})$. (The last equality is a consequence of the fact that $f : \mathsf{U}D \to C$ defines a T-algebra homomorphism from $\langle \mathsf{U}D, \mathsf{U}\beta\rangle$ to $\langle C, \alpha\rangle$, and of the fact that ν is a natural isomorphism: $\epsilon_\gamma \circ \mathsf{U}\bar{f} \circ \mathsf{U}\beta \circ \nu_D = f \circ \mathsf{U}\beta \circ \nu_D = \alpha \circ \mathsf{T}f = \alpha \circ \mathsf{T}\epsilon_\gamma \circ \mathsf{TU}\bar{f} = \epsilon_\gamma \circ \mathsf{U}\alpha' \circ \nu_C \circ \mathsf{TU}\bar{f} = \epsilon_\gamma \circ \mathsf{U}\alpha' \circ \mathsf{UT}'\bar{f} \circ \nu_D$.)

$$
\begin{array}{ccccc}
\mathsf{TU}D & \xrightarrow{\nu_D} & \mathsf{UT}'D & \xrightarrow{\mathsf{U}\beta} & \mathsf{U}D \\
\downarrow\mathsf{TU}\bar{f} & & \downarrow\mathsf{UT}'\bar{f} & & \mathsf{U}\bar{f}\downarrow \\
\mathsf{TU}C' & \xrightarrow{\nu_C} & \mathsf{UT}'C' & \xrightarrow{\mathsf{U}\alpha'} & \mathsf{U}C' \\
\downarrow\mathsf{T}\epsilon_\gamma & & & & \epsilon_\gamma\downarrow \\
\mathsf{T}C & & \xrightarrow[\alpha]{} & & C
\end{array}
\qquad (\mathsf{T}f : \mathsf{TU}D \to \mathsf{T}C,\ f : \mathsf{U}D \to C)
$$

□

We note that it is precisely the triviality of the algebraic component of lifted signature morphisms that guarantees the existence of $\mathsf{T}'_{\sigma'}$-algebra structures on the cofree coextensions of the carriers of T_σ-algebras along $\mathsf{U}_{(\tau,\xi)}$.

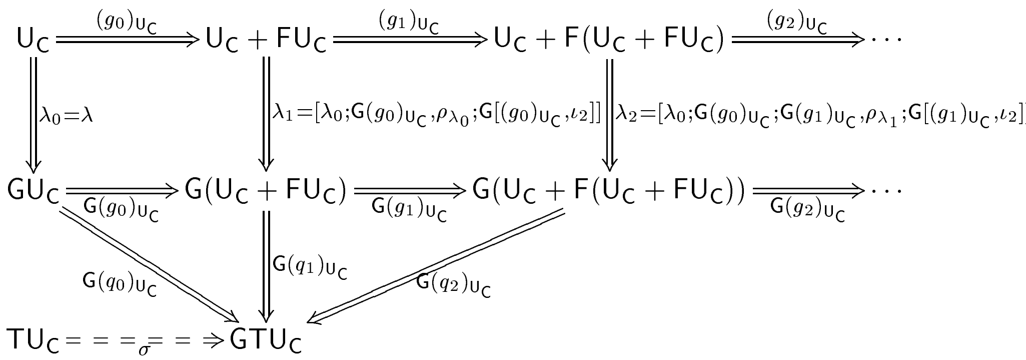
The natural transformation $\sigma : \mathsf{TU}_\mathsf{C} \Rightarrow \mathsf{GTU}_\mathsf{C}$ required by the definition of lifted signatures can be given in terms of a natural transformation $\rho : \mathsf{FU}_\mathsf{C} \Rightarrow \mathsf{G}(\mathsf{U}_\mathsf{C} + \mathsf{FU}_\mathsf{C})$ (defining the one-step observations of the results yielded by one-step computations as either zero- or one-step computations on their arguments) [13]. To see this, note that TU_C is a colimit object for the following

[13] This observation is based on the approach in [14].

ω-chain:

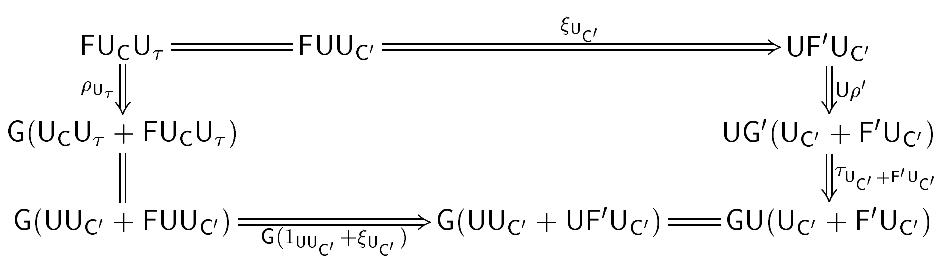
$$\mathsf{U_C} \xrightarrow{(g_0)_{\mathsf{U_C}}} \mathsf{U_C} + \mathsf{FU_C} \xrightarrow{(g_1)_{\mathsf{U_C}}} \mathsf{U_C} + \mathsf{F}(\mathsf{U_C} + \mathsf{FU_C}) \xrightarrow{(g_2)_{\mathsf{U_C}}} \cdots$$

with $g_0, g_1, \ldots$ being given by: $g_0 = \iota_1$, $g_{i+1} = 1_{\mathsf{Id_C}} + \mathsf{F}g_i$ for $i = 0, 1, \ldots$, and with $(q_0)_{\mathsf{U_C}} : \mathsf{U_C} \Rightarrow \mathsf{TU_C}$, $(q_1)_{\mathsf{U_C}} : \mathsf{U_C} + \mathsf{FU_C} \Rightarrow \mathsf{TU_C}$, $\ldots$ as colimit arrows. Then, defining $\sigma : \mathsf{TU_C} \Rightarrow \mathsf{GTU_C}$ amounts to making $\mathsf{GTU_C}$ into a cocone for this ω-chain. The following diagram defines such a cocone:



with the natural transformation $\lambda : \mathsf{U_C} \Rightarrow \mathsf{GU_C}$ being given by $\lambda_{\langle C,\gamma \rangle} = \gamma$ for $\langle C, \gamma \rangle \in |\mathsf{Coalg(G)}|$. (Commutativity of the upper squares follows by induction, using the naturality of ρ.) The colimiting property of $\mathsf{TU_C}$ then yields a natural transformation $\sigma : \mathsf{TU_C} \Rightarrow \mathsf{GTU_C}$ satisfying: $\lambda_i; \mathsf{G}(q_i)_{\mathsf{U_C}} = (q_i)_{\mathsf{U_C}}; \sigma$ for $i = 0, 1, \ldots$. Moreover, σ satisfies the conditions in Definition 3.1: $\sigma \circ (q_0)_{\mathsf{U_C}} = \mathsf{G}(q_0)_{\mathsf{U_C}} \circ \lambda$ follows immediately from the colimiting property of $\mathsf{TU_C}$, while $\sigma \circ \mu_{\mathsf{U_C}} = \mathsf{G}\mu_{\mathsf{U_C}} \circ \sigma_{\mathsf{T}_\sigma}$ follows by induction, using the naturality of ρ.

Also, if $(\mathsf{C}, \mathsf{G}, \mathsf{F}, \sigma)$ and $(\mathsf{C}', \mathsf{G}', \mathsf{F}', \sigma')$ are the lifted signatures induced by the natural transformations $\rho : \mathsf{FU_C} \Rightarrow \mathsf{G}(\mathsf{U_C} + \mathsf{FU_C})$ and respectively $\rho' : \mathsf{F}'\mathsf{U}_{\mathsf{C}'} \Rightarrow \mathsf{G}'(\mathsf{U}_{\mathsf{C}'} + \mathsf{F}'\mathsf{U}_{\mathsf{C}'})$, and if $(\mathsf{U}, \tau) : (\mathsf{C}, \mathsf{G}) \to (\mathsf{C}', \mathsf{G}')$ is a cosignature morphism and $(\mathsf{U}, \xi) : (\mathsf{C}, \mathsf{F}) \to (\mathsf{C}', \mathsf{F}')$ is a signature morphism with ξ a natural isomorphism, such that the following diagram commutes:



then (U, τ, ξ) defines a lifted signature morphism from $(\mathsf{C}, \mathsf{G}, \mathsf{F}, \sigma)$ to $(\mathsf{C}', \mathsf{G}', \mathsf{F}', \sigma')$. (The above constraint on ρ, ρ', ξ and τ ensures that ρ and ν satisfy the constraint on σ, σ', ν and τ required by Definition 3.1.)

We conclude this section by noting that an approach which involves lifting the algebraic structure of syntactic domains (of programs) to an algebraic structure on observable behaviours over these syntactic domains can be obtained essentially by dualising the definitions and results of this section. In particular, such an approach can be used for the specification of data type observers. (This will be illustrated in the next section.)

4 A Specification Formalism for Objects

Instantiating the abstract specification framework described in Section 3 to endofunctors of the form of the ones induced by many-sorted signatures and respectively cosignatures yields a formalism for the specification and verification of objects.

We begin by introducing the notions of constructor signature and respectively destructor cosignature, to be used later in specifying the computational, and respectively observational features of objects. Given a many-sorted signature (V, Ψ), a *constructor signature over* Ψ is a pair (H, Γ), with $(V \cup H, \Gamma)$ a many-sorted signature satisfying: (i) $\Psi \subseteq \Gamma$ and (ii) $(\Gamma \setminus \Psi)_v = \emptyset$. Similarly, given a many-sorted cosignature (V, Ξ), a *destructor cosignature over* Ξ is a pair (H, Δ), with $(V \cup H, \Delta)$ a many-sorted cosignature satisfying: (i) $\Xi \subseteq \Delta$ and (ii) $(\Delta \setminus \Xi)_v = \emptyset$. Also, a *constructor signature morphism* from (H, Γ) to (H', Γ') (respectively *destructor cosignature morphism* from (H, Δ) to (H', Δ')) is a many-sorted signature morphism $\phi : (V \cup H, \Gamma) \to (V \cup H', \Gamma')$ (many-sorted cosignature morphism $\psi : (V \cup H, \Delta) \to (V \cup H', \Delta')$) satisfying $\phi(H) \subseteq H'$ and $\phi\restriction_\Psi = 1_\Psi$ ($\psi(H) \subseteq H'$ and $\psi\restriction_\Xi = 1_\Xi$).

Now let D denote a V-sorted set carrying both a Ψ-structure with Ψ a V-sorted signature, and a Ξ-structure with Ξ a V-sorted cosignature. Then, given a constructor signature (H, Γ) over Ψ (respectively destructor cosignature (H, Δ) over Ξ), a Γ-*algebra over* D (respectively Δ-*coalgebra over* D) is given by a many-sorted $(V \cup H, \Gamma)$-algebra A (many-sorted $(V \cup H, \Delta)$-coalgebra C) such that $A\restriction_\Psi = D\restriction_\Psi$ ($C\restriction_\Xi = D\restriction_\Xi$). Also, an (H, Γ)-*homomorphism* between Γ-algebras A and B (respectively (H, Δ)-*homomorphism* between Δ-coalgebras C and D) over D is a many-sorted $(V \cup H, \Gamma)$-homomorphism $f : A \to B$ (many-sorted $(V \cup H, \Delta)$-homomorphism $g : C \to D$) satisfying $f_v = 1_{D_v}$ ($g_v = 1_{D_v}$) for $v \in V$.

Then, the category $\mathsf{Coalg}_D(H, \Delta)$ of Δ-coalgebras over D coincides with $\mathsf{Coalg}(\mathsf{Set}^S_D, \mathsf{G}'_\Delta)$, with $S = V \cup H$, with Set^S_D denoting the category of S-sorted sets whose visible components are given by D and S-sorted functions whose visible components are given by 1_D, and with G'_Δ being given by:

$$(\mathsf{G}'_\Delta X)_s = \begin{cases} D_s & \text{if } s \in V \\ (\mathsf{G}_\Delta X)_s & \text{if } s \in H \end{cases}, \quad X \in |\mathsf{Set}^S|,\ s \in S.$$

Also, if $\phi : (H, \Delta) \to (H', \Delta')$ denotes a destructor cosignature morphism, then $\mathsf{U}_\phi : \mathsf{Coalg}_D(H', \Delta') \to \mathsf{Coalg}_D(H, \Delta)$ is naturally isomorphic to $\mathsf{U}_{\eta'_\phi}$, where $\eta'_\phi : \mathsf{U}' \circ \mathsf{G}'_{\Delta'} \Rightarrow \mathsf{G}'_\Delta \circ \mathsf{U}'$ is given by:

$$(\eta'_{\phi,X})_s = \begin{cases} 1_{D_s} & \text{if } s \in V \\ (\eta_{\phi,X})_s & \text{if } s \in H \end{cases}, \quad s \in S,\ X \in |\mathsf{Set}_D^{S'}|$$

and where $\mathsf{U}' : \mathsf{Set}_D^{S'} \to \mathsf{Set}_D^S$ denotes the functor induced by the sort component of ϕ.

If F and G denote the endofunctors associated to a many-sorted signature Γ and respectively a many-sorted cosignature Δ, and if T and D denote the monad and respectively the comonad induced by these endofunctors, then defining natural transformations $\mathsf{TU} \Rightarrow \mathsf{GTU}$ and respectively $\mathsf{FDU} \Rightarrow \mathsf{DU}$ (with U being used to denote the functors taking G-coalgebras, respectively F-algebras to their carriers) involves, in each case, defining the values of certain program observations in terms of particular observations of the program arguments. However, specific observations of the program arguments can not be used in expressions defining observations of the results yielded by the programs before their type has been determined. The notion of *constraint* formalises this intuition.

Definition 4.1 Let (S, Γ) denote a many-sorted signature, and let (S, Δ) denote a many-sorted cosignature [14]. A (Γ, Δ)**-constraint** is given by a tuple $((c.t, Z.t'), (c_i.X_{j_i}, Z_i.Y_i)_{i=1,\ldots,n})$ (alternatively denoted $c.t = Z.t'$ if $c_1.X_{j_1} = Z_1.Y_1, \ldots, c_n.X_{j_n} = Z_n.Y_n$) with $t \in T_\Gamma(\mathcal{V})_s$ and $c \in T_\Delta[\mathcal{C}]_s$ for some $s \in S$, $t' \in T_\Gamma(\{Y_1, \ldots, Y_n\})_{s'}$ and $Z \in covar(c)_{s'}$ for some $s' \in S$, $X_{j_i} \in var(t)_{s_i}$ and $c_i \in T_\Delta[\mathcal{C}_i]_{s_i}$ for some $s_i \in S$, and $Y_i : s'_i$ and $Z_i \in covar(c_i)_{s'_i}$ for some $s'_i \in S$, for $i = 1, \ldots n$.

The conditional part of a constraint is used to extract the values of certain observations on the states/values denoted by variables in the lhs, to be used in the rhs for defining the value of the lhs.

In writing constraints, it is often the case that some of the variables used in the lhs of constraints are (indirectly) passed as arguments to the rhs, namely through conditions of form $Z.X = Z.Y$. For convenience, in forthcoming examples we shall omit such conditions from the conditional part of constraints and use the original variable X instead of its counterpart Y in the rhs of the constraint.

Then, the natural transformations σ defining lifted signatures and respectively lifted cosignatures whose underlying endofunctors correspond to many-sorted signatures and cosignatures can be specified using suitably restricted sets of constraints. In particular, if F and G denote the endofunctors induced by a many-sorted signature (S, Γ) and respectively a many-sorted cosignature (S, Δ), then natural transformations $\rho : \mathsf{F}(\mathsf{U} \times \mathsf{GU}) \Rightarrow \mathsf{FU}$ (inducing natural

[14] In particular, (S, Γ) and (S, Δ) can be given by the many-sorted signature, respectively cosignature underlying a constructor signature over (V, Ψ), respectively a destructor cosignature over (V, Ξ).

transformations σ : $\mathsf{FDU} \Rightarrow \mathsf{DU}$) can be specified using sets E of (Γ, Δ)-constraints of form:

$$[Z_1, \dots, Z_n]\delta.\gamma(X_1, \dots, X_m) = Z.t \text{ if } c_1.X_{j_1} = Z'_1.Y_1, \dots, c_k.X_{j_k} = Z'_k.Y_k$$

with $c_1, \dots, c_n$ containing at most one Δ-symbol, subject to the additional constraint that for each $s \in S$, $\gamma \in \Gamma_{s_1 \dots s_m, s}$, $\delta \in \Delta_{s, s'_1 \dots s'_n}$ and each choice $(n_{\delta,i})_{\delta \in \Delta_{s_i}, i \in \{1, \dots, m\}}$ of *one-step Δ-behaviours* for the sorts $s_1, \dots, s_m$ (with a one-step Δ-behaviour for sort s being given by a tuple $(n_\delta)_{\delta \in \Delta_s}$ with $n_\delta \in \{1, \dots, k\}$ for $\delta \in \Delta_{s, s_1 \dots s_k}$), there exists a unique (Γ, Δ)-constraint of form:

$$[Z_1, \dots, Z_n]\delta.\gamma(X_1, \dots, X_m) = Z.t \text{ if } c_1.X_{j_1} = Z'_1.Y_1, \dots, c_k.X_{j_k} = Z'_k.Y_k$$

in E whose conditions hold for the chosen one-step Δ-behaviours for $X_1, \dots, X_m$ (that is, if c_i is of form $[Z^i_1, \dots, Z^i_p]\delta_i$ and $n_{\delta_i, j_i} = l$, then $Z'_i = Z^i_l$, for $i = 1, \dots, m$).

Similarly, if F and G denote the endofunctors induced by a constructor signature (H, Γ) over Ψ and respectively a destructor cosignature (H, Δ) over Ξ, then natural transformations $\rho : \mathsf{FU} \Rightarrow \mathsf{G}(\mathsf{U} + \mathsf{FU})$ (inducing natural transformations $\sigma : \mathsf{TU} \Rightarrow \mathsf{GTU}$) can be specified using sets E of (Γ, Δ)-constraints of form:

$$[Z_1, \dots, Z_n]\delta.\gamma(X_1, \dots, X_m) = Z.t \text{ if } c_1.X_{j_1} = Z'_1.Y_1, \dots, c_k.X_{j_k} = Z'_k.Y_k$$

with $t \in T_\Gamma(\{Y_1, \dots, Y_k\})$ containing at most one $\Gamma \setminus \Psi$-symbol, subject to the additional constraint that for each $h \in H$, $\gamma \in \Gamma_{s_1 \dots s_m, h}$, $\delta \in \Delta_{h, s'_1 \dots s'_n}$ and each choice of *Δ-behaviours* $f_1, \dots, f_m$ for the sorts $s_1, \dots, s_m$ (with a Δ-behaviour for sort s being given by an element $f \in F_s$ of the final Δ-coalgebra over D), there exists a unique (Γ, Δ)-constraint of form:

$$[Z_1, \dots, Z_n]\delta.\gamma(X_1, \dots, X_m) = Z.t \text{ if } c_1.X_{j_1} = Z'_1.Y_1, \dots, c_n.X_{j_n} = Z'_n.Y_n$$

in E whose conditions hold for $f_1, \dots, f_m$ (i.e. $(c_i)_F(f_{j_i}) \in \iota_{Z'_i}(F_{s'_i})$ for $i = 1, \dots, n$).

The remaining of this section exemplifies the formalism resulting from the particular instantiation described above with a specification of (bounded) stacks of natural numbers.

First, the data manipulated by stacks is specified using visible sorts `1` and `Nat`, together with algebraic operations $* :\to 1$, $\mathtt{0} :\to \mathtt{Nat}$ and $\mathtt{s} : \mathtt{Nat} \to \mathtt{Nat}$, coalgebraic operations $\mathtt{p} : \mathtt{Nat} \to \mathtt{1} + \mathtt{Nat}$ and $\mathtt{!} : \mathtt{Nat} \to \mathtt{1}$, and constraints:

```
[Z,N]p.0 = Z.*
[Z,N]p.s(x) = N.x
[Z]!.0 = Z.*
[Z]!.s(x) = Z.*
```

These constraints define a lifted cosignature $(\mathsf{Set}^{\{1,\mathtt{Nat}\}}, \mathsf{F}, \mathsf{G}, \sigma)$, with $\mathsf{F}, \mathsf{G} : \mathsf{Set}^{\{1,\mathtt{Nat}\}} \to \mathsf{Set}^{\{1,\mathtt{Nat}\}}$ being given by: $(\mathsf{F}X)_\mathtt{1} = 1$, $(\mathsf{F}X)_\mathtt{Nat} = 1 + X_\mathtt{Nat}$, $(\mathsf{G}X)_\mathtt{1} =$

1 and $(\mathsf{G}X)_{\mathtt{Nat}} = (X_1 + X_{\mathtt{Nat}}) \times X_1$ (with 1 denoting both a sort and the empty product in Set), and with the natural transformation σ being induced by the natural transformation $\rho : \mathsf{F}(\mathsf{U} \times \mathsf{GU}) \Rightarrow \mathsf{GU}$ whose components are given by: $(\rho_A)_1(*) = *$, $\pi_1((\rho_A)_{\mathtt{Nat}}(\iota_1(*))) = \iota_1(*_A)$, $\pi_1((\rho_A)_{\mathtt{Nat}}(\iota_2(\langle a, a' \rangle))) = \iota_2(a)$ and $\pi_2((\rho_A)_{\mathtt{Nat}}(x)) = \iota_1(*_A)$, for each F-algebra A.

To facilitate the formulation of correctness properties for stacks, derived coalgebraic operations `<m` : `Nat` $\rightarrow$ `1` + `1` with `m` $\in \mathbb{N}$ (inducing G-observers) can be defined using coequations:

```
[T,F]<0 = [F]!
[T,F]<(m+1) = [T,[T,F]<m]p
```

with `m` $\in \mathbb{N}$.

Then, one can use induction to show that the coequation:

```
[T,F]<(m+1) = [T,F]<m if ([T,F]<m,T)
```

with `m` $\in \mathbb{N}$ holds, up to reachability, in all coalgebras of the cosignature of natural numbers.

We let D denote the quotient of the unique homomorphism from the initial to the final coalgebra of the above lifted cosignature.

Stacks of natural numbers are now specified using a hidden sort `Stack`, observers `first` : `Stack` $\rightarrow$ `1 Nat`, `rest` : `Stack` $\rightarrow$ `1 Stack`, constructors `empty` : $\rightarrow$ `Stack`, `push` : `Stack Nat` $\rightarrow$ `Stack`, `pop` : `Stack` $\rightarrow$ `Stack`, and constraints:

```
[Z,E]first.empty = Z.*
[Z,S]rest.empty = Z.*
[Z,E]first.push(s,e) = E.e
[Z,S]rest.push(s,e) = S.s
[Z,E]first.pop(s) = Z.* if [Z,[Z,E]first]rest.s=Z.z
[Z,E]first.pop(s) = E.e if [Z,[Z,E]first]rest.s=E.e
[Z,S]rest.pop(s) = Z.* if [Z,[Z,S]rest]rest.s=Z.z
[Z,S]rest.pop(s) = S.s if [Z,[Z,S]rest]rest.s=S.s
```

(The first and third constraint defining `pop` cover both the case when the stack denoted by `s` is empty, and the case when this stack only contains one element.)

One can then use coinduction to prove that the equations:

```
pop(empty) = empty
pop(push(s,e)) = s
```

with `s` of type `Stack` hold, up to bisimulation, in all algebras of the lifted signature of stacks.

A state invariant for stacks is captured by the coequations:

```
[Z,S]rest = [Z,S']rest if ([Z,E]first,Z)
[Z,E]first = [Z,E']first if ([Z,S]rest,Z)
```

and induction can be used to prove their satisfaction up to reachability by algebras of the lifted signature of stacks. (The proof requires some insights into the notion of reachability under `empty`, `push` and `pop`.)

The specification of stacks can now be extended to obtain a specification of bounded stacks of maximum size `m`, with $\mathtt{m} \in \mathbb{N}$. This is achieved by adding a new hidden sort `BStack` together with observers `stack` : `BStack` → `Stack`, `depth` : `BStack` → `Nat`, constructors `empty` :→ `BStack`, `push` : `BStack Nat` → `BStack`, `pop` : `BStack` → `BStack` and constraints:

```
[S]stack.empty = S.empty
[N]depth.empty = N.0
[S]stack.push(b,e) = S.push(s,e)
     if [T,F]full?.b = F.f and [S]stack.b = S.s
[S]stack.push(b,e) = S.s
     if [T,F]full?.b = T.t and [S]stack.b = S.s
[N]depth.push(b,e) = N.s(d)
     if [T,F]full?.b = F.f and [D]depth.b = D.d
[N]depth.push(b,e) = N.d
     if [T,F]full?.b = T.t and [D]depth.b = D.d
[S]stack.pop(b) = S.pop(s) if [S]stack.b = S.s
[N]depth.pop(b) = N.0 if [[F,D]p]depth.b = F.f
[N]depth.pop(b) = N.d if [[F,D]p]depth.b = D.d
```

where `full?` : `BStack` → `1 1` denotes a derived observer defined by the coequation:

```
[T,F]full? = [[F,T]<m]depth
```

The depth of a bounded stack of size `m` should not exceed `m`. This property of bounded stacks can be formalised using the coequation:

```
[[T,F]<(m+1)]depth = [T]!
```

with `!` : `BStack` → `1` denoting the constant observer defined by:

```
[T]!.b = T.*
```

gain, one can use induction to prove that the above coequation holds, up to reachability, in all algebras of the lifted signature of bounded stacks. (The fact that the equation:

```
x = *
```

holds in D is used here.)

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