On the Takagi interpolation problem

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Dedicated to Professor Paul A. Fuhrmann on the occasion of his retirement.

1 Introduction and problem statement

In this paper we consider some issues related to the following problem:

Let N distinct points λ_i in the open right-half plane be given, together with N subspaces $\mathcal{V}_i \subset \mathbb{C}^{\mathtt{m}+\mathtt{p}}$, and let

$$J := \begin{bmatrix} I_{\mathtt{m}} & 0 \\ 0 & -I_{\mathtt{p}} \end{bmatrix} \tag{1}$$

Find the smallest $k \in \mathbb{N}$ and $Y \in \mathbb{R}^{\mathtt{p}\times\mathtt{p}}[\xi]$, $U \in \mathbb{R}^{\mathtt{p}\times\mathtt{m}}[\xi]$ such that

(*a*) *U, Y are left coprime*;

(*b*) $\begin{bmatrix} U(\lambda_i) & -Y(\lambda_i) \end{bmatrix} v = 0$, *for all $v \in \mathcal{V}_i$, $1 \leq i \leq N$*;

(*c*) $\|Y^{-1}U\|_\infty < 1$;

(*d*) *Y has k singularities in the right half-plane.*

This problem is a generalization of the *tangential Takagi interpolation problem* with simple multiplicities (in the following abbreviated with *TIP*), which was first studied in [20] as an extension of the well-known Nevanlinna-Pick interpolation problem (see [4, 14, 16, 18]). The TIP and the closely related Nudel'man problem have been posed and solved following various approaches: in the discrete-time case (see [20] and [15]), in the context of interpolation with rational matrix functions in continuous-time as reported in the book [4], and with the generalized Beurling-Lax approach introduced in [5].

This paper examines some issues related to the existence and characterization of solutions of the *TIP*, following an approach based on the exact modeling

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of vector-exponential time-series illustrated in [1, 3, 11, 24, 25]. We state an algorithm which computes a special kernel representation of the MPUM of the data $\{(\lambda_i, \mathcal{V}_i)$, $1 \leq i \leq N$, and of their "duals" (see [2] for the first use of the dualization technique, and [11, 18] for further applications in the context of exact identification). We also investigate several properties of this special representation of the MPUM, some of which are related to the signature of the so-called Pick matrix of the data introduced in [18]. The special representation of the MPUM obtained by our algorithm is used in the main result of this paper in order to characterize all solutions to the *TIP*. An important original aspect of the material illustrated here with respect to that present in [11, 18] is the development of specific intermediate results that connect in a novel way the signature of the Pick matrix with the location of the roots of the determinant of certain subblocks of the MPUM representation computed by the algorithm.

The authors assume that the reader is familiar with the behavioral approach to systems and control (see [17] for a thorough introduction) and, at least for some detail of the proofs, with quadratic differential forms (for more information on this subject, see [27]). In order to make the paper as self-contained as possible, the basics of exact modeling and the notion of Most Powerful Unfalsified Model (MPUM) are introduced in section 2. The main results of this paper are illustrated in section 3. Finally, in section 4 we discuss some further research topics stemming from the work presented here.

Notation. In this paper we denote the sets of real numbers with $\mathbb{R}$, and the set of complex numbers with $\mathbb{C}$. The space of $\mathtt{n}$ dimensional real vectors is denoted by $\mathbb{R}^{\mathtt{n}}$, and the space of $\mathtt{m} \times \mathtt{n}$ real matrices, by $\mathbb{R}^{\mathtt{m}\times\mathtt{n}}$. The set of all maps from X to Y is denoted with $(Y)^X$. The powerset of X is denoted with 2^X.

If $A \in \mathbb{R}^{\mathtt{m}\times\mathtt{n}}$, then $A^T \in \mathbb{R}^{\mathtt{n}\times\mathtt{m}}$ denotes its transpose. Whenever one of the two dimensions is not specified, a bullet $\bullet$ is used; so that for example, $\mathbb{C}^{\bullet\times\mathtt{n}}$ denotes the set of complex matrices with $\mathtt{n}$ columns and an unspecified number of rows. In order to enhance readability, when dealing with a vector space $\mathbb{R}^{\bullet}$ whose elements are commonly denoted with w, we use the notation $\mathbb{R}^{\mathtt{w}}$ (note the typewriter font type!); similar considerations hold for matrices representing linear operators on such spaces. If $A_i \in \mathbb{R}^{\bullet\times\bullet}$, $i = 1, \dots, r$ have the same number of columns, $\text{col}(A_i)_{i=1,\dots,r}$ denotes the matrix obtained by stacking the A_i on top of each other. If $H \in \mathbb{C}^{\mathtt{w}\times\mathtt{w}}$ is an Hermitian matrix, i.e. $H^* := \bar{H}^T = H$,then we define its signature to be the ordered triple $\text{sign}(H) = (\nu_-(H), \nu_0(H), \nu_+(H))$, where $\nu_-(H)$ is the number (counting multiplicities) of negative eigenvalues of H, $\nu_0(H)$ is the multiplicity of the zero eigenvalue of H, and $\nu_+(H)$ is the number (counting multiplicities) of positive eigenvalues of H.

The ring of polynomials with real coefficients in the indeterminate ξ is denoted by $\mathbb{R}[\xi]$; the set of two-variable polynomials with real coefficients in the indeterminates ζ and η is denoted by $\mathbb{R}[\zeta, \eta]$. The space of all $\mathtt{n} \times \mathtt{m}$ polynomial matrices in the indeterminate ξ is denoted by $\mathbb{R}^{\mathtt{n}\times\mathtt{m}}[\xi]$, and that consisting of all $\mathtt{n} \times \mathtt{m}$ polynomial matrices in the indeterminates ζ and η by $\mathbb{R}^{\mathtt{n}\times\mathtt{m}}[\zeta, \eta]$. Given a matrix $R \in \mathbb{R}^{\mathtt{n}\times\mathtt{m}}[\xi]$, we define $R^*(\xi) := R^T(-\xi) \in \mathbb{R}^{\mathtt{m}\times\mathtt{n}}[\xi]$. If $R(\xi)$ has complex coefficients, then $R^*(\xi)$ denotes the matrix obtained from R by substituting $-\xi$ in place of ξ, transposing, and conjugating.

We denote with $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{q}})$ the set of infinitely often differentiable functions from $\mathbb{R}$ to $\mathbb{R}^{\mathtt{q}}$.

2 The Most Powerful Unfalsified Model

In this paper, we denote with $\mathcal{L}^{\mathtt{w}} \subseteq 2^{\mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})}$ the set of *linear differential behaviors*, consisting of elements $\mathfrak{B} \subseteq \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ for which there exists $R \in \mathbb{R}^{\bullet\times\mathtt{w}}[\xi]$ such that $\mathfrak{B} = \ker R(\frac{d}{dt})$. The representation

$$R(\frac{d}{dt})w = 0 \tag{2}$$

is called a *kernel representation* of the behavior $\mathfrak{B}$.

Now let $w_i \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$, $i = 1, \ldots, N$, be *polynomial vector exponential functions*, i.e.

$$w_i(t) = \sum_{j=1}^{k_i} v_{i,j} \frac{t^j}{j!} exp_{\lambda_i}(t)$$

where $v_{i,j} \in \mathbb{C}^{\mathtt{w}}$, and $\lambda_i \in \mathbb{C}$, $i =, \ldots, N$, $j = 1, \ldots, k_i$. $\mathfrak{B} \in \mathcal{L}^{\mathtt{w}}$ is an *unfalsified model* for the *data set* $\{w_i\}_{i=1,\ldots,N}$ if $w_i \in \mathfrak{B}$ for $i = 1, \ldots, N$. We call a behavior $\mathfrak{B}^*$ the *Most Powerful Unfalsified Model* (*MPUM*) in $\mathcal{L}^{\mathtt{w}}$ for the given data set, if it is unfalsified and moreover

$$[w_i \in \mathfrak{B}', I = 1, \ldots, N,\ \mathfrak{B}' \in \mathcal{L}^{\mathtt{w}}] \Longrightarrow [\mathfrak{B}^* \subseteq \mathfrak{B}']$$

i.e. if it is the smallest behavior in $\mathcal{L}^{\mathtt{w}}$ containing the data. It can be shown that the MPUM always exists and that it is unique (see [3]). Indeed, define

$$\mathfrak{B}^* := \text{span}\{w_i\}, \tag{3}$$

and observe that $\mathfrak{B}^*$ contains all trajectories w_i, $1 \leq i \leq N$. On the other hand, any other unfalsified model in $\mathcal{L}^{\mathtt{w}}$ for the data must contain their linear span, and therefore it must contain $\mathfrak{B}^*$.

Observe that the MPUM $\mathfrak{B}^*$ of a finite set of polynomial vector exponential trajectories is always autonomous, i.e. it is a finite dimensional subspace of $\mathcal{L}^{\mathtt{w}}$. Equivalently, $\mathfrak{B}^*$ can be represented as the kernel of a matrix polynomial differential operator $R(\frac{d}{dt})$, with the property that R is square and nonsingular as a polynomial matrix (see [17]).

We now show how to compute a kernel representation of the MPUM $\mathfrak{B}^*$ for a given set of vector-exponential trajectories $\{v_i \exp_{\lambda_i t}\}_{i=1,\ldots,N}$ (see [26]). Define $R_{-1} := I_q$ and proceed iteratively as follows for $k = 0, 1, \ldots, N$. At step k, define the k-th *error trajectory*

$$\varepsilon_k := R_{k-1}(\frac{d}{dt})v_k \exp_{\lambda_k t} = \underbrace{R_{k-1}(\lambda_k t)v_k}_{:=e_k} \exp_{\lambda_k t} = e_k \exp_{\lambda_k t}$$

Now compute the polynomial matrix corresponding to a kernel representation E_k of the MPUM for ε_k, i.e. $E_k(\frac{d}{dt})\varepsilon_k = 0$; one possible choice for E_k is:

$$E_k(\frac{d}{dt}) = \frac{d}{dt} I_{\mathtt{w}} - \lambda_k \frac{e_k e_k^T}{\|e_k\|^2}$$

and define $R_k := E_k R_{k-1}$. After $N+1$ steps such algorithm produces a $\mathtt{w} \times \mathtt{w}$ polynomial matrix R_N such that $R_N(\frac{d}{dt})w_i = 0$ for $i = 1, \ldots, N$, and moreover

$$\mathfrak{B}^* = \ker\ R_N(\frac{d}{dt})$$

In the next section we show that the algorithm illustrated above can be adapted to work in the case when the data trajectories need to be "explained" by a model having specific metric- and stability constraints.

3 Main result

We begin by showing that the Takagi interpolation problem can be cast in the framework of exact modeling developed in [24, 25] and sketched in the previous section.

We associate to the data $\{(\lambda_i, \mathcal{V}_i)\}_{i=1,\ldots,N}$ of the *TIP* the set of vector-exponential trajectories $\mathcal{V}_i \exp_{\lambda_i t} := \{v \exp_{\lambda_i t} \mid v \in \mathcal{V}_i\}_{i=1,\ldots,N}$. We now show that any controllable, unfalsified model for $\mathcal{V}_i \exp_{\lambda_i t} := \{v \exp_{\lambda_i t} \mid v \in \mathcal{V}_i\}_{i=1,\ldots,N}$ with $\mathtt{p}$ rows satisfies the constraints (a) and (b). Indeed, if the behavior $\ker \begin{bmatrix} U(\frac{d}{dt}) & -Y(\frac{d}{dt}) \end{bmatrix}$, with $Y \in \mathbb{R}^{\mathtt{p}\times\mathtt{p}}[\xi]$ and $U \in \mathbb{R}^{\mathtt{p}\times\mathtt{m}}[\xi]$, represents an unfalsified model for the data, it holds that

$$\begin{bmatrix} U(\frac{d}{dt}) & -Y(\frac{d}{dt}) \end{bmatrix} \mathcal{V}_i \exp_{\lambda_i t} = \begin{bmatrix} U(\lambda_i) & -Y(\lambda_i) \end{bmatrix} \mathcal{V}_i \exp_{\lambda_i t} = 0$$

and consequently $\begin{bmatrix} U(\lambda_i) & -Y(\lambda_i) \end{bmatrix} v = 0$ for all $v \in \mathcal{V}_i$, i.e. U and Y satisfy the interpolation constraint (b). It is easy to see that requirement (a) is equivalent to $\ker \begin{bmatrix} U(\frac{d}{dt}) & -Y(\frac{d}{dt}) \end{bmatrix}$ being controllable.

In order to accommodate in the MPUM framework the metric- and root location aspects of the solution to the TIP represented by the conditions (c) and (d) in section 1, we use the concept of *dualization* of the data, which we now introduce.

We define from the interpolation data $\{(\lambda_i, \mathcal{V}_i)\}_{i=1,\ldots,N}$ the set

$$\mathcal{V}_i^{\perp} := \{v \in \mathbb{C}^{\mathtt{m}+\mathtt{p}} \mid v^* J \mathcal{V}_i = 0\}$$

and the *dual* of $\mathcal{V}_i \exp_{\lambda_i t}$ as

$$\mathcal{V}_i^{\perp} \exp_{-\bar{\lambda}_i t} := \{v \exp_{-\bar{\lambda}_i t} \mid v \in \mathcal{V}_i^{\perp}\}$$

We also define the *dualized data* $\mathcal{D}$ as

$$\mathcal{D} := \cup_{i=1,\ldots,N} \{\mathcal{V}_i \exp_{\lambda_i t}, \mathcal{V}_i^{\perp} \exp_{-\bar{\lambda}_i t}\} \quad (4)$$

Finally, we define the notion of *Pick matrix associated with the data* $\{(\lambda_i, \mathcal{V}_i)\}_{1\leq i\leq N}$. Let $V_i \in \mathbb{R}^{(\mathtt{m}+\mathtt{p})\times \dim(\mathcal{V}_i)}$ be a (full column-rank) matrix such that $\mathrm{Im}(V_i) = \mathcal{V}_i$, $i = 1, \ldots, N$. The Pick matrix associated with $\{(\lambda_i, \mathcal{V}_i)\}_{1\leq i\leq N}$ is the Hermitian block-matrix

$$T_{\{(\lambda_i,V_i)\}_{1\leq i\leq N}} := \left[\frac{V_i^* J V_j}{\bar{\lambda}_i + \lambda_j}\right]_{1\leq i,j\leq N} \tag{5}$$

Observe that the matrix defined in this way depends on the particular choice of the basis matrices V_i; however, its signature does not, and since the latter will be the only feature relevant for the purposes of this paper, in the rest of this paper we will continue to call $T_{\{(\lambda_i,V_i)\}_{1\leq i\leq N}}$ *the* Pick matrix of the data.

Often in the following, when the interpolation data $\{(\lambda_i, V_i)\}_{1\leq i\leq N}$ is assumed fixed, we will drop the notation $T_{\{(\lambda_i,V_i)\}_{1\leq i\leq N}}$ for the svelter one $T_{1\leq i\leq N}$.

Now consider the following procedure:

Algorithm T

Input: $\{(\lambda_i, V_i)\}_{1\leq i\leq N}$

Output: A kernel representation of the MPUM for $\mathcal{D}$

Define $R_0 := I_{\mathtt{p}+\mathtt{m}}$;

For $i = 1, \ldots, N$

$V_i' := R_{i-1}(\lambda_i)V_i$;

$R_i(\xi) := \left[(\xi + \bar{\lambda}_i)I_{\mathtt{p}+\mathtt{m}} - V_i' T^{-1}_{\{(\lambda_i,V_i')\}}(V_i')^* J\right] R_{i-1}(\xi)$;

end;

We following result shows that the matrix R_N produced by this algorithm produces a representation of the MPUM for the dualized data $\mathcal{D}$, and relates some properties of that representation to those of the Pick matrix of the data.

Theorem 1 *Assume that the Pick matrix (5) is invertible, and denote its signature with* $(\nu_-(T), 0, \nu_+(T))$. *Assume that* $\mathtt{p} \geq k_i$, $1 \leq i \leq N$. *Then the following statements are equivalent:*

1. *The Pick matrix (5) has* $\nu_-(T)$ *negative eigenvalues;*
2. *Algorithm T produces a kernel representation of the MPUM for the dualized data set* $\mathcal{D}$ *defined in (4) induced by a matrix of the form*

$$R := \begin{bmatrix} -D^* & N^* \\ Q & -P \end{bmatrix} \tag{6}$$

where $D \in \mathbb{R}^{\mathtt{m}\times\mathtt{m}}[\xi]$, $N \in \mathbb{R}^{\mathtt{m}\times\mathtt{p}}[\xi]$, $Q \in \mathbb{R}^{\mathtt{p}\times\mathtt{m}}[\xi]$, $P \in \mathbb{R}^{\mathtt{p}\times\mathtt{p}}[\xi]$ *satisfy the following properties:*

(a) D, P *are nonsingular;*

(*b*) $QD - PN = 0$;

(*c*) $\det(P)$ *has* $\nu_-(T)$ *roots in the right-half plane;*

(*d*) $RJR^* = R^*JR = pp^*J$ *with* $p(\xi) = \Pi_{i=1}^N (\xi + \bar{\lambda}_i)$;

(*e*) $\|P^{-1}Q\|_\infty < 1$;

(*f*) $\|QD^{*-1}\|_\infty < 1$;

(*g*) $\|N^*P^{-1}\|_\infty < 1$.

Proof. Let us first prove (1) $\Rightarrow$ (2). We will prove this by induction on the number N of subspaces $\mathcal{V}_i$.

For $N = 1$, partition the basis matrix V_1 as $V_1 = \text{col}(V_{11}, V_{12})$ with $V_{11} \in \mathbb{C}^{\mathtt{m} \times k_1}$ and $V_{12} \in \mathbb{C}^{\mathtt{p} \times k_1}$, and consider the model $\mathcal{B}_1$ represented in kernel form by

$$R_1(\xi) := (\xi + \bar{\lambda}_1) I_{p+m} - V_1 T_{\{V_1\}}^{-1} V_1^* J \tag{7}$$

Note that for every $\alpha \in \mathbb{C}^{k_1 \times 1}$ it holds that

$$\left(\frac{d}{dt} + \bar{\lambda}_1\right) V_1 \alpha \exp_{\lambda_1 t} - V_1 T_{\{V_1\}}^{-1} V_1^* J V_1 \alpha \exp_{\lambda_1 t} =$$
$$(\lambda_1 + \bar{\lambda}_1) V_1 \alpha \exp_{\lambda_1 t} - V_1 \left(\frac{V_1^* J V_1}{\lambda_1 + \bar{\lambda}_1}\right)^{-1} (V_1^* J V_1) \alpha \exp_{\lambda_1 t} = 0.$$

Note also that if $V_1^\perp$ is a $(\mathtt{m} + \mathtt{p}) \times (\mathtt{m} + \mathtt{p} - k_1)$ matrix such that $\text{Im}(V_1^\perp) = \mathcal{V}_1^\perp$, and $\beta \in \mathbb{C}^{(\mathtt{m}+\mathtt{p}-k_1) \times 1}$, it holds that

$$\left(\frac{d}{dt} + \bar{\lambda}_1\right) V_1^\perp \beta \exp_{-\bar{\lambda}_1 t} - V_1 T_{\{V_1\}}^{-1} V_1^* J V_1^\perp \beta \exp_{-\bar{\lambda}_1 t} = 0.$$

Consequently, $\mathcal{V}_1 \exp_{\lambda_1 t} \subseteq \mathfrak{B}_1$ and $\mathcal{B}_1 \supseteq \mathcal{V}_1^\perp \exp_{-\bar{\lambda}_1 t}$. In order to prove that $\mathcal{B}_1$ is the MPUM, observe that the determinant of (7) has degree $\mathtt{p}+\mathtt{m}$, and therefore $\mathcal{B}_1$ contains $\mathtt{p}+\mathtt{m}$ independent trajectories. Since $\dim(\mathcal{V}_1 \exp_{\lambda_1 t} \oplus \mathcal{V}_1^\perp \exp_{-\bar{\lambda}_1 t}) = \mathtt{m} + \mathtt{p}$, the claim is proved.

In order to prove that (7) satisfies $(2a) - (2f)$, partition it according to the partition of $V_1 = \text{col}(V_{11}, V_{12})$ as

$$\begin{aligned} R_1(\xi) &:= \begin{pmatrix} -D_1^*(\xi) & N_1^*(\xi) \\ Q_1(\xi) & -P_1(\xi) \end{pmatrix} \\ &:= \begin{pmatrix} (\xi + \bar{\lambda}_1) I_{\mathtt{m}} - V_{11} T_{\{V_1\}}^{-1} V_{11}^* & V_{11} T_{\{V_1\}}^{-1} V_{12}^* \\ -V_{12} T_{\{V_1\}}^{-1} V_{11}^* & (\xi + \bar{\lambda}_1) I_{\mathtt{p}} + V_{12} T_{\{V_1\}}^{-1} V_{12}^* \end{pmatrix} \end{aligned} \tag{8}$$

Observe that D_1 and P_1 in (8) are row proper, and consequently nonsingular. $Q_1 D_1 - P_1 N_1 = 0$ follows from straightforward manipulations. This proves claims (a) and (b).

The claim ($2c$) on the number of zeros of $\det(P_1)$ in the right half-plane follows from the following result.

Lemma 2 $\det(P_1)$ *has at least* $\mathtt{p} - k_1$ *roots in* $-\overline{\lambda}_1$*. Of the remaining* k_1 *roots,* $\nu_+(T_{\{V_1\}})$ *are in the left-half plane, and* $\nu_-(T_{\{V_1\}})$ *are in the right-half plane. Among those in the left half-plane,* $k_1 - rank(V_{12})$ *are in* $-\overline{\lambda}_1$*.*

Proof. The proof of this claim is articulated in a series of intermediate results of independent interest. The first one is the following.

Lemma 3 *Denote* $k_1' := rank(V_{12})$*; then there exists a nonsingular* $C \in \mathbb{C}^{k_1 \times k_1}$ *such that*

$$V_1 C = \begin{bmatrix} V_{11}C \\ V_{12}C \end{bmatrix} = \begin{bmatrix} V_{11}^a & V_{11}^b \\ V_{12}^a & 0 \end{bmatrix}. \tag{9}$$

with V_{12}^a *and* V_{11}^b *of full column rank* k_1' *and* $k_1 - k_1'$*, respectively. Define*

$$\hat{V}_1 := V_{11}^{a*}V_{11}^a - V_{12}^{a*}V_{12}^a - V_{11}^{a*}V_{11}^b(V_{11}^{b*}V_{11}^b)^{-1}V_{11}^{b*}V_{11}^a;$$

then

$$P_1(\xi) = -(\xi + \overline{\lambda}_1)I_{\mathtt{p}} - V_{12}^a \left(\frac{\hat{V}_1}{\lambda_1 + \overline{\lambda}_1} \right)^{-1} V_{12}^{a*}$$

Proof. The existence of a nonsingular $C \in \mathbb{C}^{k_1 \times k_1}$ such that (9) holds follows from elementary linear algebra considerations. The fact that $V_{11}^b \in \mathbb{C}^{m \times (k_1 - k_1')}$ and $V_{12}^a \in \mathbb{C}^{p \times k_1'}$ are full column rank follows from the fact that the columns of V_1 are linearly independent.

Observe that

$$\begin{aligned} V_{12}T_1^{-1}V_{12}^* &= V_{12}CC^{-1}T_1^{-1}C^{*-1}C^*V_{12}^* \\ &= \begin{bmatrix} V_{12}^a & 0 \end{bmatrix} C^{-1}T_1^{-1}C^{*-1} \begin{bmatrix} V_{12}^{a*} \\ 0 \end{bmatrix} \end{aligned} \tag{10}$$

Observe also that

$$\begin{aligned} C^{-1}T_1^{-1}C^{*-1} &= (\lambda_1 + \bar{\lambda}_1)C^{-1}\left\{V_{11}^*V_{11} - V_{12}^*V_{12}\right\}^{-1}C^{*-1} \\ &= (\lambda_1 + \bar{\lambda}_1)\left\{C^*\left(V_{11}^*V_{11} - V_{12}^*V_{12}\right)C\right\}^{-1} \\ &= (\lambda_1 + \bar{\lambda}_1)\left\{C^*V_{11}^*V_{11}C - C^*V_{12}^*V_{12}C\right\}^{-1} \\ &= (\lambda_1 + \bar{\lambda}_1)\underbrace{\begin{bmatrix} V_{11}^{a*}V_{11}^a - V_{12}^{a*}V_{12}^a & V_{11}^{a*}V_{11}^b \\ V_{11}^{b*}V_{11}^a & V_{11}^{b*}V_{11}^b \end{bmatrix}^{-1}}_{=:\hat{V}^{-1}}. \end{aligned} \tag{11}$$

Observe that $V_{11}^{b*}V_{11}^b \in \mathbb{C}^{(k_1 - k_1') \times (k_1 - k_1')}$ is positive definite, since V_{11}^b is full column rank. Define the nonsingular matrix

$$S := \begin{bmatrix} I_{k_1'} & 0 \\ S_{21} & I_{(k_1 - k_1') \times (k_1 - k_1')} \end{bmatrix} := \begin{bmatrix} I_{k_1'} & 0 \\ -(V_{11}^{b*}V_{11}^b)^{-1}V_{11}^{b*}V_{11}^a & I_{(k_1 - k_1') \times (k_1 - k_1')} \end{bmatrix} \tag{12}$$

and observe that

$$\begin{aligned}\hat{V}^{-1} &= \left\{S^{*-1}S^*\hat{V}SS^{-1}\right\}^{-1} = S\left\{S^*\hat{V}S\right\}^{-1}S^* \\ &= \begin{bmatrix} I & 0 \\ S_{21} & I \end{bmatrix}\begin{bmatrix} \hat{V}_1^{-1} & 0 \\ 0 & (V_{11}^{b*}V_{11}^{b})^{-1} \end{bmatrix}\begin{bmatrix} I & S_{21}^* \\ 0 & I \end{bmatrix} \qquad (13)\end{aligned}$$

where $\hat{V}_1 := V_{11}^{a*}V_{11}^{a} - V_{12}^{a*}V_{12}^{a} - V_{11}^{a*}V_{11}^{b}(V_{11}^{b*}V_{11}^{b})^{-1}V_{11}^{b*}V_{11}^{a}$. Now using (13) and (11) we rewrite (10) as

$$\begin{aligned}V_{12}T_1^{-1}V_{12}^* &= (\lambda_1+\bar{\lambda}_1)\begin{bmatrix} V_{12}^{a} & 0 \end{bmatrix}\hat{V}^{-1}\begin{bmatrix} V_{12}^{a*} \\ 0 \end{bmatrix} \\ &= (\lambda_1+\bar{\lambda}_1)\begin{bmatrix} V_{12}^{a} & 0 \end{bmatrix}\begin{bmatrix} \hat{V}_1^{-1} & 0 \\ 0 & (V_{11}^{b*}V_{11}^{b})^{-1} \end{bmatrix}\begin{bmatrix} V_{12}^{a*} \\ 0 \end{bmatrix} \\ &= V_{12}^{a}\left(\frac{\hat{V}_1}{\lambda_1+\overline{\lambda}_1}\right)^{-1}V_{12}^{a*}\end{aligned}$$

which yields the claim of the Lemma. ∎

From Lemma 3 it follows that $\det(P_1)$ has $\mathtt{p}-k_1'$ roots in $-\overline{\lambda}_1$. Indeed, $P_1(-\overline{\lambda}_1) = -V_{12}^{a}T_1^{-1}V_{12}^{a*}$, and consequently $P_1(-\overline{\lambda}_1)v = 0$ for all directions $v \in \mathcal{V}_{12}^{a\perp} := \{v \in \mathbb{C}^m | V_{12}^{a*}v = 0\}$; note that this subspace has dimension $\mathtt{p}-k_1'$. In the following we show that the location in the complex plane of the remaining k_1' roots is associated with the signature of the "modified Pick matrix"

$$T_1' := \frac{\hat{V}_1}{\lambda_1+\overline{\lambda}_1} \in \mathbb{R}^{k_1'\times k_1'} \qquad (14)$$

In order to do this, we first prove the following result concerning the properties of the directions associated with a singularity $\mu \neq \lambda_1$ of $\det(D_1)$.

Lemma 4 *Define the set*

$$\mathcal{S} := \{v \in \mathbb{C}^{\mathtt{p}} \mid \exists \mu \in \mathbb{C}, \mu \neq -\overline{\lambda}_1 \ s.t.\ P_1(\mu)v = 0\}$$

Then

1. $v \in \mathcal{S}$ *if and only if* v *is an eigenvector of* $-(\overline{\lambda}_1 I_m - V_{12}^{a}T_1'^{-1}V_{12}^{a*})$;
2. *If* $v \in \mathcal{S}$ *then there exists* $\alpha \in \mathbb{C}^{k_1'}$ *such that* $v = V_{12}^{a}\alpha$;
3. $\mathcal{S}$ *is a set of linearly independent vectors, and so is*

$$\mathcal{S}' := \{\alpha \in \mathbb{C}^{k_1'} \mid \exists v \in \mathcal{S} \ s.t.\ v = V_{12}^{a}\alpha\}$$

Proof. We use the result of Lemma 3. Statement (1) follows from

$$\left[P_1(\mu)v = -(\mu+\overline{\lambda}_1)v - V_{12}^{a}T_1^{-1}V_{12}^{a*}v = 0\right] \Longleftrightarrow \left[\mu v = -\left(\overline{\lambda}_1 I_{\mathtt{p}} + V_{12}^{a}T_1^{-1}V_{12}^{a*}\right)v\right]$$

Statement (2) follows from the fact that $P_1(\mu)v = 0$ if and only if $(\mu+\overline{\lambda}_1)v = V_{12}^a T_1'^{-1} V_{12}^{a*} v = V_{12}^a \left(T_1'^{-1} V_{12}^{a*} v\right)$, and defining $\alpha := \frac{1}{\mu+\overline{\lambda}_1} T_1'^{-1} V_{12}^{a*} v$.

In order to prove statement (3), observe that $\overline{\lambda}_1 I_{\mathtt{p}} + V_{12}^a T_1'^{-1} V_{12}^{a*}$ is an Hermitian matrix, and consequently it has a basis of eigenvectors. The first part of the claim follows then from part (1) of this Lemma. The second part of statement (3) follows from the fact that V_{12}^a has full column rank. ■

We proceed to state a property of the modified Pick matrix T_1' defined in equation (14).

Lemma 5 *Let $\mu \neq -\overline{\lambda}_1$, and let $v = V_{11}^a \alpha$ be such that $P_1(\mu)v = 0$. Then*

$$[\alpha^* T_1' \alpha > 0] \iff [Re(\mu) < 0]$$

Proof. From

$$\begin{aligned} P_1(\mu)v &= (\mu+\overline{\lambda}_1)v + V_{12}^a T_1'^{-1} V_{12}^{a*} v = (\mu+\overline{\lambda}_1) V_{12}^a \alpha + V_{12}^a T_1'^{-1} V_{12}^{a*} V_{12}^a \alpha \\ &= V_{12}^a \left((\mu+\overline{\lambda}_1)\alpha + T_1'^{-1} V_{12}^{a*} V_{12}^a \alpha\right) = 0 \end{aligned}$$

and the fact that V_{12}^a has full column rank we conclude that $(\mu+\overline{\lambda}_1)\alpha + T_1'^{-1} V_{12}^{a*} V_{12}^a \alpha = 0$. Rewrite this equality as $(\mu+\overline{\lambda}_1) T_1' \alpha + V_{12}^{a*} V_{12}^a \alpha = 0$, and substitute the expression (14) in order to arrive at

$$(\mu+\overline{\lambda}_1)\left(V_{11}^{a*} V_{11}^a - V_{12}^{a*} V_{12}^a - V_{11}^{a*} V_{11}^b (V_{11}^{b*} V_{11}^b)^{-1} V_{11}^{b*} V_{11}^a\right)\alpha + \left(\lambda_1 + \overline{\lambda}_1\right) V_{12}^{a*} V_{12}^a \alpha = 0$$

Multiply this equality on the left by α^* in order to conclude that

$$\begin{aligned} &\mu\alpha^* \left(V_{11}^{a*} V_{11}^a - V_{12}^{a*} V_{12}^a - V_{11}^{a*} V_{11}^b (V_{11}^{b*} V_{11}^b)^{-1} V_{11}^{b*} V_{11}^a\right)\alpha \\ &= -\lambda_1 \alpha^* V_{12}^{a*} V_{12}^a \alpha - \overline{\lambda}_1 \alpha^* \left(V_{11}^{a*} V_{11}^a - V_{11}^{a*} V_{11}^b (V_{11}^{b*} V_{11}^b)^{-1} V_{11}^{b*} V_{11}^a\right)\alpha \quad (15) \end{aligned}$$

Now observe that

$$V_{11}^{a*} V_{11}^a - V_{11}^{a*} V_{11}^b (V_{11}^{b*} V_{11}^b)^{-1} V_{11}^{b*} V_{11}^a = V_{11}^{a*}\left(I_{\mathtt{p}} - V_{11}^b (V_{11}^{b*} V_{11}^b)^{-1} V_{11}^{b*}\right) V_{11}^a \geq 0$$

where the last inequality follows from the fact that $I_{\mathtt{p}} - V_{11}^b (V_{11}^{b*} V_{11}^b)^{-1} V_{11}^{b*}$ is the orthogonal projection of $\mathbb{C}^{\mathtt{p}}$ on the subspace orthogonal to the columns of V_{11}^b, and consequently is nonnegative definite. Taking the real part of both sides of (15) it follows from this argument that

$$\begin{aligned} &\mathrm{Re}(\mu)\ \alpha^* \left(V_{11}^{a*} V_{11}^a - V_{12}^{a*} V_{12}^a - V_{11}^{a*} V_{11}^b (V_{11}^{b*} V_{11}^b)^{-1} V_{11}^{b*} V_{11}^a\right)\alpha \\ &= -\mathrm{Re}(\lambda_1)\ \underbrace{\alpha^* V_{12}^{a*} V_{12}^a \alpha}_{>0} - \mathrm{Re}(\overline{\lambda}_1)\ \underbrace{\alpha^* \left(V_{11}^{a*} V_{11}^a - V_{11}^{a*} V_{11}^b (V_{11}^{b*} V_{11}^b)^{-1} V_{11}^{b*} V_{11}^a\right)\alpha}_{\geq 0} \end{aligned}$$

where the strict inequality on $\alpha^* V_{12}^{a*} V_{12}^a \alpha$ follows from the fact that V_{12}^a has full column rank. Now observe that the right-hand side of the last equality is negative, and consequently that the claim is true. ■

The next result is instrumental in associating the number of negative eigenvalues of the modified Pick matrix T_1' with the number of roots of $\det(P_1)$ in the right-half plane.

Lemma 6 *Let $Q = Q^* \in \mathbb{C}^{\bullet \times \bullet}$ be nonsingular, and define*

$$\begin{aligned} \mathcal{N}_+ &:= \{x \in \mathbb{C}^\bullet \mid x^* Q x > 0\} \\ \mathcal{N}_- &:= \{x \in \mathbb{C}^\bullet \mid x^* Q x < 0\} \end{aligned}$$

Then $dim(\mathcal{N}_+) = \nu_+(Q)$ (respectively, $dim(\mathcal{N}_-) = \nu_-(Q)$).

Proof. Let $\nu_+(Q)$, respectively $\nu_-(Q)$, be the number of positive, respectively negative, eigenvalues of Q, and denote with n_+, respectively n_-, the maximal number of linearly independent vectors in $\mathcal{N}_+$, respectively $\mathcal{N}_-$. Observe that $n_+ \geq \nu_+(Q)$, and that $n_- \geq \nu_-(Q)$. Assume by contradiction that $n_+ > \nu_+(Q)$; then $n_+ + n_- > \nu_+(Q) + \nu_-(Q)$, which by the assumption of nonsingularity of Q equals the dimension of the space on which Q acts. Since $\mathcal{N}_+ \cap \mathcal{N}_- = \{0\}$, this leads to a contradiction. The case $n_- > \nu_-(Q)$ is treated analogously. ■

We are now in the position to prove our statement about the number of roots of $\det(P_1)$ in the right-half plane.

From Lemma 5 it follows that if $\mu \in \mathbb{C}_-$ is such that $\mu \neq -\overline{\lambda}_1$ and $\det(P_1(\mu)) = 0$, then for the corresponding $\alpha \in \mathcal{S}'$ it holds that $\alpha^* T_1' \alpha > 0$; and that if $\mu \in \mathbb{C}_+$ is such that $\det(P_1(\mu)) = 0$, then for the corresponding $\alpha \in \mathcal{S}'$ it holds that $\alpha^* T_1' \alpha < 0$. From statement (3) of Lemma 4 it follows that each one of the sets $\{\alpha \in \mathcal{S}' \mid \alpha^* T_1' \alpha < 0\}$ and $\{\alpha \in \mathcal{S}' \mid \alpha^* T_1' \alpha > 0\}$ consists of linearly independent vectors. Notice that in total there are k_1' elements in the union of these two sets, since each element corresponds to a root $\mu \neq -\overline{\lambda}_1$ of $\det(P_1)$.

Now apply Lemma 6 in order to conclude that the number of elements in $\{\alpha \in \mathcal{S}' \mid \alpha^* T_1' \alpha < 0\}$ equals the number of negative eigenvalues of T_1'. In order to conclude the proof of the Lemma, observe that $\nu_+(T_1) = \nu_+(T_1')$ and that $\nu_-(T_1) = \nu_-(T_1') + k_1 - k_1'$. ■

Having proved property ($2c$), we now prove ($2d$). Observe that

$$\begin{aligned} R_1(\xi) J R_1(\xi)^* = [(\xi + \bar{\lambda}_1) I_{p+m} - V_1 T_{\{V_1\}}^{-1} V_1^* J] J [(-\xi + \lambda_1) I_{p+m} - J V_1 T_{\{V_1\}}^{-1} V_1^*] = \\ (\xi + \bar{\lambda}_1)(-\xi + \lambda_1) J - (\lambda_1 + \bar{\lambda}_1) V_1 T_{\{V_1\}}^{-1} V_1^* + (\lambda_1 + \bar{\lambda}_1) V_1 T_{\{V_1\}}^{-1} V_1^* = \\ (\xi + \bar{\lambda}_1)(-\xi + \lambda_1) J \end{aligned} \quad (16)$$

The second equality of ($2d$) can be proved analogously.

In order to prove ($2e$), observe that from ($2a$) and ($2b$) follows that $P_1^{-1} Q_1 = N_1 D_1^{-1}$. Consequently, in order to prove $\|P_1^{-1} Q_1\|_\infty < 1$, it will suffice to prove that $D_1^*(i\omega) D_1(i\omega) - N_1^*(i\omega) N_1(i\omega) > 0$ for every $\omega \in \mathbb{R}$. Note that $D_1^* D_1 - N_1^* N_1$ is the $(1,1)$-block of $R_1 J R_1^*$ and, by property ($2e$), on the imaginary axis it equals

$$(-i\omega + \bar{\lambda}_1)(i\omega + \lambda_1) I_{\mathtt{m}},$$

which is positive definite for every $\omega \in \mathbb{R}$. This implies $\det(D(i\omega)) \neq 0 \; \forall \; \omega \in \mathbb{R}$ and consequently $\|P_1^{-1} Q_1\|_\infty < 1$.

In order to prove claim ($2f$), note that $D_1(i\omega) D_1^*(i\omega) - Q_1^*(i\omega) Q_1(i\omega)$ is the $(1,1)$ block of $R_1(i\omega)^* J R_1(i\omega)$ and that by property ($2d$) this block is positive definite for all $\omega \in \mathbb{R}$. In order to prove claim ($2g$), note that $N_1(i\omega) N_1^*(i\omega) -$

$P_1^*(i\omega)P_1(i\omega)$ is the $(2,2)$ block of $R_1(i\omega)^*JR_1(i\omega)$ and that by $(2d)$ this block is negative definite for all $\omega \in \mathbb{R}$.

This concludes the proof of $(2a)-(2g)$ for the representation (7) of the MPUM for $N=1$.

Let us now assume that the claim (1) $\Rightarrow$ (2) holds for a number j of subspaces to interpolate, $1 \leq j \leq N-1$. In order to prove the claim for N subspaces, we proceed as follows. We have shown above that there exists a representation R_1 of the MPUM for $\mathcal{V}_1 \exp_{\lambda_1 t} \oplus \mathcal{V}_1^{\perp} \exp_{-\bar{\lambda}_1 t}$ that satisfies $(2a)-(2g)$. We proceed now by first determining a congruence transformation on the Pick matrix of the data which will make it easier to apply the inductive assumption. Then we will apply the inductive assumption and conclude that a representation R' of the MPUM for the error subspaces satisfying $(2a)-(2f)$ exists. Combining the representations of the two MPUMs as $R'R_1$ we obtain a representation of the MPUM for $\mathcal{D}$, and we will show that it satisfies $(2a)-(2g)$.

Assume now that a representation (8) of the MPUM for $\mathcal{V}_1 \exp_{\lambda_1 t} \oplus \mathcal{V}_1^{\perp} \exp_{-\bar{\lambda}_1 t}$ has been computed, satisfying $(2a)-(2f)$. The i-th error subspace associated to this model is $\mathcal{V}_i' \exp_{\lambda_i t} := R(\lambda_i)\mathcal{V}_i \exp_{\lambda_i t}$, $2 \leq i \leq N$ and has a basis matrix

$$V_i' := (\lambda_i + \bar{\lambda}_1)V_i - V_1 T_1^{-1} V_1^* J V_i, \qquad 2 \leq i \leq N$$

We now investigate the relationship of the signature of the Pick matrix $T'_{2\leq i\leq N} := T_{\{(\lambda_i, V_i')\}}$ associated with $\mathcal{V}_i' \exp_{\lambda_i t}$, $2 \leq i \leq N$, with the signature of the matrix $T_{1\leq i\leq N}$. Note first that for $2 \leq i, j \leq N$, the $(i-1, j-1)$-th block element of $T'_{2\leq i\leq N}$ is

$$\frac{V_i'^* J V_j'}{\bar{\lambda}_i + \lambda_j} = \frac{1}{\lambda_j + \bar{\lambda}_i}[(\bar{\lambda}_i + \lambda_1)V_i^* - V_i^* J V_1 T_1^{-1} V_1^*] J [(\lambda_j + \bar{\lambda}_1)V_j - V_1 T_1^{-1} V_1^* J V_j]. \tag{17}$$

Easy computations show that (17) equals

$$\frac{(\bar{\lambda}_i + \lambda_1)(\lambda_j + \bar{\lambda}_1)}{\bar{\lambda}_i + \lambda_j} V_i^* J V_j - V_i^* J V_1 T_1^{-1} V_1^* J V_j. \tag{18}$$

Partition now $T_{1\leq i\leq N}$ as

$$\begin{pmatrix} T_1 & \bar{b}^T \\ b & T_{2\leq i\leq N} \end{pmatrix}$$

with $b := col(\frac{V_i^* J V_1}{\bar{\lambda}_i + \lambda_1})_{2\leq i\leq N}$, and define $\Delta := diag((\bar{\lambda}_i + \lambda_1))_{2\leq i\leq N}$. Observe that

$$\begin{pmatrix} 1 & 0 \\ -\Delta b T_1^{-1} & \Delta \end{pmatrix} T_{1\leq i\leq N} \begin{pmatrix} 1 & -T_1^{-1}\bar{b}^T\bar{\Delta} \\ 0 & \bar{\Delta} \end{pmatrix} = \begin{pmatrix} T_1 & 0 \\ 0 & \Delta T_{2\leq i\leq N}\bar{\Delta} - \Delta b T_1^{-1}\bar{b}^T\bar{\Delta} \end{pmatrix} \tag{19}$$

We prove now that the $(2,2)$ block of (19) coincides with $T'_{2\leq i\leq N}$. In fact, the (i,j)-th block of $\Delta T_{2\leq i\leq N}\bar{\Delta} - \Delta b T_1^{-1}\bar{b}^T\bar{\Delta}$ equals

$$\frac{(\bar{\lambda}_i + \lambda_1)(\lambda_j + \bar{\lambda}_1)}{\bar{\lambda}_i + \lambda_j} V_i^* J V_j - V_i^* J V_1 T_1^{-1} V_1^* J V_j,$$

and, since the (i,j)-th block of $T'_{2\leq i\leq N}$ is given by (18), this proves the claim. Observe that from (19) it follows that

$$\text{sign}(T_{1\leq i\leq N}) = \text{sign}(T'_{2\leq i\leq N}) + \text{sign}(T_1)$$

and consequently that, $\nu_-(T'_{2\leq i\leq N}) = \nu_-(T_{1\leq i\leq N}) - \nu_-(T_1)$.

By inductive assumption we conclude that there exists a kernel representation $R' \in \mathbb{R}^{2\times 2}[\xi]$ of the MPUM for $\{\mathcal{V}'_i \exp_{\lambda_i t}, \mathcal{V}'^{\perp}_i \exp_{-\bar{\lambda}_i t}\}_{2\leq i\leq N}$ of the form

$$R' = \begin{pmatrix} -D'^* & N'^* \\ Q' & -P' \end{pmatrix}$$

satisfying the properties $(2a) - (2g)$ of the Theorem.

It is easily verified that the MPUM for $\mathcal{D}$ is represented by

$$\begin{pmatrix} -D^* & N^* \\ Q & -P \end{pmatrix} := \begin{pmatrix} -D'^* & N'^* \\ Q' & -P' \end{pmatrix} \begin{pmatrix} -D_1^* & N_1^* \\ Q_1 & -P_1 \end{pmatrix} \tag{20}$$

We now show that (20) satisfies $(2a) - (2g)$.

In order to prove $(2a)$, we first show that P is nonsingular. From (20) it follows that

$$-P = Q' N_1^* + P' P_1 \tag{21}$$

By inductive assumption, P' and P_1 are nonsingular, and consequently

$$-(P')^{-1} P P_1^{-1} = (P')^{-1} Q' \cdot N_1^* P_1^{-1} + I_{\mathtt{p}}$$

Conclude from the inductive assumption that $\|(P')^{-1}Q'\|_\infty < 1$ and that $\|N_1^* P_1^{-1}\|_\infty < 1$. It follows that $(P')^{-1} P P_1^{-1}$ is nonsingular on the imaginary axis, and consequently P is also nonsingular on the imaginary axis, and *a fortiori* nonsingular in $\mathbb{R}^{\mathtt{p}\times\mathtt{p}}[\xi]$.

We now show that D is nonsingular. Note from (20) that $D = D_1 D' + Q_1^* N'$. Observe from the formula (8) that $Q_1 = -N_1$, and consequently $D = D_1 D' - N_1^* N'$. Now use the contractivity of $N_1 D_1^{-1}$ and of $N'(D')^{-1}$ to show in a manner analogous to that used for the proof of the nonsingularity of P, that D is also nonsingular. This concludes the proof of $(2a)$.

Claim $(2b)$ can be proved by a straightforward computation, using equation (20) and the inductive assumption.

We now prove $(2c)$, the claim regarding the number of roots of $\det(P)$ in $\mathbb{C}_+$. Conclude from (20) that

$$-P^{-1} = P_1^{-1}(I_{\mathtt{p}} + P'^{-1} Q' \cdot N_1^* P_1^{-1})^{-1} P'^{-1}$$

and consequently that

$$-\frac{1}{\det(P)} = \frac{1}{\det(P')} \frac{1}{\det(I_{\mathtt{p}} + P'^{-1} Q' \cdot N_1^* P_1^{-1})} \frac{1}{\det(P_1)} \tag{22}$$

Now recall that the winding number wno($\cdot$) of a function f defined on the imaginary axis and admitting a meromorphic continuation in $\mathbb{C}_+$ satisfies

$$\text{wno}(f) = (\#\text{ zeros of } f \text{ in } \mathbb{C}_+) - (\#\text{ poles of } f \text{ in } \mathbb{C}_+)$$

Observe that

$$\text{wno}\left(\frac{1}{\det(I_{\mathtt{p}} + \alpha P'^{-1}Q' \cdot N_1^* P_1^{-1})}\right)$$

for $0 \leq \alpha \leq 1$ is a continuous function of α taking integer values, and consequently its value is independent of α. This fact, together with the contractivity of $N'D'^{-1}$ and $Q_1 D_1^{*-1}$, implies that

$$\text{wno}\left(\frac{1}{\det(I_{\mathtt{p}} + P'^{-1}Q' \cdot N_1^* P_1^{-1})}\right) = \text{wno}(\det(I_{\mathtt{p}})) = 0$$

Now apply the logarithmic property of wno($\cdot$) to both sides of (22) and obtain

$$\text{wno}\left(-\frac{1}{\det(P)}\right) = \text{wno}\left(\frac{1}{\det(P_1)}\right) + \text{wno}\left(\frac{1}{\det(P')}\right) \tag{23}$$

From the inductive assumption it follows that

$$\begin{aligned} \text{wno}\left(\frac{1}{\det(P_1)}\right) &= -\nu_-(T_1)) \\ \text{wno}\left(\frac{1}{\det(P')}\right) &= -\nu_-(T'_{2\leq i\leq N}) \end{aligned}$$

Consequently, $\text{wno}\left(-\frac{1}{\det(P)}\right) = -\nu_-(T_1)) - \nu_-(T'_{2\leq i\leq N})$; from equation (19) and the following discussion, it follows that $\text{wno}\left(-\frac{1}{\det(P)}\right) = -\nu_-(T)$ and claim ($2c$) is proved.

Claim ($2d$) follows easily from (20) and the inductive assumption.

In order to prove ($2e$), we show that $P^*P - Q^*Q > 0$ on the imaginary axis. Note that $P^*P - Q^*Q$ is the $(2,2)$ block-element of $R^{*\prime}R_1^* J R_1 R'$. Using ($2d$), it is easily seen that this element equals $-p(i\omega)p^*(i\omega) < 0$ for all $\omega \in \mathbb{R}$. This implies $\|P^{-1}Q\|_\infty < 1$. The proofs of ($2f$) and ($2g$) follow a similar argument.

This concludes the proof of (1) $\Rightarrow$ (2).

In order to prove the converse implication, we proceed by induction on the number N of subspaces to be interpolated, using property ($2c$) of the special MPUM representation resulting from Algorithm T.

For $N = 1$, use the expression

$$P_1(\xi) = -(\xi + \bar{\lambda}_1)I_{\mathtt{p}} - V_{12}^a \left(\frac{\hat{V}_1}{\lambda_1 + \bar{\lambda}_1}\right)^{-1} V_{12}^{a*}$$

obtained in Lemma 3 in order to conclude that $\det(P_1)$ has n_- roots in $\mathbb{C}_+$ if and only if $n_- = \nu_-(T_1)$. The claim is thus proved for $N = 1$.

We now assume the claim is true for all $1 \leq j \leq N-1$ and we prove it for $j = N$. In order to do this, consider first that the special representation R for the model for N trajectories is obtained from the model R_1 for $v_1 \exp_{\lambda_1 t}$ and the model R' for the error trajectories $R_1(\lambda_i) v_i \exp_{\lambda_i t}$, $2 \leq i \leq N$ as $R = R' R_1$. Observe that by inductive assumption, the Pick matrix of the error subspaces has as many negative eigenvalues as the number of right half-plane singularities of the $(2,2)$ block-element of R'.

It follows from equation (23) that the number of right half-plane singularities of P, the $(2,2)$ block-element of R, equals the number of such singularities of the corresponding block-element of R' plus the number of such singularities of P_1. Now observe that $T_{\{v_i\}_{1 \leq i \leq N}}$, the Pick matrix of the data, is congruent to the matrix on the right-hand side of (19). The signature of this block-diagonal matrix equals the sum of the signature of the Pick matrix $T_{\{v_1\}}$, and that of the Pick matrix $T'_{2 \leq i \leq N}$ associated to the error trajectories. This completes the proof of $(2) \Longrightarrow (1)$. $\blacksquare$

We now give two examples of the application of the algorithm.

Example 7 Consider the (frequency, vector) pairs

$$(\lambda_1, v_1) = \left(4, \begin{bmatrix} 6 \\ -7 \end{bmatrix}\right) \qquad (\lambda_2, v_2) = \left(5, \begin{bmatrix} 12 \\ -9 \end{bmatrix}\right) \qquad (\lambda_3, v_3) = \left(5, \begin{bmatrix} 20 \\ -11 \end{bmatrix}\right)$$

corresponding to the Pick matrix

$$\begin{bmatrix} -\frac{13}{8} & 1 & \frac{43}{10} \\ 1 & \frac{63}{10} & \frac{141}{11} \\ \frac{43}{10} & \frac{141}{11} & \frac{93}{4} \end{bmatrix}$$

whose eigenvalues are 30.7039, −2.78503, 0.00614669. We conclude that there exists a representation of the MPUM for the dualized data whose $(2,2)$-element has a determinant with 1 root in $\mathbb{C}^+$.

The model for the first point is

$$R_1(\xi) := \begin{bmatrix} \frac{340}{13} + \xi & \frac{336}{13} \\ -\frac{336}{13} & \xi - \frac{340}{13} \end{bmatrix}$$

As was to be expected from the fact that the Pick matrix corresponding to (λ_1, v_1) is negative definite, the $(2,2)$ entry has a singularity in $\mathbb{C}_+$.

The vector corresponding to the first error trajectory is $v_2' := R_1(5) v_2 = \begin{bmatrix} \frac{1836}{13} \\ -\frac{1557}{13} \end{bmatrix}$, with corresponding kernel representation

$$R_2'(\xi) := \begin{bmatrix} -\frac{357725}{11687} + \xi & -\frac{352920}{11687} \\ \frac{352920}{11687} & \frac{357725}{11687} + \xi \end{bmatrix}$$

Conclude that a kernel representation of the MPUM for the first two trajectories and their duals is

$$R_2(\xi) = R_2'(\xi) R_1(\xi) = \begin{bmatrix} -\frac{18020}{899} - \frac{4005}{899}\xi + \xi^2 & -\frac{24}{899}(50 + 163\xi) \\ \frac{24}{899}(-50 + 163\xi) & -\frac{18020}{899} + \frac{4005}{899}\xi + \xi^2 \end{bmatrix}$$

Observe that the $(2,2)$ entry of R_2 has a positive and a negative real root, as was to be expected from the fact that the 2×2 principal submatrix of the Pick matrix has one negative and one positive eigenvalue.

The third error trajectory is associated with the vector $v_3' := \begin{bmatrix} \frac{77672}{899} \\ \frac{23326}{899} \end{bmatrix}$. It can be shown that a kernel representation corresponding to this vector is induced by

$$R_3'(\xi) := \begin{bmatrix} -\frac{3288520930}{457403109}+\xi & \frac{1811777072}{457403109} \\ -\frac{1811777072}{457403109} & \frac{3288520930}{457403109}+\xi \end{bmatrix}$$

Conclude that a kernel representation of the MPUM for the given data is $R_3'(\xi)R_2(\xi)$, given by

$$\begin{bmatrix} \frac{70632200+14867334\xi-5924615\xi^2+508791\xi^3}{508791} & \frac{-40(887836-605418\xi+4967\xi^2)}{508791} \\ \frac{40(887836-605418\xi+4967\xi^2)}{508791} & \frac{-70632200+14867334\xi+5924615\xi^2+508791\xi^3}{508791} \end{bmatrix}$$

Observe that the roots of the $(2,2)$ element of R_3 are 2.27811, $-6.9613 \pm 3.53247i$. ■

Example 8 We solve a problem with $\mathtt{m}=1$ and $\mathtt{p}=2$. Consider the (frequency, vector) pairs

$$(\lambda_1, v_1) = \left(4, \begin{bmatrix} 6 \\ -7 \\ -11 \end{bmatrix}\right) \quad (\lambda_2, v_2) = \left(5, \begin{bmatrix} 12 \\ -9 \\ -14 \end{bmatrix}\right) \quad (\lambda_3, v_3) = \left(5, \begin{bmatrix} 20 \\ -11 \\ -17 \end{bmatrix}\right)$$

which correspond to the Pick matrix

$$\begin{bmatrix} -\frac{67}{4} & -\frac{145}{9} & -\frac{72}{5} \\ -\frac{145}{9} & -\frac{133}{10} & -\frac{97}{11} \\ -\frac{72}{5} & -\frac{97}{11} & -\frac{5}{6} \end{bmatrix}$$

This matrix has eigenvalues -38.5789, 7.69355, 0.0020285. Consequently, we expect a representation of the MPUM with a $(2,2)$ block element having one singularity in C_+.

The kernel representation corresponding to the first trajectory and its dual is

$$\begin{bmatrix} \frac{412}{67}+\xi & \frac{168}{67} & \frac{264}{67} \\ -\frac{168}{67} & \frac{72}{67}+\xi & -\frac{308}{67} \\ -\frac{264}{67} & -\frac{308}{67} & -\frac{216}{67}+\xi \end{bmatrix}$$

Proceeding with the application of Algorithm T, we obtain as kernel representation of the MPUM a matrix whose $(2,2)$ block-element is

$$\begin{bmatrix} \frac{-1721350920+3763007528\xi+607391445\xi^2+35405647\xi^3}{35405647} & \frac{330(-31781032+4310493\xi+289807\xi^2)}{35405647} \\ \frac{330(-26518552+4320147\xi+289807\xi^2)}{35405647} & \frac{-11040998520+4339543928\xi+647295195\xi^2+35405647\xi^3}{35405647} \end{bmatrix}$$

The determinant of such matrix is

$$-58053.9 - 17244.7\xi + 6638.14\xi^2 + 3467.99\xi^3 + 535.189\xi^4 + 35.4375\xi^5 + \xi^6$$

which has roots in

$$-6, -5, -4, -11.3835 \pm 8.83631i$$

and one in 2.32962. ■

The special kernel representation of the MPUM for $\mathcal{D}$ described in Theorem 1 allows us to characterize the solutions of the TIP as follows.

Theorem 9 *Assume that the Pick matrix* $T_{\{(\lambda_i,V_i)\}_{1\leq i\leq N}} := \left[\frac{V_i^* J V_j}{\lambda_i+\lambda_j}\right]_{i,j=1,\ldots,N}$ *is invertible and has* ν_- *negative eigenvalues, and let (6) be the representation of the MPUM for* $\mathcal{D}$ *computed with Algorithm T.*

Let $U \in \mathbb{R}^{\mathtt{p}\times\mathtt{m}}[\xi]$*,* $Y \in \mathbb{R}^{\mathtt{p}\times\mathtt{p}}[\xi]$ *be left coprime. Then* $\begin{bmatrix} U & -Y \end{bmatrix} \in \mathbb{R}^{\mathtt{p}\times(\mathtt{p}+\mathtt{m})}[\xi]$ *is a solution to the TIP with* $\det(Y)$ *having* ν_- *roots in* $\mathbb{C}_+$ *if and only if there exist* $\Pi, \Phi, F \in \mathbb{R}^{\bullet\times\bullet}[\xi]$*, with* Φ, F *Hurwitz, and* $\|\Phi^{-1}\Pi\|_\infty < 1$*, such that*

$$F \begin{bmatrix} U & -Y \end{bmatrix} = \begin{bmatrix} \Pi & -\Phi \end{bmatrix} \begin{bmatrix} -D^* & N^* \\ Q & -P \end{bmatrix} \tag{24}$$

Proof. We first prove sufficiency. Let $U \in \mathbb{R}^{\mathtt{p}\times\mathtt{m}}[\xi]$, $Y \in \mathbb{R}^{\mathtt{p}\times\mathtt{p}}[\xi]$ be given such that they are left coprime, and (24) holds for some $\Pi \in \mathbb{R}^{\mathtt{p}\times\mathtt{m}}[\xi]$, $\Phi \in \mathbb{R}^{\mathtt{p}\times\mathtt{p}}[\xi]$, $F \in \mathbb{R}^{\mathtt{p}\times\mathtt{p}}[\xi]$ such that $\|\Phi^{-1}\Pi\|_\infty < 1$ and Φ, F are Hurwitz. Consider that $F \begin{bmatrix} U & -Y \end{bmatrix}$ is a left multiple of a kernel representation of the MPUM for $\mathcal{D}$, and consequently it is unfalsified on $\mathcal{D}$. It follows that

$$F(\lambda_i) \begin{bmatrix} U(\lambda_i) & -Y(\lambda_i) \end{bmatrix} \mathcal{V}_i \exp_{\lambda_i t} = 0,$$

$1 \leq i \leq N$. Conclude from the fact that F is Hurwitz that this implies $\begin{bmatrix} U(\lambda_i) & -Y(\lambda_i) \end{bmatrix} \mathcal{V}_i \exp_{\lambda_i t} = 0, 1 \leq i \leq N$, and consequently that $\begin{bmatrix} U & -Y \end{bmatrix}$ is an unfalsified model for $\mathcal{V}_i \exp_{\lambda_i t}$, $1 \leq i \leq N$. The fact that $\|Y^{-1}U\|_\infty < 1$ follows from the J-unitariness of R and from the assumption that $\|\Phi^{-1}\Pi\|_\infty < 1$. Finally, the claim on the number of roots of Y in $\mathbb{C}_+$ can be proved by observing that

$$-FY = \Pi N^* + \Phi P = \Phi(\Phi^{-1}\Pi N^* P^{-1} + I_{\mathtt{p}})P$$

or equivalently

$$-Y^{-1}F^{-1} = P^{-1}(\Phi^{-1}\Pi N^* P^{-1} + I_{\mathtt{p}})^{-1}\Phi^{-1}$$

and consequently

$$-\frac{1}{\det(F)}\frac{1}{\det(Y)} = \frac{1}{\det(\Phi)}\frac{1}{\det(P)}\frac{1}{\det(\Phi^{-1}\Pi N^* P^{-1} + I_{\mathtt{p}})}$$

It follows from the fact that Φ and F are Hurwitz that $\mathrm{wno}(\frac{1}{\det(F)}) = 0 = \mathrm{wno}(\frac{1}{\det(\Phi)})$. It follows from the fact that $\|\Phi^{-1}\Pi\|_\infty < 1$ and that $\|P^{-1}N^*\|_\infty < 1$, that $\mathrm{wno}(\frac{1}{\det(\Phi^{-1}\Pi N^* P^{-1}+I_{\mathtt{p}})}) = 0$. Conclude that the number of roots of $\det(Y)$ in $\mathbb{C}_+$ equals the number of roots of $\det(P)$ in $\mathbb{C}_+$; from Theorem 1 we conclude that the latter is exactly ν_-, the number of negative eigenvalues of the Pick matrix of the data. Sufficiency is thus proved.

In order to prove necessity, we proceed as follows. Let $U \in \mathbb{R}^{\mathtt{p}\times\mathtt{m}}[\xi]$, $Y \in \mathbb{R}^{\mathtt{p}\times\mathtt{p}}[\xi]$ constitute a solution of the TIP. Choose $F \in \mathbb{R}^{\mathtt{p}\times\mathtt{p}}[\xi]$ so that $F \begin{bmatrix} U & -Y \end{bmatrix}$ also models the trajectories in $\mathcal{V}_i^{\perp} \exp_{-\bar{\lambda}_i t}$, $1 \leq i \leq N$ besides the trajectories $v_i \exp_{\lambda_i t}$, $1 \leq i \leq N$. Observe that F can be chosen to be Hurwitz, since $\begin{bmatrix} U & -Y \end{bmatrix}$ already models $\mathcal{V}_i \exp_{\lambda_i t}$, $1 \leq i \leq N$. Conclude from the fact that $F \begin{bmatrix} U & -Y \end{bmatrix}$ models $\mathcal{D}$ and from the fact that a representation of the MPUM for $\mathcal{D}$ is given, that there exist $\Pi, \Phi \in \mathbb{R}^{\bullet\times\bullet}[\xi]$ such that (24) holds. We now prove the claim regarding the contractivity of $\Phi^{-1}\Pi$ and the Hurwitzianity of Φ.

Contractivity follows easily from the J-unitariness of R and from the contractivity of $Y^{-1}U$, since

$$\begin{aligned} &F^T(-i\omega)(U^T(-i\omega)U(i\omega) - Y^T(-i\omega)Y(i\omega))F(i\omega) \\ &= (\Pi^T(-i\omega)\Pi(i\omega) - \Phi^T(-i\omega)\Phi(i\omega))\Pi_{i=1}^N(-i\omega + \bar{\lambda}_i)(i\omega - \lambda_i) \end{aligned}$$

is negative definite for all $\omega \in \mathbb{R}$ if and only if $\Pi^T(-i\omega)\Pi(i\omega) - \Phi^T(-i\omega)\Phi(i\omega) < 0$. The claim on the Hurwitzianity of Φ follows from

$$-\frac{1}{\det(F)}\frac{1}{\det(Y)} = \frac{1}{\det(\Phi)}\frac{1}{\det(P)}\frac{1}{\det(\Phi^{-1}\Pi N^* P^{-1} + I_{\mathtt{p}})}$$

and consequently

$$\begin{aligned} \underbrace{\mathrm{wno}(\frac{1}{\det(F)})}_{=0} + \underbrace{\mathrm{wno}(\frac{1}{\det(Y)})}_{=-\nu_-} = \mathrm{wno}(\frac{1}{\det(\Phi)}) + \underbrace{\mathrm{wno}(\frac{1}{\det(P)})}_{=-\nu_-} \\ + \underbrace{\mathrm{wno}(\frac{1}{\det(\Phi^{-1}\Pi N^* P^{-1} + I_{\mathtt{p}})}}_{=0} \end{aligned}$$

The proof of the Theorem is thus complete. ■

The following conclusion can be drawn easily from the results of Theorem 1 and Theorem 9.

Corollary 10 *The smallest k for which the Takagi interpolation problem has a solution is the number of negative eigenvalues of the Pick matrix $T_{\{(\lambda_i, V_i)\}_{1\leq i\leq N}}$.*

Remark 11 We now discuss briefly the relationship of the results presented in Theorem 1 and Theorem 9 with well-known results in the field of interpolation. The time-series modeling point of view on interpolation and the notion of "subspace interpolation problem" put forward in [18] and in the present paper, are germane to the approach to interpolation problems illustrated in [5]; indeed, the time-series $v \exp_{\lambda_i t}$ and $v \exp_{-\overline{\lambda_i} t}$ constituting the dualized data are associated in a natural way with the graph of a solution to the interpolation problem as considered by Ball and Helton. Another connection to more classical approaches to interpolation theory has already been mentioned in Remark 4.2 and in Remark 5.2 of [18]; namely, the polynomial matrix appearing in equation (6) is intimately related with the rational J-contractive matrix Θ of the approach of Ball, Gohberg, and Rodman illustrated in the book [4] (see Theorem 18.1 therein) from which all solutions to the Nevanlinna- and Takagi problems are characterized. Finally, the recursive computation of solutions to the Nevanlinna- and Takagi interpolation problems, which we considered in the context of the theory of exact identification of [24, 25], has been studied in [6, 8, 10, 19] in the state-space and operator-theoretic context.

4 Conclusions and further work

In this paper we have given a new look at the interpolation problem of Takagi from the perspective of time-series modeling, an approach introduced in [3] and further refined in [1, 11, 18]. That framework has been extended and refined in this paper by new results of independent interest about the relation between the roots of the determinant of the denominator of an unfalsified model, and the signature of the Pick matrix of the data (see the proof of statement ($2c$) of Theorem 1).

The work presented in this paper and the techniques used in the proof of Theorem 1 are being extended and applied in the following directions.

State-space formulas The state-space case is a special case of the results presented in this paper; however, deriving explicit state-space formulas is a task deserving interest in its own right. In this respect, see also [8, 10, 19].

Dissipativity theory The relation between storage functions and Pick matrices has been examined in detail in [22, 23]. It has been shown in Th. 6.4 of [27] that the positivity of the real symmetric matrix K inducing a storage function is related to the location of the roots of the "denominator" of a canonical symmetric factorization of a quadratic differential form; this fact has important consequences in H_∞-control in a behavioral setting (see [28]). We plan to use the results illustrated in this paper in order to generalize this results to the case of an indefinite K.

Stabilization with dissipative controllers We are in the process of using the results illustrated in this paper in order to attack the problem of *stabilization with dissipative controllers*, formulated as follows. Let J be as in (1), and let $\mathfrak{B}$ be a controllable behavior. Let $\mathfrak{B}_{\text{des}}$ be a stable, autonomous subspace of $\mathfrak{B}$ representing the desired behavior after interconnection with some controller having behavior $\mathfrak{C}$. Does there exist a J-dissipative controller such that $\mathfrak{C} \cap \mathfrak{B} = \mathfrak{B}_{\text{des}}$? Assuming such a controller exists, how many unstable poles does the transfer function associated with the controllable part of $\mathfrak{C}$ have? It is expected that the particular kernel representation obtained through Algorithm T can provide significant insight in the solution of this problem. See also [7, 9, 12, 13, 21] for the use of interpolation methods in controller design.

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