Optimal Controller Realisations with the Smallest Dynamic Range

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Outline

- Motivation for optimal finite word length controller design with the smallest dynamic range
- The proposed two-stage approach for solving this multi-objective optimal FWL controller design
- Numerical experimental investigation of the proposed technique
Motivation

- FWL effect may degrade designed closed-loop performance, and this problem is particularly serious in **fixed-point** implementation.

- Care must be exercised in implementing or **realising** designed control law so as to minimise FWL effect.

- Most existing techniques are based on maximising some FWL **closed-loop stability** measures ⇒ far from “optimal”:
  
  ☆ In fixed-point implementation, total available bits have to accommodate **dynamic range** or integer part, and remaining bits left are then used to implement **precision** or fractional part.

  ☆ Optimising a FWL closed-loop stability measure, while minimising **fractional bit length**, may not guarantee a small dynamic range.
Motivation (continue)

- Normalising with $l_2$-norm will minimise integer bit length but may not guarantee adequate FWL closed-loop stability robustness.

- True optimal FWL controller design is computationally challenging multi-objective optimisation.

  - Simultaneously maximise a FWL closed-loop stability measure and minimise a dynamic range measure.

- Our previous work: optimising combined FWL closed-loop stability measure and dynamic-range measure.

Proposed Approach

- **True** optimal controller realisation: Simultaneously achieves maximum robustness of FWL closed-loop stability and minimum dynamic range.

  We propose a computationally attractive two-step approach to solve this challenging multi-objective optimisation.

- **Step one**: Maximise FWL closed-loop stability measure
  
  - Assuming sufficient integer bit length to avoid overflow, resulting realisation achieves **maximum** robustness of FWL closed-loop stability.
  - We know great deal how to do this.
  - Solution is an infinite set of controller realisations.

- **Step two**: Search solution set of optimal FWL closed-loop stability to yield a realisation that has a **minimum** integer bit length.
System Model

Discrete-time closed-loop system with generalised operator $\rho$

$$\rho = \begin{cases} 
z, \text{ shift} \\
\delta = \frac{z-1}{h}, \text{ delta} 
\end{cases}$$
System Model (continue)

- State-space description of plant $\hat{P}$

\[
\begin{align*}
\rho x(k) &= A_\rho x(k) + B_\rho e(k) \\
y(k) &= C_\rho x(k)
\end{align*}
\]

$A_\rho \in \mathcal{R}^{n \times n}$, $B_\rho \in \mathcal{R}^{n \times p}$ and $C_\rho \in \mathcal{R}^{q \times n}$

- State-space description of controller $\hat{C}$

\[
\begin{align*}
\rho v(k) &= F_\rho v(k) + G_\rho y(k) + H_\rho e(k) \\
u(k) &= J_\rho v(k) + M_\rho y(k)
\end{align*}
\]

$F_\rho \in \mathcal{R}^{m \times m}$, $G_\rho \in \mathcal{R}^{m \times q}$, $J_\rho \in \mathcal{R}^{p \times m}$, $M_\rho \in \mathcal{R}^{p \times q}$ and $H_\rho \in \mathcal{R}^{m \times p}$

- $\hat{C}$ includes output feedback, full-order observer-based, and reduced-order observer-based controllers
Controller Realisation Set

- Given initial realisation \((F_\rho, G_\rho, J_\rho, M_\rho, H_\rho)\) by standard controller design, all realisations of \(\hat{C}\) form realisation set

\[
S_\rho = \left\{ (F_\rho, G_\rho, J_\rho, M_\rho, H_\rho) : F_\rho = T_\rho^{-1} F_\rho 0 T_\rho, G_\rho = T_\rho^{-1} G_\rho 0, 
J_\rho = J_\rho 0 T_\rho, M_\rho = M_\rho 0, H_\rho = T_\rho^{-1} H_\rho 0 \right\}
\]

\(T_\rho \in \mathcal{R}^{m \times m}\) is any real-valued nonsingular transformation matrix.

- We can also write a controller realisation in vector form

\[
w_\rho = \left[ \text{Vec}^T(F_\rho) \, \text{Vec}^T(G_\rho) \, \text{Vec}(J_\rho) \, \text{Vec}^T(M_\rho) \, \text{Vec}^T(H_\rho) \right]^T
\]

- Transition matrix of closed-loop system

\[
\overline{A}(w_\rho) = \begin{bmatrix}
A_\rho + B_\rho M_\rho C_\rho & B_\rho J_\rho \\
G_\rho C_\rho + H_\rho M_\rho C_\rho & F_\rho + H_\rho J_\rho
\end{bmatrix} = \begin{bmatrix}
I & 0 \\
0 & T_\rho^{-1}
\end{bmatrix} \overline{A}(w_\rho 0) \begin{bmatrix}
I & 0 \\
0 & T_\rho
\end{bmatrix}
\]

whose eigenvalues are \(\lambda_i = \lambda_i(\overline{A}(w_\rho)), \; \forall i \in \{1, \cdots, m+n\}\)
FWL Robustness

- **Fixed-point** format of bit length $b = 1 + b_g + b_f$: one bit for **sign**, $b_g$ bits for **integer part**, and $b_f$ bits for **fractional part**

- Assume $b_g$ is sufficient so no **overflow** occurs, i.e.

\[ \| w_\rho \|_M \leq 2^{b_g} \]

where $\| U \|_M$ denotes maximum absolute element of matrix $U$

- In FWL implementation, $w_\rho$ is perturbed into $w_\rho + \Delta$ due to **finite** $b_f$
  - With **perturbation** $\Delta$, $\lambda_i(\overline{A}(w_\rho))$ moves to $\lambda_i(\overline{A}(w_\rho + \Delta))$
  - Will $\overline{A}(w_\rho + \Delta)$ remain stable?

- Under condition of no overflow, **closed-loop stability** depends only on $\Delta$, i.e. **precision** of fractional part representation

- We want a controller **realisation** $w_\rho$ whose closed-loop stability has maximum **robustness** to controller **perturbation** $\Delta$
Optimal Realisation

- Optimal FWL **realisation** problem

\[ \nu = \min_{w_{\rho} \in S_{\rho}} f(w_{\rho}) \]

- with Frobenius-norm \( \| \cdot \|_F \), FWL closed-loop **stability** measure

\[ f(w_{\rho}) = \max_{i \in \{1, \ldots, m+n\}} \left\| \frac{\partial \lambda_i}{\partial w_{\rho}} \right\|_F \]

- **Stability margin** of \( \lambda_i(\mathbf{A}(w_{\rho})) \)

\[ SM(\lambda_i(\mathbf{A}(w_{\rho}))) = \begin{cases} 
1 - |\lambda_i(\mathbf{A}(w_z))|, & \text{if } \rho = z \\
\frac{1}{h} - |\lambda_i(\mathbf{A}(w_{\delta})) + \frac{1}{h}|, & \text{if } \rho = \delta 
\end{cases} \]

- **Note** this says nothing about \( \|w_{\rho}\|_M \) or **dynamic range** of \( w_{\rho} \)
Optimal Realisation Solution

- An optimal realisation solution $w_{\rho_{\text{opt}}}$, i.e. $(F_{\rho_{\text{opt}}}, G_{\rho_{\text{opt}}}, J_{\rho_{\text{opt}}}, M_{\rho_{\text{opt}}}, H_{\rho_{\text{opt}}})$, can readily be obtained using algorithm of


- This actually defines optimal solution set $w_{\rho_{\text{opt}}}(V)$, where $V \in \mathcal{R}^{m \times m}$ is an arbitrary orthogonal matrix, i.e.

  $$ S_{\rho_{\text{opt}}} = \{ (F_\rho, G_\rho, J_\rho, M_\rho, H_\rho) : F_\rho = V^{-1}F_{\rho_{\text{opt}}}V, G_\rho = V^{-1}G_{\rho_{\text{opt}}}, $$

  $$ J_\rho = J_{\rho_{\text{opt}}}V, M_\rho = M_{\rho_{\text{opt}}}, H_\rho = V^{-1}H_{\rho_{\text{opt}}}, V \in \mathcal{R}^{m \times m}, V^T V = I \} $$

- Any $w_{\rho_{\text{opt}}}(V)$ in $S_{\rho_{\text{opt}}}$ is a solution of optimal FWL realisation problem, but different $w_{\rho_{\text{opt}}}(V)$ have different dynamic range $\|w_{\rho_{\text{opt}}}(V)\|_M$
Minimising Dynamic Range

- Search $S_{\rho_{\text{opt}}}$ for a realisation with smallest dynamic range

$$\mu = \min_{\mathbf{V} \in \mathbb{R}^{m \times m}} d(\mathbf{w}_{\rho_{\text{opt}}} (\mathbf{V}))$$

where $d(\mathbf{w}_\rho) = \|\mathbf{w}_\rho\|_M$ is dynamic range of $\mathbf{w}_\rho$

- Using Givens rotation with $r = \frac{m(m-1)}{2}$ and $\theta_i \in [-\pi, \pi), 1 \leq i \leq r$

$$d_1(\theta_1, \cdots, \theta_r) = d(\mathbf{w}_{\rho_{\text{opt}}}(\mathbf{V}))$$

- Using optimisation algorithm relying on function value only to solve

$$\mu = \min_{\theta_1, \cdots, \theta_r \in [-\pi, \pi)} d_1(\theta_1, \cdots, \theta_r)$$

With optimal solution $\theta_{1_{\text{opt}}}, \cdots, \theta_{r_{\text{opt}}}$ \Rightarrow $\mathbf{V}_{\text{opt}}$ \Rightarrow $\mathbf{w}_{\rho_{\text{opt}1}} = \mathbf{w}_{\rho_{\text{opt}}}(\mathbf{V}_{\text{opt}})$, optimal realisation with smallest dynamic range
Numerical Example


- **Plant** $\hat{P}$ has order $n = 4$, **controller** $\hat{C}$ is output feedback one with order $m = 4$

- **Initial** controller realisation provided is denoted by $w_{\rho 0}$

- **Optimal** FWL controller realisation obtained by optimising FWL closed-loop stability measure alone is denoted by $w_{\rho_{opt}}$

- Proposed **optimal** FWL controller realisation with **smallest** dynamic range is denoted by $w_{\rho_{opt1}}$
Results

- Comparison of three realisations using $z$ operator

<table>
<thead>
<tr>
<th>Realisation</th>
<th>$f(w_z)$</th>
<th>$d(w_z)$</th>
<th>$b_{f}^{\text{min}}$</th>
<th>$b_{g}^{\text{min}}$</th>
<th>$b_{\text{min}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$w_{z0}$</td>
<td>$3.9697e + 6$</td>
<td>$1.0959e + 6$</td>
<td>20</td>
<td>21</td>
<td>42</td>
</tr>
<tr>
<td>$w_{z\text{opt}}$</td>
<td>$2.4246e + 3$</td>
<td>$1.9673e + 2$</td>
<td>8</td>
<td>8</td>
<td>17</td>
</tr>
<tr>
<td>$w_{z\text{opt1}}$</td>
<td>$2.4246e + 3$</td>
<td>$1.1799e + 2$</td>
<td>8</td>
<td>7</td>
<td>16</td>
</tr>
</tbody>
</table>

- Comparison of three realisations using $\delta$ operator with $h = 2^{-14}$

<table>
<thead>
<tr>
<th>Realisation</th>
<th>$f(w_\delta)$</th>
<th>$d(w_\delta)$</th>
<th>$b_{f}^{\text{min}}$</th>
<th>$b_{g}^{\text{min}}$</th>
<th>$b_{\text{min}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$w_{\delta0}$</td>
<td>$2.7712e + 5$</td>
<td>$1.7956e + 10$</td>
<td>15</td>
<td>35</td>
<td>51</td>
</tr>
<tr>
<td>$w_{\delta\text{opt}}$</td>
<td>$3.3740e - 1$</td>
<td>$5.1236e + 4$</td>
<td>$-4$</td>
<td>16</td>
<td>13</td>
</tr>
<tr>
<td>$w_{\delta\text{opt1}}$</td>
<td>$3.3740e - 1$</td>
<td>$2.5810e + 4$</td>
<td>$-4$</td>
<td>15</td>
<td>12</td>
</tr>
</tbody>
</table>

"−4 fractional bits": entire fractional part and first lowest 4-bit integer part are omitted
There exist optimal values of $h$ for the $\delta$ operator $\Rightarrow$ resulting optimal controller realisations $w_{\delta_{\text{opt}1}}$ achieve maximum robustness to FWL errors

<table>
<thead>
<tr>
<th>$h$</th>
<th>$f(w_{\delta_{\text{opt}1}})$</th>
<th>$d(w_{\delta_{\text{opt}1}})$</th>
<th>$b^\text{min}_f$</th>
<th>$b^\text{min}_g$</th>
<th>$b^\text{min}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2^{-7}$</td>
<td>1.9248e + 1</td>
<td>1.3349e + 3</td>
<td>1</td>
<td>11</td>
<td>13</td>
</tr>
<tr>
<td>$2^{-8}$</td>
<td>9.7758e + 0</td>
<td>1.8878e + 3</td>
<td>0</td>
<td>11</td>
<td>12</td>
</tr>
<tr>
<td>$2^{-9}$</td>
<td>5.0361e + 0</td>
<td>2.6698e + 3</td>
<td>-1</td>
<td>12</td>
<td>12</td>
</tr>
<tr>
<td>$2^{-10}$</td>
<td>2.6601e + 0</td>
<td>3.7756e + 3</td>
<td>-2</td>
<td>12</td>
<td>11</td>
</tr>
<tr>
<td>$2^{-11}$</td>
<td>1.4618e + 0</td>
<td>5.3396e + 3</td>
<td>-3</td>
<td>13</td>
<td>11</td>
</tr>
<tr>
<td>$2^{-12}$</td>
<td>8.4740e - 1</td>
<td>7.6314e + 3</td>
<td>-3</td>
<td>13</td>
<td>11</td>
</tr>
<tr>
<td>$2^{-13}$</td>
<td>5.2102e - 1</td>
<td>1.2905e + 4</td>
<td>-3</td>
<td>14</td>
<td>12</td>
</tr>
<tr>
<td>$2^{-14}$</td>
<td>3.3740e - 1</td>
<td>2.5810e + 4</td>
<td>-4</td>
<td>15</td>
<td>12</td>
</tr>
<tr>
<td>$2^{-15}$</td>
<td>2.2681e - 1</td>
<td>5.1621e + 4</td>
<td>-5</td>
<td>16</td>
<td>12</td>
</tr>
<tr>
<td>$2^{-16}$</td>
<td>1.5606e - 1</td>
<td>1.0324e + 5</td>
<td>-6</td>
<td>17</td>
<td>12</td>
</tr>
<tr>
<td>$2^{-17}$</td>
<td>1.0879e - 1</td>
<td>2.0648e + 5</td>
<td>-6</td>
<td>18</td>
<td>13</td>
</tr>
<tr>
<td>$2^{-18}$</td>
<td>7.6367e - 2</td>
<td>4.1297e + 5</td>
<td>-6</td>
<td>19</td>
<td>14</td>
</tr>
<tr>
<td>$2^{-19}$</td>
<td>5.3801e - 2</td>
<td>8.2593e + 5</td>
<td>-7</td>
<td>20</td>
<td>14</td>
</tr>
<tr>
<td>$2^{-20}$</td>
<td>3.7973e - 2</td>
<td>1.6519e + 6</td>
<td>-7</td>
<td>21</td>
<td>15</td>
</tr>
<tr>
<td>$2^{-21}$</td>
<td>2.6826e - 2</td>
<td>3.3037e + 6</td>
<td>-8</td>
<td>22</td>
<td>15</td>
</tr>
<tr>
<td>$2^{-22}$</td>
<td>1.8960e - 2</td>
<td>6.6075e + 6</td>
<td>-8</td>
<td>23</td>
<td>16</td>
</tr>
<tr>
<td>$2^{-23}$</td>
<td>1.3404e - 2</td>
<td>1.3215e + 7</td>
<td>-9</td>
<td>24</td>
<td>16</td>
</tr>
<tr>
<td>$2^{-24}$</td>
<td>9.4767e - 3</td>
<td>2.6430e + 7</td>
<td>-9</td>
<td>25</td>
<td>17</td>
</tr>
</tbody>
</table>
Conclusions

- A two-step approach to design optimal fixed-point digital controller realisations, which is multi-objective optimisation problem.
  
  ★ **Step one**: find an optimal realisation by minimising FWL closed-loop stability measure.
  
  ★ **Step two**: modifying this realisation to produce optimal realisation with smallest dynamic range.

- Approach developed within unified framework that includes both shift and delta operator parameterisations of generic controller structure.

- With appropriate $h$, optimal $\delta$-operator realisation has much better FWL closed-loop stability characteristics than optimal $z$-operator one.
THANK YOU.

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