Authors' response to the referees' reports

Paper title: Linear dynamic filtering with noisy input and output
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July 14, 2010

We thank the referees for their efforts to improve the manuscript.

In this document, we quote in **bold face** statements from the reports. Our replies follow in ordinary print. If not stated otherwise, the references are given to the old (not revised) version of the manuscript.

Associate editors' report

We revised the manuscript according to the referees' recommendations. Detailed answers and explanations to their questions and concerns are given in the following sections of this document. In view of the page limitation for the technical communique format, we abbreviated significantly the manuscript. In particular, we skipped the proof of Theorem 3 (the recursive smoothing solution), which is more technical, and changed the assumption that the noise covariances $V_{\tilde{u}}, V_{\tilde{y}}$ are time varying with the assumption that they are time invariant. This makes the notation more compact without sacrificing the strength of the results.

1) We respond to the questions of Reviewers #7 and #8 in the corresponding sections. An extensive explanation, related to the contribution and significance questions of Reviewers #7 and #8, is given in the answer to the first question of Review #6.
2) The suggested references describe real-life applications of the problems that we study. We incorporated these references in the revised version of the manuscript.
3) Our answers are given in the corresponding sections.

Review #6 (ID 9227)

I suggest that the authors better clarify the relationships between their state-space approach, and the transfer function approach adopted in the literature by Guidorzi and coworkers.

The problems treated in [2, 1] are equivalent to the noisy I/O smoothing and filtering problems of, respectively, Definitions 1 and 2, except for the following two points:

1. in [2, 1], the SISO case is considered, while we deal with the general MIMO case,

2. in [2, 1] the initial condition information is not specified, while we do so.

It is well known that in finite-time optimal smoothing and filtering problems the proper treatment of the initial conditions is crucial. In fact, without specifying the initial conditions assumption, it is not clear in what sense the solution is optimal. We guess that the algorithms of [2, 1] solve the problems under the implicit assumption that the initial conditions are unknown (in the notation introduced in Note 2, with $P_0 = \infty \cdot I$). The following statement from [1, Sec. 7], however, is rather confusing in this respect:

> All algorithms [for the noisy I/O filtering problem] are equivalent and lead thus to the same filtered sequences; the state-space algorithm, however, can give different estimates in the first steps depending on its initial state as shown, for example, in Fig. 6.

The uncertainty about the initial conditions makes hard to draw precise conclusion about the relation with our results. We guess, modulo initial conditions and restricting to the SISO case, the solution of [2, 1] is equivalent to the one we have.

Next we comment on the approach of Guidorzi and coworkers in comparison with ours. The algorithms of [2, 1] are derived from a transfer function point of view. The so called state-space algorithm does not make an exception because it uses a state space representation of the residual in a difference equation representation of the system and not of the original system, see [2, Sec. 5]. From our point of view, the result of [2, 1] is a derivation of the (modified) Kalman filter in a transfer function setting. This is well known to be difficult and cumbersome, especially in the MIMO case. The final result of the derivation, however, do give a solution of the original noisy I/O filtering problem (for an appropriate assumption about the initial conditions). If one believes that our derivation and the derivation in [2, 1] are correct, the corresponding solutions should be equivalent.

The usefulness of Section III is questionable. In particular, I do not understand why two different solutions are given in (7) and in lemma 1, respectively. Which is the relationship between them? Please explain, or else drop one of them.

We agree that the explicit solution of problem (9) is not used in the derivation of the main result. It is given for completeness and can be skipped. In the revised version of the manuscript, Section 3 is significantly abbreviated.

I do not see the analogy with (8).

Both (8) and (18) are derived by substituting the measurement error model equations (2) in, respectively, the equation after (7) and in (1) (*i.e.*, the state space representation of the system).

The resulting system (18)-(19) is not "in the form (3)", but it is exactly (3).

Except for the fact that u becomes u_d and y becomes y_d. The signals u_d and y_d are realizations of stochastic processes while u and y needed not be stochastic.

By the way, once it has been established that (1)-(2) can be rewritten as (18)-(19), the noisy i/o filtering problem boils down to standard Kalman filtering and there is not much more to say.

Our purpose is to solve the problem of Definition 2 and it is not obvious for us why the Kalman filter for (18)-(19) is the solution. This requires proof although we agree that it is rather obvious in the course of the presentation. See also our replay to the questions of Reviewer #8.

Review #7 (ID 9228)

1. The Introduction does not clearly state the purpose of the paper.

In the Introduction, we define the noisy I/O model (1–2) and state the purpose of the paper (quoting from page 2 of the manuscript):

> The considered problem is to find the least squares estimate of the state x from the measured input/output data (u_d, y_d).

We do find this (informal) statement of the problem clear. Please, clarify concretely which point of the presentation is unclear.

2. The significance of the paper is not clearly explained.

The result of the paper has the theoretical significance of solving the noisy I/O smoothing and filtering problems, which we believe are useful estimation problems. If by "significance" the reviewer means value for real-life applications, then we agree that such a discussion is missing. Following the suggestion of Reviewer #9 we added references that present applications for fault detection and data reconciliation.

3. The paper is not clearly written and well organized.

We did not really understand this. We feel that the presentation in the paper is quite logical.

4. The main contribution of the paper seems to derive a recursive smoothing algorithm for the considered problem.

We consider as main contribution of the paper the established equivalence between the optimal noisy I/O filter and an appropriately modified Kalman filter, which shows that the noisy input/output filtering problem is not fundamentally different from the classical Kalman filtering problem.

5.a) In my opinion, the materials given in Section 3 and 6 are unnecessary and could be dropped due to their limited usefulness.

Following the recommendation of Reviewer #6, Section 3 is abbreviated. Section 6 shows a small numerical example that verifies the main result of the paper. Since it does not take much space and since many readers would like to see numerical illustration of the result, we would like to keep it.

5.b) The paper presents a modified Kalman filter to solve the addressed noisy I/O filtering problem. However, this result is nothing new and can be easily inferred.

We agree that the derivation of the result is straightforward but we do not agree that it is not new. As we show in Note 5 the solution of [2] is a direct consequence of this result. Moreover, the solution using the modified Kalman filter is applicable for MIMO systems while the result of [2] is restricted to the SISO case.

Reference [1] contains equivalent to our result. The authors of this paper, however, cite our work (at this time an internal report), which proves that we first came to the result.

Is it possible to derive the result by using the approach given in Section 4?

Yes, this would be the complete and direct solution of the problem. We followed this approach in the continuous-time case, see [3]. The discrete-time case, however, is technically more complicated, so we decided to use the indirect approach of presenting the modified Kalman filter and proving that it is indeed the desired solution.

By the way, what is the noisy I/O filter in Section 5?

It is the modified Kalman filter, see Theorem 2.

5.c) We all know that the smoothing problem can also be solved by using the Kalman filter.

We guess, the reviewer is right that the noisy I/O smoothing problem can be solved by the modified Kalman filter. Formally this requires a proof, however, because the well known relation between smoothing and filter is established for the classical case when there is no measurement noise on the input.

What are the relationship and the difference between them?

Although they are derived in a different way they solve closely related problems. The difference is clearly in the fact that the smoother uses all the data while the filter uses only past data.

5.d) In filtering point of view, the contribution of the paper is minor.

We disagree. See our replay to item 5.b.

In fact, the paper did not clearly state the main effort and the significance of this work.

Quoting from the introduction:

> The solution is the Kalman filter for the system
>
> $$\begin{aligned} x(t+1) &= Ax(t) + Bu(t) + w(t), \\ y(t) &= Cx(t) + Du(t) + v(t), \end{aligned}$$
>
> where the *process noise* w and the *measurement noise* v are white with joint covariance matrix
>
> $$\begin{bmatrix} Q(t) & S(t) \\ S^\top(t) & R(t) \end{bmatrix} = \begin{bmatrix} -B & 0 \\ -D & I \end{bmatrix} \begin{bmatrix} V_{\tilde{u}}(t) & \\ & V_{\tilde{y}}(t) \end{bmatrix} \begin{bmatrix} -B & 0 \\ -D & I \end{bmatrix}^\top .$$

This concisely states the main result of the paper. Concerning the significance of the work, see our earlier replay.

A possible answer to this is to clearly state the main difference between this work and that given in (Diversi et al., 2003b).

For a comparison with the transfer function approach of Guidorzi and coworkers, see our replay to the first question of Reviewer #6. The work (Diversi et al., 2003b) of the same group is rather different in spirit. It takes over the state space approach that we use and presents equivalent result to ours, see also our comment to item 5.b.

5.e) There are some typographic errors, e.g. $u(t)$ **in (14) and (16), the range space in Lemma 1, and etc.**

We corrected the above mentioned errors.

Review #8 (ID 9229)

The paper discusses the contribution and the relation to previous work. However, this part needs to be more clear.

In the answer to the first question of Reviewer #6, we elaborated on the relation to the work of Guidorzi and coworkers. See also our answers to the questions of Reviewer #7, items 2, 4, 5.b, and 5.d.

Are the contribution twofold: A) Kalman filter solution and B) a proof that this is equivalent to the solution to the "filtering/smoothing problem"?

Yes, A), the derivation of the modified Kalman filter, is straightforward. The fact that it is the desired solution of the noisy I/O filtering problem, however, still needs a proof and this is part B). Our proof is algebraic, using the state space representation of the system (8) and its relation to the (modified) Kalman filter from one side and to the noisy I/O filter on the other side. Thus by (8), we link the modified Kalman filter with the noisy I/O filter and show their equivalence.

A) seems a minor contribution as the solution comes from the modified Kalman filter for (18)-(19) plus the relations below

$$\hat{x}(t+1|t) = E(x(t+1)|y(t), y(t-1), ..., y(0), u(t), u(t-1), ..., u(0)) \tag{1}$$

$$\hat{x}(t+1|t) = A\hat{x}(t|t) + B\hat{u}(t|t) \tag{2}$$

$$\hat{y}(t|t) = C\hat{x}(t|t) + D\hat{u}(t|t) \tag{3}$$

We do not understand the argument of the reviewer. As we noted above, A) is indeed easy but it is not a complete solution to the problem. Once we prove (and this is B)) that $\hat{x}(t+1|t)$ is obtained from the modified Kalman filter, equations (2) and (3) above show that $\hat{u}(t|t)$ and $\hat{y}(t|t)$ can be derived from $\hat{x}(t+1|t)$ and $\hat{x}(t|t)$, *i.e.*, from the modified Kalman filter, see Note 5.

If A) and B) both are considered as part of the contribution what are the individual importance then?

The noisy I/O filtering problem can be solved directly by, *e.g.*, dynamic programming or completion of squares approach, see [3]. We found, however, easier to following the indirect approach A) + B):

A) displays a (well motivated) candidate solution and

B) proves that it is indeed the optimal solution.

Review #9 (ID 9654)

Some other works have been published on the subject that the authors do not list in theirs references.

We examined the suggested references [4, 5]. They are indeed related to the topic that we study but are more application oriented. The most relevant result that we found is in [4]. It is basically the solution (7) of the smoothing problem. We added the references in the introduction commenting that they present applications of the I/O noisy estimation problem for fault detection and data reconciliation.

The covariance matrices are supposed to be positive definite. The case semi definite positive is not taken into account. Why?

The solution is simpler in the case when the covariance matrices are nonsingular. Technically the reason is that the inverses of these matrices appear in the solution. The positive semidefinite case can be treated as well but (we guess) it will significantly complicate the derivations.

Just before eq.3, the expression "the solution is the Kalman filter ..." is inelegant.

Corrected to "the optimal filter is the Kalman filter ..."

After eq.3 are defined the joint covariance matrices. The notations V_u and V_y have not been defined. I suppose that they refer to $V_{\tilde{u}}$ and $V_{\tilde{y}}$.

Yes, this is a typo, which we corrected.

The notation "blk diag" is not standard.

Now we clarify it as follows: "...the block diagonal covariance matrix $V_{\tilde{w}} = \operatorname{blk\,diag}(V_{\tilde{u}}, V_{\tilde{y}})$."

What is the meaning of $V^{-1/2}$ when V is not necessarily a diagonal matrix?

The notation $V^{-1/2} = (V^{1/2})^{-1}$ refers to the inverse of the square root of $V > 0$. In fact, all we need for $V^{1/2}$ is that $V = V^{1/2\top} V^{1/2}$, so that it could be in particular the upper triangular Cholesky factor of V. In the revised version of the manuscript, we changed the notation to $\sqrt{V}$ which is a standard notation for the matrix square root of V. (Note that in our problem, we do not need to compute the matrix square root.)

In the rest of the paper, I don't see what is the use of the result given in eq.7

It is used for the verification of the results in Section 6, because it is the most simple and explicit solution of the noisy I/O smoothing problem. In addition it shows that the noisy I/O smoothing problem is a weighted least squares problem which is already a serious indication that in the filtering case the solution can be expected to be in the form of a Kalman filter. In the revised version of the manuscript, we give only the weighted least squares problem and skip its well known solution (7).

The notations used in the beginning of the page are not very clear. For example in eq.8 and eq.9, u and y are changed into Δu and Δy. Then comparing δ and $\hat{\delta}$ shows that the notation is not good. Comparing eq.8 and eq.9, I don't understand the presence of $\hat{x}$. After, there is a mistake in the definition of $\hat{\delta}$. At last, the solution of (9) is given with other notations. In the demonstration of lemma 1, there are simultaneously δ and $\hat{d}$ in the Lagrangian.

Most of the comments above are no longer relevant because Lemma 1 is dropped from the presentation, see our answer to Reviewer #6. In the equations (8) and (9), $\Delta u, \Delta y$ are estimates of the noises $\tilde{u}, \tilde{y}$. They are applied on the measured data $u_{\rm d}, y_{\rm d}$ in order to make the corrected I/O signals $u_{\rm d} + \Delta u, y_{\rm d} + \Delta y$ consistent with the system equations. Since x is unmeasured (latent) variable, it should be estimated and its estimate is denoted by $\hat{x}$.

In eq.13, the covariance matrices have been omitted. Why?

This is a typo. We corrected it.

In eq. 14 and eq16, the measured input and output are missing.

The u and y in this equations should be the measured input $u_{\rm d}$ and output $y_{\rm d}$, respectively. The correction is done.

Nothing about the convergence of the eq. 15 is said.

The difference Riccati equation (15) is a standard observer equation. Under the assumption that the pair (C, A) is observable, (15) converges to a steady state solution. (We have as a global assumption that $V_{\tilde{y}} > 0$.) A note is added, see Note 3, in the revised version of the manuscript.

Two particular cases could be to consider and discussed : $V_{\tilde{u}}$ null and after $V_{\tilde{u}}$ infinity. What is your opinion?

These are extreme cases that has to be treated separately. The derivations are done under the assumption that $V_{\tilde{u}}$ and $V_{\tilde{u}}$ are nonsingular and finite. (The formulas do not "work" in these cases.) We agree that considering the special cases ($V_{\tilde{u}} = 0$ by the way is the classical Kalman filter case) is a useful complement of our work but it would require more space while even in the present form the paper is too long for a technical communique.

Just before eq.22, it is said that there is a link between the state estimate, the state prediction and the input estimated are linked. It would be useful to mention the time dependence for all these variables. The same remark yields for the sentence after eq.22.

The link is equation (22). The state estimate and the state prediction dynamics are governed by the modified Kalman filter. The input estimate follows from (22) and is given in (23). It does not involve an additional dynamics. In other words, the order of the noisy I/O filter with optimal estimation of the input and the output is the same as the order of the modified Kalman filter (which is equal to the order n of the system).

In the example, the coefficients in A and B are given with less that two significant numbers and in matrices C and D for or five are used. Is there any reason?

The reason is that the coefficients of A are given exactly and no more digits are needed. The other coefficients are either exact or rounded to the fifth digit.

What is the conclusion of the results given in table 1?

The error of estimation is progressively larger down the rows of the table. This is to be expected (as an average behavior of the errors) because the smoother uses more information than the filter does, and the time invariant filter is suboptimal. More important, however, is the observation that the errors obtained by the time-varying modified Kalman filter and the optimal filter (computed explicitly from the definition by solving a sequence of smoothing problems by weighted least squares) are equal. This is the desired numerical verification of the theoretical result and the actual purpose of the simulation example.

References

[1] R. Diversi, R. Guidorzi, and U. Soverini. Algorithms for optimal errors-in-variables filtering. *Control Lett.*, 48:1–13, 2003.

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[4] G. Mourot, D. Maquin, and J. Ragot. Simultaneous state and parameter estimation: Application to data validation for urban sewer network control. In *14th IFAC World Congress*, pages 165–170, Beijing, China, July 5–9 1999.

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